# HAUSDORFF DIMENSION OF THE EXCEPTIONAL SET OF INTERVAL PIECEWISE AFFINE CONTRACTIONS 

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#### Abstract

Let $I=[0,1),-1<\lambda<1$ and $f: I \rightarrow I$ be a piecewise $\lambda$-affine map of the interval $I$, i.e., there exist a partition $0=a_{0}<a_{1}<\cdots<a_{k-1}<a_{k}=1$ of the interval $I$ into $k \geq 2$ subintervals and $b_{1}, \ldots, b_{k} \in \mathbb{R}$ such that $f(x)=\lambda x+b_{i}$ for every $x \in\left[a_{i-1}, a_{i}\right)$ and $i=1, \ldots, k$. The exceptional set $\mathcal{E}_{f}$ of $f$ is the set of parameters $\delta \in \mathbb{R}$ such that $R_{\delta} \circ f$ is not asymptotically periodic, where $R_{\delta}: I \rightarrow I$ is the rotation of angle $\delta$. In this paper we prove that $\mathcal{E}_{f}$ has zero Hausdorff dimension. We derive this result from a more general theorem concerning piecewise Lipschitz contractions on $\mathbb{R}$ that has independent interest.


## 1. Introduction

Let $I=[0,1)$ and $-1<\lambda<1$. We say that an interval map $f: I \rightarrow I$ is piecewise $\lambda$-affine if there exist $k \geq 2$, real numbers $b_{1}, \ldots, b_{k}$ and a partition of the interval $I$,

$$
0=a_{0}<a_{1}<\cdots<a_{k-1}<a_{k}=1,
$$

such that $f(x)=\lambda x+b_{i}$ for every $x \in\left[a_{i-1}, a_{i}\right)$ and $i=1, \ldots, k$. Given a piecewise $\lambda$-affine map $f$, consider the one-parameter family of piecewise $\lambda$-affine maps $f_{\delta}: I \rightarrow I$ defined by

$$
f_{\delta}=R_{\delta} \circ f,
$$

where $R_{\delta}(x)=\{x+\delta\}$ is the rotation of angle $\delta \in \mathbb{R}$ and $\{\cdot\}$ denotes the fractional part. See Figure 1 for an illustration of the graph of $f_{\delta}$.

We are interested in the dimension of the exceptional set

$$
\mathcal{E}_{f}=\left\{\delta \in \mathbb{R}: f_{\delta} \text { is not asymptotically periodic }\right\} .
$$

A map $f: I \rightarrow I$ is asymptotically periodic if $f$ has at most a finite number of periodic orbits and the $\omega$-limit set $\omega(f, x)$ of any $x \in I$ is a periodic orbit of $f$. We recall that $\omega(f, x)$ is the set of accumulation points of the forward orbit of $x$ under $f$. It is known that $\mathcal{E}_{f}$ has zero Lebesgue measure [10, Theorem 1.1], but the question of the Hausdorff dimension of $\mathcal{E}_{f}$ was still open.


Figure 1. Plots of $f_{\delta}$.

In this paper, we settle this question.

## Theorem A.

$$
\operatorname{dim}_{H} \mathcal{E}_{f}=0
$$

A notable example of piecewise $\lambda$-affine maps is the family of contracted rotations, $f(x)=\{\lambda x+b\}$ with $0<\lambda<1$ and $1-\lambda<$ $b<1$. Contracted rotations have been extensively studied by many authors either as a dynamical system or related to applications, e.g. [3, 4, 9, [8, 5, 2]. In the case of contracted rotations, the exceptional set $\mathcal{E}_{f}$ has a Cantor structure and in this case Theorem A was proved by Laurent and Nogueira in [9, Theorem 5] by exploiting a combinatorial structure, associated to $\mathcal{E}_{f}$, which is reminiscent of the classical SternBrocot tree. Janson and Öberg improved in [8] the result of Laurent and Nogueira by considering other gauge functions in the definition of the Hausdorff dimension.

For general piecewise $\lambda$-affine maps, Theorem A was proved in [1] by Pires using the theory of $b$-adic expansions and under the assumptions that $f$ is injective, $\lambda^{-1}=b$ is a positive integer $\geq k$ and the connected components of $I \backslash f(I)$ have equal length. In order to remove all these assumptions and prove Theorem A for any piecewise $\lambda$-affine map $f$ we use a different strategy inspired by a recent work dealing with piecewise increasing contractions [7].

We will deduce Theorem A from a more general result, Theorem B below. We say that a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is a piecewise contraction if it has a finite number of discontinuity points and on each connected component $D$ of the domain of continuity of $f$ the restriction map $\left.f\right|_{D}$ is a Lipschitz contraction. Let $k \geq 2$ and

$$
\Omega_{k}=\left\{\left(a_{1}, \ldots, a_{k-1}\right) \in \mathbb{R}^{k-1}: a_{1}<\cdots<a_{k-1}\right\} .
$$

A finite collection of Lipschitz contractions $\Phi=\left\{\phi_{i}: \mathbb{R} \rightarrow \mathbb{R}\right\}_{i=1}^{k}$ is called an iterative function system (IFS). An IFS $\Phi=\left\{\phi_{1}, \ldots, \phi_{k}\right\}$ together with $\boldsymbol{a} \in \Omega_{k}$, determine a piecewise contraction $f=f_{\Phi, \boldsymbol{a}}: \mathbb{R} \rightarrow$
$\mathbb{R}$ defined by

$$
f(x)= \begin{cases}\phi_{1}(x), & x \in\left(-\infty, a_{1}\right)  \tag{1}\\ \phi_{i}(x), & x \in\left[a_{i-1}, a_{i}\right) \quad i \in\{2, \ldots, k-1\}, \\ \phi_{k}(x), & x \in\left[a_{k-1},+\infty\right)\end{cases}
$$

Notice that $(\Phi, \boldsymbol{a}) \mapsto f_{\Phi, \boldsymbol{a}}$ is not injective, i.e., a piecewise contraction is not uniquely determined by a pair $(\Phi, \boldsymbol{a})$. In [10] the authors prove the following result.

