An Overview of the Martingale Representation Theorem

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Background and notation

Definition

Denote by $\{\mathcal{F}_t\}_{t\geq 0}$ the filtration generated by the one-dimensional Brownian motion B_t and by $\mathcal{B}$ the Borel σ-algebra on $[0,\infty)$.
Let $\mathcal{V} = \mathcal{V}(S,T)$ be the class of functions $f : [0,\infty) \times \Omega \to \mathbb{R}$ such that

(i) $(t,\omega) \to f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable.

(ii) $f(t,\omega)$ is $\mathcal{F}_t$-adapted.

(iii) $E\left[\int_S^T (f(t,\omega))^2 \mathrm{d}t\right] < \infty$.

Background and notation

Definition (The Itô integral)

Let $f \in \mathcal{V}(S, T)$. Then the *Itô integral* of f (from S to T) is defined by

$$\int_S^T f(t,\omega)\mathrm{d}B_t(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega)\mathrm{d}B_t(\omega) \quad \text{limit in } L^2(P) \;, \tag{1}$$

where $\{\phi_n\}$ is a sequence of elementary functions such that

$$E\left[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2 \mathrm{d}t\right] \to 0 \quad \text{as } n \to \infty \;, \tag{2}$$

where the limit in (1) exists and does not depend on the choice of $\{\phi_n\}$, as long as (2) holds.

Background and notation

Properties of the Itô integral:

- Itô isometry:
  $$E\left[\left(\int_S^T f(t,\omega)\mathrm{d}B_t(\omega)\right)^2\right] = E\left[\int_S^T (f(t,\omega))^2\,\mathrm{d}t\right] \quad \text{for all } f \in \mathcal{V}(S,T)$$
- Linearity
- $E\left[\int_S^T f\,\mathrm{d}B_t\right] = 0$
- $\int_S^T f\,\mathrm{d}B_t$ is $\mathcal{F}_T$-measurable
- Existence of a continuous version
- Martingale property

Definition (Martingale)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X = \{X_t : t \geq 0\}$ a stochastic process on it.

We say that X_t is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if

(i) X is adapted to $\mathcal{F}_t$.

(ii) $E[|X_t|] < \infty$ for all t.

(iii) $E[X_s|\mathcal{F}_t] = X_t$ for all $s \geq t$.

Background and notation

Theorem (Itô formula)

Let X_t be an Itô process given by

$$\mathrm{d}X_t = u\mathrm{d}t + v\mathrm{d}B_t$$

and let $g(t,x) \in C^2([0,\infty) \times \mathbb{R})$. Then

$$Y_t = g(t, X_t)$$

is again an Itô process, and

$$\mathrm{d}Y_t = \frac{\partial g}{\partial t}(t, X_t)\mathrm{d}t + \frac{\partial g}{\partial x}(t, X_t)\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)(\mathrm{d}X_t)^2 \ ,$$

where $(\mathrm{d}X_t)^2 = (\mathrm{d}X_t).(\mathrm{d}X_t)$ is computed according to the "multiplication table"

$$\mathrm{d}t.\mathrm{d}t = \mathrm{d}t.\mathrm{d}B_t = \mathrm{d}B_t.\mathrm{d}t = 0 \ , \qquad \mathrm{d}B_t.\mathrm{d}B_t = \mathrm{d}t \ .$$

Martingale Representation Theorem

- Let $B(t) = (B_1(t), \ldots, B_n(t))$ be a n-dimensional Brownian motion.
- We know that if $v \in \mathcal{V}^n$ then the Itô integral

$$X_t = X_0 + \int_0^t v(s, w) dB(s); \quad t \geq 0$$

  is always a martingale w.r.t. filtration $\mathcal{F}_t^{(n)}$.
- In this talk we will prove that the converse is also true:
  - any $\mathcal{F}_t^{(n)}$-martingale (w.r.t. P) can be represented as an Itô integral.
  - this result is know as *martingale representation theorem*.

