HAUSDORFF DIMENSION OF THE EXCEPTIONAL SET OF INTERVAL PIECEWISE AFFINE CONTRACTIONS

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ABSTRACT. Let $I = [0, 1), -1 < \lambda < 1$ and $f: I \to I$ be a piecewise λ -affine map of the interval I, i.e., there exist a partition $0 = a_0 < a_1 < \cdots < a_{k-1} < a_k = 1$ of the interval I into $k \geq 2$ subintervals and $b_1, \ldots, b_k \in \mathbb{R}$ such that $f(x) = \lambda x + b_i$ for every $x \in [a_{i-1}, a_i)$ and $i = 1, \ldots, k$. The exceptional set \mathcal{E}_f of f is the set of parameters $\delta \in \mathbb{R}$ such that $R_\delta \circ f$ is not asymptotically periodic, where $R_\delta: I \to I$ is the rotation of angle δ . In this paper we prove that \mathcal{E}_f has zero Hausdorff dimension. We derive this result from a more general theorem concerning piecewise Lipschitz contractions on \mathbb{R} that has independent interest.

1. INTRODUCTION

Let I = [0, 1) and $-1 < \lambda < 1$. We say that an interval map $f : I \to I$ is *piecewise* λ -affine if there exist $k \ge 2$, real numbers b_1, \ldots, b_k and a partition of the interval I,

$$0 = a_0 < a_1 < \dots < a_{k-1} < a_k = 1,$$

such that $f(x) = \lambda x + b_i$ for every $x \in [a_{i-1}, a_i)$ and $i = 1, \ldots, k$. Given a piecewise λ -affine map f, consider the one-parameter family of piecewise λ -affine maps $f_{\delta} \colon I \to I$ defined by

$$f_{\delta} = R_{\delta} \circ f,$$

where $R_{\delta}(x) = \{x + \delta\}$ is the rotation of angle $\delta \in \mathbb{R}$ and $\{\cdot\}$ denotes the fractional part. See Figure 1 for an illustration of the graph of f_{δ} . We are interested in the dimension of the exceptional set

 $\mathcal{E}_f = \{ \delta \in \mathbb{R} : f_\delta \text{ is not asymptotically periodic} \}.$

A map $f: I \to I$ is asymptotically periodic if f has at most a finite number of periodic orbits and the ω -limit set $\omega(f, x)$ of any $x \in I$ is a periodic orbit of f. We recall that $\omega(f, x)$ is the set of accumulation points of the forward orbit of x under f. It is known that \mathcal{E}_f has zero Lebesgue measure [10, Theorem 1.1], but the question of the Hausdorff dimension of \mathcal{E}_f was still open.

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FIGURE 1. Plots of f_{δ} .

In this paper, we settle this question.

Theorem A.

$$\dim_H \mathcal{E}_f = 0.$$

A notable example of piecewise λ -affine maps is the family of contracted rotations, $f(x) = \{\lambda x + b\}$ with $0 < \lambda < 1$ and $1 - \lambda < b < 1$. Contracted rotations have been extensively studied by many authors either as a dynamical system or related to applications, e.g. [3, 4, 9, 8, 5, 2]. In the case of contracted rotations, the exceptional set \mathcal{E}_f has a Cantor structure and in this case Theorem A was proved by Laurent and Nogueira in [9, Theorem 5] by exploiting a combinatorial structure, associated to \mathcal{E}_f , which is reminiscent of the classical Stern-Brocot tree. Janson and Öberg improved in [8] the result of Laurent and Nogueira by considering other gauge functions in the definition of the Hausdorff dimension.

For general piecewise λ -affine maps, Theorem A was proved in [1] by Pires using the theory of *b*-adic expansions and under the assumptions that *f* is injective, $\lambda^{-1} = b$ is a positive integer $\geq k$ and the connected components of $I \setminus f(I)$ have equal length. In order to remove all these assumptions and prove Theorem A for any piecewise λ -affine map *f* we use a different strategy inspired by a recent work dealing with piecewise increasing contractions [7].

We will deduce Theorem A from a more general result, Theorem B below. We say that a function $f \colon \mathbb{R} \to \mathbb{R}$ is a *piecewise contraction* if it has a finite number of discontinuity points and on each connected component D of the domain of continuity of f the restriction map $f|_D$ is a Lipschitz contraction. Let $k \geq 2$ and

$$\Omega_k = \{ (a_1, \dots, a_{k-1}) \in \mathbb{R}^{k-1} \colon a_1 < \dots < a_{k-1} \}.$$

A finite collection of Lipschitz contractions $\Phi = \{\phi_i \colon \mathbb{R} \to \mathbb{R}\}_{i=1}^k$ is called an iterative function system (IFS). An IFS $\Phi = \{\phi_1, \ldots, \phi_k\}$ together with $a \in \Omega_k$, determine a piecewise contraction $f = f_{\Phi,a} \colon \mathbb{R} \to$ $\mathbb R$ defined by

$$f(x) = \begin{cases} \phi_1(x), & x \in (-\infty, a_1), \\ \phi_i(x), & x \in [a_{i-1}, a_i) \quad i \in \{2, \dots, k-1\}, \\ \phi_k(x), & x \in [a_{k-1}, +\infty). \end{cases}$$
(1)

Notice that $(\Phi, \mathbf{a}) \mapsto f_{\Phi, \mathbf{a}}$ is not injective, i.e., a piecewise contraction is not uniquely determined by a pair (Φ, \mathbf{a}) . In [10] the authors prove the following result.

