On the Gap Lemma

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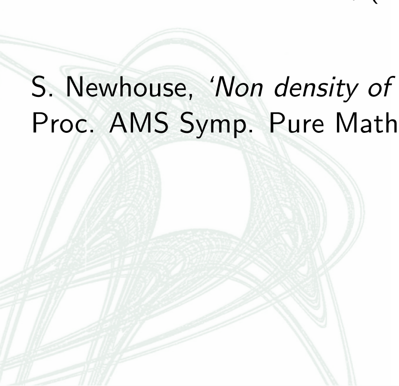
February, 2011

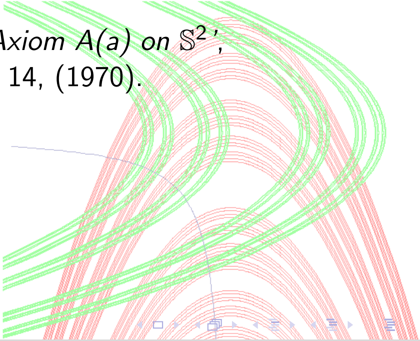
Thickness, concept's origin

$$K_1 + K_2 = \{ \, t \in \mathbb{R} \, : \, K_1 \cap (t - K_2) \neq \emptyset \, \}$$

M. Hall, *'On the sum and products of continued fractions'*, Annals of Mathematics 48, (1947).



S. Newhouse, *'Non density of Axiom A(a) on* $\mathbb{S}^2$*'*, Proc. AMS Symp. Pure Math. 14, (1970).



Cantor Sets

A topological Hausdorff space K is called a **Cantor Set** iff K is

- compact,
- perfect,
- and totally disconnected.

From now on we shall consider Cantor sets on the real line

$$K \subset \mathbb{R} .$$

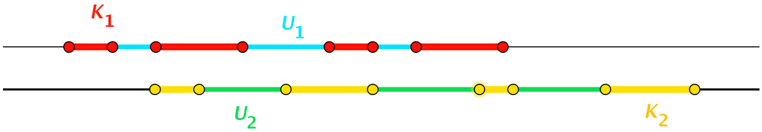
Gaps of a Cantor Set

A connected component of $\mathbb{R} - K$ is called a **gap** of K.

The complement of the unbounded connected component of $\mathbb{R} - K$ is called the **supporting interval** of K.

Let $K_1, K_2 \subset \mathbb{R}$ be Cantor sets.

A pair of gaps U_1 of K_1 and U_2 of K_2 is said to be **linked** iff $U_1 \cap U_2 \neq \emptyset$, $U_1 \not\subseteq U_2$ and $U_2 \not\subseteq U_1$.



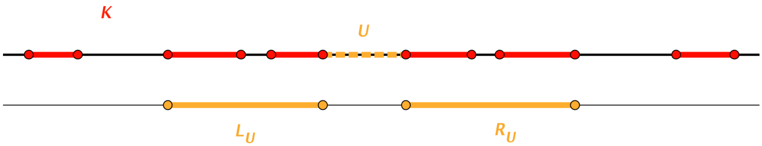
The pair of Cantor sets K_1, K_2 is said to be **linked** iff their unbounded gaps are linked.

Thickness, Geometric Definition

Given a bounded gap U of K define its **right** and **left bridges**

$L_U =$ largest interval left adjacent to U that contains no gap of K of length $\geq |U|$.

$R_U =$ largest interval right adjacent to U that contains no gap of K of length $\geq |U|$.



Thickness, Geometric Definition

The **thickness** of K at a bounded gap U of K is

$$\tau_U(K) = \min\left\{ \frac{|L_U|}{|U|}, \frac{|R_U|}{|U|} \right\} .$$

The **thickness** of the Cantor set K is defined by

$$\tau(K) = \inf\{\, \tau_U(K) \,:\, U \text{ is a bounded gap of } K \,\} .$$

Thickness and Hausdorff Dimension

Thick Cantor sets have Hausdorff dimension close to one.

$$\dim_{\mathrm{H}}(K) \geq \frac{\log 2}{\log\left(2+\tau(K)^{-1}\right)} .$$

In particular, $\dim_{\mathrm{H}}(K) \to 1$, as $\tau(K) \to \infty$.

