Existence and uniqueness of SDE solutions

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Background and notation

- We will study the question of existence and uniqueness of solutions to a non-linear stochastic differential equation

$$\mathrm{d}x(t) = f(x(t), t)\mathrm{d}t + g(x(t), t)\mathrm{d}B(t), \quad t \in [t_0, T]$$

  with initial value $x(t_0) = x_0$, where $0 \leq t_0 < T < \infty$.
- Some of the main (mathematical) questions regarding such equations:
  - Is there a solution?
  - If there is a solution, is it unique?
  - What kind of properties do solutions have?
  - How can solutions be obtained in practice?

Background and notation

- Let
  - $(\Omega, \mathcal{F}, P)$ be a probability space.
  - $B(t) = (B_1(t), \dots, B_m(t))^T$ be an m-dimensional Brownian motion.
  - x_0 be an $\mathcal{F}_{t_0}$-measurable (where $0 \leq t_0 < T < \infty$) $\mathbb{R}^d$-valued random variable such that $E|x_0|^2 < \infty$.
  - $f : R^d \times [t_0, T] \to R^d$ and $g : R^d \times [t_0, T] \to R^{d \times m}$ be Borel measurable.
- Consider the d-dimensional stochastic differential equation of Itô type

$$\mathrm{d}x(t) = f(x(t), t)\mathrm{d}t + g(x(t), t)\mathrm{d}B(t), \quad t_0 \leq t \leq T,$$

  with initial value $x(t_0) = x_0$.
- The initial value problem above is equivalent to the following stochastic integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)\mathrm{d}s + \int_{t_0}^{t} g(x(s), s)\mathrm{d}B(s), \quad t_0 \leq t \leq T.$$

Background and notation

Definition (SDE solution)

We say that the stochastic process $\{x(t)\}_{t_0 \le t \le T}$ is a *solution* of the stochastic differential equation

$$\mathrm{d}x(t) = f(x(t), t)\mathrm{d}t + g(x(t), t)\mathrm{d}B(t)$$

with initial condition $x(t_0) = x_0$ if the following conditions hold:

(i) $\{x(t)\}$ is continuous and $\mathcal{F}_t$-adapted;

(ii) $\{f(x(t), t)\} \in \mathcal{L}^1([t_0, T]; \mathbb{R}^d)$ and $\{g(x(t), t)\} \in \mathcal{L}^2([t_0, T]; \mathbb{R}^{d \times m})$;

(iii) the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)\mathrm{d}s + \int_{t_0}^{t} g(x(s), s)\mathrm{d}B(s)$$

holds for every $t \in [t_0, T]$ with probability 1.

Background and notation

Definition (Solution uniqueness)

A solution $\{x(t)\}$ is said to be unique if any other solution $\{\overline{x}(t)\}$ is indistinguishable from $\{x(t)\}$, that is, almost all their sample paths agree

$$P\{x(t) = \overline{x}(t) \text{ for all } t_0 \leq t \leq T\} = 1 \ .$$

Background and notation

Example

- Let us consider the the stochastic differential equation given by

$$\mathrm{d}N_t = rN_t\mathrm{d}t + \alpha N_t \mathrm{d}B_t \ .$$

- Equivalently, we have that

$$\frac{\mathrm{d}N_t}{N_t} = r\mathrm{d}t + \alpha \mathrm{d}B_t \ .$$

- Hence

$$\int_0^t \frac{\mathrm{d}N_s}{N_s} = rt + \alpha B_t \quad (B_0 = 0) \ .$$

Background and notation

Example

- To evaluate the integral on the left hand side, we use the Itô formula for the function

$$g(t,x) = \ln x \ , \quad x > 0$$

to obtain

$$\begin{aligned} \mathrm{d}(\ln N_t) &= \frac{1}{N_t} \cdot \mathrm{d}N_t + \frac{1}{2}\Big(-\frac{1}{N_t^2}\Big)(\mathrm{d}N_t)^2 \\ &= \frac{\mathrm{d}N_t}{N_t} - \frac{1}{2N_t^2} \cdot \alpha^2 N_t^2 \mathrm{d}t = \frac{\mathrm{d}N_t}{N_t} - \frac{1}{2}\alpha^2 \mathrm{d}t \ . \end{aligned}$$