Theorem 1.1 ([10, Theorem 1.4]). Let $\Phi = \{\phi_1, \ldots, \phi_k\}$ be an IFS. There is a Lebesgue full measure set $W \subset \mathbb{R}$ such that for every $\mathbf{a} \in \Omega_k \cap W^{k-1}$, the map $f_{\Phi,\mathbf{a}}$ is asymptotically periodic and has at most k periodic orbits.

We consider a very specific perturbation of $f_{\Phi,a}$. To simplify the notation, we shall write $\mathbf{a} + \delta = (a_1 + \delta, \dots, a_{k-1} + \delta)$. Notice that $\mathbf{a} + \delta \in \Omega_k$ for every $\delta \in \mathbb{R}$.

We say that an IFS is *injective* if all its contractions are injective functions. Under the assumption that the IFS is injective, we prove the following result.

Theorem B. Let $\Phi = \{\phi_1, \ldots, \phi_k\}$ be an injective IFS and $\mathbf{a} \in \Omega_k$. Then

$$\dim_H \{ \delta \in \mathbb{R} : f_{\Phi, \boldsymbol{a} + \delta} \text{ is not asymptotically periodic} \} = 0.$$

The injectivity assumption of Theorem B can be weakened to the assumption that the functions of the IFS have a finite number of local extrema. This hypothesis guarantees that the pre-image of any point is a finite set, a crucial property that we use in our arguments to prove Theorem B. By Sard's theorem for Lipschitz functions, the pre-image of almost every point is a finite set. However, this property is not sufficient for our arguments to work in the general situation. The injectivity of the IFS is a natural condition, and piecewise contractions appearing in applications satisfy this condition. Moreover, assuming the injectivity of the IFS, the authors in [6] have obtained a spectral decomposition of the attractor of a piecewise contraction, i.e., the attractor is a finite union of periodic orbits together with a finite union of Cantor sets. Our Theorem B excludes the Cantor attractors in a very strong metric sense, i.e., piecewise contractions can only have Cantor attractors for a parameter set of zero Hausdorff dimension.

As a final remark, the definition (1) of f at the points $\{a_1, \ldots, a_{k-1}\}$ is not relevant for proving Theorem B, i.e., the proof of Theorem B can be adapted to any other choice of values $f(a_i) \in \{f(a_i^-), f(a_i^+)\}$.

The rest of the paper is organized as follows. In Section 2 we collect the lemmas that we need to prove Theorem B and complete its proof in Section 3. Then, in Section 4, we use Theorem B to deduce Theorem A.

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2. Preliminary results

Throughout this section, let $\Phi = \{\phi_1, \ldots, \phi_k\}$ be an injective IFS and $\boldsymbol{a} = (a_1, \ldots, a_{k-1}) \in \Omega_k$ with $k \ge 2$. Define

$$\lambda_{\Phi} := \max_{i} \operatorname{Lip}(\phi_{i}) \quad \text{and} \quad r_{\Phi} := \frac{1 + \lambda_{\Phi}}{1 - \lambda_{\Phi}} \max_{i} |z_{i}|,$$

where $\operatorname{Lip}(\phi_i)$ denotes the Lipschitz constant of ϕ_i and z_i is the unique fixed point of ϕ_i . Clearly, $0 \leq \lambda_{\Phi} < 1$.

Our first observation is that the recurrent dynamics of $f_{\Phi,a}$ occurs inside an attracting compact interval

$$K_{\Phi} := [-2r_{\Phi}, 2r_{\Phi}].$$

Given $n \in \mathbb{N}$ and $\omega \in \{1, \ldots, k\}^n$, let

$$\phi_{\omega} := \phi_{\omega_n} \circ \cdots \circ \phi_{\omega_1}$$

Lemma 2.1. The following holds:

- (1) $\phi_{\omega}(K_{\Phi}) \subset K_{\Phi}$ for every $\omega \in \{1, \ldots, k\}^n$ and $n \in \mathbb{N}$,
- (2) $f_{\Phi,\boldsymbol{a}}(K_{\Phi}) \subset K_{\Phi},$
- (3) For every $x \in \mathbb{R}$ there is $n \ge 0$ such that $f_{\Phi,a}^n(x) \in K_{\Phi}$,
- (4) $\omega(f_{\Phi,a}, x) \subset K_{\Phi}$ for every $x \in \mathbb{R}$.