Gap Lemma *Given two linked Cantor sets $K_1, K_2 \subset \mathbb{R}$,*

$$\tau(K_1)\,\tau(K_2) > 1 \quad \Rightarrow \quad K_1 \cap K_2 \neq \emptyset.$$

Proof

$$K_1 \cap K_2 = \emptyset$$

$$\Downarrow$$

$\exists$ sequence $(U_n^{(1)}, U_n^{(2)})$ of pairs of linked gaps,
$U_n^{(1)}$ of K_1 and $U_n^{(2)}$ of K_2 such that
$\left|U_n^{(1)}\right|, \left|U_n^{(2)}\right| \to 0$, as $n \to +\infty$.

$$\Downarrow$$

The limit point lies in $K_1 \cap K_2$. $\square$

Left and Right Thickness

The **left thickness** of K is defined by

$$\tau_L(K) = \inf \left\{ \frac{|L_U|}{|U|} \; : \; U \text{ is a bounded gap of } U \right\} .$$

The **right thickness** of K is defined by

$$\tau_R(K) = \inf \left\{ \frac{|R_U|}{|U|} \; : \; U \text{ is a bounded gap of } U \right\} .$$

Gustavo Moreira's Gap Lemma

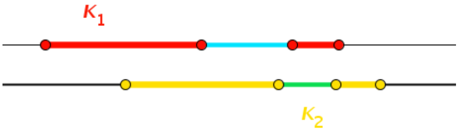
Left-Right Gap Lemma *Given linked Cantor sets* $K_1, K_2 \subset \mathbb{R}$,

$$\tau_L(K_1)\,\tau_R(K_2) > 1 \text{ and } \tau_R(K_1)\,\tau_L(K_2) > 1 \quad \Rightarrow \quad K_1 \cap K_2 \neq \emptyset.$$

Moreira's Gap Lemma $\Rightarrow$ Newhouse's Gap Lemma

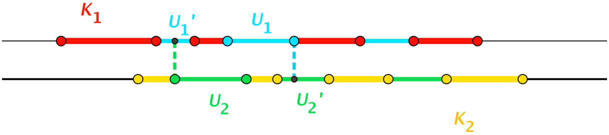
$$\tau(K) \leq \tau_L(K) \quad \text{and} \quad \tau(K) \leq \tau_R(K)\,.$$

$$\tau(K_1)\,\tau(K_2) \leq \min\{\,\tau_L(K_1)\,\tau_R(K_2),\; \tau_R(K_1)\,\tau_L(K_2)\,\}$$



Gap Lemma's Proof

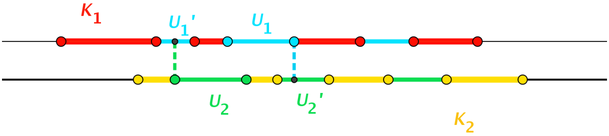
Given a pair of linked gaps (U_1, U_2),



if the left and right endpoints $U_1 \cup U_2$ are not in $K_1 \cap K_2$, then they are inside gaps U_1' of K_1 and U_2' of K_2.

(U_1', U_2) and (U_1, U_2') are new pairs of linked gaps, and either $|U_1'| < |U_1|$ or $|U_2'| < |U_2|$.

