The Cramér-Lundberg and the dual risk models: Ruin, dividend problems and duality features

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Abstract

In the present paper we study some existing duality features between two very known models in Risk Theory. The *classical* Cramér–Lundberg risk model with application to insurance, and the dual risk model with (some) financial application. For simplicity the former will be referred as the *primal* model. The former has been of extensive treatment in the literature, it assumes that a given surplus process has constant deterministic gains (premiums) and random loses (claims) that come at random times. On the other hand, the latter, called as *dual* model, works in opposite direction, losses (costs) are constant and deterministic, and the gains (earnings) are random and come at random times. Sometimes this one is called the negative model. Similar quantities, with similar mathematical properties, work in opposite direction and have different meanings. There is however an important feature that makes the two models quite distinct, either in their application or in their nature: the loading condition, positive or negative, respectively.

The primal model has been worked extensively and focuses essentially in ruin problems (in many different aspects) whereas the dual model has developed more recently and focuses on dividend payments. I most cases, they have been worked apart, however they have connection points that allow us to use methods and results from one to another. basically form the first to the second. Identifying the right connection, or duality, is crucial so that we *transport* methods and results. In the work by Afonso *et al.* (2013) this connection is first addressed in the case when the times between claims/gains follow an exponential distribution.

We can easily understand that the ruin time in the primal has a correspondence to the dividend time in the latter. On the opposite side the time to hit an upper barrier in the primal model has a correspondence to the time to ruin in the dual model. Another interesting feature is the severity of ruin in the former and the size of the dividend payment in the latter.

Keywords: Cramér–Lundberg risk model, Dual risk model; Erlang(n) interclaim times; Phase–Type distribution, generalized Lundberg's equation; ruin probability; time of ruin; expected discounted dividends.

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1 Introduction

The well-known Cramér–Lundberg risk model, with application to insurance, is driven by equation

$$U_P(t) = u + ct - \sum_{i=0}^{N(t)} X_i,\ t \geq 0,\ u \geq 0 \tag{1.1}$$

where $X_0 \equiv 0$, $U_P(t)$ represents the surplus of an insurance portfolio accumulated up to time t, $u = U_P(0)$ is the initial surplus or the surplus known at a giving or starting instant, c is the premium income per unit time, assumed deterministic and fixed. We will call this model as the **primal model** for simplicity and as opposed to the dual model introduced below. The index $_P$ refers to that. $\{X_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables with common cumulative distribution and density functions $P(x)$, with $P(0) = 0$, and $p(x)$, respectively. We assume the existence of $\mu_1 = E[X_1]$. We denote the Laplace transform of $p(x)$ by $\hat{p}(s)$. By $N(t) = \max\{k : W_0 + W_1 + \cdots + W_k \leq t,\ W_0 = 0\}$ we denote the number of claims occurring before or at a given time t, where the random variable W_i denotes the interarrival time between jumps $i-1$ and i (≥ 1). We assume that $\{N(t), t \geq 0\}$ is a renewal process, independent of the sequence $\{X_i\}_{i=0}^{\infty}$, so that $\{W_i\}_{i=0}^{\infty}$ is a sequence of i.i.d. random variables (also independent from $\{X_i\}$). The process $\left\{S(t) = \sum_{i=0}^{N(t)} X_i,\ t \geq 0\right\}$ is then a compound renewal process. Denote by $K(t)$ and $k(t)$ the distribution and the density function of W_1, respectively. Similarly to $\hat{p}(s)$, we denote by $\hat{k}(s)$ the Laplace transform of $k(x)$. We will assume particular distributions in some sections of this manuscript and we state it appropriate and clearly. An important condition for the model is the so called income condition, in the case positive loading condition: $cE(W_1) > E(X_1)$. It brings an economical sense to the model: it is expected that the income until the next claim is greater than the size of the next claim. The net income between the $(i-1)$-th and the i-th claims (inc.) is $cW_i - X_i$. In this model it is well known the notion of the adjustment coefficient, provided that the moment generating function of X_1 exists, in that case it is denoted by $M_X(.)$. The existence on the adjustment coefficient is a condition in some situations and we will state it clearly.

Still for this model, let the time to ruin be denoted by $T_P = \inf\{t > 0 : \ U_P(t) < 0|U_P(0) = u\}$, and $T_P = \infty$ if and only if $U_P(t) \geq 0\ \forall, t > 0$. The ultimate ruin probability is defined as $\psi_P(u) = \Pr(T_P < \infty)$ and the corresponding non–ruin probability (or survival probability) is $\phi_P(u) = 1 - \psi_P(u)$.

Consider an upper level $\beta \geq u$ and define the first time the surplus process reaches level β, irrespective of ruin.

$$\tau_{P,\beta} = min\{t \geq 0 : U_P(t) = \beta|U_P(0) = u\}.$$

Let us now define the distribution of the severity of ruin, it plays an important role in relation to distribution of a single dividend amount in the dual model. The probability that ruin occurs and that the deficit at ruin is at most y is given by $G_P(u, y) = \Pr(T_P < \infty, U_P(T_P) \geq -y|U_P(0) = u)$. For a given u, this is a defective distribution function, clearly $\lim_{y\to\infty} G_P(u, y) = \psi_P(u)$. The corresponding (defective) density is denoted as $g_P(u, y)$.