Theorem 1.1 ([10, Theorem 1.4]). Let $\Phi=\left\{\phi_{1}, \ldots, \phi_{k}\right\}$ be an IFS. There is a Lebesgue full measure set $W \subset \mathbb{R}$ such that for every $\boldsymbol{a} \in$ $\Omega_{k} \cap W^{k-1}$, the map $f_{\Phi, a}$ is asymptotically periodic and has at most $k$ periodic orbits.

We consider a very specific perturbation of $f_{\Phi, \boldsymbol{a}}$. To simplify the notation, we shall write $\boldsymbol{a}+\delta=\left(a_{1}+\delta, \ldots, a_{k-1}+\delta\right)$. Notice that $\boldsymbol{a}+\delta \in \Omega_{k}$ for every $\delta \in \mathbb{R}$.

We say that an IFS is injective if all its contractions are injective functions. Under the assumption that the IFS is injective, we prove the following result.

Theorem B. Let $\Phi=\left\{\phi_{1}, \ldots, \phi_{k}\right\}$ be an injective IFS and $\boldsymbol{a} \in \Omega_{k}$. Then

$$
\operatorname{dim}_{H}\left\{\delta \in \mathbb{R}: f_{\Phi, a+\delta} \text { is not asymptotically periodic }\right\}=0 .
$$

The injectivity assumption of Theorem B can be weakened to the assumption that the functions of the IFS have a finite number of local extrema. This hypothesis guarantees that the pre-image of any point is a finite set, a crucial property that we use in our arguments to prove Theorem B By Sard's theorem for Lipschitz functions, the pre-image of almost every point is a finite set. However, this property is not sufficient for our arguments to work in the general situation. The injectivity of the IFS is a natural condition, and piecewise contractions appearing in applications satisfy this condition. Moreover, assuming the injectivity of the IFS, the authors in [6] have obtained a spectral decomposition of the attractor of a piecewise contraction, i.e., the attractor is a finite union of periodic orbits together with a finite union of Cantor sets. Our Theorem B excludes the Cantor attractors in a very strong metric sense, i.e., piecewise contractions can only have Cantor attractors for a parameter set of zero Hausdorff dimension.

As a final remark, the definition (1) of $f$ at the points $\left\{a_{1}, \ldots, a_{k-1}\right\}$ is not relevant for proving Theorem $\bar{B}$, i.e., the proof of Theorem $B$ can be adapted to any other choice of values $f\left(a_{i}\right) \in\left\{f\left(a_{i}^{-}\right), f\left(a_{i}^{+}\right)\right\}$.

The rest of the paper is organized as follows. In Section 2 we collect the lemmas that we need to prove Theorem B and complete its proof in Section 3. Then, in Section 4, we use Theorem B to deduce Theorem A.

## 2. Preliminary results

Throughout this section, let $\Phi=\left\{\phi_{1}, \ldots, \phi_{k}\right\}$ be an injective IFS and $\boldsymbol{a}=\left(a_{1}, \ldots, a_{k-1}\right) \in \Omega_{k}$ with $k \geq 2$. Define

$$
\lambda_{\Phi}:=\max _{i} \operatorname{Lip}\left(\phi_{i}\right) \quad \text { and } \quad r_{\Phi}:=\frac{1+\lambda_{\Phi}}{1-\lambda_{\Phi}} \max _{i}\left|z_{i}\right|
$$

where $\operatorname{Lip}\left(\phi_{i}\right)$ denotes the Lipschitz constant of $\phi_{i}$ and $z_{i}$ is the unique fixed point of $\phi_{i}$. Clearly, $0 \leq \lambda_{\Phi}<1$.

Our first observation is that the recurrent dynamics of $f_{\Phi, a}$ occurs inside an attracting compact interval

$$
K_{\Phi}:=\left[-2 r_{\Phi}, 2 r_{\Phi}\right]
$$

Given $n \in \mathbb{N}$ and $\omega \in\{1, \ldots, k\}^{n}$, let

$$
\phi_{\omega}:=\phi_{\omega_{n}} \circ \cdots \circ \phi_{\omega_{1}} .
$$

Lemma 2.1. The following holds:
(1) $\phi_{\omega}\left(K_{\Phi}\right) \subset K_{\Phi}$ for every $\omega \in\{1, \ldots, k\}^{n}$ and $n \in \mathbb{N}$,
(2) $f_{\Phi, a}\left(K_{\Phi}\right) \subset K_{\Phi}$,
(3) For every $x \in \mathbb{R}$ there is $n \geq 0$ such that $f_{\Phi, a}^{n}(x) \in K_{\Phi}$,
(4) $\omega\left(f_{\Phi, \boldsymbol{a}}, x\right) \subset K_{\Phi}$ for every $x \in \mathbb{R}$.