Martingale Representation Theorem

Theorem (The martingale representation theorem)

Let $B(t) = (B_1(t), \ldots, B_n(t))$ be n-dimensional. Suposse M_t is an $\mathcal{F}_t^{(n)}$-martingale (w.r.t. P) and that $M_t \in L^2(P)$ for all $t \geq 0$. Then there exists a unique stochastic process $g(s, \omega)$ such that $g \in \mathcal{V}^{(n)}(0, t)$ for all $t \geq 0$ and

$$M_t(\omega) = E[M_0] + \int_0^t g(s, \omega) \mathrm{d}B(s) \quad a.s., \textit{for all } t \geq 0.$$

- The key result for the proof of the martingale representation theorem is the Itô representation theorem

Martingale Representation Theorem

Theorem (The Itô representation theorem)

Let $F \in L^2(\mathcal{F}_T^{(n)}, P)$*. Then there exists a unique stochastic process* $f(t,\omega) \in \mathcal{V}^n(0,T)$ *such that*

$$F(\omega) = E[F] + \int_0^T f(t,\omega)\mathrm{d}B(t).$$

- For the proof of the Itô representation theorem we need to prove some auxiliary lemmas.

Martingale Representation Theorem

Lemma (Doob-Dinkyn Lemma)

Let (Ω, Σ) and $(S, \mathcal{A})$ be measurable spaces and $f : \Omega \to S$ be measurable, i.e., $f^{-1}(\mathcal{A}) \subset \Sigma$. Then a function $g : \Omega \to \mathbb{R}$ is measurable relative to the σ-algebra $f^{-1}(\mathcal{A})$ [i.e., $g^{-1}(\mathcal{B}) \subset f^{-1}(\mathcal{A})$] if and only if there is a measurable function $h : S \to \mathbb{R}$ such that $g = h \circ f$.

Proof.

$(\Leftarrow)$

- Let $g = h \circ f : \Omega \to \mathbb{R}$ be measurable.
- Then

$$g^{-1}(\mathcal{B}) = (h \circ f)^{-1}(\mathcal{B}) = f^{-1}(h^{-1}(\mathcal{B})) \subset f^{-1}(\mathcal{A})$$

  since $h^{-1}(\mathcal{B}) \subset \mathcal{A}$.

Martingale Representation Theorem

Proof.

($\Rightarrow$)

- Let g be $f^{-1}(\mathcal{A})$-measurable, where $f^{-1}(\mathcal{A})$ is a σ-algebra contained in Σ.
- We start by checking that it is enough to prove the result for simple functions

$$g = \sum_{i=1}^{n} a_i \chi_{A_i}, A_i \in f^{-1}(\mathcal{A}) \ .$$

- Recall that for a measurable function g w.r.t. the σ-algebra $f^{-1}(\mathcal{A})$, there exists a sequence of simple functions g_n, measurable w.r.t. $f^{-1}(\mathcal{A})$, such that

$$g_n(\omega) \to g(\omega)$$

as $n \to \infty$ for each $\omega \in \Omega$.

- Assuming the result holds for simple functions, there is an $\mathcal{A}$-measurable $h_n : S \to \mathbb{R}$, $g_n = h_n \circ f$, for each $n \geq 1$.
- Define $S_0 = \{s \in S : h_n(s) \to \tilde{h}(s), n \to \infty\}$. Then:
  - $S_0 \in \mathcal{A}$ and $f(\Omega) \subset S$.

Martingale Representation Theorem

Proof.

- Let $h(s) = \tilde{h}(s)$ if $s \in S_0$ and $h(s) = 0$ if $s \in S - S_0$
  - Then h is $\mathcal{A}$-measurable and $g(\omega) = h(f(\omega)), \quad \omega \in \Omega$, as required
- Assume now that $g = \sum_{i=1}^{n} a_i \chi_{A_i}, A_i = f^{-1}(B_i) \in f^{-1}(\mathcal{A})$ for a $B_i \in \mathcal{A}$.
- Define $h = \sum_{i=1}^{n} a_i \chi_{B_i}$
  - Then $h : S \to \mathbb{R}$ is $\mathcal{A}$-measurable and simple.
- Thus

$$
\begin{aligned}
h(f(\omega)) &= \sum_{i=1}^{n} a_i \chi_{B_i}(f(\omega)) = \sum_{i=1}^{n} a_i \chi_{f^{-1}(B_i)}(\omega) \\
&= \sum_{i=1}^{n} a_i \chi_{A_i}(\omega) = g(\omega), \quad \omega \in \Omega,
\end{aligned}
$$

and $h \circ f = g$

□

Martingale Representation Theorem

- If $S = \mathbb{R}^n$ and $\mathcal{A}$ is the Borel σ-algebra of $\mathbb{R}^n$, then there is an $h : \mathbb{R}^n \to \mathbb{R}$, Borel measurable, which satisfies the requirements.