Let us now consider the **dual model**. It assumes that the surplus or equity of the company is commonly described by the equation

$$U_D(t) = u - ct + \sum_{i=0}^{N(t)} X_i,\ t \geq 0,\ u \geq 0. \tag{1.2}$$

where $U_D(t)$ is the surplus accumulated up to time t for this model. We give the same notation for the quantities that have a similar mathematical meaning to those of the primal model (1.1), noting however that they have different interpretation: c is now the rate of expenses, assumed equally deterministic and fixed, and the sequence $\{X_i\}_{i=1}^{\infty}$ is the gains sequence, still a sequence of i.i.d. random variables with common cumulative distribution and density functions $P(x)$, with $P(0) = 0$, and $p(x)$, respectively. Likewise, we assume the existence of $\mu_1 = E[X_1]$. The mathematical properties of sequences $\{X_i\}$, $\{W_i\}$ and processes $\{N(t)\}$, $\{S(t)\}$ as well as their stochastic relationships remain unchanged. Obviously, here $N(t)$ and $S(t)$ are the number of gains and the accumulated gains occurred up to time t, respectively. We may particularize the distribution of W_1 to come from the Erlang(n) family, with density

$$k_n(t) = \lambda^n t^{n-1} e^{-\lambda t}/(n-1)!\,,\ \ t \geq 0,\ \ \lambda > 0\ \ , n \in \mathbb{N}^+\,.$$

A crucial difference between the two models is the income condition: in the dual model the income condition is reversed. Here it is assumed the existence of the negative loading condition, i.e. $cE(W_1) < E(X_1)$, again, giving an economic sense to the model: on average gains are greater than expenses, per unit time.

This model has been of increasing interest in ruin theory in recent times. There are many possible interpretations for the model. We can look at the surplus as the amount of capital of a business engaged in research and development, where gains are random, at random instants, and costs are certain. More precisely, the company pays expenses which occur continuously along time for the research activity, gets occasional revenues along time according to some predefined distribution $K(.)$. Revenues can be interpreted as values of future gains from an invention or discovery, the decrease of surplus can represent costs of production, payments to employees, maintenance of equipment, etc. Invested capital will be rewarded through occurring future dividends. For that we will need to set up an upper dividend barrier, beyond which a dividend will be payable.

For now let's consider the model free of dividend barrier. Let

$$\tau_x = \inf\left\{t > 0 : U_D(t) = 0 | U_D(0) = x\right\},$$

be the time to ruin, this is the usual definition for the model free of the dividend barrier ($\tau_x = \infty$ if $U_D(t) \geq 0\ \forall t \geq 0$). Let

$$\psi_D(x, \delta) = E\left[e^{-\delta \tau_x} I(\tau_x < \infty) | U_D(0) = x\right],$$

where δ is a non negative constant. $\psi_D(u, \delta)$ is the Laplace transform of time to ruin τ_x. If $\delta = 0$ it reduces to the probability of ultimate ruin of the process free of the dividend barrier, when $\delta > 0$ we can see $\psi_D(u, \delta)$ *as the present value of a contingent claim of 1 payable at* τ_x*, evaluated under a given valuation force of interest* δ [see Ng (2010)]. For simplicity we write $\psi_D(x) = \lim_{\delta \to\ 0^+} \psi_D(x, \delta)$.

Let's now consider an arbitrary upper level $\beta \geq u \geq 0$ in the model, see the upper graph of Figure 1, we don't call it yet a dividend barrier. Let

$$T_x = \inf \{t > 0 : U_D(t) > \beta | U_D(0) = x\}$$

be the time to reach an upper level $\beta \geq x \geq 0$ for the process which we allow to continue even if it crosses the ruin level "0". Due to the income condition T_x is a proper random variable since the probability of crossing β is one.

Let's now introduce into the model the barrier $\beta = b$ as a dividend barrier, and the ruin barrier "0", respectively reflecting and absorbing, such that if the process isn't ruined it will reach the level b. Here, an immediate dividend is paid by an amount in excess of b, the surplus is restored to level b and the process resumes. We will be mostly working the case $0 < u \leq b$. Dividend will only be due if $T_x < \tau_x$ and ruin will only occur prior to that upcross otherwise. Whenever we refer to conditional random variables, or distributions, we will denote them by adding a "tilde", like $\tilde{T}_x$ for $T_x | T_x < \tau_x$.

Let $\chi(u, b)$ denote the probability of reaching b before ruin occurring, for a process with initial surplus u, and $\xi(u, b) = 1 - \chi(u, b)$ is the probability of ruin before reaching b. We have $\chi(u, b) = \Pr(T_u < \tau_u)$.

Because of the existence of the barrier b ultimate ruin has probability 1. The ruin level can be attained before or after the process is reflected on b. Then the probability of ultimate ruin is $\chi(u, b) + \xi(u, b) = 1$.

Let $D_u = \{U_D(T_u) - b \text{ and } T_u < \tau_u\}$ be the dividend amount and its distribution function be denoted as

$$G_D(u, b; x) = \Pr(T_u < \tau_u \text{ and } U_D(T_u) \leq b + x) | u, b)$$

with density $g_D(u, b; x) = \frac{d}{dx} G_D(u, b; x)$. $G_D(u, b; x)$ is a defective distribution function, clearly $G_D(u, b; \infty) = \chi(u, b)$.

We refer now to the upper graph in Figure 1. If the process crosses b for the first time before ruin at a random instant, say $T_{(1)}$, then a random amount, denoted as $D_{(1)}$ is paid. The process repeats, now from level b. The random variables $D_{(i)}$ and $T_{(i)}$, $i = 1, 2, ...$, respectively dividend amount i and waiting time until that dividend, make a bivariate sequence of independent random variables $\left\{\left(T_{(i)}, D_{(i)}\right)\right\}_{i=1}^{\infty}$. We mean, $D_{(i)}$ and $T_{(i)}$ are dependent in general but $D_{(i)}$ and $T_{(j)}$, $i \neq j$, are independent. Furthermore, if we take the subset $\left\{\left(T_{(i)}, D_{(i)}\right)\right\}_{i=2}^{\infty}$ we have now a sequence of independent and jointly identically distributed random variables (and independent of the $\left(T_{(1)}, D_{(1)}\right)$, the bivariate random variables only have the same joint distribution if $u = b$). To simplify notations we set that $\left(T_{(i)}, D_{(i)}\right)$ is distributed as (T_b, D_b), $i = 2, 3, ...$, and $\left(T_{(1)}, D_{(1)}\right)$ is distributed as (T_u, D_u).