Proof. To show (1) let $|x| \leq 2 r_{\Phi}$. Then

$$
\begin{aligned}
\left|\phi_{i}(x)\right|=\left|\phi_{i}(x)-z_{i}+z_{i}\right| & \leq \lambda_{\Phi}\left|x-z_{i}\right|+\left|z_{i}\right| \\
& \leq 2 \lambda_{\Phi} r_{\Phi}+\left|z_{i}\right|\left(1+\lambda_{\Phi}\right) \\
& \leq 2 \lambda_{\Phi} r_{\Phi}+\max _{i}\left|z_{i}\right|\left(1+\lambda_{\Phi}\right) \\
& =2 \lambda_{\Phi} r_{\Phi}+\left(1-\lambda_{\Phi}\right) r_{\Phi} \\
& =\left(1+\lambda_{\Phi}\right) r_{\Phi} \\
& <2 r_{\Phi}
\end{aligned}
$$

which proves (1). Item (2) follows immediately from (1). Next, to prove (3), let $|x| \leq r_{\Phi}$. Repeating the above estimates we get $\left|\phi_{i}(x)\right| \leq r_{\Phi}$. For any $\omega \in\{1, \ldots, k\}^{n}$ and $n \in \mathbb{N}$, this shows that $\left|\phi_{\omega}(x)\right| \leq r_{\Phi}$ whenever $|x| \leq r_{\Phi}$. Consequently, the unique fixed point $z_{\omega}$ of $\phi_{\omega}$ satisfies $\left|z_{\omega}\right| \leq r_{\Phi}$. Now, given any $x \in \mathbb{R}$, choose $n \geq 1$ such that $\lambda_{\Phi}^{n}<\frac{r_{\Phi}}{|x|+r_{\Phi}}$. Then, for any $\omega \in\{1, \ldots, k\}^{n}$

$$
\begin{aligned}
\left|\phi_{\omega}(x)\right| & =\left|\phi_{\omega}(x)-z_{\omega}+z_{\omega}\right| \\
& \leq \lambda_{\Phi}^{n}\left|x-z_{\omega}\right|+\left|z_{\omega}\right| \\
& \leq \lambda_{\Phi}^{n}\left(|x|+r_{\Phi}\right)+r_{\Phi} \\
& <2 r_{\Phi} .
\end{aligned}
$$

This proves (3). Finally, (4) follows immediately from (2) and (3).
Let

$$
S_{a}:=\left\{a_{1}, \ldots, a_{k-1}\right\}
$$

Definition 2.1. We say that the pair $(\Phi, \boldsymbol{a})$ has a singular connection if there exist $n \in \mathbb{N}$ and $\omega \in\{1, \ldots, k\}^{n}$ such that

$$
\phi_{\omega}\left(S_{\boldsymbol{a}}\right) \cap S_{\boldsymbol{a}} \neq \emptyset
$$

Lemma 2.2. The set

$$
\{\delta \in \mathbb{R}:(\Phi, \boldsymbol{a}+\delta) \text { has a singular connection }\}
$$

is countable.
Proof. Given $\omega \in\{1, \ldots, k\}^{n}$, the map $\phi_{\omega}$ is a Lipschitz contraction on $\mathbb{R}$. Given $(i, j) \in\{1, \ldots, k-1\}^{2}$, the map $\mathbb{R} \ni \delta \mapsto \phi_{\omega}\left(a_{i}+\delta\right)-a_{j}$ is also a Lipschitz contraction on $\mathbb{R}$, hence it has a unique fixed point, say $z_{\omega, i, j} \in \mathbb{R}$. Let

$$
\Delta:=\bigcup_{n \in \mathbb{N}} \bigcup_{\omega \in\{1, \ldots, k\}^{n}} \bigcup_{i=1}^{k-1} \bigcup_{j=1}^{k-1}\left\{z_{\omega, i, j}\right\}
$$

Observe that $(\Phi, \boldsymbol{a}+\delta)$ has a singular connection if and only if $\delta \in \Delta$. Since $\Delta$ is a countable set, the claim follows.

Definition 2.2. Given $x \in K_{\Phi}$ and $n \in \mathbb{N}$, we say that $x$ is an $n$-regular point of $(\Phi, \boldsymbol{a})$ if $f_{\Phi, \boldsymbol{a}}^{j}(x) \notin S_{\boldsymbol{a}}$ for every $0 \leq j<n$.

Let $D_{a}^{(0)}:=S_{\boldsymbol{a}}, D_{a}^{(n)}:=f_{\Phi, \boldsymbol{a}}^{-1}\left(D_{a}^{(n-1)}\right)$ for $n \geq 1$ and

$$
Q_{a}^{(n)}:=\bigcup_{i=0}^{n-1} D_{\boldsymbol{a}}^{(i)}, \quad n \in \mathbb{N} .
$$

Notice that the sets $\left\{Q_{a}^{(n)}\right\}_{n \geq 0}$ are finite. Indeed, this follows from the fact that $S_{a}$ is finite and $\Phi$ is injective. Given $x \in K_{\Phi}$, it is also clear that $x$ is an $n$-regular point of $(\Phi, \boldsymbol{a})$ if and only if $x \notin Q_{a}^{(n)}$.

Let $X_{1}:=\left(-\infty, a_{1}\right), X_{i}:=\left(a_{i-1}, a_{i}\right)$ with $i=2, \ldots, k-1$, and $X_{k}:=\left(a_{k-1},+\infty\right)$. By construction, these open intervals are disjoint and their union equals $\mathbb{R} \backslash S_{a}$.

Definition 2.3. Given $n \in \mathbb{N}$, a tuple $\left(\omega_{0}, \omega_{1}, \ldots, \omega_{n-1}\right) \in\{1, \ldots, k\}^{n}$ is called an itinerary of order $n$ of $(\Phi, \boldsymbol{a})$ if there is an $n$-regular point $x$ of $(\Phi, \boldsymbol{a})$ such that $f_{\Phi, \boldsymbol{a}}^{j}(x) \in X_{\omega_{j}}$ for every $0 \leq j<n$.

