# ROTATION NUMBER OF 2-INTERVAL PIECEWISE AFFINE MAPS 

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#### Abstract

We study maps of the unit interval whose graph is made up of two increasing segments and which are injective in an extended sense. Such maps $f_{p}$ are parametrized by a quintuple $\boldsymbol{p}$ of real numbers satisfying inequations. Viewing $f_{\boldsymbol{p}}$ as a circle map, we show that it has a rotation number $\rho\left(f_{\boldsymbol{p}}\right)$ and we compute $\rho\left(f_{\boldsymbol{p}}\right)$ as a function of $\boldsymbol{p}$ in terms of Hecke-Mahler series. As a corollary, we prove that $\rho\left(f_{\boldsymbol{p}}\right)$ is a rational number when the components of $\boldsymbol{p}$ are algebraic numbers.


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## 1. Introduction

Let $\mathbb{R} / \mathbb{Z}$ denote the unit circle and $f: \mathbb{R} / \mathbb{Z} \rightarrow \mathbb{R} / \mathbb{Z}$ be an orientation-preserving circle homeomorphism. Any continuous lift $F: \mathbb{R} \rightarrow \mathbb{R}$ of $f$ is strictly increasing and $F$ - id is one-periodic. In order to study the dynamics of orientation preserving circle homeomorphisms, Poincaré introduced an invariant quantity $\rho(f) \in[0,1)$, known as the rotation number of $f$ that measures

[^0]the average rotation along any orbit of $f$. Given any continuous lift $F: \mathbb{R} \rightarrow \mathbb{R}$ of $f$ and $x \in \mathbb{R}$, the rotation number of $f$ is defined as
$$
\rho(f):=\lim _{n \rightarrow \infty} \frac{F^{n}(x)}{n} \quad(\bmod 1) .
$$

This limit exists and is independent of $x$ and the lift $F$. Moreover, $\rho(f)$ is rational if and only if $f$ has a periodic point.

The theory of rotation number was extended to orientation-preserving circle maps which are not continuous, neither surjective, in particular by Rhodes and Thompson [27, 28]. Let $\mathcal{M}$ denote the set of circle maps $f$ whose lifts $F$ are strictly increasing and $F$ - id is one-periodic. Rhodes and Thompson have shown that the rotation number is well defined for all maps $f \in \mathcal{M}$ and it varies continuously as a function of $f$ on any continuous one parameter family contained in $\mathcal{M}$.

By identifying the unit interval $I:=[0,1)$ with $\mathbb{R} / \mathbb{Z}$ through the canonical bijection $I \hookrightarrow \mathbb{R} \rightarrow$ $\mathbb{R} / \mathbb{Z}$, we may view any circle map in $\mathcal{M}$ as an orientation preserving injective map of $I$. Through this identification, we associate a rotation number to any orientation preserving injective map.

The study of the rotation number of injective piecewise affine increasing contractions with only one discontinuity point on $\mathbb{R} / \mathbb{Z}$ and a unique slope was introduced in Canaiello's neuronic equations which are 2 -interval piecewise contractions, see [16] for a discussion of the topic. The dynamics of piecewise contractions has been studied by many authors, in particular [8, 13, 15, $24,25,26]$. A detailed study of contracted rotations (meaning that the two branches have equal slopes) can be found in $[22,11,12,3,4,5,16,18,23,17,2,7,6,14,1]$. In [18], Laurent and Nogueira were the first to relate transcendental properties of the parameters of these maps to the irrationality of their rotation number. Later in [19], Laurent and Nogueira extended their work to allow injective maps with two different slopes and a single discontinuity point. However, they also studied the dynamics of maps which are not piecewise contractions, in particular see [19, Corollary, page 36], where the rotation number of a 2 -interval piecewise affine circle homeomorphism is obtained.

This family of maps is also known in the literature as contractive piecewise linear Lorenz maps. It is sometimes claimed, see for instance [10, page 237], that the dynamics of such a contractive map is trivial, meaning that all its orbits converge to a periodic cycle. This is almost true, but not always. In fact, for an uncountable set of parameters, the corresponding maps have singularly continuous invariant probability measures and all the orbits converge to a Cantor set sharing fine arithmetical properties which are investigated in [5] and [6].

In the present work, we extend the framework of [19] by allowing an additional discontinuity to the circle map $f$ at the break point (denoted $\eta$ below) and even allowing $f$ to be non-injective in some cases. Moreover, the family of maps we consider are not necessarily piecewise contractions, i.e., one of its branches may expand.

The article is organized as follows. In Section 2 we introduce our family of maps whose set of parameters is described in Section 3. In Section 4 we prove that the maps we consider have a well-defined rotation number. Our main results will be stated in Section 5. Theorem 5.2 describes the rotation number of $f=f_{\boldsymbol{p}}$ as a function of the parameter $\boldsymbol{p}$ and Theorem 5.5 makes explicit the semi-conjugacy of $f$ to a rotation following the approach given in [9]. In order to prove our main results the dynamics of our map is reduced to that of a map on an invariant interval. Up to an isomorphism, this map belongs to the family of maps that have been studied in [19]. In Corollary 5.3, using transcendence results on Hecke-Mahler series, it is proved that the rotation number takes a rational value when all the components of $\boldsymbol{p}$ are algebraic numbers. In Section 6, we explain our strategy for the proof of the results in Section 5.

The properties of two functions $\boldsymbol{a}$ and $\phi$ describing the dynamics of $f$ are displayed in Section 7 . The proofs of the main results from Section 5 are given in Section 8.

## 2. The SETting

Let $I=[0,1)$ be the unit interval identified with the circle $\mathbb{R} / \mathbb{Z}$ as in Section 1 . We are concerned with the dynamics of maps $f: I \rightarrow I$ as pictured in Figure 1 below. Namely, the graph of $f$ is made up with two increasing segments such that $f$ has no fixed point and is injective when restricted to a certain invariant subinterval of $I$. Such a map $f$ has a rotation number $\rho(f) \in I$ and is semi-conjugated to the rotation $R_{\rho}: x \rightarrow x+\rho(\bmod 1)$. Our goal is to make explicit these two assertions in terms of the parameters defining $f$. The precise statements are given in Theorem 5.2 and 5.5 below.


