Positivity of the Lyapunov exponent for Schrödinger cocycles defined by skew-shift dynamics and large, non-analytic potential functions

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The discrete, 1-dim, quasi-periodic Schrödinger equation

$$-(u_{n+1} + u_n) + \lambda\, v\,(T^n x) \cdot u_n = E \cdot u_n$$

- $u = [u_n]_{n \in \mathbb{Z}} \subset \mathbb{R}$ (state)
- $E \in \mathbb{R}$ (energy)
- $\lambda \in \mathbb{R}$ (coupling constant)
- T is an ergodic transformation on $X = \mathbb{T},\ \mathbb{T}^2,\ \mathbb{T}^d$
  T^n its nth iteration
- $v \colon x \in X \mapsto v(x) \in \mathbb{R}$ potential function.

The associated Schrödinger cocycle

Consider the $SL_2(\mathbb{R})$ matrix

$$A_E = A_E(x) = \begin{bmatrix} \lambda v(x) - E & -1 \\ 1 & 0 \end{bmatrix}$$

The skew-product mapping:

$$(T, A_E) \colon X \times \mathbb{R}^2 \to X \times \mathbb{R}^2$$

$$(x, \begin{bmatrix} u_1 \\ u_0 \end{bmatrix}) \mapsto (Tx, \ A_E(x) \cdot \begin{bmatrix} u_1 \\ u_0 \end{bmatrix})$$

is called a Schrödinger cocycle.

Its forward iterates have the form $(T^n x, A_E^n(x))$, where

$$A_E^n(x) = A_E(T^{n-1}\,x) \cdot \ldots \cdot A_E(x) =$$

$$= \begin{bmatrix} \lambda v(T^{n-1}\,x) - E & -1 \\ 1 & 0 \end{bmatrix} \cdot \ldots \cdot \begin{bmatrix} \lambda v(x) - E & -1 \\ 1 & 0 \end{bmatrix}$$

are called the transfer / fundamental matrices.

Relationship between Schrödinger eqn, transfer matrices

Transfer matrices solve the Schrödinger equation.
Indeed, if $u = [u_n]_{n\in\mathbb{Z}}$, E are a formal solution to

$$-\left(u_{n+1} + u_n\right) + \lambda\, v\left(T^n x\right) \cdot u_n = E \cdot u_n$$

then

$$\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = \begin{bmatrix} \lambda v(T^n x) - E & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix}$$

so

$$\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = A_E^{n+1}(x) \cdot \begin{bmatrix} u_1 \\ u_0 \end{bmatrix}$$

It is then natural to study the growth of the norms of these transfer matrices $A_E^n(x)$ as $n \to \infty$.

The Lyapunov exponent

The average exponential growth of the norms of the the transfer matrices

$$A_E^n(x) = \prod_{j=n}^{1} \begin{bmatrix} \lambda v(T_j\, x) - E & -1 \\ 1 & 0 \end{bmatrix}$$

is called the (maximal) Lyapunov exponent:

$$L(E) := \lim_{n\to\infty} \int_X \frac{1}{n} \log ||A_E^n(x)||\, dx$$

The limit above exists due to sub-additivity.
In fact, due to the ergodicity of the transformation T,

$$\frac{1}{n} \log ||A_E^n(x)|| \to L(E) \text{ as } n \to \infty \text{ for a.e. } x \in X$$

Positivity of the Lyapunov exponent

$$L(E) = \lim_{n\to\infty} \int_X \frac{1}{n} \log ||A_E^n(x)|| \, dx$$

Since $A_E^n(x) \in SL_2(\mathbb{R})$, the Lyapunov exponent is non-negative.
Goal: prove positivity of the Lyapunov exponent for *certain* Schrödinger cocycles, uniformly in the energy E.

In other words, we want to show

$$L(E) \geq c > 0$$

for *fixed* potential v but for all energies E (in $\mathbb{R}$ or in an interval).

This property is relevant in mathematical physics $\rightsquigarrow$ correlation between positivity of the Lyapunov exponent and absence of absolutely continuous spectrum, or even Anderson localization, continuity properties of the integrated density of states etc.