Let M denote the number of dividends of the process. Total amount of discounted dividends at a force of interest $\delta > 0$ is denoted as $D(u, b, \delta)$ and $D(u, b) = D(u, b, 0^+)$ is the undiscounted total amount. Their n-th moments are denoted as $V_n(u; b, \delta)$ and $V_n(u; b)$, respectively. For simplicity denote as $V(u; b, \delta) = V_1(u; b, \delta)$. We have

$$\begin{aligned} D(u, b, \delta) &= \sum_{i=1}^{\infty} e^{-\delta\left(\sum_{j=1}^{i} T_{(j)}\right)} D_{(i)} \\ V_n(u; b, \delta) &= \mathbb{E}[D(u, b, \delta)^n] . \end{aligned}$$

2 Connecting the primal and the dual model

If we first consider the model without barriers we can relate the two models by setting the primal surplus process driven in the following way

$$U_P(t) = u^* + ct - S(t) = (\beta - u) + ct - S(t), \;\; t \geq 0, \beta > u, \tag{2.1}$$

where $u^* = (\beta - u)$. When we consider the dividend problem and put the barriers back in order to establish the wanted relation between the primal and the dual models. We refer to Figure1. In the dual we consider it with an upper dividend barrier and a ruin barrier. The first is reflecting and the second is absorbing. In the corresponding primal model, the corresponding dividend barrier, $\beta = b$, can be seen as the ruin barrier of a surplus starting from initial surplus $\beta - u$. The other mentioned barrier usually is not considered in the standard primal model, and it may just correspond to an upper line at level β. See again Figure 1.

An important issue that we have to take into account is the income condition. The models use opposite conditions, in order to relate results from one to the other we have to set which condition we are in, in most situations. The primal model has been widely studied, we favor to find results for the dual model adapting from the former. So, let's consider as a basis the negative loading condition used for the dual model.

With that condition, as far as the dual model (DM) is considered, we note that if the ruin level wasn't absorbing, i.e., the process would continue if the ruin level "0" was achieved, then the upper level b would be reached with probability 1, due to the income condition. However, we follow the model defined by Avanzi *et al.* (2007) where we should only pay dividends if the process isn't ruined. Perhaps we could work with negative capital, but that is out of scope in this work for now [that kind problem is addressed by Cheung (2012)]. We are only interested working over the set of the sample paths of the surplus process that do not lead to ruin. Hence, we need to calculate the probability of the surplus process reaches the barrier b before crossing the level zero. Note that this probability does not correspond to the survival probability, from initial level u.

Look at Figure 1, upper graph again. If we turn it upside down (rotate 180º) and look at it from right to left we get the *classical model shape*, where level "$\beta = b$" is the ruin level, "u" is the initial surplus, becoming $\beta - u$, and the level "0" is an upper barrier. $\left\{D_{(i)}\right\}_{i=2}^{\infty}$ is viewed as a sequence of *i.i.d.* severity of ruin random variables from initial surplus zero and $D_{(1)}$ the independent, but not identically distributed, severity of ruin random variable from initial surplus "$\beta - u$". Similarly, we have that $\left\{T_{(i)}\right\}_{i=2}^{\infty}$ can be viewed as a sequence of i.i.d. random variables meaning time of ruin from initial surplus 0, independent of $T_{(1)}$ which in turn represents the time of ruin from initial surplus $\beta - u$. The connection between the two models is briefly mentioned by Avanzi (2009) (Section 3.1), however not clearly. It is implicit here that in the case of the *classical* model whenever ruin occurs, the surplus is replaced at level "0".