Corollary

Let (Ω, Σ) and $(\mathbb{R}^n, \mathcal{A})$ be measurable spaces, and $f : \Omega \to \mathbb{R}^n$ be measurable. Then $g : \Omega \to \mathbb{R}$ is $f^{-1}(\mathcal{A})$-measurable if and only if there is a Borel measure function $h : \mathbb{R}^n \to \mathbb{R}$ such that $g = h(f_1, f_2, \ldots, f_n) = h \circ f$ where $f = (f_1, \ldots, f_n)$.

Lemma

Fix $T > 0$. The set of random variables

$$\{\phi(B_{t_1}, \ldots, B_{t_n});\ t_i \in [0, T], \phi \in C_0^\infty(\mathbb{R}^n),\ n = 1, 2, \ldots\}$$

is dense in $L^2(\mathcal{F}_T, P)$.

Martingale Representation Theorem

Proof.

- Let $\{t_i\}_{i=1}^{\infty}$ be a dense subset of $[0, T]$;
- Let $\mathcal{H}_n$ be the σ-algebra generated by $B_{t_1}(\cdot), \ldots, B_{t_n}(\cdot)$.
  - Then clearly $\mathcal{H}_n \subset \mathcal{H}_{n+1}$.
  - $\mathcal{F}_T$ is the smallest σ-algebra containing all the $\mathcal{H}_n$.
- Choose $g \in L^2(\mathcal{F}_T, P)$.
  - then by the martingale convergence theorem, we have that

$$g = E[g|\mathcal{F}_T] = \lim_{n\to\infty} E[g|\mathcal{H}_n].$$

- The limit is pointwise a.e. (P) and in $L^2(\mathcal{F}_T, P)$.
  - By the Doob-Dynkin Lemma we can write, for each n,

$$E[g|\mathcal{H}_n] = g_n(B_{t_1}, \ldots, B_{t_n})$$

  for some Borel measurable function $g_n : \mathbb{R}^n \to \mathbb{R}$.

- Each such $g_n(B_{t_1}, \ldots, B_{t_n})$ can be approximated in $L^2(\mathcal{F}_T, P)$ by functions $\phi_n(B_{t_1}, \ldots, B_{t_n})$, where $\phi_n \in C_0^\infty(\mathbb{R}^n)$ and the result follows.

□

Martingale Representation Theorem

Lemma

The linear span of random variables of the type

$$\exp\left\{ \int_0^T h(t)dB_t(\omega) - \frac{1}{2}\int_0^T h^2(t)dt \right\}; \quad h \in L^2[0,T]\,(deterministic)$$

is dense in $L^2(\mathcal{F}_T, P)$.

Proof.

- Suppose $g \in L^2(\mathcal{F}_T, P)$ is orthogonal (in $L^2(\mathcal{F}_T, P)$) to all functions of the form $\exp\left\{ \int_0^T h(t)dB_t(\omega) - \frac{1}{2}\int_0^T h^2(t)dt \right\}$.
  - Then in particular

$$G(\lambda) := \int_\Omega \exp\{\lambda_1 B_{t_1}(\omega) + \ldots + \lambda_n B_{t_n}(\omega)\} g(\omega) dP(\omega) = 0$$

for all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and all $t_1, \ldots, t_n \in [0, T]$.

Martingale Representation Theorem

Proof.

- The function $G(\lambda)$ is real analytic in $\lambda \in \mathbb{R}^n$ and hence G has an analytic extension to the complex space $\mathbb{C}^n$ given by

$$G(z) := \int_{\Omega} \exp\{z_1 B_{t_1}(\omega) + \ldots + z_n B_{t_n}(\omega)\} g(\omega) dP(\omega)$$

  for all $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.

- Since $G = 0$ on $\mathbb{R}^n$ and G is analytic, $G = 0$ on $\mathbb{C}^n$.
  - In particular, $G(iy_1, \ldots, iy_n) = 0$ for all $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

Martingale Representation Theorem

Proof.

