The Avalanche Principle

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Meaning and use of the Avalanche Principle

Given a long list of $SL_2(\mathbb{R})$ matrices $A_1, A_2, \ldots A_n$, the Avalanche Principle gives (quantitative) conditions under which:

$$||A_n \cdot \ldots \cdot A_j \cdot \ldots \cdot A_1|| \approx ||A_n|| \cdot \ldots ||A_j|| \cdot \ldots \cdot ||A_1||$$

[First appeared in a 2001 Annals paper by M. Goldstein, W. Schlag]

It is used to prove:

- Positivity of the Lyapunov exponent at *large* disorder
- Inductive step of large deviation estimates (LDT)
- Hölder continuity / fine continuity properties of the Lyapunov exponent as a function of energy E and / or jointly (energy, frequency), (E, ω)

Meaning and use of the Avalanche Principle

The matrices A_j the principle is applied to are typically iterates of a Schrödinger cocycle:

Given the $SL_2(\mathbb{R})$ matrix

$$A(x) = A(E, x) = \begin{bmatrix} \lambda v(x) - E & -1 \\ 1 & 0 \end{bmatrix}$$

The skew-product mapping:

$$(T, A(E)) \colon X \times \mathbb{R}^2 \to X \times \mathbb{R}^2$$

$$(x, \begin{bmatrix} u_1 \\ u_0 \end{bmatrix}) \mapsto (Tx, \, A(E, x) \cdot \begin{bmatrix} u_1 \\ u_0 \end{bmatrix})$$

is called a Schrödinger cocycle.

Its forward iterates have the form $(T^n x, A_n(E, x))$, where

$$A_n(E, x) = A(E, T^{n-1} x) \cdot \ldots \cdot A(E, x) =$$

$$= \begin{bmatrix} \lambda v(T^{n-1} x) - E & -1 \\ 1 & 0 \end{bmatrix} \cdot \ldots \cdot \begin{bmatrix} \lambda v(x) - E & -1 \\ 1 & 0 \end{bmatrix}$$

are called the transfer / fundamental matrices.

Avalanche Principle I

Let $A_1, \ldots, A_n$ be a sequence of arbitrary $SL_2(\mathbb{R})$ matrices. Suppose that for all j we have:

$$||A_j|| \geq \mu$$

where $\mu > n$

$$\frac{||A_{j+1} \cdot A_j||}{||A_{j+1}|| \cdot ||A_j||} > \frac{1}{\sqrt{\mu}}$$

Then

$$|\log ||A_n \cdot \ldots \cdot A_1|| - \sum_{j=1}^{n} \log ||A_j|| - \sum_{j=1}^{n-1} \log |b^{++}(A_j, A_{j+1})|| \leq C \frac{n}{\mu}$$

and

$$|\log ||A_n \cdot \ldots \cdot A_1|| + \sum_{j=2}^{n-1} \log ||A_j|| - \sum_{j=1}^{n-1} \log ||A_{j+1} A_j||| \leq C \frac{n}{\mu}$$

Main technical steps in the proof of the AP

Definition: $b^{++}(K, M) := v_K^+ \cdot u_M^+$

where for any $SL_2(\mathbb{R})$ matrix A, if we consider the polar decomposition $A = U \cdot \sqrt{A^* A}$, and if $u_A^\pm$ are two normalized eigenvectors of $\sqrt{A^* A}$, then there are two *unit* vectors $v_A^\pm$ so that:

$$A\, u_A^+ = \|A\|\, v_A^+ \quad \text{and} \quad A\, u_A^- = \|A\|^{-1}\, v_A^-$$

Lemma 1: Given $M, K \in SL_2(\mathbb{R})$ we have:

$$\frac{||M \cdot K||}{||M|| \cdot ||K||} - \frac{1}{||K||^2} - \frac{1}{||M||^2} \leq |b^{++}(K, M)| \leq \frac{||M \cdot K||}{||M|| \cdot ||K||} + \frac{1}{||M||^2}$$

From here we get, using the assumptions of the AP:

$$|b^{++}(A_j, A_{j+1})| \cdot \frac{||A_j|| \cdot ||A_{j+1}||}{||A_{j+1} \cdot A_j||} = 1 + O(\frac{1}{\mu^{3/2}})$$

$$|b^{++}(A_j, A_{j+1})| > \frac{1}{2\sqrt{\mu}}$$

Main technical steps in the proof of the AP

Lemma 2: Given $M, K \in SL_2(\mathbb{R})$ and u a unit vector, we have:

$$MKu = \sum_{\epsilon_1,\epsilon_2=\pm 1} ||M||^{\epsilon_2} \cdot ||K||^{\epsilon_1} \cdot b^{\epsilon_1,\epsilon_2}(K, M) \cdot (u_K^{\epsilon_1} \cdot u)\, v_M^{\epsilon_2}$$

By induction on n we get:

Lemma 3: Given $A_1, A_2, \dots A_n \ \in SL_2(\mathbb{R})$ and u a unit vector, we have:

$$A_n \cdot \ldots \cdot A_1 \ u = \sum_{\epsilon_1,\dots,\epsilon_n=\pm 1} ||A_n||^{\epsilon_n} \cdot \ldots \cdot ||A_1||^{\epsilon_1} \cdot$$

$$\cdot \prod_{j=1}^{n-1} b^{\epsilon_j,\epsilon_{j+1}}(A_j, A_{j+1}) \cdot (u_{A_1}^{\epsilon_1} \cdot u)\, v_{A_n}^{\epsilon_n}$$

Refinements

Avalanche Principle II:
The same conclusion holds with the same proof if we make the same assumptions but instead of $SL_2(\mathbb{R})$ matrices we take 2×2 matrices satisfying

$$\max_{1 \leq j \leq n} \; |\det(A_j)| \leq \kappa$$

Avalanche Principle III:
The condition $\mu > n$ can be replaced by: μ sufficiently large (say $\mu > 1000$) and large relative to the integer *factors* of n:

$n = n_1 \cdot \ldots \cdot n_s$, where $3 \leq n_j < \frac{\mu}{2}$ for $1 \leq j \leq s-1$ and $n_s < \mu$.