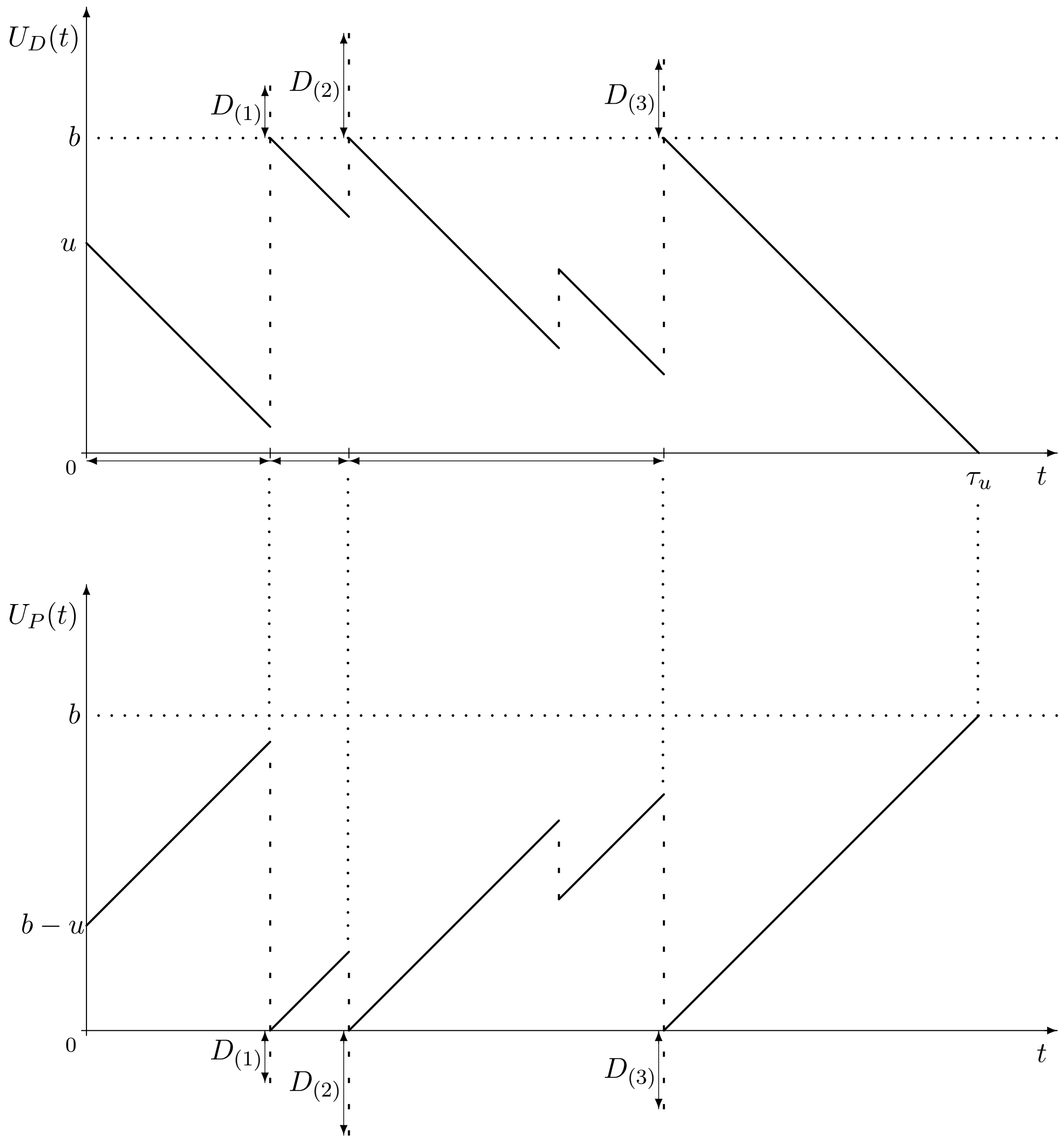


Figure 1: Classical *vs* dual model

3 Lundberg's equations

The adjustment coefficient, denoted as R, is the unique positive real root of the fundamental Lundberg's equation, developed as follows,

$$E\left[e^{-r(cW_1-X_1)}\right]=1 \Leftrightarrow E\left[e^{-rcW_1}\right]E\left[e^{rX_1}\right]=1 \tag{3.1}$$

We note that expectation $E\left[e^{rX_1}\right]$ exists at least for $r<0$, however $E\left[e^{rX_1}\right]$ is a moment generating function if expectation exists for a neighbourhood of zero. In that case R exists and expectation $E\left[e^{-RcW_1}\right]$ exists, it is a Laplace transform. The lefthand side of the equation above can be regarded as the expected discounted profit for each *waiting arrival period.* So that the adjustment coefficient R, provided that it exists, makes the expected discounted profit even (considering that premium income and claim costs come together). Constant R can then seen as an interest force. That equation is known as the fundamental Lundberg's equation.

For practical purposes we find the fundamental Lundberg's equation in the literature written in a different way. We make a change of variable $s=-r$ and extend the domain for $s\in\mathbb{C}$. Then, we get $E\left[e^{rX_1}\right]=\hat{p}(s)$ and $E\left[e^{-rcW_1}\right]=\hat{k}(-cs)$, so that Equation (3.1) takes the form

$$\hat{k}(-cs)\hat{p}(s)=1\,. \tag{3.2}$$

In particular if W_1 follows an Erlang(n) distribution Equation (3.2) takes the form

$$\hat{p}(s)=\left(1-\left(\frac{c}{\lambda}\right)s\right)^n,\ s\in\mathbb{C}\,. \tag{3.3}$$

Consider now the **dual model** case driven by equation (1.2). The equation corresponding to the fundamental Lundberg's equation in (3.1), taking a similar reasoning, is now given by

$$E\left[e^{-s(X_1-cW_1)}\right]=1 \Leftrightarrow E\left[e^{scW_1}\right]E\left[e^{-sX_1}\right]=1 \Leftrightarrow \hat{k}(-cs)\hat{p}(s)=1\ , \tag{3.4}$$

where the corresponding net income per *waiting arrival period* i is given by the reversed difference X_i-cW_i. Equation (3.4) looks the same as (3.3), after the change of variable in the the latter.

If W_1 follows an Erlang(n) distribution Equation (3.4) takes the form of (3.3), i.e.,

$$\hat{p}(s)=\left(1-\left(\frac{c}{\lambda}\right)s\right)^n,\ s\in\mathbb{C}\,. \tag{3.5}$$

Similarly to the primal model, and for the the case when W_1 follows an Erlang(1) distribution, Gerber (1979) defines a unique positive solution, say ρ, of Equation (3.5) so that

$$\psi_D(x,0)=e^{-\rho x}.$$

However this definition might look similar to the one of adjustment coefficient for the primal model, it has a different nature and importance and its uniqueness property may not be true for other distributions of W_1, as shown by Rodríguez-Martínez *et al.* (2013).