Proof. To show (1) let $|x| \leq 2r_{\Phi}$. Then

$$\begin{aligned} |\phi_i(x)| &= |\phi_i(x) - z_i + z_i| \leq \lambda_{\Phi} |x - z_i| + |z_i| \\ &\leq 2\lambda_{\Phi} r_{\Phi} + |z_i|(1 + \lambda_{\Phi}) \\ &\leq 2\lambda_{\Phi} r_{\Phi} + \max_i |z_i|(1 + \lambda_{\Phi}) \\ &= 2\lambda_{\Phi} r_{\Phi} + (1 - \lambda_{\Phi}) r_{\Phi} \\ &= (1 + \lambda_{\Phi}) r_{\Phi} \\ &< 2r_{\Phi}, \end{aligned}$$

which proves (1). Item (2) follows immediately from (1). Next, to prove (3), let $|x| \leq r_{\Phi}$. Repeating the above estimates we get $|\phi_i(x)| \leq r_{\Phi}$. For any $\omega \in \{1, \ldots, k\}^n$ and $n \in \mathbb{N}$, this shows that $|\phi_{\omega}(x)| \leq r_{\Phi}$ whenever $|x| \leq r_{\Phi}$. Consequently, the unique fixed point z_{ω} of ϕ_{ω} satisfies $|z_{\omega}| \leq r_{\Phi}$. Now, given any $x \in \mathbb{R}$, choose $n \geq 1$ such that $\lambda_{\Phi}^n < \frac{r_{\Phi}}{|x|+r_{\Phi}}$. Then, for any $\omega \in \{1, \ldots, k\}^n$

$$\begin{aligned} |\phi_{\omega}(x)| &= |\phi_{\omega}(x) - z_{\omega} + z_{\omega}| \\ &\leq \lambda_{\Phi}^{n} |x - z_{\omega}| + |z_{\omega}| \\ &\leq \lambda_{\Phi}^{n} (|x| + r_{\Phi}) + r_{\Phi} \\ &< 2r_{\Phi}. \end{aligned}$$

This proves (3). Finally, (4) follows immediately from (2) and (3). \Box Let

$$S_{\boldsymbol{a}} := \{a_1, \ldots, a_{k-1}\}.$$

Definition 2.1. We say that the pair (Φ, a) has a singular connection if there exist $n \in \mathbb{N}$ and $\omega \in \{1, \ldots, k\}^n$ such that

 $\phi_{\omega}(S_{\boldsymbol{a}}) \cap S_{\boldsymbol{a}} \neq \emptyset.$

Lemma 2.2. The set

$$\{\delta \in \mathbb{R} : (\Phi, \boldsymbol{a} + \delta) \text{ has a singular connection}\}$$

is countable.

Proof. Given $\omega \in \{1, \ldots, k\}^n$, the map ϕ_{ω} is a Lipschitz contraction on \mathbb{R} . Given $(i, j) \in \{1, \ldots, k-1\}^2$, the map $\mathbb{R} \ni \delta \mapsto \phi_{\omega}(a_i + \delta) - a_j$ is also a Lipschitz contraction on \mathbb{R} , hence it has a unique fixed point, say $z_{\omega,i,j} \in \mathbb{R}$. Let

$$\Delta := \bigcup_{n \in \mathbb{N}} \bigcup_{\omega \in \{1, \dots, k\}^n} \bigcup_{i=1}^{k-1} \bigcup_{j=1}^{k-1} \{z_{\omega, i, j}\}.$$

Observe that $(\Phi, \mathbf{a} + \delta)$ has a singular connection if and only if $\delta \in \Delta$. Since Δ is a countable set, the claim follows.

Definition 2.2. Given $x \in K_{\Phi}$ and $n \in \mathbb{N}$, we say that x is an *n*-regular point of (Φ, \mathbf{a}) if $f_{\Phi, \mathbf{a}}^{j}(x) \notin S_{\mathbf{a}}$ for every $0 \leq j < n$.

Let
$$D_{a}^{(0)} := S_{a}, D_{a}^{(n)} := f_{\Phi,a}^{-1}(D_{a}^{(n-1)})$$
 for $n \ge 1$ and
 $Q_{a}^{(n)} := \bigcup_{i=0}^{n-1} D_{a}^{(i)}, \quad n \in \mathbb{N}.$

Notice that the sets $\{Q_{a}^{(n)}\}_{n\geq 0}$ are finite. Indeed, this follows from the fact that S_{a} is finite and Φ is injective. Given $x \in K_{\Phi}$, it is also clear that x is an n-regular point of (Φ, a) if and only if $x \notin Q_{a}^{(n)}$.

Let $X_1 := (-\infty, a_1), X_i := (a_{i-1}, a_i)$ with $i = 2, \ldots, k-1$, and $X_k := (a_{k-1}, +\infty)$. By construction, these open intervals are disjoint and their union equals $\mathbb{R} \setminus S_a$.

Definition 2.3. Given $n \in \mathbb{N}$, a tuple $(\omega_0, \omega_1, \ldots, \omega_{n-1}) \in \{1, \ldots, k\}^n$ is called an *itinerary of order* n of (Φ, \mathbf{a}) if there is an n-regular point x of (Φ, \mathbf{a}) such that $f_{\Phi, \mathbf{a}}^j(x) \in X_{\omega_j}$ for every $0 \leq j < n$.

