Theory of shadowing and its applications.

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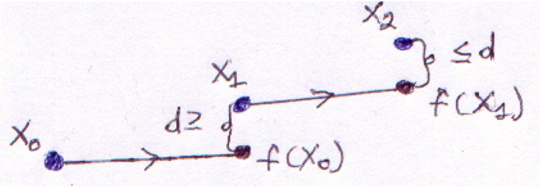
d-pseudotrajectory

- M manifold, dist, $f \in \mathrm{Diff}^1(M)$, $O(p,f) = \{f^n(p)\}_{n\in\mathbb{Z}}$
- $\xi = \{x_n\}$ is a d-pseudotrajectory, if

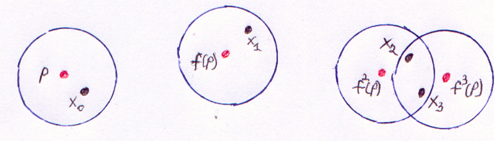
$$\mathrm{dist}(x_{n+1}, f(x_n)) \le d \quad \forall n \in \mathbb{Z}.$$



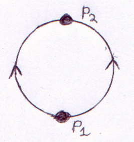
POTP

- $\forall \epsilon > 0\ \exists d > 0$ such that $\forall d$-pseudotrajectory ξ there exists an exact trajectory $\{p_n = f^n(p)\}$ such that

$$\mathrm{dist}(x_n, p_n) < \epsilon \quad \forall n \in \mathbb{Z}.$$



- Example, where POTP holds:



OSP and WSP

- $f \in \text{OSP} \Leftrightarrow \forall \epsilon > 0 \; \exists d > 0$ such that $\forall d$-pseudotrajectory ξ there exists an exact trajectory $O(p, f)$ of a point p such that

$$\xi \subset N(\epsilon, O(p, f)) \quad \text{and} \quad O(p, f) \subset N(\epsilon, \xi).$$

- Example where OSP holds and POTP does not hold: an irrational rotation of the circle

$$x \mapsto g(x) = x + \alpha \quad \text{for } \alpha \notin \mathbb{Q}.$$

- $f \in \text{WSP} \Leftrightarrow \forall \epsilon > 0 \; \exists d > 0$ such that $\forall d$-pseudotrajectory ξ there exists an exact trajectory $O(p, f)$ of a point p such that

$$\xi \subset N(\epsilon, O(p, f)).$$

Main problems

- $H(M)$ with C^0-metric, $\mathrm{Diff}^1(M)$ with C^1-metric
- a set is **generic** if it contains a countable intersection of open and dense sets

(P1) Is the set of mappings having some shadowing property generic in Baire sense (for space $H(M)$, $\mathrm{Diff}^1(M)$)?

(P2) Characterisation of sets of diffeomorphisms having some shadowing property in terms of hyperbolic theory.

Results

	POTP	OSP	WSP
C^0-topology	generic, Pilyugin, Plamenevskaya 1999	generic	generic
C^1-topology	nondense, Bonatti, Diaz, Turcat 2000	nondense, Osipov, 2010	generic, Crovisier 2006

Hyperbolic set, definition.

A is a **hyperbolic set** for $f \in \mathrm{Diff}^1(M) \Leftrightarrow$

A is a compact f-invariant set and
there exist constants C and λ such that
for all $p \in A$ there exist complementary linear subspaces
$E^s(p)$ and $E^u(p)$ of T_pM and

$$|Df^k(p)v| \le C\lambda^k|v|, \quad \forall v \in E^s(p), k \ge 0,$$
$$|Df^{-k}(p)v| \le C\lambda^k|v|, \quad \forall v \in E^u(p), k \ge 0.$$

Hyperbolic set, stable and unstable manifolds.

- If A is a hyperbolic set then $\forall p \in A$ there exist manifolds $W^s(p)$ and $W^u(p)$ (stable and unstable manifolds) such that

$$\dim W^s(p) = s, \quad \dim W^u(p) = u,$$

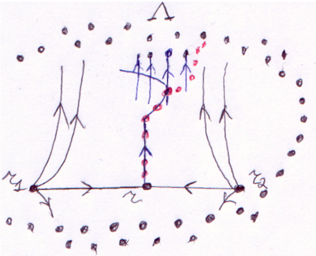
$$f^n(x) \longrightarrow p, \quad \text{as } n \to +\infty \qquad \forall x \in W^s(p),$$

$$f^{-n}(x) \longrightarrow p, \quad \text{as } n \to +\infty \qquad \forall x \in W^u(p).$$

- The number u is called an **index** of a hyperbolic set A.
- Hyperbolic periodic point.