We define the set of all itineraries of order $n$ of $(\Phi, \boldsymbol{a})$,
$\mathcal{I}_{\Phi, a}^{(n)}:=\left\{\omega \in\{1, \ldots, k\}^{n}: \omega\right.$ is an itinerary of order $n$ of $\left.(\Phi, \boldsymbol{a})\right\}$.
The set $\mathcal{I}_{\Phi, \boldsymbol{a}}^{(n)}$ is in a one-to-one correspondence with the set of connected components of $K_{\Phi} \backslash Q_{a}^{(n)}$. Indeed, for each connected component $J$ of $K_{\Phi} \backslash Q_{a}^{(n)}$, all points in $J$ are $n$-regular points of $(\Phi, \boldsymbol{a})$ and because the sets $J, f_{\Phi, \boldsymbol{a}}(J), \ldots, f_{\Phi, \boldsymbol{a}}^{n-1}(J)$ are subintervals of $K_{\Phi}$ not intersecting $S_{\boldsymbol{a}}$, we conclude that there is an itinerary of order $n$ of $(\Phi, \boldsymbol{a})$, say $\left(\omega_{0}, \omega_{1}, \ldots, \omega_{n-1}\right) \in \mathcal{I}_{\Phi, \boldsymbol{a}}^{(n)}$, such that $f_{\Phi, \boldsymbol{a}}^{j}(J) \subset X_{\omega_{j}}$ for every $0 \leq j<n$.

Given $\varepsilon \geq 0$, we enlarge the set $\mathcal{I}_{\Phi, a}^{(n)}$ as follows,

$$
\mathcal{I}_{\Phi, \boldsymbol{a}}^{(n)}(\varepsilon):=\bigcup_{|\delta| \leq \varepsilon} \mathcal{I}_{\Phi, a+\delta}^{(n)}
$$

The following result establishes that the number of itineraries grows subexponentially, a crucial property to prove Theorem B.

Lemma 2.3. Suppose that $(\Phi, \boldsymbol{a})$ has no singular connections. Then

$$
\lim _{\varepsilon \rightarrow 0^{+}} \limsup _{n \rightarrow+\infty} \frac{1}{n} \log \# \mathcal{I}_{\Phi, \boldsymbol{a}}^{(n)}(\varepsilon)=0
$$

Proof. Given $\rho>1$, let $m=\lceil\log 2 / \log \rho\rceil$ and $\tau(m, \boldsymbol{a})>0$ be the minimum distance between any pair of distinct points of $Q_{a}^{(m)}$. Notice that the sets $\left\{D_{a}^{(n)}\right\}_{n \geq 0}$ are pairwise disjoint. Indeed, suppose that there is $x \in D_{a}^{\left(n_{1}\right)} \cap D_{a}^{\left(n_{2}\right)}$ with $n_{2}>n_{1} \geq 0$. Then $f_{\Phi, \boldsymbol{a}}^{n_{1}}(x)=a_{i}$ and $f_{\Phi, a}^{n_{2}}(x)=a_{j}$ for some $i, j \in\{1, \ldots, k-1\}$. This implies that $f_{\Phi, \boldsymbol{a}}^{n_{2}-n_{1}}\left(a_{i}\right)=a_{j}$, contradicting the assumption that $(\Phi, \boldsymbol{a})$ has no singular connections. Since $Q_{a}^{(m)}=D_{a}^{(0)} \cup D_{a}^{(1)} \cup \cdots \cup D_{a}^{(m-1)}$ and $\Phi$ is injective, there is an $\varepsilon_{0}=\varepsilon_{0}(m)>0$ such that the set-valued map $\left(-\varepsilon_{0}, \varepsilon_{0}\right) \ni \delta \mapsto Q_{a+\delta}^{(m)}$ varies continuously in the Hausdorff metric of compact subsets of $\mathbb{R}$, and for every $|\delta|<\varepsilon_{0}$, the set $Q_{a+\delta}^{(m)}$ has the same number of elements of $Q_{a}^{(m)}$ and

$$
\tau(m):=\inf _{|\delta|<\varepsilon_{0}} \tau(m, \boldsymbol{a}+\delta)>0
$$

Let $\alpha_{n}(\varepsilon):=\# \mathcal{I}_{\Phi, \boldsymbol{a}}^{(n)}(\varepsilon)$ with $0 \leq \varepsilon<\varepsilon_{0}$. Clearly, $\alpha_{m}(\varepsilon)=\alpha_{m}(0)=$ $\# \mathcal{I}_{\Phi, a}^{(m)}$. Now, choose $n_{0} \geq 0$ sufficiently large so that for every $|\delta|<\varepsilon_{0}$, every $n \geq n_{0}$ and every connected component $J$ of $K_{\Phi} \backslash Q_{a+\delta}^{(n)}$, the length of the interval $f_{\Phi, a+\delta}^{n}(J)$ is smaller than $\tau(m)$. Notice that this is possible since $\left|f_{\Phi, a+\delta}^{n}(J)\right| \leq \lambda_{\Phi}^{n}|J| \leq 4 \lambda_{\Phi}^{n} r_{\Phi}$. Since, for every $|\delta|<\varepsilon_{0}$, any interval $J \subset K_{\Phi}$ with length $<\tau(m)$ will intersect $Q_{a+\delta}^{(m)}$ in at most a single point, we conclude that

$$
\alpha_{n+m}(\varepsilon) \leq 2 \alpha_{n}(\varepsilon), \quad \forall n \geq n_{0}
$$