- Hence

$$\frac{\mathrm{d}N_t}{N_t} = \mathrm{d}(\ln N_t) + \frac{1}{2}\alpha^2 \mathrm{d}t \ .$$

Background and notation

Example

- Therefore, from $\int_0^t \frac{\mathrm{d}N_s}{N_s} = rt + \alpha B_t$ we conclude that

$$\ln \frac{N_t}{N_0} = (r - \frac{1}{2}\alpha^2)t + \alpha B_t$$

or

$$N_t = N_0 \exp((r - \frac{1}{2}\alpha^2)t + \alpha B_t).$$

- The solution N_t is a process of the form

$$X_t = X_0 \exp(\mu t + \alpha B_t), \quad \mu, \alpha \text{ constants}$$

  - We call such processes *geometric Brownian motion*.

Remark

- It seems reasonable that if B_t is independent of N_0 we should have

$$E[N_t] = E[N_0]e^{rt} \ .$$

- To see that this is indeed the case, we let

$$Y_t = e^{\alpha B_t} \ .$$

- Apply Itô's formula to obtain

$$\mathrm{d}Y_t = \alpha e^{\alpha B_t}\mathrm{d}B_t + \frac{1}{2}\alpha^2 e^{\alpha B_t}\mathrm{d}t$$

or

$$Y_t = Y_0 + \alpha \int_0^t e^{\alpha B_s}\mathrm{d}B_s + \frac{1}{2}\alpha^2 \int_0^t e^{\alpha B_s}\mathrm{d}s \ .$$

Remark

- Since $E[\int_0^t e^{\alpha B_s} \mathrm{d}B_s] = 0$, we get

$$E[Y_t] = E[Y_0] + \frac{1}{2}\alpha^2 \int_0^t E[Y_s]\mathrm{d}s$$

  i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}E[Y_t] = \frac{1}{2}\alpha^2 E[Y_t], \quad E[Y_0] = 1 \ .$$

- Therefore, we get that

$$E[Y_t] = e^{\frac{1}{2}\alpha^2 t} \ .$$

- We conclude that

$$E[N_t] = E[N_0]e^{rt} \ .$$

Background and notation

If we take $g(x,t) \equiv 0$, then the SDE above reduces to

$$\dot{x}(t) = f(x(t), t), \quad t \in [t_0, T] \ .$$

Note that the initial condition $x(t_0) = x_0$ may still be a random variable.

Example

Consider the following classical example

$$\dot{x} = 3x^{2/3}, \quad t \in [t_0, T]$$

with initial condition $x(t_0) = 1_A$, where $A \in \mathcal{F}_{t_0}$. It is possible to check that for each $0 < \alpha < T - t_0$, the stochastic process

$$x(t) = x(t,\omega) = \begin{cases} (t - t_0 + 1)^3 & \text{para } t_0 \leq t \leq T, \, \omega \in A \\ 0 & \text{para } t_0 \leq t \leq t_0 + \alpha, \, \omega \notin A \\ (t - t_0 - \alpha)^3 & \text{para } t_0 + \alpha < t \leq T, \, \omega \notin A \end{cases}$$

is a solution of the equation above.
This initial value problem has an infinite number of solutions.

Background and notation

Example

Consider yet another simple equation

$$\dot{x} = x^2, \quad t \in [t_0, T]$$

with initial condition given by $x(t_0) = x_0$, a random variable which takes values larger than $1/(T - t_0)$.

It is possible to check that the initial value problem above has a unique solution

$$x(t) = \left(\frac{1}{x_0} - (t - t_0)\right)^{-1}$$

for $t_0 \le t < t_0 + 1/x_0 < T$.

However, there is no solution for this initial value problem which is defined for all $t \in [t_0, T]$.