C^1-nondensity of POTP

DA-diffeomorphism of Williams on $\mathbb{T}^2$:

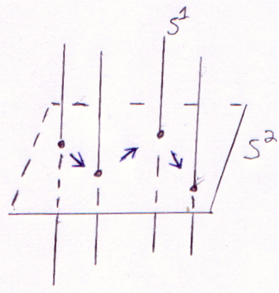


r_1, r_2 are hyperbolic repellers, r — saddle, Λ — hyperbolic attractor.

C^1-nondensity of OSP

Example of Ilyashenko and Gorodetski: domain $W \subset \mathrm{Diff}^1(S^2 \times S^1)$

- partially hyperbolic set S homeomorphic to the product of a Cantor set and a circle

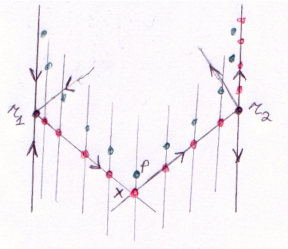


- hyperbolic points with different indices are dense in S

C^1-nondensity of OSP, case (A1)

There exist two hyperbolic periodic points r_1 and r_2 with $\dim(W^s(r_1)) = 2 = \dim(W^u(r_2))$ and $\dim(W^u(r_1)) = 1 = \dim(W^s(r_2))$ such that

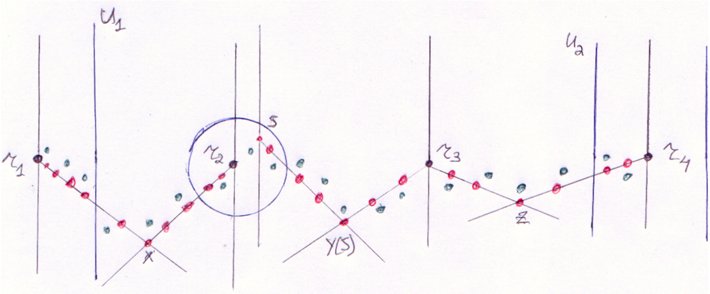
$$W^u(r_1) \cap W_s(r_2) \neq \emptyset.$$



C^1-nondensity of OSP, case (A2)

Not case (A1). Fix hyperbolic periodic points $r_1, \dots, r_4$ such that

$$\dim W^u(r_1) = \dim W^u(r_2) = 1, \quad \dim W^s(r_3) = \dim W^s(r_4) = 1.$$



Nonwandering set

- A point p is wandering for f if $\exists U \ni p$ such that $f^n(U) \cap U = \emptyset$ for all n with sufficiently large $|n|$.
- Alternative definition: a point p is nonwandering for f if $\exists \{p_k\}_{k \geq 0}, \{n_k\}_{k \geq 0}$ such that $n_k \to \infty$ and

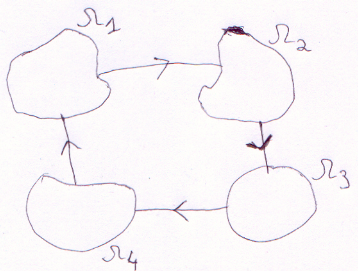
$$p_k \to p, \quad f^{n_k}(p_k) \to p.$$

Structural stability

- $f \in \mathbb{S} \Leftrightarrow \exists U$ such that $\forall g \in U \ \exists h \in H(M)$ such that $hf = gh$.
- (Robbin,Robinson,*Mañé*, 1988) **structural stability** is equivalent to **Axiom A** (the nonwandering set is hyperbolic and is the closure of periodic points) and **strong transversality condition**
- Spectral decomposition theorem (Smale): Axiom A implies $\Omega(f) = \Omega_1 \cup \ldots \cup \Omega_m$, Ω_j is hyperbolic and has a dense semi-trajectory.