Thus, $\alpha_{n}(\varepsilon) \leq 2^{\frac{n-n_{0}}{m}} \alpha_{n_{0}}(\varepsilon)$ for every $n \geq n_{0}$. By our choice of $m$, we get $\alpha_{n}(\varepsilon) \leq C \rho^{n}$ for every $n \geq n_{0}$ where $C:=2^{-n_{0} / m} \alpha_{n_{0}}\left(\varepsilon_{0}\right)$. This shows that

$$
\limsup _{n \rightarrow \infty} \frac{\alpha_{n}(\varepsilon)}{n} \leq \log \rho, \quad \forall \varepsilon \in\left[0, \varepsilon_{0}\right)
$$

As $\rho>1$ is arbitrary, the claim follows.
Let

$$
Q_{a}:=\left(\bigcup_{n \geq 0} Q_{a}^{(n)}\right) \cap K_{\Phi}
$$

The following result is adapted from the results of 10] (cf. [7, Theorem 20]). We include here a proof for the convenience of the reader.

Lemma 2.4. Suppose that $(\Phi, \boldsymbol{a})$ has no singular connections. If $Q_{a}$ is finite, then $f_{\Phi, a}$ is asymptotically periodic.

Proof. To simplify the notation, let $f=f_{\Phi, a}$. Let $\mathcal{P}=\left\{J_{\ell}\right\}_{\ell=1}^{m}$ denote the collection of connected components of $K_{\Phi} \backslash Q_{a}$. This collection is finite by hypothesis. Notice that there is a map $\sigma:\{1, \ldots, m\} \rightarrow$ $\{1, \ldots, m\}$ such that $f\left(J_{\ell}\right) \subset J_{\sigma(\ell)}$ for every $\ell \in\{1, \ldots, m\}$. Indeed, suppose by contradiction that there is $J_{\ell} \in \mathcal{P}$ such that $f\left(J_{\ell}\right) \cap Q_{\boldsymbol{a}} \neq \emptyset$. Then $J_{\ell} \cap f^{-1}\left(Q_{a}\right) \neq \emptyset$. But $f^{-1}\left(Q_{a}\right) \cap K_{\Phi} \subset Q_{a}$, which implies that $J_{\ell} \cap Q_{a} \neq \emptyset$, thus a contradiction.

Now we show that $\omega(f, x)$ is a periodic orbit for every $x \in \mathbb{R}$. By Lemma 2.1, we may assume that $x \in K_{\Phi}$. We split the proof in two cases:
(1) When $x \in K_{\Phi} \backslash Q_{\boldsymbol{a}}$, then $x \in J_{\ell_{0}}$ for some $\ell_{0} \in\{1, \ldots, m\}$. Thus, $f^{n}(x) \in J_{\ell_{n}}$ for every $n \geq 0$ where $\left(\ell_{n}\right)_{n \geq 0}$ is the sequence in $\{1, \ldots, m\}$ defined by $\ell_{n+1}=\sigma\left(\ell_{n}\right)$ for every $n \geq 0$. Clearly, the sequence $\left(\ell_{n}\right)_{n \geq 0}$ is eventually periodic, i.e., there must exist $q \geq 0$ and $p \geq 1$ such that $\ell_{n+p}=\ell_{n}$ for every $n \geq q$. We assume that $p$ is the smallest positive integer with that property, i.e., $p$ is the period. In particular, we have $f^{p}\left(J_{\ell_{q}}\right) \subset J_{\ell_{q}}$. Let $\omega \in\{1, \ldots, k\}^{p}$ such that $\left.f^{p}\right|_{J_{\ell_{q}}}=\left.\phi_{\omega}\right|_{J_{\ell_{q}}}$. Then, $f^{n p+q}(x) \rightarrow z_{\omega}$ as $n \rightarrow \infty$ where $z_{\omega}$ is the unique fixed point of $\phi_{\omega}$ which belongs to the closure of the interval $J_{\ell_{q}}$. By hypothesis, $(\Phi, \boldsymbol{a})$ has no singular connections, which implies that $z_{\omega} \in J_{\ell_{q}}$, and thus $z_{\omega}$ is a periodic point of $f$ of period $p$. Therefore, $\omega(f, x)=$ $\left\{z_{\omega}, f\left(z_{\omega}\right), \ldots, f^{p-1}\left(z_{\omega}\right)\right\}$.
(2) When $x \in Q_{\boldsymbol{a}}$, then two situations can happen. Either the forward orbit of $x$ under $f$ belongs to $Q_{a}$ and thus it is periodic or the forward orbit of $x$ under $f$ eventually leaves $Q_{\boldsymbol{a}}$. The former situation cannot happen, because $(\Phi, \boldsymbol{a})$ has no singular connections. Thus, there exists $n_{0} \geq 1$ such that $y:=f^{n_{0}}(x) \in$ $K_{\Phi} \backslash Q_{a}$. But then $\omega(f, x)=\omega(f, y)$ and we know from the previous case that $\omega(f, y)$ is a periodic orbit.
At last, because $\mathcal{P}$ is finite, $f$ has only a finite number of periodic orbits and the claim follows.