This is also the *opposite* of reducibility: the energies for which the Schrödinger cocycle is not uniformly hyperbolic and it is reducible live inside the set of zero Lyapunov exponent.

"*Certain* Schrödinger cocycles " made explicit

$$(T, A_E)\colon X \times \mathbb{R}^2 \to X \times \mathbb{R}^2$$

$$(x, \vec{v}) \mapsto (\,Tx, \begin{bmatrix} \lambda v(x) - E & -1 \\ 1 & 0 \end{bmatrix} \cdot \vec{v}\ )$$

- Transformations T considered:
  ◇ 1-dim shift: $T\colon \mathbb{T} \to \mathbb{T},\ Tx = x + \omega$, ω irrational
  ◇ 2-dim shift: $T\colon \mathbb{T}^2 \to \mathbb{T}^2,\ T(x_1, x_2) = (x_1 + \omega_1, x_2 + \omega_2)$,
  where $\omega_1,\ \omega_2$ are rationally independent
  ◇ skew-shift: $T\colon \mathbb{T}^2 \to \mathbb{T}^2,\ T(x_1, x_2) = (x_1 + x_2, x_2 + \omega)$,
  where ω is irrational
- In general, arithmetic (Diophantine) conditions on ω.
- Large potential / coupling constant: $\lambda \gg 1$
- "Smooth" potential function $v(x) \leftrightsquigarrow$ "smooth" cocycle
- Generic "transversality" condition on the function $v(x)$.

Summary of relevant results and methods

- M. R. Herman: 1-dim shift model with $v(x) = \cos x$, $\lambda > 2$ and ω any irrational number.

$$L(E) \geq \log \frac{\lambda}{2} \quad \text{for all } E$$

Use of *subharmonicity method* (Jensen's inequality).
Method also works for trigonometric polynomials but not for analytic functions.

- E. Sorets, T. Spencer: 1-dim shift model with $v(x)$ real analytic, non-constant function, ω any irrational frequency, $\lambda > \lambda_0(v)$:

$$L(E) \geq \frac{1}{2} \log \lambda \quad \text{for all } E$$

Use of *complexification* that allows one to avoid the set $[\underline{x}\colon v(\underline{x}) \approx E]$.
Method does not work in higher dimensions or for the skew shift, or (obviously) for non-analytic functions.

Summary of relevant results and methods

- L.H. Eliasson: 1-dim shift model with $v(x)$ Gevrey & satisfying a generic transversality condition, ω Diophantine, $\lambda > \lambda_0$ that depends on v and on the Diophantine condition:

$$L(E) > 0 \quad \text{for a.e } E$$

Use of KAM methods - conjugating the Schrödinger operator to a diagonal operator through an iterative procedure.

- J. Bourgain, M. Goldstein, W. Schlag: one, multi-dim and skew shift models, Diophantine frequency, analytic, non-constant potential function $v(x)$, large λ that depends on v and on the Diophantine condition:

$$L(E) \gtrsim \log \lambda \quad \text{for all } E$$

$\rightarrow$ Analytic methods (e.g. estimates on Fourier coefficients of subharmonic functions, averages of shifts of s.h. functions ...)

Summary of relevant results and methods

- K. Bjerklöv, J. Chan: 1-dim shift, more general potential functions (e.g. C^n) but results require *eliminating a positive set of energies* that depends on frequency, coupling constant.

  Use of dynamics methods à la Benedicks-Carleson (K. Bjerklöv), KAM approach & Sard-type arguments (J. Chan).

- S. Klein: all three models (one and multi-dim shift, skew-shift) $v(x)$ Gevrey, satisfying a generic transversality condition, ω Diophantine, $\lambda > \lambda_0$ that depends on v and on the Diophantine condition:

$$L(E) \gtrsim \log \lambda \quad \text{for all } E$$

  Use of analytic methods similar to those of Bourgain, Goldstein, Schlag & polynomial approximation, Sard-type arguments, semi-algebraic sets.

Gevrey regularity class, generic transversality condition

Given $v\colon X \to \mathbb{R}$, where $X = \mathbb{T},\ \mathbb{T}^2$.