Figure 1. A plot of $f_{p}$

We parametrize those maps $f$ by a quintuple $\boldsymbol{p}=(\lambda, \mu, a, b, c)$ as follows.
Definition 2.1. Let us denote by $\mathcal{P}$ the set of quintuple parameters $\boldsymbol{p}=(\lambda, \mu, a, b, c)$ satisfying the inequalities

$$
\begin{gather*}
0<\lambda<1, \mu>0,0 \leq c<b \leq 1, \lambda \mu \leq 1 \text { or }(1-b) \mu \leq 1-c  \tag{1}\\
b-b \lambda<a<b-c \lambda  \tag{2}\\
(1-\lambda)(c-\mu b)<(1-\mu) a \tag{3}
\end{gather*}
$$

Set $\eta=\frac{b-a}{\lambda}$ and define $f_{p}: I \rightarrow I$ by the splitted formula

$$
f_{\boldsymbol{p}}(x)= \begin{cases}\lambda x+a, & \text { if } 0 \leq x<\eta \\ \lambda \mu(x-\eta)+c, & \text { if } \eta \leq x<1\end{cases}
$$

The left branch of the graph has slope $\lambda$ and endpoints $(0, a)$ and $(\eta, b)$, while the right branch has slope $\lambda \mu$ and has for origin the point $(\eta, c)$. Assumption (2) ensures that the break point $\eta$ belongs to the open interval $(c, b)$. It implies that $f_{\boldsymbol{p}}$ has no fixed point. Assumption (3) ensures that $f_{p}$ is injective when restricted to the interval $J:=[c, b)$. Indeed, we have

$$
f_{\boldsymbol{p}}(c)=\lambda c+a \quad \text { and } \quad f_{\boldsymbol{p}}\left(b^{-}\right)=\lambda \mu(b-\eta)+c=\lambda \mu b+\mu(a-b)+c
$$

so that the inequality

$$
\begin{equation*}
f_{\boldsymbol{p}}(c)>f_{\boldsymbol{p}}\left(b^{-}\right) \tag{4}
\end{equation*}
$$

is equivalent to (3). Equivalently, (3) ensures that $f_{\boldsymbol{p}}$ contracts on average in $[c, b)$, i.e.,

$$
\begin{equation*}
\lambda \frac{\eta-c}{b-c}+\lambda \mu \frac{b-\eta}{b-c}<1 \tag{5}
\end{equation*}
$$

It remains to show that the image $f_{\boldsymbol{p}}(I)$ is contained in $I$. Looking at Figure 1 , it is sufficient to prove that $f_{\boldsymbol{p}}\left(1^{-}\right)<1$. We postpone the proof to Section 3 which makes use of the assumptions, $\lambda \mu \leq 1$ or $\mu \leq \frac{1-c}{1-b}$, occurring in (1).

There is no loss of generality assuming $0<\lambda<1$ in Assumption (1), since the homeomorphism $(x, y) \in(0,1)^{2} \mapsto(1-x, 1-y) \in(0,1)^{2}$ exchanges the two branches and at least one slope must be less than 1 by (5).

## 3. Description of the parameter set $\mathcal{P}$

In order to study the dynamics of $f_{\boldsymbol{p}}$, we shall basically view the four parameters $\lambda, \mu, b, c$ as fixed and regard the fifth parameter $a$ as a variable. It turns out that $a$ ranges over an interval once $(\lambda, \mu, b, c)$ has been fixed. The description of this interval is the following.

Set

$$
\begin{equation*}
\mathcal{Q}:=\{(\lambda, \mu, b, c): 0<\lambda<1, \mu>0,0 \leq c<b \leq 1, \lambda \mu \leq 1 \text { or } \mu(1-b) \leq 1-c\} \tag{6}
\end{equation*}
$$

and define

$$
d_{\lambda, \mu, b, c}:= \begin{cases}b-c \lambda, & \text { if } \quad \lambda \mu<1  \tag{7}\\ \frac{(1-\lambda)(\mu b-c)}{\mu-1}, & \text { if } \quad \lambda \mu \geq 1\end{cases}
$$

Notice that $b-c \lambda=\frac{(1-\lambda)(\mu b-c)}{\mu-1}$ when $\lambda \mu=1$.
Lemma 3.1. For every $(\lambda, \mu, b, c) \in \mathcal{Q}$, the inequality $b-b \lambda<d_{\lambda, \mu, b, c}$ holds and

$$
\mathcal{P}=\left\{(\lambda, \mu, a, b, c):(\lambda, \mu, b, c) \in \mathcal{Q}, a \in\left(b-b \lambda, d_{\lambda, \mu, b, c}\right)\right\} .
$$

Proof. Given $(\lambda, \mu, b, c) \in \mathcal{Q}$ it is a simple exercise to show that $b-b \lambda<d_{\lambda, \mu, b, c}$. Let $(\lambda, \mu, a, b, c) \in$ $\mathcal{P}$. By (2), we have the inequalities $b-b \lambda<a<b-c \lambda$. Now, we solve inequality (3) with respect to $a$. We have three cases:

- When $\mu<1$, then (3) reads

$$
a>\frac{(1-\lambda)(c-\mu b)}{1-\mu}=(1-\lambda) b-\frac{(1-\lambda)(b-c)}{1-\mu}
$$

which is greater than $b-b \lambda$, since $\frac{(1-\lambda)(b-c)}{1-\mu}>0$. We conclude in this case that $(2)$ and (3) together give

$$
b-b \lambda<a<b-c \lambda=d_{\lambda, \mu, b, c}
$$

as required, since $\lambda \mu<1$.