A generalization of each of Equations (3.1) and (3.4) were introduced to the actuarial literature and became known as the generalized Lundberg's equation. They take the following form, respectively for the primal and the dual model, for a constant $\delta > 0$ (see e.g. Landriault and Willmot (2008)):

$$E\left[e^{-\delta W_1}e^{-r(cW_1-X_1)}\right]=1 \text{ and } E\left[e^{-\delta W_1}e^{-s(X_1-cW_1)}\right]=1,$$

equivalent to, setting $s=-r$,

$$\hat{k}(\delta-cs)\hat{p}(s)=1,\ s\in\mathbb{C}\,. \tag{3.6}$$

This equation can be found in Gerber and Shiu (2005) and Ren (2007). The positive constant δ is often regarded as an interest force and we can think of (3.4) as the limiting case of (3.7) when $\delta\to 0^+$.

We remark that although the generalized Lunderg's equation is presented the same for both models, solutions, on s, aren't the same as the income condition is reversed for the models, that implies a different parameter choice. In the following section we discuss the solutions of both the fundamental and generalized Lundberg's equations under this particularization.

When W_1 follows an Erlang(n) distribution Equation (3.6) takes the following form:

$$\hat{p}(s) \quad = \quad \left(1+\frac{\delta}{\lambda}-\left(\frac{c}{\lambda}\right)s\right)^n. \tag{3.7}$$

4 Solutions of the Lundberg's equations

In this section we discuss the solutions of Lundberg equations for the case when W_1 follows an Erlang(n) distribution.

According to Theorem 2 and Remark 1 of Li and Garrido (2004a), in a Sparre–Andersen risk model with Erlang(n) distributed interclaim times, Equation (3.7) has n roots with positive real parts and Equation (3.5) has $n-1$ roots with positive real parts.

In the dual model, we can use Rouché's theorem, as in Theorem 2 of Li and Garrido (2004a), to prove that both equations have exactly n roots with positive real parts. Let $\rho_1(\delta),\ldots,\rho_n(\delta)$ denote these roots. Moreover, if n is an odd number only one of these roots is real, say $\rho_n(\delta)$, and if n is even there are always two real roots, say $\rho_n(\delta)$ and $\rho_{n-1}(\delta)$ such that $0<\rho_n(\delta)<\frac{\lambda+\delta}{c}<\rho_{n-1}(\delta)$. The other roots form pairs of conjugate complex numbers on each situation. We note that Remark1 of that theorem does not totally apply to the dual model since it needs the loading condition in its point 2, which is reversed in this case.

The difference between Li and Garrido (2004a)'s conclusion for the primal model and the dual one concerning the number of roots in the limiting case $\delta\to 0^+$ lies on the loading condition. To understand this we can proceed as in Li and Garrido (2004a). Let's define the function

$$h(s)=\left(\frac{\lambda}{c}\right)^n\hat{p}(s)-\left(\frac{\lambda+\delta}{c}-s\right)^n.$$

Since $h(0)<0$ and $\lim_{s\to-\infty}h(s)=+\infty$, for a sufficiently smooth density $p(x)$ (it is sufficient that $\hat{p}(s)$ is continuous) we will have at least one negative real root, we denote the largest by $-R(\delta)$. Also, we have

$$h'(0)=-\left(\frac{\lambda}{c}\right)^n\mu_1+n\left(\frac{\lambda+\delta}{c}\right)^{n-1}=\left(\frac{\lambda}{c}\right)^{n-1}\left(-\frac{\lambda}{c}\mu_1\right)+n\left(\frac{\lambda+\delta}{c}\right)^{n-1}<0,$$

due to the negative loading condition ($cn < \lambda\mu_1$) and for a sufficiently small δ. Note that $h(s)$ has a local minimum between 0 and $\rho_n(\delta)$. Therefore, $\lim_{\delta\to 0^+}(-R(\delta)) = 0$ because $\lim_{\delta\to 0^+} h(0) = 0$. In the limit only the root $-R(\delta)$ equals zero, all the others remain nonzero, since the $\lim_{\delta\to 0^+} \rho_n(\delta) > 0$. Note that if we considered the loading condition to be reversed, which makes economical sense for the primal model, we would have $\lim_{\delta\to 0^+} \rho_n(\delta) = 0$ [see Remark 1 of Theorem 2 in Li and Garrido (2004a)].

Following Ji and Zhang (2012) we note that roots $\rho_1(\delta), \ldots, \rho_n(\delta)$ are all distinct for $\delta \geq 0$, see end of their Section 1, p. 75. This remark was originally described for the primal model, but it remains valid in the case of the dual (their equation corresponding to (3.7) although dependent of c is irrespective of the loading condition). This feature will be very important later on this manuscript.

For simplicity we will denote $\rho_i(\delta)$ by ρ_i, $i = 1, ..., n$, unless stated otherwise.

5 On the time to ruin and the first hitting time to reach an upper barrier

In this section we explore the duality between the time to ruin in the dual model and the time to reach a given upper barrier in the primal model. First, we start with the dual model, specifically we study the Laplace transform of the time of ruin. Then, we relate it to the Laplace transform of the first time when the surplus reaches a given barrier, say β, in the primal model. Throughout this section we consider that W_1 follows an Erlang(n) distribution.

5.1 The Laplace transform of the time to ruin in the dual risk model

Developments done in this subsection can be found in Rodríguez-Martínez *et al.* (2013). For the Erlang(n) case, the Laplace transform of the time to ruin satisfies the renewal equation

$$\psi_D(u,\delta) = (1 - K_n(t_0))\, e^{-\delta t_0} + \int_0^{t_0} k_n(t)\, e^{-\delta t} \int_0^\infty p(x)\psi_D(u - ct + x, \delta)dxdt. \tag{5.1}$$

with $t_0 = u/c$. The following theorem shows an integro–differential equation for $\psi_D(u,\delta)$.