Existence and uniqueness of solutions

Theorem (Existence and uniqueness of solution)

Assume that there exist two positive constants $\overline{K}$ and K such that the following two conditions hold:

(i) *Lipschitz condition: for all $x, y \in \mathbb{R}^d$ and $t \in [t_0, T]$*

$$\max\{|f(x,t) - f(y,t)|^2, |g(x,t) - g(y,t)|^2\} \leq \overline{K}|x - y|^2 \ ;$$

(ii) *Linear growth condition: for all $(x, t) \in \mathbb{R}^d \times [t_0, T]$*

$$\max\{|f(x,t)|^2, |g(x,t)|^2\} \leq K(1 + |x|^2) \ .$$

Let x_0 be a random variable which is independent of the σ-algebra $\mathcal{F}_\infty^{(m)}$ generated by $B_s(\cdot)$, $s \geq 0$ and such that $E|x_0|^2 < \infty$.
Then there exists a unique t-continuous solution $X_t(\omega)$ of the initial value problem

$$\mathrm{d}x(t) = f(x(t), t)\mathrm{d}t + g(x(t), t)\mathrm{d}B(t) \ , \ t_0 \leq t \leq T \ , \ x(t_0) = x_0$$

with the property that $X_t(\omega)$ is adapted to the filtration $\mathcal{F}_t^{x_0}$ generated by x_0 and $B_s(\cdot)$, $s \leq t$. Furthermore, such solution belongs to $\mathcal{M}^2([t_0, T]; \mathbb{R}^d)$.

Existence and uniqueness of solutions

- We start by proving some auxiliary lemmas to prepare for the proof of the theorem above.

Theorem

Let $p \geq 2$ *and let* $g \in \mathcal{M}^2([0,T]; R^{d\times m})$ *be such that*

$$E\left[\int_0^T |g(s)|^p \mathrm{d}s\right] < \infty \, .$$

Then

$$E\Big|\int_0^T g(s)\mathrm{d}B(s)\Big|^p \leq \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} T^{\frac{(p-2)}{2}} E\left[\int_0^T |g(s)|^p \mathrm{d}s\right] \, .$$

In particular, the equality holds for $p = 2$.

Existence and uniqueness of solutions

Proof.

- For $0 \le t \le T$, set

$$x(t) = \int_0^t g(s)\mathrm{d}B(s)\ .$$

- Using Itô's formula (and Itô's integral properties), one can obtain

$$\begin{aligned} E|x(t)|^p &= \frac{p}{2}E\int_0^t \left(|x(s)|^{p-2}|g(s)|^2 + (p-2)|x(s)|^{p-4}|x^T(s)g(s)|^2\right)\mathrm{d}s \\ &\le \frac{p(p-1)}{2}E\int_0^t |x(s)|^{p-2}|g(s)|^2\mathrm{d}s. \end{aligned}$$

- Recall the Hölder's inequality, for $1 \le p, q \le \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $X \in L^p$, $Y \in L^q$ we have

$$E|XY| \le (E|X|^p)^{\frac{1}{p}}(E|Y|^q)^{\frac{1}{q}}$$

Existence and uniqueness of solutions

Proof.

- Using the previous inequality one then sees that

$$\begin{aligned} E|x(t)|^p &\le \frac{p(p-1)}{2}\Big(E\int_0^t |x(s)|^p \mathrm{d}s\Big)^{\frac{p-2}{p}}\Big(E\int_0^t |g(s)|^p \mathrm{d}s\Big)^{\frac{2}{p}} \\ &= \frac{p(p-1)}{2}\Big(\int_0^t E|x(s)|^p \mathrm{d}s\Big)^{\frac{p-2}{p}}\Big(E\int_0^t |g(s)|^p \mathrm{d}s\Big)^{\frac{2}{p}}. \end{aligned}$$

- Noting that $E|x(t)|^p$ is nondecreasing in t, we obtain

$$E|x(t)|^p \le \frac{p(p-1)}{2}\Big[tE|x(t)|^p\Big]^{\frac{p-2}{p}}\Big(E\int_0^t |g(s)|^p \mathrm{d}s\Big)^{\frac{2}{p}} .$$

- This last inequality yields

$$E|x(t)|^p \le \Big(\frac{p(p-1)}{2}\Big)^{\frac{p}{2}} t^{\frac{p-2}{2}} E\int_0^t |g(s)|^p \mathrm{d}s ,$$

concluding the proof. □

Existence and uniqueness of solutions

Theorem

Under the assumptions of the previous theorem,

$$E\Big[\sup_{0\le t\le T}\Big|\int_0^t g(s)\mathrm{d}B(s)\Big|^p\Big] \le \Big(\frac{p^3}{2(p-1)}\Big)^{\frac{p}{2}} T^{\frac{p-2}{2}} E\int_0^T |g(s)|^p \mathrm{d}s.$$

Proof.