We define the set of all itineraries of order n of (Φ, a) ,

 $\mathcal{I}_{\Phi,\boldsymbol{a}}^{(n)} := \left\{ \omega \in \{1, \dots, k\}^n \colon \omega \text{ is an itinerary of order } n \text{ of } (\Phi, \boldsymbol{a}) \right\}.$

The set $\mathcal{I}_{\Phi,\boldsymbol{a}}^{(n)}$ is in a one-to-one correspondence with the set of connected components of $K_{\Phi} \setminus Q_{\boldsymbol{a}}^{(n)}$. Indeed, for each connected component J of $K_{\Phi} \setminus Q_{\boldsymbol{a}}^{(n)}$, all points in J are *n*-regular points of (Φ, \boldsymbol{a}) and because the sets $J, f_{\Phi,\boldsymbol{a}}(J), \ldots, f_{\Phi,\boldsymbol{a}}^{n-1}(J)$ are subintervals of K_{Φ} not intersecting $S_{\boldsymbol{a}}$, we conclude that there is an itinerary of order n of (Φ, \boldsymbol{a}) , say $(\omega_0, \omega_1, \ldots, \omega_{n-1}) \in \mathcal{I}_{\Phi,\boldsymbol{a}}^{(n)}$, such that $f_{\Phi,\boldsymbol{a}}^j(J) \subset X_{\omega_j}$ for every $0 \leq j < n$. Given $\varepsilon \geq 0$, we enlarge the set $\mathcal{I}_{\Phi,a}^{(n)}$ as follows,

$$\mathcal{I}_{\Phi, \boldsymbol{a}}^{(n)}(\varepsilon) := \bigcup_{|\delta| \leq \varepsilon} \mathcal{I}_{\Phi, \boldsymbol{a} + \delta}^{(n)}$$

The following result establishes that the number of itineraries grows subexponentially, a crucial property to prove Theorem B.

Lemma 2.3. Suppose that (Φ, a) has no singular connections. Then

$$\lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log \# \mathcal{I}_{\Phi, \boldsymbol{a}}^{(n)}(\varepsilon) = 0.$$

Proof. Given $\rho > 1$, let $m = \lceil \log 2/\log \rho \rceil$ and $\tau(m, a) > 0$ be the minimum distance between any pair of distinct points of $Q_{a}^{(m)}$. Notice that the sets $\{D_{a}^{(n)}\}_{n\geq 0}$ are pairwise disjoint. Indeed, suppose that there is $x \in D_{a}^{(n_{1})} \cap D_{a}^{(n_{2})}$ with $n_{2} > n_{1} \geq 0$. Then $f_{\Phi,a}^{n_{1}}(x) = a_{i}$ and $f_{\Phi,a}^{n_{2}}(x) = a_{j}$ for some $i, j \in \{1, \ldots, k-1\}$. This implies that $f_{\Phi,a}^{n_{2}-n_{1}}(a_{i}) = a_{j}$, contradicting the assumption that (Φ, a) has no singular connections. Since $Q_{a}^{(m)} = D_{a}^{(0)} \cup D_{a}^{(1)} \cup \cdots \cup D_{a}^{(m-1)}$ and Φ is injective, there is an $\varepsilon_{0} = \varepsilon_{0}(m) > 0$ such that the set-valued map $(-\varepsilon_{0}, \varepsilon_{0}) \ni \delta \mapsto Q_{a+\delta}^{(m)}$ varies continuously in the Hausdorff metric of compact subsets of \mathbb{R} , and for every $|\delta| < \varepsilon_{0}$, the set $Q_{a+\delta}^{(m)}$ has the same number of elements of $Q_{a}^{(m)}$ and

$$\tau(m) := \inf_{|\delta| < \varepsilon_0} \tau(m, \boldsymbol{a} + \delta) > 0.$$

Let $\alpha_n(\varepsilon) := \#\mathcal{I}_{\Phi,\mathbf{a}}^{(n)}(\varepsilon)$ with $0 \leq \varepsilon < \varepsilon_0$. Clearly, $\alpha_m(\varepsilon) = \alpha_m(0) = \#\mathcal{I}_{\Phi,\mathbf{a}}^{(m)}$. Now, choose $n_0 \geq 0$ sufficiently large so that for every $|\delta| < \varepsilon_0$, every $n \geq n_0$ and every connected component J of $K_{\Phi} \setminus Q_{\mathbf{a}+\delta}^{(n)}$, the length of the interval $f_{\Phi,\mathbf{a}+\delta}^n(J)$ is smaller than $\tau(m)$. Notice that this is possible since $|f_{\Phi,\mathbf{a}+\delta}^n(J)| \leq \lambda_{\Phi}^n |J| \leq 4\lambda_{\Phi}^n r_{\Phi}$. Since, for every $|\delta| < \varepsilon_0$, any interval $J \subset K_{\Phi}$ with length $< \tau(m)$ will intersect $Q_{\mathbf{a}+\delta}^{(m)}$ in at most a single point, we conclude that

$$\alpha_{n+m}(\varepsilon) \le 2\alpha_n(\varepsilon), \quad \forall n \ge n_0.$$

Thus, $\alpha_n(\varepsilon) \leq 2^{\frac{n-n_0}{m}} \alpha_{n_0}(\varepsilon)$ for every $n \geq n_0$. By our choice of m, we get $\alpha_n(\varepsilon) \leq C\rho^n$ for every $n \geq n_0$ where $C := 2^{-n_0/m} \alpha_{n_0}(\varepsilon_0)$. This shows that

$$\limsup_{n \to \infty} \frac{\alpha_n(\varepsilon)}{n} \le \log \rho, \quad \forall \, \varepsilon \in [0, \varepsilon_0).$$

As $\rho > 1$ is arbitrary, the claim follows.