- Gevrey regularity:

$$\sup_{x\in X} |\partial^{\underline{m}}\, v(x)| \le M K^{|\underline{m}|} (\underline{m}!)^s \quad \forall\, \underline{m} \in \mathbb{N} \text{ or } \mathbb{N}^2$$

which is equivalent to the following *sub*-exponential decay of its Fourier coefficients:

$$|\hat{v}(\underline{l})| \le M e^{-\rho |\underline{l}|^{1/s}} \quad \forall\, \underline{l} \in \mathbb{Z} \text{ or } \mathbb{Z}^2$$

The exponent $s > 1$ is the order of the Gevrey class.

- Transversality condition (TC):

$$\forall\, x \in X \ \ \exists\, \underline{m} \in \mathbb{N} \text{ or } \mathbb{N}^2, \ \ |\underline{m}| \ne 0 \ \text{such that} \ \ \partial^{\underline{m}}\, v(x) \ne 0$$

Gevrey + TC natural extension of analytic, non-constant.

Outline of the analytic method of proof

Main technical result needed is a large deviation estimate (LDT) for logarithmic averages of transfer matrices. Due to ergodicity,

$$x \text{ a.s.} \quad u_n(\underline{x}) = \frac{1}{n} \log ||A_E^n(x)|| \to L(E) \quad \text{as } n \to \infty$$

The LDT provides a *quantitative* version of this convergence:

$$\text{mes } [x \in X : |u_n(x) - < u_n > | > \epsilon] < \delta(n, \epsilon)$$

where $\epsilon = o(1)$ and $\delta(n, \epsilon) \to 0$ as $n \to \infty$
We get the above estimate with $\epsilon \approx n^{-\tau}$ and $\delta \approx e^{-n^{\sigma}}$

The proof of this LDT is done by induction on the scale n.

- First step requires non degeneracy of the potential & a large coupling constant. Crucial ingredient: a Łojasiewicz type estimate.
- Inductive step requires regularity of the potential (analyticity or Gevrey) & the arithmetic condition on the frequency. Main ingredients: averages of shifts of subharmonic functions, avalanche principle for $SL_2(\mathbb{R})$ matrices.

Inductive step of the LDT in the 1-dim shift, analytic case

$v(x)$ analytic

$\Downarrow$

$v(x)$ has bounded holomorphic extension to a strip of width $\rho > 0$

$\Downarrow$

$$u_n(x) := \frac{1}{n} \log ||A_E^n(x)||$$

has a bounded (by B) subharmonic extension to this strip.

$\Downarrow$

the Fourier coefficients of $u_n(x)$ have the decay :

$$|\hat{u}_n(k)| \lesssim \frac{B}{\rho} \frac{1}{|k|} \quad \text{for all } k \neq 0$$

The decay above is <u>uniform</u> in n.

Averages of shifts of subharmonic functions

If $u \colon X \to \mathbb{R}$ has a subharmonic extension $u(z)$ to a strip of width ρ and it is bounded by a number B on that strip, then assuming a Diophantine condition on the frequency ω, for some explicit constants $\sigma, \tau > 0$ & for R large enough we have:

$$\text{mes } [x \in X : | \frac{1}{R} \sum_{j=0}^{R-1} u(T^j x) - < u > | > \frac{B}{\rho} R^{-\tau}] < e^{-R^\sigma}$$

Idea / moral of the poof / statement: due to ergodicity of the transformation T, the forward orbit points $x, Tx, \ldots T^{R-1}x$ tend to be fairly well uniformly distributed throughout the phase space X for all points x.

Hence the averages of shifts above resemble Riemann sums, which converge to the Riemann integral / mean of the function $u(x)$.

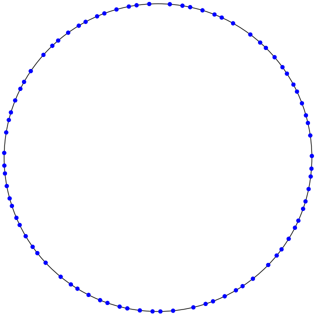
A *quantitative* convergence to the mean $< u >$ should presumably follow from a quantitative description of ergodicity $\leftrightsquigarrow$ arithmetic condition on the frequency.