Theorem 5.1. *In the Erlang(n) dual risk model the Laplace transform of the time of ruin satisfies the integro–differential equation*

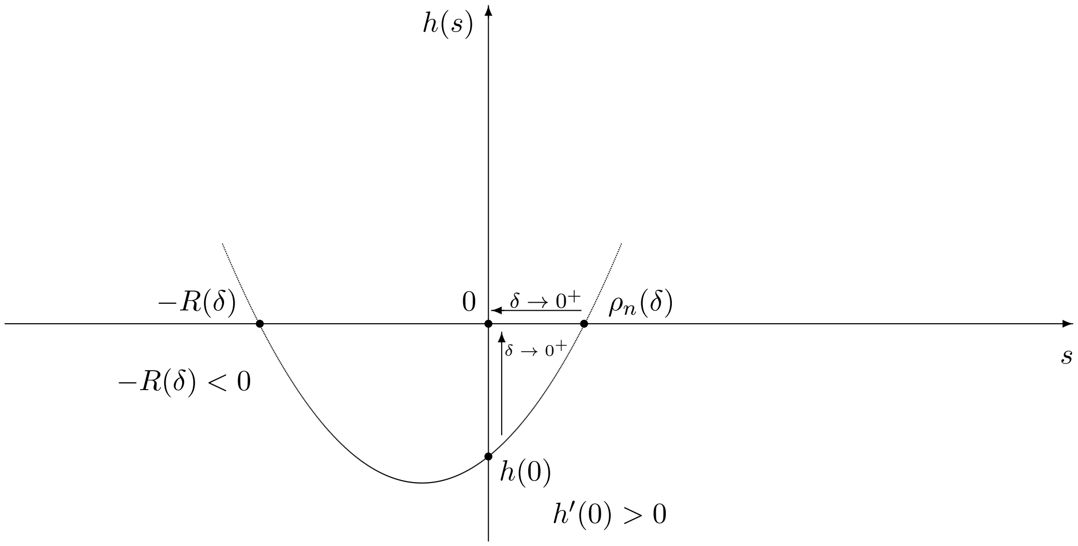
$$\left(\left(1 + \frac{\delta}{\lambda}\right)\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n \psi_D(u,\delta) = \int_0^\infty p(x)\psi_D(u + x, \delta)dx, \tag{5.2}$$

with boundary conditions

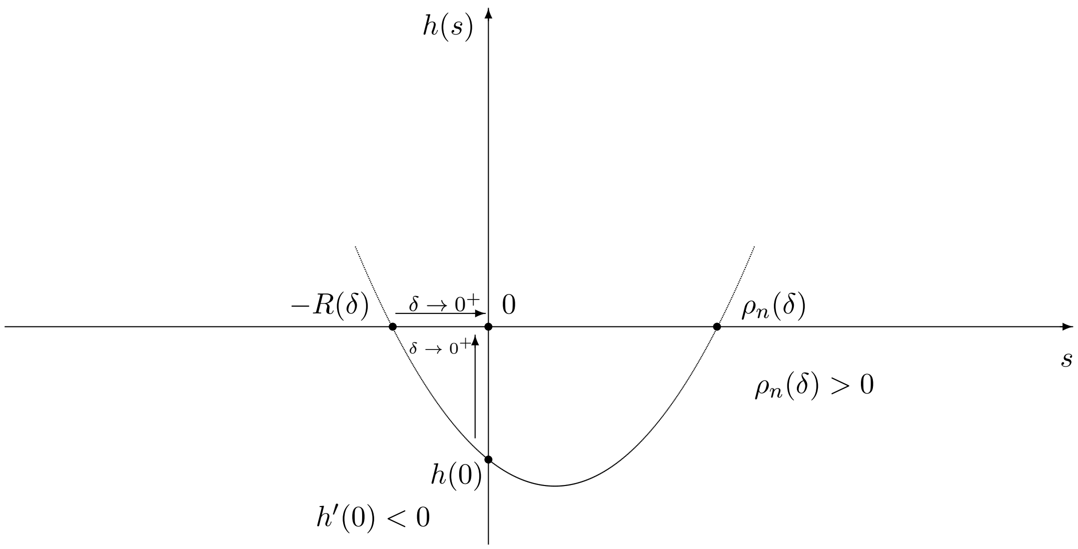
$$\psi_D(0,\delta) = 1, \quad \left.\frac{d^i}{du^i}\psi_D(u,\delta)\right|_{u=0} = (-1)^i \left(\frac{\delta}{c}\right)^i, \; i = 1, \ldots, n-1. \tag{5.3}$$

Proof. We take successive derivatives of (5.1). Then, changing variable the renewal equation can be rewritten in the form

$$\psi_D(u,\delta) = \left(1 - K_n\left(\frac{u}{c}\right)\right) e^{-\delta(\frac{u}{c})} + \frac{1}{c}\int_0^u k_n\left(\frac{u-s}{c}\right) e^{-\delta(\frac{u-s}{c})} W_\delta(s)ds,$$



Primal Model: $h(s) = \left(\frac{\lambda}{c}\right)^n \hat{p}(s) - \left(\frac{\lambda+\delta}{c} - s\right)^n$ when $\delta \to 0^+$.



Dual Model: $h(s) = \left(\frac{\lambda}{c}\right)^n \hat{p}(s) - \left(\frac{\lambda+\delta}{c} - s\right)^n$ when $\delta \to 0^+$.

Figure 2: Roots of Lundberg's equations

where $W_\delta(s) = \int_0^\infty \psi_D(s+x,\delta)p(x)dx$.