- When $\mu>1$, then (3) reads

$$
a<\frac{(1-\lambda)(\mu b-c)}{\mu-1}=(1-\lambda) b+\frac{(1-\lambda)(b-c)}{\mu-1} .
$$

Thus, (2) and (3) together give the interval

$$
b-b \lambda<a<\min \left\{b-c \lambda,(1-\lambda) b+\frac{(1-\lambda)(b-c)}{\mu-1}\right\} .
$$

In this case, a simple computation shows that

$$
\min \left\{b-c \lambda,(1-\lambda) b+\frac{(1-\lambda)(b-c)}{\mu-1}\right\}= \begin{cases}b-c \lambda, & \text { if } \quad \lambda \mu<1 \\ \frac{(1-\lambda)(\mu b-c)}{\mu-1}, & \text { if } \quad \lambda \mu \geq 1\end{cases}
$$

- When $\mu=1$, the inequality (3) reads $(1-\lambda)(c-b)<0$, which is obviously satisfied. We thus get the interval

$$
b-b \lambda<a<b-c \lambda=d_{\lambda, 1, b, c}
$$

Therefore, we find the interval $b-b \lambda<a<d_{\lambda, \mu, b, c}$ in the three cases.

The assumptions, $\lambda \mu \leq 1$ or $(1-b) \mu \leq 1-c$, occurring in the above definition of $\mathcal{Q}$ are needed to obtain the

Lemma 3.2. For every $\boldsymbol{p} \in \mathcal{P}$, the inequality $f_{\boldsymbol{p}}\left(1^{-}\right)<1$ holds true.

Proof. We write

$$
f_{\boldsymbol{p}}\left(1^{-}\right)=\lambda \mu(1-\eta)+c=\lambda \mu-\mu b+\mu a+c
$$

and use the upper bound $a<d_{\lambda, \mu, b, c}$. When $\lambda \mu \leq 1$, by (7) we have $d_{\lambda, \mu, b, c}=b-c \lambda$, so that

$$
f_{\boldsymbol{p}}\left(1^{-}\right)<\lambda \mu-\mu b+\mu(b-c \lambda)+c=\lambda \mu+(1-\lambda \mu) c=1-(1-\lambda \mu)(1-c) \leq 1
$$

as required. When $\lambda \mu>1$, by (7) we have $d_{\lambda, \mu, b, c}=\frac{(1-\lambda)(\mu b-c)}{\mu-1}$, so that
$f_{\boldsymbol{p}}\left(1^{-}\right)<\lambda \mu-\mu b+\frac{\mu(1-\lambda)(\mu b-c)}{\mu-1}+c=\lambda \mu+\left(-1+\frac{\mu(1-\lambda)}{\mu-1}\right)(\mu b-c)=1+(\lambda \mu-1)\left(1-\frac{\mu b-c}{\mu-1}\right)$.
Notice finally that the factor $1-\frac{\mu b-c}{\mu-1}$ is $\leq 0$, since $1<\mu \leq \frac{1-c}{1-b}$ by assumption, while $\lambda \mu-1>0$.

Remark 3.3. In [19], a 2-interval piecewise affine map depending on a triple of parameters $(\lambda, \mu, \delta)$ was studied. The parameters $(\lambda, \mu, \delta)$ were assumed to satisfy the inequalities:

$$
0<\lambda<1, \quad \mu>0 \quad \text { and } \quad 1-\lambda<\delta<d_{\lambda, \mu}:=\left\{\begin{array}{lll}
1, & \text { if } \quad \lambda \mu<1  \tag{8}\\
\frac{(1-\lambda) \mu}{\mu-1}, & \text { if } \quad \lambda \mu \geq 1
\end{array}\right.
$$

In case $b=1$ and $c=0$, we recover the description of the set of parameters in (8). To see it, let

$$
\begin{equation*}
\mathcal{P}_{0}:=\{(\lambda, \mu, a, b, c) \in \mathcal{P}: b=1, c=0\} . \tag{9}
\end{equation*}
$$

Taking into account the definition of $\mathcal{Q}$ in (6), we have $(\lambda, \mu, 1,0) \in \mathcal{Q}$ if and only if $0<\lambda<1$ and $\mu>0$. Moreover, given $(\lambda, \mu, 1,0) \in \mathcal{Q}$ we have $d_{\lambda, \mu, 1,0}=d_{\lambda, \mu}$. Indeed, this follows from the identity

$$
\begin{equation*}
d_{\lambda, \mu, b, c}=d_{\lambda, \mu}(b-c)+c(1-\lambda) . \tag{10}
\end{equation*}
$$

Thus,

$$
\mathcal{P}_{0}=\left\{(\lambda, \mu, a, 1,0): 0<\lambda<1, \mu>0,1-\lambda<a<d_{\lambda, \mu}\right\} .
$$

## 4. Rotation number of $f_{\boldsymbol{p}}$

Let $\boldsymbol{p} \in \mathcal{P}$ be given. Define a real function $F=F_{\boldsymbol{p}}: \mathbb{R} \rightarrow \mathbb{R}$ by

$$
F(x)=\left\{\begin{array}{lll}
\lambda x+a+(1-\lambda)\lfloor x\rfloor, & \text { if } \quad 0 \leq\{x\}<\eta \\
\lambda \mu(x-\eta)+c+1+(1-\lambda \mu)\lfloor x\rfloor, & \text { if } \quad \eta \leq\{x\}<1
\end{array}\right.
$$

Let $f=f_{\boldsymbol{p}}$. Clearly, $F$ is a lift of $f$, i.e., for every $x \in \mathbb{R}$,
(1) $F(x+1)=F(x)+1$,
(2) $\{F(x)\}=f(\{x\})$.

Notice that $F$ is not always strictly increasing, so we cannot apply immediately the theory of Rhodes and Thompson developed in [27] that generalizes the classical theory of the rotation number of Poincaré to circle maps having some strictly increasing lift. Nevertheless, we show in Lemma 4.2 below, that $f$ has a well-defined rotation number.

Let

$$
X=\{x \in \mathbb{R}: c \leq\{x\}<b\}
$$

Lemma 4.1. F satisfies the following properties:
(1) $F(X) \subseteq X$,
(2) $\left.F\right|_{X}$ is strictly increasing,
(3) for every $x \in \mathbb{R}$ there exists $n \geq 0$ such that $F^{n}(x) \in X$.