For instance if $n = 3^s$ and μ were large, the condition would be met.

Typical application of the Avalanche Principle

We make the following notations:

$$u_n(x) = u_n(E, x) := \frac{1}{n} \log \| A_n(E, x) \|$$

$$L_n(E) \; = < u_n > := \int_X \; u_n(x) \; dx$$

Then the maximal Lyapunov exponent is $L(E) = \lim_{n \to \infty} \; L_n(E)$ and the functions $u_n(x)$ have bounded s.h. extensions.

We want to show:

LDT: $\text{mes} \; [x \in X : |u_n(x) - < u_n > | > n^{-\tau} \;] < e^{-n^\sigma}$

Lower bound for Lyapunov exp: $L_n(E) > c \log \lambda$ for all n, E

These estimates can be obtained at a large enough initial scale n_0 using non-transversality of the potential + large coupling λ.

To obtain them at every scale n, we use an inductive procedure whose main ingredients are: AP + two estimates on s.h. functions.

Two estimates on subharmonic (s.h.) functions

Assume $u \colon \mathbb{T} \to \mathbb{R}$ has a bounded s.h. extension $u(z)$ to a strip.

Averages of shifts of s.h. functions: Assuming a Diophantine condition on the frequency ω, for some explicit constants $\sigma, \tau > 0$ and for n large enough we have:

$$\text{mes}\,[x \in \mathbb{T} : |\, \frac{1}{n} \sum_{j=0}^{n-1} u(T^j x) - < u > | > n^{-\tau}] < e^{-n^\sigma} \qquad (♪)$$

Boosting mean oscillations (via John-Nirenberg's inequality and BMO estimates on s.h. functions): If for $\epsilon_1 \ll \epsilon_0 \ll 1$ we have

$$\text{mes}\,[x \in \mathbb{T} : |\, u(x) - < u > | > \epsilon_0] < \epsilon_1 \qquad (\flat)$$

then for an absolute constant $c > 0$,

$$\text{mes}\,[x \in \mathbb{T} : |\, u(x) - < u > | > \sqrt{\epsilon_0}] < e^{-c\left(\sqrt{\epsilon_0}\right)^{-1}} \qquad (\sharp)$$

The two estimates above have higher dimensional analogues.

Inductive step of LDT and lower bound for Lyapunov exp

As indicated above, these estimates hold at an initial scale n_0. Choose n such that $n_0 \ll n \ll e^{n_0^\sigma}$ and denote $n_1 := n \cdot n_0$.

Then for x outside of a set of measure $< e^{-n_0^{\sigma_1}}$, due to the estimate on "averages of shifts of s.h. functions" we get:

$$L_{n_0} = < u_{n_0} > \approx \frac{1}{n} \sum_{j=0}^{n-1} u_{n_0}(T^j x) \qquad (♪)$$

The AP applies to $A_j = A_j(x) := \|A_{n_0}(T^{j \cdot n_0} x)\|$ and since $A_{n-1} \cdot \ldots \cdot A_0 = A_{n \cdot n_0} = A_{n_1}$, we obtain:

$$\frac{1}{n_1} \log\|A_{n_1}(x)\| = \frac{1}{n_0} \frac{1}{n} \log\|A_{n-1} \cdot \ldots \cdot A_0\| \overset{(AP)}{\approx} \frac{1}{n_0} \frac{1}{n} \sum_{j=0}^{n-1} \log\|A_j\|$$

$$= \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{n_0} \log\|A_{n_0}(T^{j \cdot n_0} x)\| \approx \frac{1}{n} \sum_{j=0}^{n-1} u_{n_0}(T^j x) \overset{(♪)}{\approx} L_{n_0}$$

Inductive step of LDT and lower bound for Lyapunov exp

We then have that for x outside of a set of measure $< e^{-{n_0}^{\sigma_1}}$, the following holds:

$$u_{n_1}(x) = \frac{1}{n_1} \log\|A_{n_1}(x)\| \approx L_{n_0}$$

hence averaging in x we get:

$$L_{n_1} \approx L_{n_0} \qquad \text{and}$$

$$\text{mes } [x \in X : |u_{n_1}(x) - < u_{n_1} > | > n_1^{-\tau} \,] < e^{-{n_0}^{\sigma_1}}$$

To be able to continue this procedure to a larger scale $n_2 \gg n_1$, we need to *boost* the above estimate using (♯), and we obtain:

$$\text{mes } [x \in X : |u_{n_1}(x) - < u_{n_1} > | > n_1^{-\tau_1} \,] < e^{-{n_1}^{\sigma_1}}$$

We therefore get scales $\ldots n_{k+1} \gg n_k \gg \ldots n_2 \gg n_1 \gg n_0$ such that the LDT holds at each scale and

$$\ldots L_{n_{k+1}} \approx L_{n_k} \approx \ldots L_{n_2} \approx L_{n_1} \approx L_{n_0} > c \log \lambda$$

which establishes positivity of the Lyapunov exponent.