- Recall that the stochastic integral $\int_0^t g(s)\mathrm{d}B(s)$ is a martingale.
- By the Doob martingale inequality we have that

$$E\Big[\sup_{0\le t\le T}\Big|\int_0^t g(s)\mathrm{d}B(s)\Big|^p\Big] \le \Big(\frac{p}{p-1}\Big)^p E\Big|\int_0^T g(s)\mathrm{d}B(s)\Big|^p .$$

- Using the previous theorem, we then obtain the desired inequality.

□

Existence and uniqueness of solutions

Theorem (Gronwall's inequality)

Let $T > 0$ and $c \geq 0$. Let $u(\cdot)$ be a Borel measurable bounded nonnegative function on $[0, T]$, and let $v(\cdot)$ be a nonnegative integrable function on $[0, T]$. If

$$u(t) \leq c + \int_0^t v(s)u(s)\mathrm{d}s, \quad \textit{for all} \quad 0 \leq t \leq T,$$

then

$$u(t) \leq c \exp\left(\int_0^t v(s)\mathrm{d}s\right), \quad \textit{for all} \quad 0 \leq t \leq T.$$

Proof.

- Without loss of generality we may assume that $c > 0$.
- Set
$$z(t) = c + \int_0^t v(s)u(s)\mathrm{d}s, \quad \text{for} \quad 0 \leq t \leq T.$$
- Then $u(t) \leq z(t)$.

Existence and uniqueness of solutions

Proof.

- Clearly, we have that

$$\log(z(t)) = \log(c) + \int_0^t \frac{v(s)u(s)}{z(s)}\mathrm{d}s \leq \log(c) + \int_0^t v(s)\mathrm{d}s \ .$$

- This imples

$$z(t) \leq c\exp(\int_0^t v(s)\mathrm{d}s), \quad \text{for} \quad 0 \leq t \leq T \ .$$

- The required inequality follows since $u(t) \leq z(t)$.

□

Existence and uniqueness of solutions

Lemma

Assume that the linear growth condition holds. If $x(t)$ is a solution of equation

$$\mathrm{d}x(t) = f(x(t), t)\mathrm{d}t + g(x(t), t)\mathrm{d}B(t),$$

then

$$E\Big(\sup_{t_0 \le t \le T} |x(t)|^2\Big) \le (1 + 3E|x_0|^2)e^{3K(T-t_0)(T-t_0+4)}.$$

In particular, $x(t)$ belongs to $\mathcal{M}^2([t_0, T; \mathbb{R}^d])$.

Proof.

- For every integer $n \ge 1$, define the stopping time

$$\tau_n = \min\{T, \inf\{t \in [t_0, T] : |x(t)| \ge n\}\}.$$

- Set $x_n(t) = x(\min\{t, \tau_n\})$ for $t \in [t_0, T]$.

Existence and uniqueness of solutions

Proof.

- Then $x_n(t)$ satisfies the equation

$$x_n(t) = x_0 + \int_{t_0}^{t} f(x_n(s), s) I_{[[t_0,\tau_n]]}(s)\mathrm{d}s + \int_{t_0}^{t} g(x_n(s), s) I_{[[t_0,\tau_n]]}(s)\mathrm{d}s.$$

- Using the elementary inequality

$$|a+b+c|^2 \leq 3(|a|^2 + |b|^2 + |c|^2),$$

  the Hölder inequality and the linear growth condition, one can show that

$$|x_n(t)|^2 \leq 3|x_0|^2 + 3K(t-t_0)\int_{t_0}^{t}(1+|x_n(s)|^2)\mathrm{d}s + 3\left|\int_{t_0}^{t} g(x_n(s), s) I_{[[t_0,\tau_n]]}(s)\mathrm{d}s.\right|^2$$

Existence and uniqueness of solutions

Proof.