The 1-dim shift transformation on $\mathbb{T}$:

$$Tx := x + \omega$$

Its nth iteration:

$$T^n x = x + n\,\omega$$

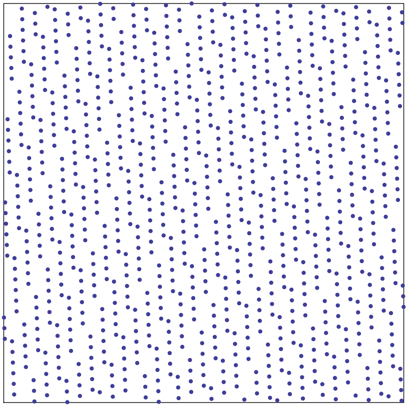


The 2-dim shift transformation on $\mathbb{T}$:

$$T\underline{x} := (x_1 + \omega_1, x_2 + \omega_2)$$

Its *n*th iteration:

$$T^n\underline{x} = (x_1 + n\,\omega_1, x_2 + n\,\omega_2)$$

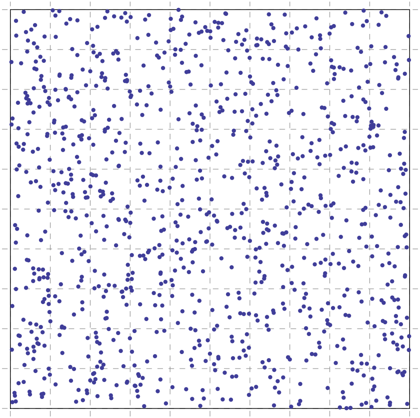


The skew-shift transformation on $\mathbb{T}^2$:

$$T(x_1, x_2) := (x_1 + x_2, x_2 + \omega)$$

Its *n*th iteration:

$$T^n(x_1, x_2) = (x_1 + nx_2 + \frac{n(n-1)}{2}\omega, x_2 + n\omega)$$



The avalanche principle in $SL_2(\mathbb{R})$

Let $A_1, \ldots, A_R$ be a sequence of arbitrary $SL_2(\mathbb{R})$ matrices. Suppose that for all j

$$||A_j|| \geq \mu \geq R$$

$$[\log ||A_{j+1}|| + \log ||A_j|| - \log ||A_{j+1}A_j||] \leq \frac{1}{2} \log \mu$$

Then

$$|\log ||A_R \cdot \ldots \cdot A_1|| + \sum_{j=2}^{R-1} \log ||A_j|| - \sum_{j=1}^{R-1} \log ||A_{j+1}A_j|| \,| \leq C\frac{R}{\mu}$$

Large deviation theorem: first step

Recall the transversality condition (TC) on the potential $v(\underline{x})$:

$$\forall\, \underline{x} \in \mathbb{T}^2 \; \exists\, \underline{m} \in \mathbb{N}^2 \; |\underline{m}| \neq 0 : \quad \partial^{\underline{m}}\, v(\underline{x}) \neq 0$$

This means that $v(\underline{x})$ is not flat at any point. The key ingredient here is describing this non-flatness in a quantitative way.

Łojasiewicz type estimate:

Assume $v(\underline{x})$ is a smooth function on $[0,1]^2$ satisfying the (TC). Then for every $\epsilon > 0$

$$\sup_{E \in \mathbb{R}} \; \text{mes}\, [\underline{x} \in [0,1]^2 : |v(\underline{x}) - E| < \epsilon] < C \cdot \epsilon^{b}$$

where $C, b > 0$ depend only on v.

In other words, $[\underline{x} \colon v(\underline{x}) \approx E]$ is a set of small measure.
This combined with choosing a large coupling constant leads to LDT at a large enough initial scale n_0.