Given $\varepsilon \geq 0$, let

$$
\Omega_{\Phi, a}^{\varepsilon}:=\bigcap_{m \geq 1} \overline{\bigcup_{n \geq m} \bigcup_{\omega \in \mathcal{I}_{\Phi, a}^{(n)}(\varepsilon)}\left\{\phi_{\omega}(0)\right\}} .
$$

Notice that $\Omega_{\Phi, a}^{\varepsilon} \subset K_{\Phi}$ and $\Omega_{\Phi, a}^{\varepsilon}$ is compact (see Lemma 2.1).

Lemma 2.5. For every $n \in \mathbb{N}$, the set $\Omega_{\Phi, a}^{\varepsilon}$ can be covered by finitely many intervals of length $2\left(1+2 r_{\Phi}\right) \lambda_{\Phi}^{n}$ centered at the points $\phi_{\omega}(0)$ with $\omega \in \mathcal{I}_{\Phi, \boldsymbol{a}}^{(n)}(\varepsilon)$.

Proof. Let $x \in \Omega_{\Phi, a}^{\varepsilon}$ and $n \in \mathbb{N}$. By definition of $\Omega_{\Phi, a}^{\varepsilon}$, there is an increasing sequence of positive integers $m_{j} \rightarrow \infty$ and a sequence $\omega^{(j)} \in$ $\mathcal{I}_{\Phi, a}^{\left(m_{j}\right)}(\varepsilon)$ such that $\phi_{\omega^{(j)}}(0) \rightarrow x$ as $j \rightarrow \infty$. Let $j \geq 1$ be sufficiently large such that $m_{j} \geq n$ and $\left|x-\phi_{\omega^{(j)}}(0)\right| \leq \lambda_{\Phi}^{n}$. Denote by $\omega^{(j, n)}=$ $\left(\omega_{m_{j}-n+1}^{(j)}, \ldots, \omega_{m_{j}}^{(j)}\right) \in \mathcal{I}_{\Phi, a}^{(n)}(\varepsilon)$ the last $n$ entries of $\omega_{j}$. Then

$$
\left|\phi_{\omega^{(j)}}(0)-\phi_{\omega^{(j, n)}}(0)\right|=\left|\phi_{\omega^{(j, n)}}(y)-\phi_{\omega^{(j, n)}}(0)\right| \leq \lambda_{\Phi}^{n}|y| \leq 2 r_{\Phi} \lambda_{\Phi}^{n},
$$

where $y:=\phi_{\omega_{m_{j}-n}^{(j)}} \circ \cdots \circ \phi_{\omega_{1}^{(j)}}(0)$ and, by Lemma 2.1. $|y| \leq 2 r_{\Phi}$. Thus,

$$
\begin{aligned}
\left|x-\phi_{\omega^{(j, n)}}(0)\right| & \leq\left|x-\phi_{\omega^{(j)}}(0)\right|+\left|\phi_{\omega^{(j)}}(0)-\phi_{\omega^{(j, n)}}(0)\right| \\
& \leq\left(1+2 r_{\Phi}\right) \lambda_{\Phi}^{n} .
\end{aligned}
$$

Lemma 2.6. Let $|\delta|<\varepsilon$. If $\Omega_{\Phi, a}^{\varepsilon} \cap S_{a+\delta}=\emptyset$, then $Q_{a+\delta}$ is finite.
Proof. By hypothesis, there is $\tau>0$ such that

$$
\begin{equation*}
\min _{1 \leq i<k}\left|a_{i}+\delta-\phi_{\omega}(0)\right| \geq \tau, \quad \forall \omega \in \mathcal{I}_{\Phi, \boldsymbol{a}}^{(n)}(\varepsilon), n \in \mathbb{N} \tag{2}
\end{equation*}
$$

Suppose, by contradiction, that $Q_{a+\delta}$ is not finite. Then,

$$
Q_{a+\delta}^{(1)} \cap K_{\Phi} \nsubseteq Q_{a+\delta}^{(2)} \cap K_{\Phi} \varsubsetneqq Q_{a+\delta}^{(3)} \cap K_{\Phi} \nsubseteq \cdots
$$

So, we can pick a sequence $\left(x_{n}\right)_{n \geq 1}$ in $Q_{a+\delta}$ having the property that $x_{n} \in Q_{a+\delta}^{(n+1)} \cap K_{\Phi} \backslash Q_{a+\delta}^{(n)}$ for every $n \in \mathbb{N}$. Thus, $x_{n}$ is an $n$-regular point of $(\Phi, \boldsymbol{a}+\delta)$ and $f_{\Phi, \boldsymbol{a}+\delta}^{n}\left(x_{n}\right)=a_{j_{n}}+\delta$ for some $j_{n} \in\{1, \ldots, k-1\}$. Let $\omega^{(n)} \in \mathcal{I}_{\Phi, \boldsymbol{a}}^{(n)}(\varepsilon)$ denote the itinerary of order $n$ of $(\Phi, \boldsymbol{a}+\delta)$ associated to $x_{n}$.

Now, choose $n \in \mathbb{N}$ sufficiently large so that $2 r_{\Phi} \lambda_{\Phi}^{n}<\tau$. Then, taking into account that $x_{n} \in K_{\Phi}$, we have

$$
\begin{aligned}
\left|a_{j_{n}}+\delta-\phi_{\omega^{(n)}}(0)\right| & =\left|f_{\Phi, a+\delta}^{n}\left(x_{n}\right)-\phi_{\omega^{(n)}}(0)\right| \\
& =\left|\phi_{\omega^{(n)}}\left(x_{n}\right)-\phi_{\omega^{(n)}}(0)\right| \\
& \leq \lambda_{\Phi}^{n}\left|x_{n}\right| \\
& \leq 2 r_{\Phi} \lambda_{\Phi}^{n} \\
& <\tau,
\end{aligned}
$$

contradicting (2).