After applying the operator $\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)$ to the Laplace transform we get

$$\begin{aligned}\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)\psi_D(u,\delta) &= \left(1 - K_{n-1}\left(\frac{u}{c}\right)\right)e^{-\delta(\frac{u}{c})} \\ &+ \frac{1}{c}\int_0^u k_{n-1}\left(\frac{u-s}{c}\right)e^{-\delta(\frac{u-s}{c})}W_\delta(s)ds.\end{aligned}$$

Similarly, following an inductive argument, we show that

$$\begin{aligned}\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^i\psi_D(u,\delta) &= \left(1 - K_{n-i}\left(\frac{u}{c}\right)\right)e^{-\delta(\frac{u}{c})} \\ &+ \frac{1}{c}\int_0^u k_{n-i}\left(\frac{u-s}{c}\right)e^{-\delta(\frac{u-s}{c})}W_\delta(s)ds,\end{aligned}$$

for $i = 1, \ldots, n-1$. In particular, we obtain

$$\begin{aligned}\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^{n-1}\psi_D(u,\delta) &= \left(1 - K_1\left(\frac{u}{c}\right)\right)e^{-\delta(\frac{u}{c})} \\ &+ \frac{1}{c}\int_0^u k_1\left(\frac{u-s}{c}\right)e^{-\delta(\frac{u-s}{c})}W_\delta(s)ds.\end{aligned}$$

Applying once more the operator gives

$$\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n\psi_D(u,\delta) = W_\delta(u).$$

This proves equation (5.2).

For the boundary conditions, clearly $\psi_D(0,\delta) = 1$. We find the remaining conditions computing directly the derivatives of $\psi_D(u,\delta)$ and evaluate at $u = 0$,

$$\begin{aligned}\frac{d^i}{du^i}\psi_D(u,\delta) &= \left[\left(-\frac{\delta}{c}\right)^i\left(1 - K_n\left(\frac{u}{c}\right)\right) - \frac{1}{c^i}\sum_{j=1}^{i}\binom{i}{j}(-\delta)^{i-j}k_n^{(j-1)}\left(\frac{u}{c}\right)\right]e^{-\delta(\frac{u}{c})} \\ &+ \left(\frac{1}{c}\right)\int_0^u\left[\frac{1}{c^i}\sum_{j=0}^{i}\binom{i}{j}(-\delta)^{i-j}k_n^{(j)}\left(\frac{u-s}{c}\right)\right]e^{-\delta(\frac{u-s}{c})}W_\delta(s)ds,\end{aligned}$$

for $i = 1, \ldots, n-1$, so that we get $d^i\psi_D(u,\delta)/du^i\big|_{u=0} = (-\delta/c)^i$, $i = 1, \ldots, n-1$. □

The solution for $\psi_D(u,\delta)$ is given in the following theorem.

Theorem 5.2. *The Laplace transform of the time of ruin can be written as a combination of exponential functions*

$$\psi_D(u,\delta) = \sum_{k=1}^{n}\left[\prod_{i=1,i\neq k}^{n}\frac{\left(\rho_i - \frac{\delta}{c}\right)}{(\rho_i - \rho_k)}\right]e^{-\rho_k u}, \tag{5.4}$$

where $\rho_1, \ldots, \rho_n$ *are the only roots of the generalized Lundberg's equation* (3.7) *which have positive real parts.*

Proof. All the functions $e^{-\rho_k u}$, $k = 1, ..., n$, are particular solutions of the integro–differential equation

$$\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n f(u)=\int_0^\infty p(x)f(u+x)dx. \tag{5.5}$$

Since these functions are linearly independent, we can write every solution of (5.5) as a linear combination of them. Therefore,

$$\psi_D(u,\delta)=\sum_{i=1}^n a_i e^{-\rho_i u},$$

where a_i, $i = 1, ...n$, are constants and solutions of the system of equations

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \rho_1 & \rho_2 & \cdots & \rho_n \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1^{n-1} & \rho_2^{n-1} & \cdots & \rho_n^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \frac{\delta}{c} \\ \vdots \\ \left(\frac{\delta}{c}\right)^{n-1} \end{pmatrix} \Leftrightarrow \mathbf{a}=\mathbf{P}^{-1}\mathbf{\Lambda},$$

in matrix form, where $\mathbf{P} = \mathbf{P}(\rho_1, \ldots, \rho_n)$ is a Vandermonde matrix, $\mathbf{a}' = (a_1, a_2, \ldots, a_n)$ and $\mathbf{\Lambda}' = (1, \delta/c, \ldots, (\delta/c)^{n-1})$.

Finally, we get expressions for the coefficients

$$\begin{aligned} a_k &= \frac{(-1)^{k-1}(\prod_{i=1,\, i\neq k}^n(\rho_i-\frac{\delta}{c}))(\prod_{1\le i<j\le n,\, i\neq k,\, j\neq k}(\rho_j-\rho_i))}{\prod_{1\le i<j\le n}(\rho_j-\rho_i)} \\ &= \frac{(-1)^{k-1}(\prod_{i=1,\, i\neq k}^n(\rho_i-\frac{\delta}{c}))}{(\prod_{i=1}^{k-1}(\rho_k-\rho_i))(\prod_{j=k+1}^n(\rho_j-\rho_k))} = \prod_{i=1,i\neq k}^n \frac{(\rho_i-\frac{\delta}{c})}{(\rho_i-\rho_k)}. \end{aligned}$$

We note that δ/c is not a root of equation (3.7). Hence, we get the result. □

Remark:

Note that for the limiting case $\delta \to 0^+$ we obtain the ruin probability, since

$$\lim_{\delta\to 0^+}\psi_D(u,\delta)=E[I(\tau_u<\infty)\mid U(0)=u]=\psi_D(u).$$