- Hence, using again the linear growth condition and the previous theorem, we obtain that

$$\begin{aligned} E\Big(\sup_{t_0\le s\le t}|x_n(s)|^2\Big) &\le 3E|x_0|^2+3K(T-t_0)\int_{t_0}^{t}(1+E|x_n(s)|^2)\mathrm{d}s \\ &+ 12E\int_{t_0}^{t}|g(x_n(s),s|^2 I_{[}[t_0,\tau_n]](s)\mathrm{d}s \\ &\le 3E|x_0|^2+3K(T-t_0+4)\int_{t_0}^{t}(1+E|x_n(s)|^2)\mathrm{d}s. \end{aligned}$$

- Consequently

$$1+E\Big(\sup_{t_0\le s\le t}|x_n(s)|^2\Big)$$

$$\le 1+3E|x_0|^2+3K(T-t_0+4)\int_{t_0}^{t}\Big[1+E(\sup_{t_0\le r\le s}|x_n(r)|)^2)\Big]\mathrm{d}s.$$

Existence and uniqueness of solutions

Proof.

- Now, Gronwall inequality implies that

$$1 + E\Big(\sup_{t_0 \le t \le T} |x_n(t)|^2\Big) \le (1 + 3E|x_0|^2)e^{3K(T-t_0)(T-t_0+4)}.$$

- Thus

$$E\Big(\sup_{t_0 \le t \le \tau_n} |x_n(t)|^2\Big) \le (1 + 3E|x_0|^2)e^{3K(T-t_0)(T-t_0+4)}$$

- The required inequality follows by letting $n \to \infty$.

□

Existence and uniqueness of solutions

Proof of theorem of existence and uniqueness of solutions.

Uniqueness

- Let $x(t)$ and $\overline{x}(t)$ be two solutions.
- By the previous lemma, both of them belong to $\mathcal{M}^2([t_0, T]; \mathbb{R}^d)$.
- Note that

$$x(t) - \overline{x}(t) = \int_{t_0}^{t} f(x(s), s) - f(\overline{x}(s), s)\mathrm{d}s + \int_{t_0}^{t} g(x(s), s) - g(\overline{x}(s), s)\mathrm{d}B(s).$$

- Using the Hölder inequality, the previous theorem and Lipschitz condition, one can show (in the same way as in the proof of the previous lemma) that

$$E\Big(\sup_{t_0 \leq s \leq t} |x(s) - \overline{x}(s)|^2\Big) \leq 2\overline{K}(T+4)\int_{t_0}^{t} E\Big(\sup_{t_0 \leq r \leq s} |x(r) - \overline{x}(r)|^2\Big)\mathrm{d}s.$$

Existence and uniqueness of solutions

Proof.

- The Gronwall inequality then yields that

$$E\left(\sup_{t_0\le t\le T}|x(t)-\overline{x}(t)|^2\right)=0.$$

- Hence, $x(t)=\overline{x}(t)$ for all $t_0\le t\le T$ almost surely, concluding the proof of uniqueness of solutions.

Existence and uniqueness of solutions

Proof.

Existence

- Set $x_0(t) \equiv x_0$ and, for $n = 1, 2, \ldots$, define the Picard iterations

$$x_n(t) = x_0 + \int_{t_0}^{t} f(x_{n-1}(s), s)\mathrm{d}s + \int_{t_0}^{t} g(x_{n-1}(s), s)\mathrm{d}B(s)$$

  for $t \in [t_0, T]$.
  - Note that $x(\cdot) \in \mathcal{M}^2([t_0, T]; \mathbb{R}^d)$.
- It is easy to see by induction that $x_n(\cdot) \in \mathcal{M}^2([t_0, T]; \mathbb{R}^d)$, because we have that

$$E|x_n(t)|^2 \leq c_1 + 3K(T+1)\int_{t_0}^{t} E|x_{n-1}(s)|^2\mathrm{d}s$$

  where $c_1 = 3E|x_0|^2 + 3KT(T+1)$.