- But then we get, for $\phi \in C_0^\infty(\mathbb{R}^n)$

$$\int_\Omega \phi(B_{t_1}, \ldots, B_{t_n}) g(\omega) \mathrm{d}P(\omega)$$

$$\begin{aligned}
&= \int_\Omega (2\pi)^{-\frac{n}{2}} \Big(\int_{\mathbb{R}^n} \widehat{\phi}(y) e^{i(y_1 B_{t_1} + \ldots + y_n B_{t_n})} \mathrm{d}y \Big) g(\omega) \mathrm{d}P(\omega) \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \widehat{\phi}(y) \Big(\int_\Omega e^{i(y_1 B_{t_1} + \ldots + y_n B_{t_n})} g(\omega) \mathrm{d}P(\omega) \Big) \mathrm{d}y \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \widehat{\phi}(y) G(iy) \mathrm{d}y = 0 \ ,
\end{aligned}$$

where

$$\widehat{\phi}(y) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \phi(x) e^{-ix\cdot y} \mathrm{d}x$$

is the Fourier transform of ϕ.

Martingale Representation Theorem

Proof.

- We have used the inverse Fourier transform theorem

$$\phi(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \widehat{\phi}(y) e^{-ix\cdot y} \mathrm{d}y.$$

- Since

$$\int_{\Omega} \phi(B_{t_1}, \ldots, B_{t_n}) g(\omega) \mathrm{d}P(\omega) = 0$$

  and from the previous lemma g is orthogonal to a dense subset of $L^2(\mathcal{F}_T, P)$, we conclude that $g = 0$.

- Therefore the linear span of the functions

$$\exp\Big\{ \int_0^T h(t) dB_t(\omega) - \frac{1}{2} \int_0^T h^2(t) dt \Big\},$$

  must be dense in $L^2(\mathcal{F}_T, P)$.

□

Martingale Representation Theorem

Proof (Itô representation theorem).

- We consider only the case n = 1 (the proof in the general case is similar).
- First assume that F is of the form

$$F(\omega) = \exp\left\{\int_0^T h(t)\mathrm{d}B_t(\omega) - \frac{1}{2}\int_0^T h^2(t)\mathrm{d}t\right\}$$

  for some $h(t) \in L^2[0, T]$.
- Define

$$Y_t(\omega) = \exp\left\{\int_0^t h(s)\mathrm{d}B_s(\omega) - \frac{1}{2}\int_0^t h^2(s)\mathrm{d}s\right\}; \quad 0 \le t \le T.$$

- Then by Itô's formula

$$\mathrm{d}Y_t = Y_t(h(t)\mathrm{d}B_t - \frac{1}{2}h^2(t)\mathrm{d}t) + \frac{1}{2}Y_t(h(t)\mathrm{d}B_t)^2 = Y_t h(t)\mathrm{d}B_t$$

Martingale Representation Theorem

Proof.

- So that

$$Y_t = 1 + \int_0^t Y_s h(s) \mathrm{d}B_s; \quad t \in [0, T].$$

- Therefore

$$F = Y_T = 1 + \int_0^T Y_s h(s) \mathrm{d}B_s$$

  - $E[F] = 1.$

- $F(\omega) = E[F] + \int_0^T f(t, \omega) \mathrm{d}B(t)$ holds in this case.
- By linearity $F(\omega) = E[F] + \int_0^T f(t, \omega) \mathrm{d}B(t)$ also holds for linear combinations of functions of the form

$$\exp\left\{ \int_0^T h(t) dB_t(\omega) - \frac{1}{2} \int_0^T h^2(t) dt \right\}.$$

Martingale Representation Theorem

Proof.

- If $F \in L^2(\mathcal{F}_T, P)$ is arbitrary, we approximate F in $L^2(\mathcal{F}_T, P)$ by linear combinations F_n of functions of the form

$$\exp\left\{ \int_0^T h(t)dB_t(\omega) - \frac{1}{2}\int_0^T h^2(t)dt \right\}.$$

- Then for each n we have

$$F_n(\omega) = E[F_n] + \int_0^T f_n(s,\omega)\mathrm{d}B_s(\omega)$$

where $f_n \in \mathcal{V}(0, T)$.

Martingale Representation Theorem

Proof.