Let

$$Q_{\boldsymbol{a}} := \left(\bigcup_{n \ge 0} Q_{\boldsymbol{a}}^{(n)}\right) \cap K_{\Phi}.$$

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The following result is adapted from the results of [10] (cf. [7, Theorem 20]). We include here a proof for the convenience of the reader.

Lemma 2.4. Suppose that (Φ, a) has no singular connections. If Q_a is finite, then $f_{\Phi,a}$ is asymptotically periodic.

Proof. To simplify the notation, let $f = f_{\Phi,a}$. Let $\mathcal{P} = \{J_\ell\}_{\ell=1}^m$ denote the collection of connected components of $K_{\Phi} \setminus Q_a$. This collection is finite by hypothesis. Notice that there is a map $\sigma : \{1, \ldots, m\} \rightarrow$ $\{1, \ldots, m\}$ such that $f(J_\ell) \subset J_{\sigma(\ell)}$ for every $\ell \in \{1, \ldots, m\}$. Indeed, suppose by contradiction that there is $J_\ell \in \mathcal{P}$ such that $f(J_\ell) \cap Q_a \neq \emptyset$. Then $J_\ell \cap f^{-1}(Q_a) \neq \emptyset$. But $f^{-1}(Q_a) \cap K_{\Phi} \subset Q_a$, which implies that $J_\ell \cap Q_a \neq \emptyset$, thus a contradiction.

Now we show that $\omega(f, x)$ is a periodic orbit for every $x \in \mathbb{R}$. By Lemma 2.1, we may assume that $x \in K_{\Phi}$. We split the proof in two cases:

- (1) When $x \in K_{\Phi} \setminus Q_{a}$, then $x \in J_{\ell_{0}}$ for some $\ell_{0} \in \{1, \ldots, m\}$. Thus, $f^{n}(x) \in J_{\ell_{n}}$ for every $n \geq 0$ where $(\ell_{n})_{n\geq 0}$ is the sequence in $\{1, \ldots, m\}$ defined by $\ell_{n+1} = \sigma(\ell_{n})$ for every $n \geq 0$. Clearly, the sequence $(\ell_{n})_{n\geq 0}$ is eventually periodic, i.e., there must exist $q \geq 0$ and $p \geq 1$ such that $\ell_{n+p} = \ell_{n}$ for every $n \geq q$. We assume that p is the smallest positive integer with that property, i.e., p is the period. In particular, we have $f^{p}(J_{\ell_{q}}) \subset J_{\ell_{q}}$. Let $\omega \in \{1, \ldots, k\}^{p}$ such that $f^{p}|_{J_{\ell_{q}}} = \phi_{\omega}|_{J_{\ell_{q}}}$. Then, $f^{np+q}(x) \to z_{\omega}$ as $n \to \infty$ where z_{ω} is the unique fixed point of ϕ_{ω} which belongs to the closure of the interval $J_{\ell_{q}}$. By hypothesis, (Φ, a) has no singular connections, which implies that $z_{\omega} \in J_{\ell_{q}}$, and thus z_{ω} is a periodic point of f of period p. Therefore, $\omega(f, x) =$ $\{z_{\omega}, f(z_{\omega}), \ldots, f^{p-1}(z_{\omega})\}$.
- (2) When $x \in Q_a$, then two situations can happen. Either the forward orbit of x under f belongs to Q_a and thus it is periodic or the forward orbit of x under f eventually leaves Q_a . The former situation cannot happen, because (Φ, a) has no singular connections. Thus, there exists $n_0 \geq 1$ such that $y := f^{n_0}(x) \in K_{\Phi} \setminus Q_a$. But then $\omega(f, x) = \omega(f, y)$ and we know from the previous case that $\omega(f, y)$ is a periodic orbit.

At last, because \mathcal{P} is finite, f has only a finite number of periodic orbits and the claim follows.

Given $\varepsilon \geq 0$, let

$$\Omega_{\Phi,\boldsymbol{a}}^{\varepsilon} := \bigcap_{m \ge 1} \overline{\bigcup_{n \ge m} \bigcup_{\omega \in \mathcal{I}_{\Phi,\boldsymbol{a}}^{(n)}(\varepsilon)} \{\phi_{\omega}(0)\}}.$$

Notice that $\Omega_{\Phi,a}^{\varepsilon} \subset K_{\Phi}$ and $\Omega_{\Phi,a}^{\varepsilon}$ is compact (see Lemma 2.1).

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Lemma 2.5. For every $n \in \mathbb{N}$, the set $\Omega^{\varepsilon}_{\Phi,a}$ can be covered by finitely many intervals of length $2(1+2r_{\Phi})\lambda^{n}_{\Phi}$ centered at the points $\phi_{\omega}(0)$ with $\omega \in \mathcal{I}^{(n)}_{\Phi,a}(\varepsilon)$.