Therefore

$$\psi_D(u)=\lim_{\delta\to 0^+}\sum_{k=1}^n\left[\prod_{i=1,i\neq k}^n\frac{(\rho_i-\frac{\delta}{c})}{(\rho_i-\rho_k)}\right]e^{-\rho_k u}=\sum_{k=1}^n\left[\prod_{i=1,i\neq k}^n\frac{\rho_i}{(\rho_i-\rho_k)}\right]e^{-\rho_k u}. \tag{5.6}$$

5.2 The Laplace transform of first hitting time to reach an upper level in the primal model

For $\delta \geq 0$ define the Laplace transform of first hitting time to reach an upper level in the primal model

$$R(u,\beta)=E[e^{-\delta\tau_{P,\beta}}|U(0)=u],$$

An expression for $R(u,\beta)$ was found by Li (2008a) for the case when the interclaim times follow a Phase–Type distribution. In particular when the interclaim times are Erlang(n) distributed he obtained the following formula

$$R(u,\beta)=\sum_{k=1}^{n}\left[\prod_{i=1,i\neq k}^{n}\frac{\left(\rho_i-\frac{\delta}{c}\right)}{(\rho_i-\rho_k)}\right]e^{-\rho_k(\beta-u)} \tag{5.7}$$

Note that for the limiting case $\delta\to 0^+$ we obtain 1,

$$\lim_{\delta\to 0^+}R(u,\beta)=\lim_{\delta\to 0^+}E[e^{-\delta T_{P,\beta}}|U(0)=u]=1.$$

This means that for the primal model the process attains an upper level, starting from a given initial surplus, with probability one.

The Laplace transform (5.4), in the dual model, shows an interesting form, it corresponds to formula (5.7) concerning the primal model. This result enhances the duality between the two models as explained by Afonso *et al.* (2013) who worked the compound Poisson, or Erlang(1), model. We mean, the first hitting time in the primal model corresponds to the ruin time in the dual model. It is interesting that the duality features shown for the classical Erlang(1) model can be extended [see beginning of Section 3 of Afonso *et al.* (2013)]. Note, however, that the loading conditions in the two models are reversed. We refer to the explanations for the Lundberg's equations in Sections 3 and 4. Formulae (5.4) above and (5.7) show the same *appearance* but parameter c have different admissible values.

Formula (5.6) is a limiting case, as $\delta\to 0^+$, of (5.4). As described above the same behaviour does not happen concerning formula (5.7) derived for the primal model. This is due to the reversed loading condition. The first hitting time in the primal model is a proper random variable whereas the time to ruin in the dual model is a defective one.

6 On the single dividends amount

In this section we consider that the interarrival time W_1 follows an Erlang(1) distribution. As for the dual model with a a dividend barrier $\beta=b$, Afonso *et al.* (2013) shows the computation of now we get back to the usual model with an integro-differential equation for the distribution and the density of the single dividend amount , $G_D(u,b;x)$ and $g_D(u,b;x)$.

Using a standard procedure, conditioning on the first gain we get, where t_0 is such that $u-ct_0=0$,

$$G_D(u,b;x)=\int_0^{t_0}\lambda e^{-\lambda t}\left[\int_0^{b-(u-ct)}p(y)G(u-ct+y,b;x)\,dy+\int_{b-(u-ct)}^{b-(u-ct)+x}p(y)dy\right]dt.$$

Rearranging and differentiating with respect to u, we get the following integro-differential equation

$$\lambda G_D(u,b;x)+c\,\frac{\partial}{\partial u}\,G_D(u,b;x)=\lambda\int_u^b p(y-u)G_D(y,b;x)\,dy+\lambda\left[P(b-u+x)-P(b-u)\right], \tag{6.1}$$

with boundary condition $G_D(0,b;x)=0$. We get, differentiating with respect to x,

$$\lambda g_D(u,b;x)+c\frac{\partial}{\partial u}g_D(u,b;x)=\lambda\int_0^{b-u}p(y)g_D(u+y,b;x)\,dy+\lambda p(b-u+x). \tag{6.2}$$

Like is shown in Figure1 the dividend payment in the dual model can have a correspondence with the amount of the severity of ruin in the primal. Indeed, the single dividend distribution can be gotten from the formula for the distribution of the severity of ruin, previously adapted with the reversed income condition [see Afonso *et al.* (2013)].

For that, first consider the process continuing even if ruin occurs. The process can cross for the first time the upper dividend level before or after having ruined. Then we can write the (proper) distribution of the amount by which the process first upcrosses b, denoted as $H(u,b;x)=\Pr\left[U_D(T_u)\le b+x\right]$. We have

$$\begin{aligned} H(u,b;x) &= H(u,b;x|T_u<\tau_u)\chi(u,b)+H(u,b;x|T_u>\tau_u)\xi(u,b)\\ &= G_D(u,b;x)+\xi(u,b)H(0,b;x). \end{aligned}$$

The second equation above simply means that the probability of the amount by which the process first upcrosses b is less or equal than x, equals the probability of a dividend claim less or equal than x plus the probability of a similar amount but in that case it cannot be a dividend. This second probability can be computed through the probability of first reaching the level "0", $\xi(u,b)$, times the probability of an upcrossing of level b by the same amount ($\le x$) but restarting from 0, $H(0,b;x)$.

We can compute $H(u,b;x)$ through expressions known for the distribution of the severity of ruin from the primal risk model (recall that the income condition is reversed, making it a proper distribuition function). Then we get

$$G_P(b-u;x)=G_D(u,b;x)+\xi(u,b)G_P(b;x)$$

equivalent to

$$G_D(u,b;x)=G_P(b-u;x)-\xi(u,b)G_P(b;x). \tag{6.3}$$

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