Figure 2. Iterated values of $f$ in blue and red
Proof. Assertion (1) follows from the obvious inclusion $f([c, b)) \subseteq[c, b)$, while Assertion (2) follows from the inequality (4) by reduction modulo one. Indeed, for any integer $p$, we have

$$
F\left(b^{-}+p\right)=f\left(b^{-}\right)+p+1<f(c)+p+1=F(c+p+1) .
$$

For Assertion (3), the smallest value of $n$ depends on the location of $\{x\}$ on the unit interval. If $\{x\} \in[c, b)$, we can obviously choose $n=0$. If $b \leq\{x\}<1$, we have $c \leq f(\{x\})<\{x\}$. Then, if $f(\{x\})<b$, we choose $n=1$. Otherwise, we have $b \leq f(\{x\})<1$ and we iterate, so that $c \leq f^{2}(\{x\})<f(\{x\})$. If $f^{2}(\{x\}<b$, we choose $n=2$. Otherwise, we iterate once again and so
on. We thus obtain a finite sequence of decreasing points $\{x\}>f(\{x\})>f^{2}(\{x\})>\cdots$, which necessarily falls in $[c, b)$ at the end. The argumentation for $\{x\} \in[0, c)$ is similar, with now an increasing sequence of iterates. See Figure 2.
Lemma 4.2. $f$ has a well-defined rotation number, i.e., the following limit exists and is independent of $x \in \mathbb{R}$,

$$
\rho(f):=\lim _{n \rightarrow+\infty} \frac{F^{n}(x)}{n} \quad(\bmod 1) .
$$



Figure 3. A plot of $F$ in blue and of $\bar{F}-1$ in red.
Proof. Using $F$, we define a new piecewise-affine function $\bar{F}: \mathbb{R} \rightarrow \mathbb{R}$ which equals $F$ inside $X$, but outside $X$ has a graph which is obtained by connecting with a straight segment the end points $\left(b+p, F\left(b^{-}\right)+p\right)$ and $(c+p+1, F(c)+p+1)$ for every $p \in \mathbb{Z}$. See Figure 3. It follows from Lemma 4.1, that the function $\bar{F}$ is strictly increasing and $\bar{F}-$ id is 1 -periodic. Therefore, by [27, Corollary 1], the limit $\bar{\rho}:=\lim _{n \rightarrow+\infty} \frac{\bar{F}^{n}(x)}{n}(\bmod 1)$ exists and is independent of $x$. Now, again by Lemma 4.1, $F(X) \subset X$ and for every $x \in \mathbb{R}$ there is $n_{0}=n_{0}(x) \geq 0$ such that $F^{n_{0}}(x) \in X$. Since $\left.\bar{F}\right|_{X}=\left.F\right|_{X}$, we conclude that $F^{n+n_{0}}(x)=\bar{F}^{n}(y)$ for every $n \geq 0$ where $y=F^{n_{0}}(x)$. Therefore,

$$
\lim _{n \rightarrow+\infty} \frac{F^{n}(x)}{n}=\lim _{n \rightarrow+\infty} \frac{F^{n+n_{0}}(x)}{n+n_{0}}=\lim _{n \rightarrow+\infty} \frac{\bar{F}^{n}(y)}{n+n_{0}}=\lim _{n \rightarrow+\infty} \frac{\bar{F}^{n}(y)}{n} \frac{n}{n+n_{0}}=\bar{\rho} .
$$

## 5. The main results

Given $0<\lambda<1$ and $\mu>0$, let

$$
r_{\lambda, \mu}:=\left\{\begin{array}{lll}
1, & \text { if } & \lambda \mu<1 \\
-\frac{\log \lambda}{\log \mu}, & \text { if } & \lambda \mu \geq 1
\end{array} .\right.
$$

Notice that $0<r_{\lambda, \mu} \leq 1$.
Definition 5.1. Let $(\lambda, \mu, b, c) \in \mathcal{Q}$ and assume that $0<\rho<r_{\lambda, \mu}$. Set

$$
\boldsymbol{a}(\lambda, \mu, b, c, \rho):=\frac{(1-\lambda)(b+(\mu b-c) \sigma)}{1+(\mu-1) \sigma},
$$

where

$$
\sigma=\sigma(\lambda, \mu, \rho):=\sum_{k \geq 1}(\lfloor(k+1) \rho\rfloor-\lfloor k \rho\rfloor) \lambda^{k} \mu^{\lfloor k \rho\rfloor} .
$$

Note that, by [19, Lemma 8],

$$
\Psi_{\rho}(\lambda, \mu)=\frac{\lambda \mu \sigma(\lambda, \mu, \rho)}{1-\lambda}
$$

where

$$
\begin{equation*}
\Psi_{\rho}(\lambda, \mu):=\sum_{k \geq 1} \sum_{1 \leq h \leq k \rho} \lambda^{k} \mu^{h}, \tag{11}
\end{equation*}
$$

viewed as a power series in the two variables $\lambda$ and $\mu$, is a Hecke-Mahler series.
It will be proved in Proposition 7.1 that the map $\rho \mapsto \boldsymbol{a}(\lambda, \mu, b, c, \rho)$ is increasing in the interval $0<\rho<r_{\lambda, \mu}$ and that it has a left discontinuity at any rational value and is right continuous everywhere. Our main result is the following.

Theorem 5.2. Let $(\lambda, \mu, b, c) \in \mathcal{Q}$. Then, the function $a \mapsto \rho\left(f_{\lambda, \mu, a, b, c}\right)$ is continuous and non-decreasing, mapping the interval ( $b-b \lambda, d_{\lambda, \mu, b, c}$ ) onto the interval $\left(0, r_{\lambda, \mu}\right)$ and satisfies the following properties:
(1) For every irrational number $\rho$ with $0<\rho<r_{\lambda, \mu}$, the rotation number $\rho\left(f_{\lambda, \mu, a, b, c}\right)$ equals $\rho$ if and only if $a=\boldsymbol{a}(\lambda, \mu, b, c, \rho)$;
(2) Let $p / q$ be a rational number with co-prime positive integers $p<q$ and $0<p / q<r_{\lambda, \mu}$. Then, $\rho\left(f_{\lambda, \mu, a, b, c}\right)=p / q$ if and only if

$$
\boldsymbol{a}\left(\lambda, \mu, b, c,(p / q)^{-}\right) \leq a \leq \boldsymbol{a}(\lambda, \mu, b, c, p / q),
$$

where

$$
\boldsymbol{a}\left(\lambda, \mu, b, c,(p / q)^{-}\right)=\frac{(1-\lambda)\left(b+(\mu b-c) \sigma\left(\lambda, \mu,(p / q)^{-}\right)\right.}{1+(\mu-1) \sigma\left(\lambda, \mu,(p / q)^{-}\right)}
$$

and

$$
\begin{aligned}
\sigma\left(\lambda, \mu,(p / q)^{-}\right) & =\sigma(\lambda, \mu, p / q))-\frac{\lambda^{q-1} \mu^{p-1}(1-\lambda)}{1-\lambda^{q} \mu^{p}} \\
\sigma(\lambda, \mu, p / q) & =\frac{1}{1-\lambda^{q} \mu^{p}} \sum_{k=1}^{q}\left(\left\lfloor(k+1) \frac{p}{q}\right\rfloor-\left\lfloor k \frac{p}{q}\right\rfloor\right) \lambda^{k} \mu^{\left\lfloor k^{\frac{p}{q}}\right\rfloor} .
\end{aligned}
$$