- By the Itô isometry

$$\begin{aligned} E[(F_n - F_m)^2] &= E[(E[F_n - F_m] + \int_0^T (f_n - f_m)\mathrm{d}B)^2] \\ &= E([F_n - F_m])^2 + \int_0^T E[(f_n - f_m)^2]\mathrm{d}t \to 0 \quad n, m \to \infty \end{aligned}$$

  - $\{f_n\}$ is a Cauchy sequence in $L^2([0,T] \times \Omega)$.
  - and hence converges to some $f \in L^2([0,T] \times \Omega)$.
  - since $f_n \in \mathcal{V}(0,T)$ we have $f \in \mathcal{V}(0,T)$.

- Again using the Itô isometry we see that

$$F = \lim_{n\to\infty} F_n = \lim_{n\to\infty}\left(E[F_n] + \int_0^T f_n \mathrm{d}B\right) = E[F] + \int_0^T f\mathrm{d}B$$

  - Hence the representation $F(\omega) = E[F] + \int_0^T f(t,\omega)\mathrm{d}B(t)$ holds for all $F \in L^2(\mathcal{F}_T, P)$.

Martingale Representation Theorem

Proof.

- The uniqueness follows from the Itô isometry.
- Suppose

$$F(\omega) = E[F] + \int_0^T f_1(t,\omega)\mathrm{d}B_t(\omega) = E[F] + \int_0^T f_2(t,\omega)\mathrm{d}B_t(\omega)$$

  with $f_1, f_2 \in \mathcal{V}(0, T)$.
- Then

$$0 = E[(\int_0^T (f_1(t,\omega) - f_2(t,\omega))\mathrm{d}B_t(\omega))^2] = \int_0^T E[(f_1(t,\omega) - f_2(t,\omega))^2]\mathrm{d}t.$$

  - and therefore $f_1(t,\omega) = f_2(t,\omega)$ for a.a. $(t,\omega) \in [0, T] \times \Omega$.

□

Martingale Representation Theorem

Proof (Martingale Representation Theorem).

- $n = 1$.
- By the Itô representation theorem applied to $T = t$ and $F = M_t$, we have that
  - for all t there exists a unique $f^{(t)}(s,\omega) \in L^2(\mathcal{F}_T, P)$ such that

$$M_t(\omega) = E[M_t] + \int_0^t f^{(t)}(s,\omega)\mathrm{d}B_s(\omega) = E[M_0] + \int_0^t f^{(t)}(s,\omega)\mathrm{d}B_s(\omega).$$

- Now assume $0 \leq t_1 \leq t_2$.
- Then

$$\begin{aligned} M_{t_1} &= E[M_{t_2}|\mathcal{F}_{t_1}] = E[M_0] + E\Big[\int_0^{t_2} f^{(t_2)}(s,\omega)\mathrm{d}B_s(\omega)|\mathcal{F}_{t_1}\Big] \\ &= E[M_0] + \int_0^{t_1} f^{(t_2)}(s,\omega)\mathrm{d}B_s(\omega). \end{aligned} \quad (3)$$

Martingale Representation Theorem

Proof.

- But we also have

$$M_{t_1} = E[M_0] + \int_0^{t_1} f^{(t_1)}(s,\omega)\mathrm{d}B_s(\omega). \tag{4}$$

- Hence, comparing (3) and (4) we get that

$$0 = E\Big[\Big(\int_0^{t_1} (f^{(t_2)} - f^{(t_1)})\mathrm{d}B\Big)^2\Big] = \int_0^{t_1} E[(f^{(t_2)} - f^{(t_1)})]\mathrm{d}s$$

- Therefore

$$f^{(t_1)}(s,\omega) = f^{(t_2)}(s,\omega)$$

for a.a. $(s,\omega) \in [0, t_1] \times \Omega$.

Martingale Representation Theorem

Proof.

- So we can define $f(s,\omega)$ for a.a. $s \in [0,\infty) \times \Omega$ by setting $f(s,\omega) = f^{(N)}(s,\omega)$, if $s \in [0,N]$.
- Then we get

$$M_t = E[M_0] + \int_0^t f^{(t)}(s,\omega)\mathrm{d}B_s(\omega) = E[M_0] + \int_0^t f(s,\omega)\mathrm{d}B_s(\omega),$$

for all $t \geq 0$.

□