Proof. Let $x \in \Omega_{\Phi,a}^{\varepsilon}$ and $n \in \mathbb{N}$. By definition of $\Omega_{\Phi,a}^{\varepsilon}$, there is an increasing sequence of positive integers $m_j \to \infty$ and a sequence $\omega^{(j)} \in \mathcal{I}_{\Phi,a}^{(m_j)}(\varepsilon)$ such that $\phi_{\omega^{(j)}}(0) \to x$ as $j \to \infty$. Let $j \ge 1$ be sufficiently large such that $m_j \ge n$ and $|x - \phi_{\omega^{(j)}}(0)| \le \lambda_{\Phi}^n$. Denote by $\omega^{(j,n)} = (\omega_{m_j-n+1}^{(j)}, \ldots, \omega_{m_j}^{(j)}) \in \mathcal{I}_{\Phi,a}^{(n)}(\varepsilon)$ the last n entries of ω_j . Then

$$|\phi_{\omega^{(j)}}(0) - \phi_{\omega^{(j,n)}}(0)| = |\phi_{\omega^{(j,n)}}(y) - \phi_{\omega^{(j,n)}}(0)| \le \lambda_{\Phi}^{n} |y| \le 2r_{\Phi}\lambda_{\Phi}^{n},$$

where $y := \phi_{\omega_{m_j-n}^{(j)}} \circ \cdots \circ \phi_{\omega_1^{(j)}}(0)$ and, by Lemma 2.1, $|y| \leq 2r_{\Phi}$. Thus,

$$\begin{aligned} |x - \phi_{\omega^{(j,n)}}(0)| &\leq |x - \phi_{\omega^{(j)}}(0)| + |\phi_{\omega^{(j)}}(0) - \phi_{\omega^{(j,n)}}(0)| \\ &\leq (1 + 2r_{\Phi})\lambda_{\Phi}^{n}. \end{aligned}$$

Lemma 2.6. Let $|\delta| < \varepsilon$. If $\Omega_{\Phi,a}^{\varepsilon} \cap S_{a+\delta} = \emptyset$, then $Q_{a+\delta}$ is finite.

Proof. By hypothesis, there is $\tau > 0$ such that

$$\min_{1 \le i < k} |a_i + \delta - \phi_{\omega}(0)| \ge \tau, \quad \forall \, \omega \in \mathcal{I}_{\Phi, a}^{(n)}(\varepsilon), \, n \in \mathbb{N}.$$
(2)

Suppose, by contradiction, that $Q_{a+\delta}$ is not finite. Then,

$$Q_{\boldsymbol{a}+\delta}^{(1)} \cap K_{\Phi} \subsetneq Q_{\boldsymbol{a}+\delta}^{(2)} \cap K_{\Phi} \subsetneq Q_{\boldsymbol{a}+\delta}^{(3)} \cap K_{\Phi} \varsubsetneq \cdots$$

So, we can pick a sequence $(x_n)_{n\geq 1}$ in $Q_{\boldsymbol{a}+\delta}$ having the property that $x_n \in Q_{\boldsymbol{a}+\delta}^{(n+1)} \cap K_{\Phi} \setminus Q_{\boldsymbol{a}+\delta}^{(n)}$ for every $n \in \mathbb{N}$. Thus, x_n is an *n*-regular point of $(\Phi, \boldsymbol{a}+\delta)$ and $f_{\Phi,\boldsymbol{a}+\delta}^n(x_n) = a_{j_n} + \delta$ for some $j_n \in \{1,\ldots,k-1\}$. Let $\omega^{(n)} \in \mathcal{I}_{\Phi,\boldsymbol{a}}^{(n)}(\varepsilon)$ denote the itinerary of order n of $(\Phi, \boldsymbol{a}+\delta)$ associated to x_n .

Now, choose $n \in \mathbb{N}$ sufficiently large so that $2r_{\Phi}\lambda_{\Phi}^n < \tau$. Then, taking into account that $x_n \in K_{\Phi}$, we have

$$|a_{j_n} + \delta - \phi_{\omega^{(n)}}(0)| = |f_{\Phi, \mathbf{a}+\delta}^n(x_n) - \phi_{\omega^{(n)}}(0)|$$
$$= |\phi_{\omega^{(n)}}(x_n) - \phi_{\omega^{(n)}}(0)|$$
$$\leq \lambda_{\Phi}^n |x_n|$$
$$\leq 2r_{\Phi}\lambda_{\Phi}^n$$
$$< \tau,$$

contradicting (2).

3. Proof of Theorem B

We are now ready to prove Theorem B. Let $\Phi = \{\phi_1, \ldots, \phi_k\}$ be an injective IFS and $\boldsymbol{a} = (a_1, \ldots, a_{k-1}) \in \Omega_k$ with $k \ge 2$. By Lemma 2.2, the set

$$E = \{\delta \in \mathbb{R} : (\Phi, \boldsymbol{a} + \delta) \text{ has a singular connection} \}$$

is countable. Thus, it is sufficient to show that

 $Z := \{ \delta \in \mathbb{R} \setminus E \colon f_{\Phi, \boldsymbol{a} + \delta} \text{ is not asymptotically periodic} \}$

has zero Hausdorff dimension. By Lemma 2.4, $Z \subset Z'$ where

$$Z' := \{ \delta \in \mathbb{R} \setminus E \colon Q_{\Phi, \boldsymbol{a} + \delta} \text{ is not finite} \}.$$

Given $\delta \in \mathbb{R}$ and $\varepsilon > 0$, let $\Delta_{\varepsilon}(\delta) := (\delta - \varepsilon, \delta + \varepsilon)$ and $Z'_{\varepsilon}(\delta) := Z' \cap \Delta_{\varepsilon}(\delta)$. We claim that for every d > 0 and $\delta \in Z'$ there is an $\varepsilon = \varepsilon(\delta, d) > 0$ such that

$$\mathcal{H}^d(Z'_\varepsilon(\delta)) = 0,$$

where \mathcal{H}^d is the *d*-dimensional Hausdorff measure. This claim is sufficient to conclude the proof of Theorem B, since by Lindelöf Lemma, Z' is a countable union of sets $Z'_{\varepsilon_i}(\delta_i)$, each having zero *d*-dimensional Hausdorff measure. Hence, $\mathcal{H}^d(Z') = 0$ for every d > 0. This implies that $\dim_H Z' = 0$.