Proof of the Łojasiewicz estimate

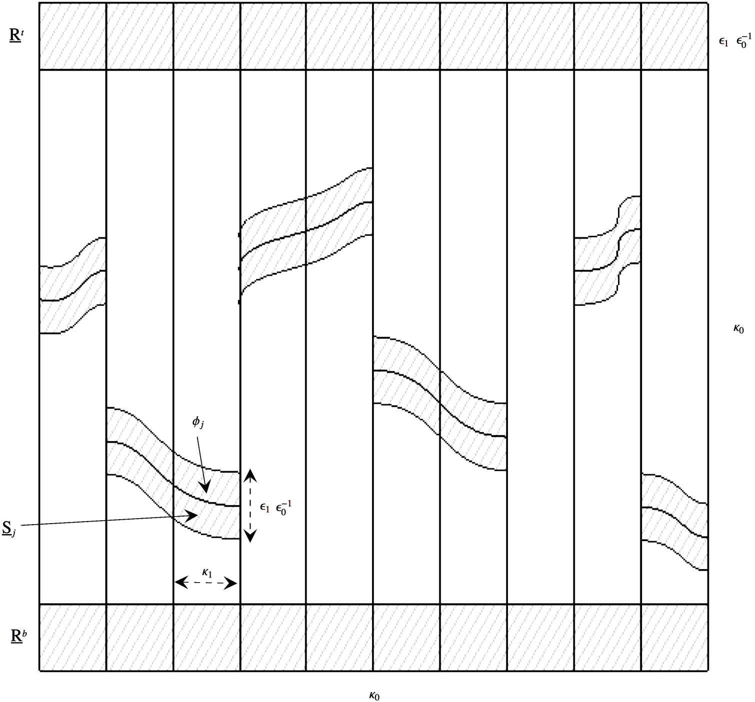
The proof of this estimate is based on repeatedly applying a *quantitative* implicit function theorem, also used by J. Chan, M. Goldstein, W. Schlag.

Quantitative implicit function theorem:
Given a C^1 function $f(\underline{x})$ on a rectangle $\underline{\mathcal{R}} = I \times J \subset [0,1]^2$ s.t.

$$\min_{\underline{x} \in \underline{\mathcal{R}}} |\partial_{x_2} f(\underline{x})| =: \epsilon_0 > 0$$

and a small enough constant $\epsilon_1 \ll \epsilon_0$, the points $[(x_1, x_2) \in \underline{\mathcal{R}} \colon |f(x_1, x_2)| \le \epsilon_1]$ are in a narrow strip at the top or at the bottom of the rectangle $\underline{\mathcal{R}}$, or near the graphs of some functions $\phi_j(x_1)$ - i.e. $x_2 \approx \phi_j(x_1)$.

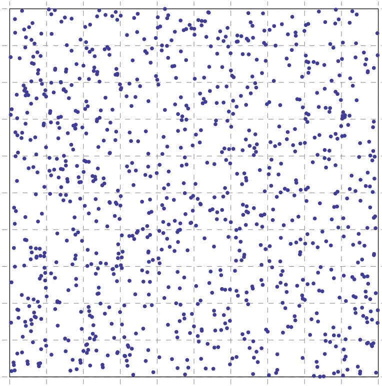
We have estimates on the number of such functions $\phi_j(x_1)$ and on their slopes, hence we have estimates on the measure and 'complexity' of the 'bad' set $[(x_1, x_2) \in \underline{\mathcal{R}} \colon |f(x_1, x_2)| \le \epsilon_1]$.



$\underline{\mathbf{R}}^t$
$\epsilon_1 \;\; \epsilon_0^{-1}$
κ_0
ϕ_j
$\epsilon_1 \;\; \epsilon_0^{-1}$
$\underline{\mathbf{S}}_j$
κ_1
$\underline{\mathbf{R}}^b$
κ_0

Important open problem for the skew-shift transformation

$$T(x_1, x_2) := (x_1 + x_2, x_2 + \omega)$$



Due to the weekly mixing properties of the skew-shift, this model is expected to behave more like the one dimensional *random* model. Therefore, one expects positivity of the Lyapunov exponent for all energies and for all coupling constants.

In other words, the size of the potential function $v(x)$ that defines the cocycle should not matter.