As a consequence of Theorem 5.2 and a classical result due to Loxton and Van der Poorten [20, 21], we obtain the following result:
Corollary 5.3. Let $(\lambda, \mu, a, b, c) \in \mathcal{P}$. If $\lambda, \mu, b, c$ and $a$ are algebraic numbers, then the rotation number $\rho\left(f_{\lambda, \mu, a, b, c}\right)$ takes a rational value.

Given a quintuple $\boldsymbol{p}=(\lambda, \mu, a, b, c) \in \mathcal{P}$, let $f=f_{\boldsymbol{p}}$ and

$$
C=C_{\boldsymbol{p}}:=\bigcap_{n \geq 0} f^{n}(I)
$$

be the limit set of $f$ and, given $x \in I$,

$$
\omega(x)=\omega\left(f_{\boldsymbol{p}}, x\right):=\bigcap_{n \geq 0} \overline{\bigcup_{k \geq n} f^{k}(x)}
$$

be the $\omega$-limit set of $x$ under $f$.
Definition 5.4. Let $\boldsymbol{p}=(\lambda, \mu, a, b, c) \in \mathcal{P}$ and $0<\rho<1$ be such that $\lambda \mu^{\rho}<1$. Let $\phi=$ $\phi_{\boldsymbol{p}, \rho}: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$
\phi(y)=\lfloor y\rfloor+\frac{a}{1-\lambda}+\frac{(1-\lambda)(c-\mu b)+a(\mu-1)}{\lambda} \Phi_{\rho}(\lambda, \mu,-\{y\}),
$$

where

$$
\Phi_{\rho}(\lambda, \mu, y)=\sum_{k \geq 0} \sum_{0 \leq l<k \rho+y} \lambda^{k} \mu^{l},
$$

with the convention that a sum indexed by an empty set equals zero.

The following result describes the dynamics of $f$ on the limit set $C$.
Theorem 5.5. Let $\boldsymbol{p}=(\lambda, \mu, a, b, c) \in \mathcal{P}$ and let $\rho=\rho\left(f_{\boldsymbol{p}}\right)$ be the rotation number of $f_{\boldsymbol{p}}$. Set $\phi=\phi_{\boldsymbol{p}, \rho}$ where $\phi_{\boldsymbol{p}, \rho}$ is defined in Definition 5.4. The following holds:
(1) If $\rho$ is irrational, then $C=\phi(I), \bar{C}$ is a Cantor set and $\omega(x)=\bar{C}$ for every $x \in I$. Moreover, $\left.f\right|_{C}$ is conjugated by $\phi$ to the rotation $R_{\rho}: x \mapsto x+\rho(\bmod 1)$, i.e., the following diagram commutes:

(2) If $\rho=p / q$ is rational with $p$ and $q$ co-prime positive integers and

$$
\boldsymbol{a}\left(\lambda, \mu, b, c,(p / q)^{-}\right) \leq a<\boldsymbol{a}(\lambda, \mu, b, c, p / q)
$$

then, for every $x \in I$,

$$
C=\omega(x)=\phi(I)=\{\phi(m / q): 0 \leq m<q\}
$$

is a cycle of order $q$ and the following diagram commutes,

where $R_{p / q}$ denotes the rotation of angle $p / q$.
(3) When $a=\boldsymbol{a}(\lambda, \mu, b, c, p / q)$, the limit set $C$ is empty, $\phi(I)$ is a finite set containing $b$ and $\omega(x)=\phi(I)$ for every $x \in I$.

## 6. Reduction of parameters

The overall idea of the proof of the results in Section 5 is that, for any $\boldsymbol{p} \in \mathcal{P}$, the dynamics of the map $f_{\boldsymbol{p}}$ is determined by its restriction $\left.f_{\boldsymbol{p}}\right|_{J}$ to the invariant interval $J:=[c, b)$. It turns out that, up to an isomorphism, the dynamics of $\left.f_{\boldsymbol{p}}\right|_{J}$ has already been studied in [19]. See Figure 4 below.

Define the map $\Theta: \mathcal{P} \rightarrow \mathbb{R}^{5}$ by

$$
\Theta(\lambda, \mu, a, b, c)=\left(\lambda, \mu, \Delta_{\lambda, b, c}(a), 1,0\right),
$$

where

$$
\begin{equation*}
\Delta_{\lambda, b, c}(a):=\frac{a-c(1-\lambda)}{b-c} \tag{12}
\end{equation*}
$$

A simple computation shows that

$$
\begin{equation*}
\Delta_{\lambda, b, c}(b-b \lambda)=1-\lambda \quad \text { and } \quad \Delta_{\lambda, b, c}\left(d_{\lambda, \mu, b, c}\right)=d_{\lambda, \mu} \tag{13}
\end{equation*}
$$

for every $(\lambda, \mu, b, c) \in \mathcal{Q}$. Notice that the second equality follows from the identity (10).
Let $h: J \rightarrow I$ be the affine map $x \mapsto(x-c) /(b-c)$. Recall that $\mathcal{P}_{0}$ is the set of parameters defined in (9) which coincides with the set of parameters defined in [19].