Ω-stability

- $f \in \Omega\mathbb{S} \Leftrightarrow f$ and any diffeomorphism from its small neighborhood are topologically conjugate on nonwandering sets
- (Palis, 1987) Ω-**stability** is equivalent to **Axiom A** and **no cycle condition**



Results

- $$\mathrm{POTP} \neq \mathbb{S},$$
  $$\mathrm{Int}^1(\mathrm{POTP}) = \mathrm{Int}^1(\mathrm{OSP}) = \mathbb{S},$$
- **LipSP, Lipschitz shadowing:** (POTP with $\epsilon = Ld$) $\exists L, d_0$ such that for any d-pseudotrajectory ξ with $d \leq d_0$ there exists a point p such that
  $$\mathrm{dist}(x_k, f^k(p)) \leq Ld \quad \forall k \in \mathbb{Z}.$$
- (Pilyugin, Tikhomirov, 2010) LipSP $= \mathbb{S}$.

Periodic shadowing

- **Periodic shadowing** (PerSh)
  $\forall \epsilon > 0\ \exists d > 0$ such that $\forall$ periodic d-pseudotrajectory ξ there exists a periodic exact trajectory $\{p_n\}$ such that
  $$\text{dist}(x_n, p_n) < \epsilon \quad \forall n \in \mathbb{Z}.$$
- **Lipschitz periodic shadowing** (LipPerSh)
  PerSh with $\epsilon = Ld$.
- (Pilyugin, Tikhomirov, Osipov, 2010)
  $$\text{Int}^1(\text{PerSh}) = \text{LipPerSh} = \Omega\mathbb{S}.$$

General Scheme of the Proof

- $\Omega S \subset \mathrm{LipPerSh}$
- $\mathrm{Int}^1(\mathrm{PerSh}) \subset \Omega S$
- $f \in \mathrm{LipPerSh} \Rightarrow f \in \Omega S$
    - Step 1. hyperbolicity of periodic points
    - Step 2. uniform hyperbolicity of periodic points
    - Step 3. f has the Axiom A
    - Step 4. f satisfies the no-cycle condition

Proof of $\Omega S \subset$ LipPerSh

- Spectral decomposition theorem:
  $\Omega(f) = \Omega_1 \cup \ldots \cup \Omega_m$, Ω_j is hyperbolic and has a dense semi-trajectory
- ξ is a periodic d-pseudotrajectory, $\xi \subset U(\Omega_j)$ for some j
- Shadowing lemma:
  if Λ is hyperbolic then f has LipSh and is expansive in some $U(\Lambda)$
- Expansivity means that a d-pseudotrajectory can be shadowed only by one p

Proof of $\mathrm{Int}^1(\mathrm{PerSh}) \subset \Omega S$

- HP — set of diffeomorphisms f such that every periodic point of f is hyperbolic
  Lemma (Aoki, 1992, Hayashi, 1992). $\mathrm{Int}^1(\mathrm{HP}) = \Omega S$
- It is enough to prove that $\mathrm{Int}^1(\mathrm{PerSh}) \subset \mathrm{HP}$
- h is a C^1-small pertubation of f that is linear in $U(p)$, p is a nonhyperbolic periodic point for h

Proof of LipPerSh $\subset \Omega S$, Steps 1 and 2

- $f, f^{-1} \in$ LipPerSh with $L > 1$
- Lemma: Every periodic point is hyperbolic
- Key lemma: Set of all periodic points of f has all properties of a standard hyperbolic set except compactness.

$$|Df^j(p)v_s| \leq C\lambda^j|v_s|, \quad |Df^{-j}(p)v_u| \leq C\lambda^j|v_u|,$$

where $j \geq 0$, $v_s \in S(p)$, $v_u \in U(p)$

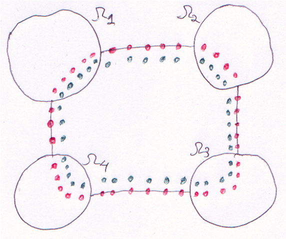
- p is an m-periodic point, let $v_0 = v_u \in U(p)$, $v_{i+1} = Df^i(p)v_i$,

$$\lambda_i = |v_{i+1}|/|v_i|, \quad a_0 = \tau, \quad a_{i+1} = \lambda_i a_i - 1,$$

  where τ is chosen such that $a_m = 0$
- $w_i = a_i v_i/|v_i|$ for $0 \leq i \leq m-1$, $\{w_i\}$ is an $m(n+1)$-periodic
- $|Df^i(p)v_u| = \lambda_0 \cdot \ldots \lambda_{i-1} > \frac{1}{16L}\left(1+\frac{1}{8L}\right)^i |v_u|, \quad 0 \leq i \leq m-1.$