Existence and uniqueness of solutions

Proof.

- For any $k \geq 1$

$$\begin{aligned}
\max_{1\leq n\leq k} E|x_n(t)|^2 &\leq c_1 + 3K(T+1)\int_{t_0}^{t} \max_{1\leq n\leq k} E|x_{n-1}(s)|^2 \mathrm{d}s \\
&\leq c_1 + 3K(T+1)\int_{t_0}^{t} \left(E|x_0|^2 + \max_{1\leq n\leq k} E|x_n(s)|^2\right)\mathrm{d}s \\
&\leq c_2 + 3K(T+1)\int_{t_0}^{t} \max_{1\leq n\leq k} E|x_n(s)|^2,
\end{aligned}$$

  where $c_2 = c_1 + 3KT(T+1)E|x_0|^2$.
- Gronwall inequality implies that

$$\max_{1\leq n\leq k} E|x_n(t)|^2 \leq c_2 e^{3KT(T+1)}.$$

- Since k is arbitrary, we must have

$$E|x_n(t)|^2 \leq c_2 e^{3KT(T+1)} \quad \text{for all } \; t_0 \leq t \leq T, n \geq 1.$$

Existence and uniqueness of solutions

Proof.

- Note that

$$|x_1(t) - x_0(t)|^2 = |x_1(t) - x_0|^2 \leq 2\left| \int_{t_0}^{t} f(x_0, s)\mathrm{d}s \right|^2 + 2\left| \int_{t_0}^{t} g(x_0, s)\mathrm{d}B(s) \right|^2.$$

- Taking the expectation and using the linear growth condition we get

$$E|x_1(t) - x_0(t)|^2 \leq 2K(t - t_0)^2(1 + E|x_0|^2) + 2K(t - t_0)(1 + E|x_0|^2) \leq C,$$

  where $C = 2K(T - t_0 + 1)(T - t_0)(1 + E|x_0|^2)$.

- We now claim that for $n \geq 0$,

$$E|x_{n+1}(t) - x_n(t)|^2 \leq \frac{C[M(t - t_0)]^n}{n!}, \quad \text{for } t_0 \leq t \leq T,$$

  where $M = 2\overline{K}(T - t_0 + 1)$.

Existence and uniqueness of solutions

Proof.

- By indution, we shall show that $E|x_{n+1}(t) - x_n(t)|^2 \leq \frac{C[M(t-t_0)]^n}{n!}$ still holds for $n+1$.
- Note that

$$\begin{aligned} |x_{n+2}(t) - x_{n+1}(t)|^2 &\leq 2\left|\int_{t_0}^{t}[f(x_{n+1}(s),s) - f(x_n(s),s)]\mathrm{d}s\right|^2 \\ &+ 2\left|\int_{t_0}^{t}[g(x_{n+1}(s),s) - g(x_n(s),s)]\mathrm{d}B(s)\right|^2. \end{aligned}$$

- Taking the expectation and using the Lipschitz condition we derive that

$$\begin{aligned} E|x_{n+2}(t) - x_{n+1}(t)|^2 &\leq 2\overline{K}(T - t_0 + 1)E\int_{t_0}^{t}|x_{n+1}(s) - x_n(s)|^2\mathrm{d}s \\ &\leq M\int_{t_0}^{t}E|x_{n+1}(s) - x_n(s)|^2\mathrm{d}s \\ &\leq M\int_{t_0}^{t}\frac{C[M(s-t_0)]^n}{n!}\mathrm{d}s = \frac{C[M(t-t_0)]^{n+1}}{(n+1)!}. \end{aligned}$$

Existence and uniqueness of solutions

Proof.