Lemma 6.1. $\Theta$ is a projection of $\mathcal{P}$ onto $\mathcal{P}_{0}$. Moreover, if $\boldsymbol{p} \in \mathcal{P}$, then $f_{\boldsymbol{p}}(J) \subset J$ and the following diagram commutes,


Proof. Let $\boldsymbol{p} \in \mathcal{P}$. By Remark 3.3, $\Theta(\boldsymbol{p}) \in \mathcal{P}_{0}$ if and only if $\Delta_{\lambda, b, c}(a) \in\left(1-\lambda, d_{\lambda, \mu}\right)$. Since $b-b \lambda<a<d_{\lambda, \mu, b, c}$ and the fact that $a \mapsto \Delta_{\lambda, b, c}(a)$ is increasing, we see that $\Delta_{\lambda, b, c}(b-b \lambda)<$ $\Delta_{\lambda, b, c}(a)<\Delta_{\lambda, b, c}\left(d_{\lambda, \mu, b, c}\right)$. By (13) we conclude that $1-\lambda<\Delta_{\lambda, b, c}(a)<d_{\lambda, \mu}$. Therefore, $\Theta(\boldsymbol{p}) \in \mathcal{P}_{0}$. Because $\Theta(\Theta(\boldsymbol{p}))=\Theta(\boldsymbol{p})$, we see that $\Theta$ is a projection of $\mathcal{P}$ onto $\mathcal{P}_{0}$. Now, taking into account that $\eta=(b-a) / \lambda$, we have that

$$
f_{\boldsymbol{p}}(\eta)=c<\eta<b=f_{\boldsymbol{p}}\left(\eta^{-}\right)
$$

Therefore, $f_{\boldsymbol{p}}(J) \subset J$ (see Figure 4). Finally, checking that the diagram above commutes is a simple exercise.


Figure 4. (A) Plot of $f_{\boldsymbol{p}}$ and the square $J^{2}$ in red. (B) Zoom of the square $J^{2}$ using the affine map $h$ and plot of $f_{\Theta(\boldsymbol{p})}$.

Let $\boldsymbol{p}=(\lambda, \mu, a, b, c) \in \mathcal{P}$. Using Lemma 6.1, we can reduce the study of the 2-interval piecewise affine map $f_{\boldsymbol{p}}$ to the study of the 2 -interval piecewise affine map $f_{\Theta(\boldsymbol{p})}$ that has just 3 parameters. Because the rotation number is invariant by conjugacy we have the following result.

Lemma 6.2. Let $\boldsymbol{p}=(\lambda, \mu, a, b, c) \in \mathcal{P}$. Then $\rho\left(f_{\boldsymbol{p}}\right)=\rho\left(f_{\Theta(\boldsymbol{p})}\right)$.
Proof. Follows from Lemma 4.1 and Lemma 6.1.
The map $f_{\Theta(\boldsymbol{p})}$ belongs to a family of maps already studied in [19] and whose rotation number has been described as a function of the parameters $(\lambda, \mu, \delta)$ where $\delta=\Delta_{\lambda, b, c}(a)$.

## 7. Properties of $\boldsymbol{a}$ and $\phi$

Recalling Definition 5.1, the following result describes $\boldsymbol{a}$ as a function of $\rho$.
Proposition 7.1. Let $(\lambda, \mu, b, c) \in \mathcal{Q}$ and assume that $0<\rho<r_{\lambda, \mu}$. The function $\rho \mapsto$ $\boldsymbol{a}(\lambda, \mu, b, c, \rho)$ is strictly increasing and right continuous on the interval $\left(0, r_{\lambda, \mu}\right)$, continuous at every irrational $\rho \in\left(0, r_{\lambda, \mu}\right)$ and maps the interval $\left(0, r_{\lambda, \mu}\right)$ inside the interval $\left(b-b \lambda, d_{\lambda, \mu, b, c}\right)$, with limit values

$$
\boldsymbol{a}\left(\lambda, \mu, b, c, 0^{+}\right)=b-b \lambda \quad \text { and } \quad \boldsymbol{a}\left(\lambda, \mu, b, c, r_{\lambda, \mu}^{-}\right)=d_{\lambda, \mu, b, c} .
$$



Figure 5. Plot of the map $\rho \mapsto \boldsymbol{a}(2 / 3,1 / 2,3 / 4,1 / 4, \rho)$ for $\rho \in(0,1)$. The image is contained in the interval $(b-b \lambda, b-c \lambda)=(1 / 4,7 / 12)$.

Proof. Recalling $\Delta_{\lambda, b, c}$ from (12), notice that

$$
\begin{equation*}
\boldsymbol{a}(\lambda, \mu, b, c, \rho)=\Delta_{\lambda, b, c}^{-1}(\boldsymbol{\delta}(\lambda, \mu, \rho))=\boldsymbol{\delta}(\lambda, \mu, \rho)(b-c)+c(1-\lambda) \tag{14}
\end{equation*}
$$

where

$$
\boldsymbol{\delta}(\lambda, \mu, \rho)=\frac{(1-\lambda)(1+\mu \sigma(\lambda, \mu, \rho))}{1+(\mu-1) \sigma(\lambda, \mu, \rho)} .
$$

By [19, Corollary on page 40], for every $0<\lambda<1$ and $\mu>0$, the function $\left(0, r_{\lambda, \mu}\right) \ni \rho \mapsto$ $\boldsymbol{\delta}(\lambda, \mu, \rho)$ is strictly increasing and its image is contained inside the interval $\left(\boldsymbol{\delta}\left(\lambda, \mu, 0^{+}\right), \boldsymbol{\delta}\left(\lambda, \mu, r_{\lambda, \mu}^{-}\right)\right)$ with limit values

$$
\boldsymbol{\delta}\left(\lambda, \mu, 0^{+}\right)=1-\lambda \quad \text { and } \quad \boldsymbol{\delta}\left(\lambda, \mu, r_{\lambda, \mu}^{-}\right)=d_{\lambda, \mu} .
$$

Taking into account (14) and the identities (13), Proposition 7.1 follows.