## 3. Proof of Theorem B

We are now ready to prove Theorem B, Let $\Phi=\left\{\phi_{1}, \ldots, \phi_{k}\right\}$ be an injective IFS and $\boldsymbol{a}=\left(a_{1}, \ldots, a_{k-1}\right) \in \Omega_{k}$ with $k \geq 2$. By Lemma 2.2, the set

$$
E=\{\delta \in \mathbb{R}:(\Phi, \boldsymbol{a}+\delta) \text { has a singular connection }\}
$$

is countable. Thus, it is sufficient to show that

$$
Z:=\left\{\delta \in \mathbb{R} \backslash E: f_{\Phi, a+\delta} \text { is not asymptotically periodic }\right\}
$$

has zero Hausdorff dimension. By Lemma 2.4, $Z \subset Z^{\prime}$ where

$$
Z^{\prime}:=\left\{\delta \in \mathbb{R} \backslash E: Q_{\Phi, a+\delta} \text { is not finite }\right\} .
$$

Given $\delta \in \mathbb{R}$ and $\varepsilon>0$, let $\Delta_{\varepsilon}(\delta):=(\delta-\varepsilon, \delta+\varepsilon)$ and $Z_{\varepsilon}^{\prime}(\delta):=$ $Z^{\prime} \cap \Delta_{\varepsilon}(\delta)$. We claim that for every $d>0$ and $\delta \in Z^{\prime}$ there is an $\varepsilon=\varepsilon(\delta, d)>0$ such that

$$
\mathcal{H}^{d}\left(Z_{\varepsilon}^{\prime}(\delta)\right)=0,
$$

where $\mathcal{H}^{d}$ is the $d$-dimensional Hausdorff measure. This claim is sufficient to conclude the proof of Theorem B, since by Lindelöf Lemma, $Z^{\prime}$ is a countable union of sets $Z_{\varepsilon_{i}}^{\prime}\left(\delta_{i}\right)$, each having zero $d$-dimensional Hausdorff measure. Hence, $\mathcal{H}^{d}\left(Z^{\prime}\right)=0$ for every $d>0$. This implies that $\operatorname{dim}_{H} Z^{\prime}=0$.

Now we prove the claim. Let $d>0, \delta_{0} \in Z^{\prime}$ and $\varepsilon>0$ that will be chosen later in the proof. By Lemma 2.6,

$$
Z_{\varepsilon}^{\prime}\left(\delta_{0}\right) \subset\left\{\delta \in \Delta_{\varepsilon}\left(\delta_{0}\right): \Omega_{\Phi, a_{0}}^{\varepsilon} \cap S_{a+\delta} \neq \emptyset\right\}
$$

where $\boldsymbol{a}_{0}:=\boldsymbol{a}+\delta_{0}$. According to Lemma 2.5, for each $n \in \mathbb{N}$, the set $\Omega_{\Phi, a_{0}}^{\varepsilon}$ can be covered by $\# \mathcal{I}_{\Phi, a_{0}}^{(n)}(\varepsilon)$ intervals of length $\ell_{n}:=2\left(1+2 r_{\Phi}\right) \lambda_{\Phi}^{n}$ centered at the points $\phi_{\omega}(0)$ with $\omega \in \mathcal{I}_{\Phi, a_{0}}^{(n)}(\varepsilon)$. Thus, for each $n \in \mathbb{N}$, we can also cover $Z_{\varepsilon}^{\prime}\left(\delta_{0}\right)$ using finitely many intervals of length $\ell_{n}$,

$$
Z_{\varepsilon}^{\prime}\left(\delta_{0}\right) \subset \bigcup_{\omega \in \mathcal{I}_{\Phi, a_{0}}^{(n)}(\varepsilon)} \bigcup_{i=1}^{k-1} W_{\omega, i}
$$

where $W_{\omega, i}=\left[y_{\omega, i}-\frac{\ell_{n}}{2}, y_{\omega, i}+\frac{\ell_{n}}{2}\right], n=|\omega|$ and $y_{\omega, i}=\phi_{\omega}(0)-a_{i}$. Using this cover, it is easy to see that there is an $\varepsilon>0$ such that the $d$-dimensional Hausdorff measure of $Z_{\varepsilon}^{\prime}\left(\delta_{0}\right)$ is zero. Indeed, for every
$n \in \mathbb{N}$,

$$
\begin{aligned}
\mathcal{H}_{2 \ell_{n}}^{d}\left(Z_{\varepsilon}^{\prime}\left(\delta_{0}\right)\right) & =\inf \left\{\sum_{i=1}^{\infty}\left(\operatorname{diam} U_{i}\right)^{d}: \bigcup_{i=1}^{\infty} U_{i} \supset Z_{\varepsilon}^{\prime}\left(\delta_{0}\right), \operatorname{diam} U_{i}<2 \ell_{n}\right\} \\
& \leq \sum_{\omega \in \mathcal{I}_{\Phi, a_{0}}^{(n)}(\varepsilon)} \sum_{i=1}^{k-1}\left(\operatorname{diam} W_{\omega, i}\right)^{d} \\
& =\sum_{\omega \in \mathcal{I}_{\Phi, a_{0}}^{(n)}(\varepsilon)} \sum_{i=1}^{k-1} \ell_{n}^{d} \\
& =(k-1) 2^{d}\left(1+2 r_{\Phi}\right)^{d}\left(\# \mathcal{I}_{\Phi, a_{0}}^{(n)}(\varepsilon)\right) \lambda_{\Phi}^{n d} .
\end{aligned}
$$