Proof of LipPerSh $\subset \Omega S$, Steps 3 and 4

- Lemma: f satisfies the Axiom A
  - P_l — the set of periodic points of index l
  - $\mathrm{Cl} P_l$ is a hyperbolic set.
  - density of periodic points in $\Omega(f)$
- Lemma: f has no cycles
  - any cycle is approximated by periodic pseudotrajectories
  - any cycle is approximated by periodic exact trajectories
  -



Inverse shadowing

- A d-**pseudomethod** is a sequence of continuous mappings $\{\Psi_k\}_{k\in\mathbb{Z}}$ such that

$$\mathrm{dist}(\Psi_k(x), f(x)) \leq d \quad \forall k \in \mathbb{Z}.$$

- We say that a sequence $\{x_k\}$ is **a pseudotrajectory generated by a** d-**pseudomethod** $\{\Psi_k\}_{k\in\mathbb{Z}}$ if

$$x_{k+1} = \Psi_k(x_k) \quad \forall k \in \mathbb{Z}.$$

- $f \in \mathrm{InvSh} \Leftrightarrow \forall\epsilon > 0\ \exists d > 0$ such that for any point p and for any d-pseudomethod $\{\Psi_k\}_{k\in\mathbb{Z}}$ there exists a pseudotrajectory $\{x_k\}$ generated by this method such that

$$\mathrm{dist}(x_k, f^k(p)) < \epsilon, \quad \forall k \in \mathbb{Z}.$$

Inverse periodic shadowing

- $\mathrm{Int}^1(\mathrm{InvSh}) = \mathrm{LipInvSh} = \mathbb{S}$.
- Inverse periodic shadowing (InvPerSh) = inverse shadowing for periodic points.
- LipInvPerSh:
  InvPerSh with $\epsilon = Ld$.
- Theorem:
  1) $\mathrm{Int}^1(\mathrm{InvPerSh}) = \Omega\mathbb{S}$,
  2) LipInvPerSh is equivalent to hyperbolicity of $Cl(Per(f))$.

Shadowing for flows

- Φ is the flow, $\Phi : \mathbb{R} \times M \mapsto M$
- **standard shadowing:** $\forall \epsilon > 0 \; \exists d > 0$ such that for any increasing $\{\Delta_k\}_{k\in\mathbb{Z}}$ such that

$$|\Delta_{k+1} - \Delta_k| \leq 1 \; \forall k \in \mathbb{Z}, \quad \lim \Delta_k = \infty \text{ for } k \to \infty$$

  $\forall \{x_k\}_{k\in\mathbb{Z}}$ such that

$$|x_{k+1} - \Phi(\Delta_{k+1} - \Delta_k, x_k)| \leq d \quad \forall k \in \mathbb{Z}$$

  $\exists p$ and $\exists \{t_k\}$ such that

$$|x_k - \Phi(t_k, p)| \leq \epsilon \quad \forall k \in \mathbb{Z}$$

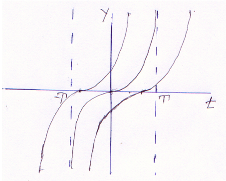
$$\text{and } |(\Delta_{k+1} - \Delta_k) - (t_{k+1} - t_k)| \leq \epsilon.$$

- for **oriented shadowing** the last condition is changed to

$$(\Delta_{k+1} - \Delta_k)/(t_{k+1} - t_k) > 0$$

Finite time blow-up

- $\dot{y} = a(t)y^2 + b(t)y + c(t)$



- $\forall \epsilon > 0\ \exists d > 0$ and $\{d_k\} \leq d$ such that for any increasing $\{\Delta_k\}_{k \geq 0}$ such that $\lim \Delta_k$ is finite $\exists K\ \forall \{x_k\}_{k \geq 0}$

$$|x_{k+1} - \Phi(\Delta_{k+1} - \Delta_k, x_k)| \leq d_k \leq d \quad \forall k \leq K$$

$\exists p$ and $\exists \{t_k\}$ such that

$$|x_k - \Phi(t_k, p)| \leq \epsilon \quad \forall k \geq K$$

$$\text{and } \sum_{k \geq K} |(\Delta_{k+1} - \Delta_k) - (t_{k+1} - t_k)| \leq \epsilon.$$

Thank you very much for your attention!