Gap Lemma's Proof



If $L_{U_1} \not\subseteq U_2 \quad \Rightarrow \quad |U_1'| < |U_1|$

If $R_{U_2} \not\subseteq U_1 \quad \Rightarrow \quad |U_2'| < |U_2|$

If $L_{U_1} \subseteq U_2$ and $R_{U_2} \subseteq U_1 \quad \Rightarrow$

$$1 < \tau_L(K_1)\,\tau_R(K_2) \leq \frac{|L_{U_1}|}{|U_1|}\,\frac{|R_{U_2}|}{|U_2|} = \frac{|L_{U_1}|}{|U_2|}\,\frac{|R_{U_2}|}{|U_1|} \leq 1\,.$$

Dynamically Defined Cantor Sets

A **dynamically defined Cantor set** is a pair (K, ψ) where

- $\psi : K \to K$ extends to a $C^{1+\alpha}$ expanding map $\psi : I_1 \dot\cup \ldots \dot\cup I_m \to I$,
- K is a Cantor set with supporting interval I,
- $I_1, \ldots, I_m$ are pairwise disjoint subintervals of I,
- $K = \bigcap_{n \geq 0} \psi^{-n}(I_1 \cup \ldots \cup I_m)$,
- $\{I_1, \ldots, I_m\}$ is a Markov Partition for (K, ψ).

Thickness, Dynamic Definition

Let (K, ψ) be a dynamically defined Cantor set, defined by a $C^{1+\alpha}$ expanding map $\psi : I_1 \dot{\cup} \ldots \dot{\cup} I_m \to I$.

Gaps of order 0 are the connected compon. of $I - (I_1 \cup \ldots \cup I_m)$.

Bounded connected components of the complement of $\cap_{i=0}^{n} \psi^{-i}(I_1 \cup \ldots \cup I_m)$, which are not gaps of order $n-1$, are called **gaps of order** n.

$$
\begin{aligned}
L_U = & \text{ largest interval left adjacent to } U \text{ that} \\
& \quad \text{contains no gap of } K \text{ of order smaller than } U. \\
R_U = & \text{ largest interval right adjacent to } U \text{ that} \\
& \quad \text{contains no gap of } K \text{ of order smaller than } U.
\end{aligned}
$$

Thickness, Dynamic Definition

Let (K, ψ) be a dynamically defined Cantor set.

The **left** and **right thickness** of (K, ψ) are defined by

$$\tau_L(K,\psi) = \inf_{n\geq 0} \tau_L^{(n)}(K,\psi) \quad \text{and} \quad \tau_R(K,\psi) = \inf_{n\geq 0} \tau_R^{(n)}(K,\psi)\,,$$

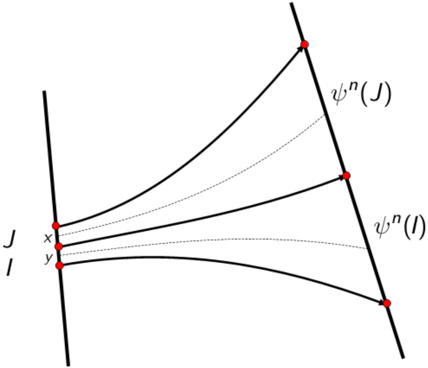
where

$$\tau_L^{(n)}(K,\psi) = \min\left\{ \frac{|L_U|}{|U|} \,:\, U \text{ is a gap of order } n \right\}.$$

$$\tau_R^{(n)}(K,\psi) = \min\left\{ \frac{|R_U|}{|U|} \,:\, U \text{ is a gap of order } n \right\}.$$

Distortion of a Dynamically Defined Cantor Set

Given intervals J and I there are points $x \in J$ and $y \in I$ such that



$$\frac{|\psi^n(J)|}{|\psi^n(I)|} : \frac{|J|}{|I|} = \frac{|\psi^n(J)|}{|J|} : \frac{|\psi^n(I)|}{|I|} = \frac{|(\psi^n)'(x)|}{|(\psi^n)'(y)|} .$$

Distortion of a Dynamically Defined Cantor Set

Given $\varepsilon > 0$ the ε-**scale distortion** of (K, ψ) is defined by

$$\mathrm{Dist}_\varepsilon(\mathrm{K}, \psi) = \sup_{\mathrm{n} \geq 1} \mathrm{Dist}_{\mathrm{n},\varepsilon}(\mathrm{K}, \psi) \ ,$$

where

$$\mathrm{Dist}_{\mathrm{n},\varepsilon}(\mathrm{K}, \psi) = \sup_{\mathrm{x,y}} \left| \log \frac{|(\psi^{\mathrm{n}})'(\mathrm{x})|}{|(\psi^{\mathrm{n}})'(\mathrm{y})|} \right| \ .$$

The sup is taken over all pairs of points (x, y) such that $[\psi^i(x), \psi^i(y)] \subseteq I_1 \cup \ldots \cup I_m$, for $i = 0, 1, \ldots, n-1$ and $|\psi^n(x) - \psi^n(y)| \leq \varepsilon$.