- Furthermore, replacing n with $n-1$ we see that

$$\sup_{t_0\le t\le T}|x_{n+1}(t)-x_n(t)|^2 \le 2\overline{K}(T-t_0)\int_{t_0}^{T}|x_n(s)-x_{n-1}(s)|^2\mathrm{d}s$$
$$+2\sup_{t_0\le t\le T}\left|\int_{t_0}^{T}[g(x_n(s),s)-g(x_{n-1}(s),s)]\mathrm{d}B(s)\right|^2.$$

- Taking the expectation and using the previous theorem, we find that

$$\begin{aligned}
E\left(\sup_{t_0\le t\le T}|x_{n+1}(t)-x_n(t)|^2\right) &\le 2\overline{K}(T-t_0+4)\int_{t_0}^{T}E|x_n(s)-x_{n-1}(s)|^2\mathrm{d}s \\
&\le 4M\int_{t_0}^{T}\frac{C[M(s-t_0)]^{n-1}}{(n-1)!}\mathrm{d}s \\
&= \frac{4C[M(T-t_0)]^n}{n!}.
\end{aligned}$$

Existence and uniqueness of solutions

Proof.

- Hence

$$P\left\{ \sup_{t_0 \le t \le T} |x_{n+1}(t) - x_n(t)| > \frac{1}{2^n} \right\} \le \frac{4C[4M(T-t_0)]^n}{n!}.$$

  - Since $\sum_{n=0}^{\infty} \frac{4C[4M(T-t_0)]^n}{n!} < \infty$, the Borel-Cantelli lemma yields that for almost all $\omega \in \Omega$ there exists a positive integer $n_0 = n_0(\omega)$ such that

$$\sup_{t_0 \le t \le T} |x_{n+1}(t) - x_n(t)| \le \frac{1}{2^n}, \quad n \ge n_0.$$

- It follows that, with probability 1, the partial sums

$$x_0(t) + \sum_{i=0}^{n-1} [x_{i+1}(t) - x_i(t)] = x_n(t)$$

are convergent uniformly in $t \in [0, T]$.

Existence and uniqueness of solutions

Proof.

- Denote the limit by $x(t)$.
  - Clearly, $x(t)$ is continuous and $\mathcal{F}_t$-adapted.
  - For every t, $\{x_n(t)_{n\geq 1}\}$ is a Cauchy sequence in L^2.
  - Hence $x_n(t) \to x(t)$ in Ł^2.
- Letting $n \to \infty$ in

$$E|x_n(t)|^2 \leq c_2 e^{3KT(T+1)}$$

gives

$$E|x(t)|^2 \leq c_2 e^{3KT(T+1)}, \quad \text{for all } t_0 \leq t \leq T.$$

- Therefore $x(\cdot) \in \mathcal{M}^2([t_0, T]; \mathbb{R}^d)$.
- It remains to show that $x(t)$ satisfies equation

$$x(t) = \int_{t_0}^{t} f(x(s), s)\mathrm{d}s + \int_{t_0}^{t} g(x(s), s)\mathrm{d}B(s).$$

Existence and uniqueness of solutions

Proof.

- Note that

$$\begin{aligned} E & \Big| \int_{t_0}^{t} f(x_n(s), s)\mathrm{d}s - \int_{t_0}^{t} f(x(s), s)\mathrm{d}s \Big| \\ + & E\Big| \int_{t_0}^{t} g(x_n(s), s)\mathrm{d}B(s) - \int_{t_0}^{t} g(x(s), s)\mathrm{d}B(s) \Big|^2 \\ \leq & \overline{K}(T - t_0 + 1) \int_{t_0}^{T} E|x_n(s) - x(s)|^2 \mathrm{d}s \to 0 \end{aligned}$$

- Hence we can let $n \to \infty$ in

$$x_n(t) = x_0 + \int_{t_0}^{t} f(x_{n-1}(s), s)\mathrm{d}s + \int_{t_0}^{t} g(x_{n-1}(s), s)\mathrm{d}B(s)$$

Existence and uniqueness of solutions

Proof.

- We obtain that

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)\mathrm{d}s + \int_{t_0}^{t} g(x(s), s)\mathrm{d}B(s), \quad \text{on } t_0 \leq t \leq T$$

as desired. □

- In the proof above we show that the Picard iterations $x_n(t)$ converge to the unique solution $x(t)$ of the equation

$$\mathrm{d}x(t) = f(x(t), t)\mathrm{d}t + g(x(t), t)\mathrm{d}B(t)$$