Now we prove the claim. Let d > 0, $\delta_0 \in Z'$ and $\varepsilon > 0$ that will be chosen later in the proof. By Lemma 2.6,

$$Z_{\varepsilon}'(\delta_0) \subset \{\delta \in \Delta_{\varepsilon}(\delta_0) \colon \Omega_{\Phi, a_0}^{\varepsilon} \cap S_{a+\delta} \neq \emptyset\}$$

where $\boldsymbol{a}_0 := \boldsymbol{a} + \delta_0$. According to Lemma 2.5, for each $n \in \mathbb{N}$, the set $\Omega_{\Phi,\boldsymbol{a}_0}^{\varepsilon}$ can be covered by $\#\mathcal{I}_{\Phi,\boldsymbol{a}_0}^{(n)}(\varepsilon)$ intervals of length $\ell_n := 2(1+2r_{\Phi})\lambda_{\Phi}^n$ centered at the points $\phi_{\omega}(0)$ with $\omega \in \mathcal{I}_{\Phi,\boldsymbol{a}_0}^{(n)}(\varepsilon)$. Thus, for each $n \in \mathbb{N}$, we can also cover $Z'_{\varepsilon}(\delta_0)$ using finitely many intervals of length ℓ_n ,

$$Z_{\varepsilon}'(\delta_0) \subset \bigcup_{\omega \in \mathcal{I}_{\Phi, \mathbf{a}_0}^{(n)}(\varepsilon)} \bigcup_{i=1}^{k-1} W_{\omega, i}$$

where $W_{\omega,i} = \left[y_{\omega,i} - \frac{\ell_n}{2}, y_{\omega,i} + \frac{\ell_n}{2}\right]$, $n = |\omega|$ and $y_{\omega,i} = \phi_{\omega}(0) - a_i$. Using this cover, it is easy to see that there is an $\varepsilon > 0$ such that the *d*-dimensional Hausdorff measure of $Z'_{\varepsilon}(\delta_0)$ is zero. Indeed, for every

$$n \in \mathbb{N},$$

$$\mathcal{H}^{d}_{2\ell_{n}}(Z'_{\varepsilon}(\delta_{0})) = \inf\left\{\sum_{i=1}^{\infty} (\operatorname{diam} U_{i})^{d} \colon \bigcup_{i=1}^{\infty} U_{i} \supset Z'_{\varepsilon}(\delta_{0}), \operatorname{diam} U_{i} < 2\ell_{n}\right\}$$
$$\leq \sum_{\omega \in \mathcal{I}^{(n)}_{\Phi, a_{0}}(\varepsilon)} \sum_{i=1}^{k-1} (\operatorname{diam} W_{\omega, i})^{d}$$
$$= \sum_{\omega \in \mathcal{I}^{(n)}_{\Phi, a_{0}}(\varepsilon)} \sum_{i=1}^{k-1} \ell_{n}^{d}$$
$$= (k-1)2^{d}(1+2r_{\Phi})^{d}(\#\mathcal{I}^{(n)}_{\Phi, a_{0}}(\varepsilon))\lambda_{\Phi}^{nd}.$$

Notice that, (Φ, \mathbf{a}_0) has no singular connections because $\delta_0 \in Z'$. Hence, by Lemma 2.3, there is an $\varepsilon = \varepsilon(\delta_0, d) > 0$ such that

$$\lim_{n \to \infty} (\# \mathcal{I}_{\Phi, \boldsymbol{a}_0}^{(n)}(\varepsilon)) \lambda_{\Phi}^{nd} = 0,$$

from which it follows that $\mathcal{H}^d(Z'_{\varepsilon}(\delta_0)) = \lim_{n \to \infty} \mathcal{H}^d_{2\ell_n}(Z'_{\varepsilon}(\delta_0)) = 0$, thus proving the claim.

4. Proof of Theorem A

Recall that I = [0, 1) and let $f: I \to I$ be a piecewise λ -affine map as defined in the introduction with $-1 < \lambda < 1$. We may suppose that $\lambda \neq 0$, otherwise $\mathcal{E}_f = \emptyset$ and the result trivially holds. Let $f_{\delta} = R_{\delta} \circ f$ where $R_{\delta}(x) = \{x + \delta\}$ is the rotation map of angle $\delta \in \mathbb{R}$. Notice that f_{δ} is also a piecewise λ -affine map and

$$\mathcal{E}_f = \mathcal{E}_{f_\delta} + \delta.$$

Therefore, to prove Theorem A it is sufficient to prove that for any piecewise λ -affine map f we have $\dim_H(\mathcal{E}_f \cap (-\delta_0, \delta_0)) = 0$ for some $\delta_0 > 0$.