Distortion and Thickness

Lemma Given $\varepsilon > 0$ and $p \geq 1$ such that every gap and bridge of order p has length $\leq \varepsilon$, then for every $n \geq 0$,

$$e^{-\mathrm{Dist}_{n,\varepsilon}(K,\psi)} \leq \frac{\tau_{L,R}^{(p)}(K,\psi)}{\tau_{L,R}^{(n+p)}(K,\psi)} \leq e^{\mathrm{Dist}_{n,\varepsilon}(K,\psi)} \,.$$

Proof U is gap of order $n+p$ $\Rightarrow$ $\psi^n(U)$ is a gap of order p
$\Rightarrow$ $L_{\psi^n(U)} = \psi^n(L_U)$

By the Mean Value Theorem, there are $x \in L_U$ and $y \in U$,

$$\left| \log \frac{\frac{|L_{\psi^n(U)}|}{|\psi^n(U)|}}{\frac{|L_U|}{|U|}} \right| = \left| \log \frac{\frac{|\psi^n(L_U)|}{|L_U|}}{\frac{|\psi^n(U)|}{|U|}} \right| = \left| \log \frac{|(\psi^n)'(x)|}{|(\psi^n)'(y)|} \right| \leq \mathrm{Dist}_{n,\varepsilon}(K,\psi) \,.$$

□

Distortion and Thickness

Thickness can be estimated at gaps of order 0 when the distortion is bounded.

Corollary

$$e^{-\mathrm{Dist}(\mathrm{K},\psi)} \leq \frac{\tau_{L,R}^{(0)}(K,\psi)}{\tau_{L,R}(K,\psi)} \leq e^{\mathrm{Dist}(\mathrm{K},\psi)} \; ,$$

where

$$\mathrm{Dist}(\mathrm{K},\psi) = \mathrm{Dist}_{\varepsilon}(\mathrm{K},\psi) \quad \text{with} \quad \varepsilon = |\mathrm{I}| \; .$$

Bounds on Distortion

Because ψ is expanding of class $C^{1+\alpha}$ with $0 < \alpha \leq 1$, there are constants $\lambda > 1$ and $\gamma > 0$ such that

(a) $|\psi'(x)| \geq \lambda$

(b) $|\log|\psi'(x)| - \log|\psi'(y)|| \leq \gamma\, |x - y|^\alpha$.

Lemma $\quad \mathrm{Dist}_{\mathrm{n},\varepsilon}(\mathrm{K}, \psi) \leq \dfrac{\gamma\, \varepsilon^\alpha}{\lambda^\alpha - 1} \quad (\to 0 \text{ as } \varepsilon \to 0)$

Corollary $\quad \mathrm{Dist}(\mathrm{K}, \psi) \leq \dfrac{\gamma\, |\mathrm{I}|^\alpha}{\lambda^\alpha - 1}$

Bounds on Distortion

Assume $[\psi^i(x), \psi^i(y)] \subseteq I_1 \cup \ldots \cup I_m$, for $i = 0, 1, \ldots, n-1$, and $|\psi^n(x) - \psi^n(y)| \leq \varepsilon$.