Recalling Definition 5.4, the following result enumerates some properties of the conjugacy $\phi$.
Proposition 7.2. Let $\boldsymbol{p}=(\lambda, \mu, a, b, c) \in \mathcal{P}$ and let $0<\rho<r_{\lambda, \mu}$ be the rotation number of $f_{\boldsymbol{p}}$. Then the function $\phi=\phi_{\boldsymbol{p}, \rho}$ satisfies the following properties:
(1) $\phi$ - id is 1-periodic,
(2) $\phi$ is right continuous and non-decreasing,
(3) $\phi$ is strictly increasing if $\rho$ is irrational,
(4) $\phi$ is constant on each interval $\left[\frac{n}{q}, \frac{n+1}{q}\right), n \in \mathbb{Z}$ provided $\rho=\frac{p}{q}$ is rational,
(5) $\phi(0) \geq c$ and $\phi\left(1^{-}\right) \leq b$. Moreover equality holds in both inequalities when $\rho$ is irrational,
(6) for any $y \in \mathbb{R}$, we have the relations

$$
\lfloor\phi(y)\rfloor=\lfloor y\rfloor, \quad\{\phi(y)\}=\phi(\{y\}) \quad \text { and } \quad \phi(y+\rho)=F(\phi(y)) .
$$



Figure 6. Graph of the function $\phi_{\boldsymbol{p}, \rho}(y)$ for the parameters

$$
\lambda=0.8, \mu=0.9, b=0.9, c=0.1, a=\boldsymbol{a}(0.8,0.9,0.9,0.1, \sqrt{2}-1)=0.43557 \ldots
$$

and $\rho=\sqrt{2}-1=0.414 \ldots$ in the interval $y \in(0,1)$. The function $\phi$ increases from $\phi\left(0^{+}\right)=c=0.1$ to $\phi\left(1^{-}\right)=b=0.9$.

Proof. Recall from Section 6 the affine map $h(x)=(x-c) /(b-c)$. For any $y \in \mathbb{R}$, a straightforward computation shows that

$$
\phi(y)=\lfloor y\rfloor+\phi(\{y\})=\lfloor y\rfloor+h^{-1}(\varphi(\{y\}))=\lfloor y\rfloor+(b-c) \varphi(\{y\})+c
$$

where $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is the function

$$
\varphi(y)=\lfloor y\rfloor+\frac{\delta}{1-\lambda}-\frac{\delta-\mu(\lambda+\delta-1)}{\lambda} \Phi_{\rho}(\lambda, \mu,-\{y\})
$$

as defined in [19, Lemma 9$]$ and $\delta:=\Delta_{\lambda, b, c}(a)$. Since $1-\lambda<\delta<d_{\lambda, \mu}$, all claimed properties follow from [19]. We only give a detailed proof of the assertion (6), the others being simpler. The equalities $\lfloor\phi(y)\rfloor=\lfloor y\rfloor$ and $\{\phi(y)\}=\phi(\{y\})$ clearly follow from (1), (2) and (5). It remains to prove the functional equation $\phi(y+\rho)=F(\phi(y))$.

Observe first that $\phi(\{y\})<\eta$ if $\{y\}<1-\rho$, while $\phi(\{y\}) \geq \eta$ if $\{y\} \geq 1-\rho$. Indeed [19, Lemma 11 and 12], applied to $f_{\Theta(p)}$, shows that the inequalities

$$
\varphi(x)<\frac{1-\delta}{\lambda}=\frac{\eta-c}{b-c}=h(\eta) \quad \text { and } \quad 0 \leq x<1
$$

hold if and only if $0 \leq x<1-\rho$. We make use of the functional equation for $\varphi$, given by [19, Lemma 14], which reads

$$
\varphi(x+\rho)= \begin{cases}\lambda \varphi(x)+\delta+(1-\lambda)\lfloor x\rfloor, & \text { if } \quad\{x\}<1-\rho  \tag{15}\\ \lambda \mu \varphi(x)+\mu(\delta-1)+1+(1-\lambda \mu)\lfloor x\rfloor, & \text { if } \quad\{x\} \geq 1-\rho\end{cases}
$$

for any $x \in \mathbb{R}$.
Assume first that $\{y\}<1-\rho$. Then, $\lfloor y+\rho\rfloor=\lfloor y\rfloor$ and $\{y+\rho\}=\{y\}+\rho$, so that by (15)

$$
\begin{aligned}
\phi(y+\rho) & =\lfloor y\rfloor+(b-c) \varphi(\{y\}+\rho)+c=\lfloor y\rfloor+(b-c)(\lambda \varphi(\{y\})+\delta)+c \\
& =\lfloor y\rfloor+\lambda(b-c) \varphi(\{y\})+a+c \lambda=\lfloor y\rfloor+\lambda(\phi(y)-\lfloor y\rfloor-c)+a+c \lambda \\
& =\lambda \phi(y)+a+(1-\lambda)\lfloor y\rfloor=F(\phi(y))
\end{aligned}
$$

since $\{\phi(y)\}<\eta$. Assume now that $\{y\} \geq 1-\rho$. In this case, we have $\lfloor y+\rho\rfloor=\lfloor y\rfloor+1$ and $\{y+\rho\}=\{y\}+\rho-1$, so that by (15)

$$
\begin{aligned}
\phi(y+\rho) & =\lfloor y\rfloor+1+(b-c) \varphi(\{y\}-1+\rho)+c=\lfloor y\rfloor+1+(b-c)(\varphi(\{y\}+\rho)-1)+c \\
& =\lfloor y\rfloor+1+(b-c)(\lambda \mu \varphi(\{y\})+\mu(\delta-1))+c \\
& =\lfloor y\rfloor+(b-c) \lambda \mu \varphi(\{y\})+\mu(a-b+c \lambda)+c+1 \\
& =\lfloor y\rfloor+\lambda \mu(\phi(y)-\lfloor y\rfloor-c)+\mu(a-b+c \lambda)+c+1 \\
& =\lambda \mu \phi(y)-\lambda \mu \eta+c+1+(1-\lambda \mu)\lfloor y\rfloor=F(\phi(y)),
\end{aligned}
$$

since $\{\phi(y)\} \geq \eta$, replacing $\delta=\frac{a-c(1-\lambda)}{b-c}$ and $\eta=\frac{b-a}{\lambda}$.