Since $|\lambda| < 1$, the map f has a gap, i.e., $I \setminus f(I) \neq \emptyset$. This gap also has non-empty interior, so we can choose a point $c \in I \setminus f(I)$ such that $\ell := \operatorname{dist}(c, f(I)) > 0$. Here, dist denotes the distance in I induced from the circle \mathbb{R}/\mathbb{Z} through the canonical identification $I \hookrightarrow \mathbb{R} \to \mathbb{R}/\mathbb{Z}$.

Now, we define $g: I \to I$ by

$$g = R_{-c} \circ f \circ R_c.$$

The map g is again a piecewise λ -affine map of I. Clearly, $\ell = \text{dist}(0, g(I))$ since R_c is an isometry of (I, dist). We claim that

$$\dim_H(\mathcal{E}_g \cap (-\ell, \ell)) = 0. \tag{3}$$

From (3) we conclude the proof of Theorem A, because f and g are conjugated through R_c , which implies that $\mathcal{E}_f = \mathcal{E}_g$.

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FIGURE 2. Plots of f and $g = R_{-c} \circ f \circ R_c$. The shaded region in (A) illustrates the gap of f.

In order to prove claim (3), notice that, for every $0 \leq \delta < \ell$, we have $\operatorname{dist}(0, g_{\delta}(I)) > 0$ where $g_{\delta} = R_{\delta} \circ g$. Equivalently,

$$\overline{g_{\delta}(I)} \subset (0,1), \quad \forall \, \delta \in [0,\ell).$$
(4)

Next, extend g_{δ} in a canonical way to a piecewise λ -affine map G_{δ} defined on the whole of \mathbb{R} . We call G_{δ} the affine extension of g_{δ} . The map G_{δ} has the property that $G_{\delta}(x) = G(x) + \delta$ where $G \colon \mathbb{R} \to \mathbb{R}$ is the affine extension of g. Notice that this property holds because of (4). Moreover, since the orbit of any $x \in \mathbb{R}$ under G_{δ} eventually enters I, we conclude that G_{δ} is asymptotically periodic if and only if g_{δ} is asymptotically periodic. Thus, (3) is equivalent to

 $\dim_H \{ \delta \in (-\ell, \ell) \colon G_\delta \text{ is not asymptotically periodic} \} = 0.$ (5)

Let (Φ, \mathbf{a}) be a pair defining G, i.e., an injective IFS $\Phi = \{\phi_1, \ldots, \phi_k\}$ and $\mathbf{a} = (a_1, \ldots, a_{k-1}) \in \Omega_k$ such that $G = f_{\Phi, \mathbf{a}}$. Notice that the ϕ_i 's are λ -affine maps. By Theorem B, we know that

 $\dim_H \{ \delta \in \mathbb{R} \colon f_{\Phi, \boldsymbol{a}+\delta} \text{ is not asymptotically periodic} \} = 0.$

It is easy to see that G_{δ} and $f_{\Phi, \boldsymbol{a}-\delta/(1-\lambda)}$ are conjugated by the affine map $x \mapsto x + \delta/(1-\lambda)$ (cf. [10, Reduction lemma]). This shows (5), thus proving Theorem A.

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References

- B. Bires, *Piecewise contractions and b-adic expansions*, C. R. Math. Rep. Acad. Sci. Canada 42 (2020), no. 1, 1–9.
- J. Bowman and S. Sanderson, Angel's staircase, sturmian sequences, and trajectories on homothetic surfaces, Journal of Modern Dynamics 16 (2020), 109–153.
- Y. Bugeaud, Dynamique de certaines applications contractantes, linéaires par morceaux, sur [0,1], C. R. Acad. Sci. de Paris 317 (1993), no. Série I, 575–578.
- Y. Bugeaud and J.-P. Conze, Calcul de la dynamique d'une classe de transformations linéaires contractantes mod 1 et arbre de farey, Acta Arithmetica LXXXVIII.3 (1999), 201–218.
- 5. Y. Bugeaud, D. H. Kim, M. Laurent, and A. Nogueira, On the Diophantine nature of the elements of Cantor sets arising in the dynamics of contracted rotations, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 22 (2021), no. 4, 1691–1704. MR 4360601
- A. Calderon, E. Catsigeras, and P. Guiraud, A spectral decomposition of the attractor of piecewise-contracting maps of the interval, Ergodic Theory Dynam. Systems 41 (2021), no. 7, 1940–1960. MR 4266357
- J. P. Gaivão and A. Nogueira, Dynamics of piecewise increasing contractions, Bulletin of the London Mathematical Society 54 (2022), no. 2, 482–500.
- S. Janson and C. Öberg, A piecewise contractive dynamical system and election methods, Bull. Soc. Math. France 147 (2019), no. 3, 395–441.
- M. Laurent and A. Nogueira, *Rotation number of contracted rotations*, Journal of Modern Dynamics **12** (2018), 175.
- 10. A. Nogueira, B. Pires, and R. A. Rosales, *Topological dynamics of piecewise* λ -affine maps, Ergodic Theory Dynam. Systems **38** (2018), no. 5, 1876–1893. MR 3820005

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