$$\begin{aligned}\left|\log \frac{|(\psi^n)'(x)|}{|(\psi^n)'(y)|}\right| &\leq \sum_{i=0}^{n-1} \left|\log\left|\psi'(\psi^i(x))\right| - \log\left|\psi'(\psi^i(y))\right|\right| \\ &\leq \gamma \sum_{i=0}^{n-1} \left|\psi^i(x) - \psi^i(y)\right|^\alpha \\ &\leq \gamma \sum_{i=0}^{n-1} \frac{1}{\lambda^{\alpha(n-i)}} \, |\psi^n(x) - \psi^n(y)|^\alpha \\ &\leq \frac{\gamma\, \varepsilon^\alpha}{\lambda^\alpha - 1} \,. \qquad \square\end{aligned}$$

Continuity of the Thickness

Theorem *The left, right and bilateral thicknesses are continuous functions of a dynamically defined Cantor set* (K,ψ) *with respect to the* $C^{1+\alpha}$*-topology.*

Proof For each $n \geq 1$, the gaps and bridges of order n of K depend continuously on (K,ψ).
Hence $(K,\psi) \mapsto \tau^{(n)}_{L,R}(K,\psi)$ is continuous.

$$\begin{gathered}\varepsilon\text{-scale distortion is small,} \quad \text{as } \varepsilon \to 0 \\ \Downarrow \\ \tau_{L,R}(K,\psi) \sim \min_{0\leq i\leq n} \tau^{(i)}_{L,R}(K,\psi), \quad \text{as } n \to +\infty \\ \Downarrow \\ (K,\psi) \mapsto \tau_{L,R}(K,\psi) \text{ is continuous.} \qquad \square\end{gathered}$$

Bounds on Thickness

The bound on distortion $\mathrm{Dist}(\mathrm{K}, \psi) \leq \dfrac{\gamma\ |\mathrm{I}|^\alpha}{\lambda^\alpha - 1}$ tends to $+\infty$ as $\lambda \to 1$.

Weakly expanding dynamically defined Cantor sets may, or not, have large distortion.

Bounds on Thickness

Let $\psi : \cup_i I_i \to I$ be an expanding of class $C^{1+\alpha}$ with $0 < \alpha \leq 1$. Assume there are constants $\lambda_i > 1$ and $\gamma > 0$ such that $\forall\, x \in I_i$

(a) $|\psi'(x)| \geq \lambda_i$

(b) $|\log|\psi'(x)| - \log|\psi'(y)|| \leq \gamma \left(1 - \frac{1}{\lambda_i^\alpha}\right) |\psi(x) - \psi(y)|^\alpha$.

Lemma $\quad \mathrm{Dist}_{n,\varepsilon}(K, \psi) \leq \gamma\, \varepsilon^\alpha \quad$ ($\to 0$ as $\varepsilon \to 0$)

Corollary $\quad \mathrm{Dist}(K, \psi) \leq \gamma\, |I|^\alpha$

Bounds on Distortion

Assume $[\psi^i(x), \psi^i(y)] \subseteq I_{\beta_i}$, for $i = 0, 1, \ldots, n-1$,
and $|\psi^n(x) - \psi^n(y)| \leq \varepsilon$.

$$\begin{aligned}
\left| \log \frac{|(\psi^n)'(x)|}{|(\psi^n)'(y)|} \right| &\leq \sum_{i=0}^{n-1} \left| \log \left| \psi'(\psi^i(x)) \right| - \log \left| \psi'(\psi^i(y)) \right| \right| \\
&\leq \sum_{i=0}^{n-1} \gamma \left(1 - \frac{1}{\lambda_{\beta_i}^{\alpha}} \right) \left| \psi^{i+1}(x) - \psi^{i+1}(y) \right|^{\alpha} \\
&\leq \gamma \sum_{i=0}^{n-1} \left(1 - \frac{1}{\lambda_{\beta_i}^{\alpha}} \right) \frac{1}{\lambda_{\beta_{i+1}}^{\alpha}} \cdots \frac{1}{\lambda_{\beta_{n-1}}^{\alpha}} \left| \psi^n(x) - \psi^n(y) \right|^{\alpha} \\
&\leq \gamma \, \varepsilon^{\alpha} . \qquad \square
\end{aligned}$$