## 8. Proofs of main Results

8.1. Proof of Theorem 5.2. Let $(\lambda, \mu, b, c) \in \mathcal{Q}$. For every $a \in\left(b-b \lambda, d_{\lambda, \mu, b, a}\right)$, Lemma 6.2 asserts that $\rho\left(f_{\boldsymbol{p}}\right)=\rho\left(f_{\Theta(\boldsymbol{p})}\right)$ where $\boldsymbol{p}=(\lambda, \mu, a, b, c)$ and $\Theta(\boldsymbol{p})=\left(\lambda, \mu, \Delta_{\lambda, b, c}(a), 1,0\right)$. By Lemma 6.1, we know that $1-\lambda<\Delta_{\lambda, b, c}(a)<d_{\lambda, \mu}$, i.e., the triple $\left(\lambda, \mu, \Delta_{\lambda, b, c}(a)\right)$ belongs to the parameter set of [19]. By [19, Theorem 3], the mapping $\delta \in\left(1-\lambda, d_{\lambda, \mu}\right) \mapsto \rho\left(f_{\lambda, \mu, \delta, 1,0}\right)$ is continuous, non-decreasing and its image equals the interval $\left(0, r_{\lambda, \mu}\right)$. Since $a \mapsto \Delta_{\lambda, b, c}(a)$ is an increasing affine map sending $\left(b-b \lambda, d_{\lambda \mu, b, c}\right)$ onto $\left(1-\lambda, d_{\lambda, \mu}\right)$ and $\rho\left(f_{\lambda, \mu, a, b, c}\right)=\rho\left(f_{\lambda, \mu, \Delta_{\lambda, b, c}(a), 1,0}\right)$, we conclude that the function $\left(b-b \lambda, d_{\lambda, \mu, b, a}\right) \ni a \mapsto \rho\left(f_{\lambda, \mu, a, b, c}\right)$ is continuous, non-decreasing and its image equals the interval $\left(0, r_{\lambda, \mu}\right)$. Finally, to prove properties (1) and (2), observe that

$$
\boldsymbol{\delta}(\lambda, \mu, \rho)=\Delta_{\lambda, b, c}(\boldsymbol{a}(\lambda, \mu, b, c, \rho))
$$

where

$$
\boldsymbol{\delta}(\lambda, \mu, \rho)=\frac{(1-\lambda)(1+\mu \sigma(\lambda, \mu, \rho))}{1+(\mu-1) \sigma(\lambda, \mu, \rho)}
$$

is the function defined in [19, Definition 2]. Therefore, (1) and (2) in the statement follow from (1) and (2) of [19, Theorem 3].
8.2. Proof of Corollary 5.3. Assume, by contradiction, that the rotation number $\rho=\rho\left(f_{\lambda, \mu, a, b, c}\right)$ is irrational. Then, by Theorem 5.2 (1),

$$
a=\frac{(1-\lambda)\left(b \lambda \mu+(1-\lambda)(\mu b-c) \Psi_{\rho}(\lambda, \mu)\right)}{\lambda \mu+(\mu-1)(1-\lambda) \Psi_{\rho}(\lambda, \mu)}
$$

where $\Psi_{\rho}$ is the Hecke-Mahler series defined in (11). As the coefficients $\lambda, \mu, b, c$ and $a$ are algebraic numbers, we conclude that $\Psi_{\rho}(\lambda, \mu)$ also takes an algebraic value. However, by [19, Theorem 10], in this case $\Psi_{\rho}(\lambda, \mu)$ has to be transcendental, thus it is a contradiction. So $\rho=\rho\left(f_{\lambda, \mu, a, b, c}\right)$ takes a rational value.
8.3. Proof of Theorem 5.5. Let $\boldsymbol{p}=(\lambda, \mu, a, b, c)$ with $(\lambda, \mu, b, c) \in \mathcal{Q}$ and $a \in\left(b-b \lambda, d_{\lambda, \mu, b, a}\right)$. By Lemma 6.1, $f_{\boldsymbol{p}}$ restricted to the interval $J$ is conjugated by the affine map $h: J \rightarrow I$ to the map $f_{\Theta(\boldsymbol{p})}$ with $\Theta(\boldsymbol{p})=\left(\lambda, \mu, \Delta_{\lambda, b, c}(a), 1,0\right) \in \mathcal{P}_{0}$ where $\Delta_{\lambda, b, c}$ is defined in (12). Let $g_{\lambda, \mu, \delta}=f_{\Theta(\boldsymbol{p})}$ where $\delta:=\Delta_{\lambda, b, c}(a)$. Because $\Theta(\boldsymbol{p}) \in \mathcal{P}_{0}$, we have $1-\lambda<\delta<d_{\lambda, \mu}$ (See Remark 3.3). Hence, the parameters $(\lambda, \mu, \delta)$ belong to the parameter set of [19]. Let $\rho=\rho\left(f_{\boldsymbol{p}}\right)$ and notice that $\rho=\rho\left(g_{\lambda, \mu, \delta}\right)$ by Lemma 6.2. Assume that $\rho$ is irrational. Applying [19, Theorem $6]$ to the map $g_{\lambda, \mu, \delta}$, we obtain the following commutative diagram,

where $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is the function

$$
\varphi(y)=\lfloor y\rfloor+\frac{\delta}{1-\lambda}-\frac{\delta-\mu(\lambda+\delta-1)}{\lambda} \Phi_{\rho}(\lambda, \mu,-\{y\})
$$

as defined in [19, Lemma 9], and $\overline{\varphi(I)}$ is a Cantor set which is equal to the $\omega$-limit set $\omega\left(f_{\Theta(\boldsymbol{p})}, x\right)$ of every $x \in I$. Recall from Section 6 the affine map $h: J \rightarrow I$. For any $y \in \mathbb{R}$, we have that $\phi(\{y\})=h^{-1}(\varphi(\{y\}))$. Taking into account that $\overline{h^{-1}(\varphi(I))}$ is also a Cantor set and $\omega\left(f_{\boldsymbol{p}}, x\right)=h^{-1}\left(\omega\left(f_{\Theta(\boldsymbol{p})}, h(x)\right)\right.$ for every $x \in J$, we obtain assertion (1) of Theorem 5.5. The assertions (2) and (3) regarding the rational case, i.e., when $\rho$ is rational, can be deduced in a similar way from [19, Theorem 6]. Thus, we conclude the proof of the theorem.

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