Notice that, $\left(\Phi, \boldsymbol{a}_{0}\right)$ has no singular connections because $\delta_{0} \in Z^{\prime}$. Hence, by Lemma 2.3, there is an $\varepsilon=\varepsilon\left(\delta_{0}, d\right)>0$ such that

$$
\lim _{n \rightarrow \infty}\left(\# \mathcal{I}_{\Phi, a_{0}}^{(n)}(\varepsilon)\right) \lambda_{\Phi}^{n d}=0
$$

from which it follows that $\mathcal{H}^{d}\left(Z_{\varepsilon}^{\prime}\left(\delta_{0}\right)\right)=\lim _{n \rightarrow \infty} \mathcal{H}_{2 \ell_{n}}^{d}\left(Z_{\varepsilon}^{\prime}\left(\delta_{0}\right)\right)=0$, thus proving the claim.

## 4. Proof of Theorem A

Recall that $I=[0,1)$ and let $f: I \rightarrow I$ be a piecewise $\lambda$-affine map as defined in the introduction with $-1<\lambda<1$. We may suppose that $\lambda \neq 0$, otherwise $\mathcal{E}_{f}=\emptyset$ and the result trivially holds. Let $f_{\delta}=R_{\delta} \circ f$ where $R_{\delta}(x)=\{x+\delta\}$ is the rotation map of angle $\delta \in \mathbb{R}$. Notice that $f_{\delta}$ is also a piecewise $\lambda$-affine map and

$$
\mathcal{E}_{f}=\mathcal{E}_{f_{\delta}}+\delta
$$

Therefore, to prove Theorem $A$ it is sufficient to prove that for any piecewise $\lambda$-affine map $f$ we have $\operatorname{dim}_{H}\left(\mathcal{E}_{f} \cap\left(-\delta_{0}, \delta_{0}\right)\right)=0$ for some $\delta_{0}>0$.

Since $|\lambda|<1$, the map $f$ has a gap, i.e., $I \backslash f(I) \neq \emptyset$. This gap also has non-empty interior, so we can choose a point $c \in I \backslash f(I)$ such that $\ell:=\operatorname{dist}(c, f(I))>0$. Here, dist denotes the distance in $I$ induced from the circle $\mathbb{R} / \mathbb{Z}$ through the canonical identification $I \hookrightarrow \mathbb{R} \rightarrow \mathbb{R} / \mathbb{Z}$.

Now, we define $g: I \rightarrow I$ by

$$
g=R_{-c} \circ f \circ R_{c} .
$$

The map $g$ is again a piecewise $\lambda$-affine map of $I$. Clearly, $\ell=$ dist $(0, g(I))$ since $R_{c}$ is an isometry of ( $I$, dist). We claim that

$$
\begin{equation*}
\operatorname{dim}_{H}\left(\mathcal{E}_{g} \cap(-\ell, \ell)\right)=0 \tag{3}
\end{equation*}
$$

From (3) we conclude the proof of Theorem A, because $f$ and $g$ are conjugated through $R_{c}$, which implies that $\mathcal{E}_{f}=\mathcal{E}_{g}$.


Figure 2. Plots of $f$ and $g=R_{-c} \circ f \circ R_{c}$. The shaded region in (A) illustrates the gap of $f$.

In order to prove claim (3), notice that, for every $0 \leq \delta<\ell$, we have $\operatorname{dist}\left(0, g_{\delta}(I)\right)>0$ where $g_{\delta}=R_{\delta} \circ g$. Equivalently,

$$
\begin{equation*}
\overline{g_{\delta}(I)} \subset(0,1), \quad \forall \delta \in[0, \ell) . \tag{4}
\end{equation*}
$$

Next, extend $g_{\delta}$ in a canonical way to a piecewise $\lambda$-affine map $G_{\delta}$ defined on the whole of $\mathbb{R}$. We call $G_{\delta}$ the affine extension of $g_{\delta}$. The map $G_{\delta}$ has the property that $G_{\delta}(x)=G(x)+\delta$ where $G: \mathbb{R} \rightarrow \mathbb{R}$ is the affine extension of $g$. Notice that this property holds because of (4). Moreover, since the orbit of any $x \in \mathbb{R}$ under $G_{\delta}$ eventually enters $I$, we conclude that $G_{\delta}$ is asymptotically periodic if and only if $g_{\delta}$ is asymptotically periodic. Thus, (3) is equivalent to

$$
\begin{equation*}
\operatorname{dim}_{H}\left\{\delta \in(-\ell, \ell): G_{\delta} \text { is not asymptotically periodic }\right\}=0 . \tag{5}
\end{equation*}
$$

Let $(\Phi, \boldsymbol{a})$ be a pair defining $G$, i.e., an injective $\operatorname{IFS} \Phi=\left\{\phi_{1}, \ldots, \phi_{k}\right\}$ and $\boldsymbol{a}=\left(a_{1}, \ldots, a_{k-1}\right) \in \Omega_{k}$ such that $G=f_{\Phi, a}$. Notice that the $\phi_{i}$ 's are $\lambda$-affine maps. By Theorem B we know that

$$
\operatorname{dim}_{H}\left\{\delta \in \mathbb{R}: f_{\Phi, a+\delta} \text { is not asymptotically periodic }\right\}=0
$$

It is easy to see that $G_{\delta}$ and $f_{\Phi, a-\delta /(1-\lambda)}$ are conjugated by the affine map $x \mapsto x+\delta /(1-\lambda)$ (cf. [10, Reduction lemma]). This shows (5), thus proving Theorem A.

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