# An overview of the Itô formula

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December 2, 2009

# The Itô Integral definition and properties

## Definition

Denote by $\{\mathcal{F}_t\}_{t\geq 0}$ the filtration generated by the one-dimensional Brownian motion $B_t$ and by $\mathcal{B}$ the Borel $\sigma$-algebra on $[0,\infty)$.
Let $\mathcal{V} = \mathcal{V}(S,T)$ be the class of functions $f : [0,\infty) \times \Omega \to \mathbb{R}$ such that

(i) $(t,\omega) \to f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable.

(ii) $f(t,\omega)$ is $\mathcal{F}_t$-adapted.

(iii) $E\left[\int_S^T (f(t,\omega))^2 \mathrm{d}t\right] < \infty$.

- We have already defined the Itô integral for functions $f \in \mathcal{V}$ (as well as for larger classes of functions) and studied some of its properties.

# The Itô Integral definition and properties

## Definition (Elementary function)

A function $\phi \in \mathcal{V}(S,T)$ is called *elementary* if it has the form

$$\phi(t,\omega) = \sum_j e_j(\omega) I_{[t_j, t_{j+1})}(t) \ ,$$

where the points $t_j$ define a partition of the interval $[S,T]$.
We have been using a partition defined by the points

$$t_j = t_j^{(n)} = \begin{cases} j2^{-n} & \text{if } S \leq j2^{-n} \leq T \\ S & \text{if } j2^{-n} < S \\ T & \text{if } j2^{-n} > T \end{cases} \ .$$

- Note that since $\phi \in \mathcal{V}$ then $e_j(\omega)$ must be $\mathcal{F}_{t_j}$-measurable.
- For elementary functions $\phi(t,\omega)$ we define the stochastic integral as

$$\int_S^T \phi(t,\omega) \mathrm{d}B_t(\omega) = \sum_{j \geq 0} e_j(\omega) [B_{t_{j+1}} - B_{t_j}](\omega) \ ,$$

# The Itô Integral definition and properties

## Definition (The Itô integral)

Let $f \in \mathcal{V}(S,T)$. Then the *Itô integral* of $f$ (from $S$ to $T$) is defined by

$$\int_S^T f(t,\omega)\mathrm{d}B_t(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega)\mathrm{d}B_t(\omega) \quad \text{limit in } L^2(P) \ , \tag{1}$$

where $\{\phi_n\}$ is a sequence of elementary functions such that

$$E\left[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2 \mathrm{d}t\right] \to 0 \quad \text{as } n \to \infty \ , \tag{2}$$

where the limit in (1) exists and does not depend on the choice of $\{\phi_n\}$, as long as (2) holds.

# The Itô Integral definition and properties

- Properties of the Itô integral:
  - Itô isometry:
    $E\left[\left(\int_S^T f(t,\omega)\mathrm{d}B_t(\omega)\right)^2\right] = E\left[\int_S^T (f(t,\omega))^2 \,\mathrm{d}t\right]$ for all $f \in \mathcal{V}(S,T)$
  - Linearity
  - $E\left[\int_S^T f\mathrm{d}B_t\right] = 0$
  - $\int_S^T f\mathrm{d}B_t$ is $\mathcal{F}_T$-measurable
  - Existence of a continuous version
  - The martingale property

# Motivation for the Itô formula

Let us recall the computation of the Itô integral $\int_0^t B_s \mathrm{d}B_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$.

## Example

Assume $B_0 = 0$. We consider the sequence of elementary functions

$$\phi_n(t,\omega) = \sum_j B_j(\omega) I_{[t_j,t_{j+1})}(t),$$

where $B_j = B_{t_j}$. Then

$$\begin{aligned} E\left[\int_0^t (\phi_n - B_s)^2 \,\mathrm{d}s\right] &= E\left[\sum_j \int_{t_j}^{t_{j+1}} (B_j - B_s)^2 \,\mathrm{d}s\right] \\ &= \sum_j \int_{t_j}^{t_{j+1}} (s - t_j)\,\mathrm{d}s \\ &= \sum_j \frac{1}{2}(t_{j+1} - t_j)^2 \to 0 \quad \text{as } \Delta t_j \to 0. \end{aligned}$$

# Motivation for the Itô formula

## Example

Thus,

$$\int_0^t B_s \mathrm{d}B_s = \lim_{\Delta t_j \to 0} \int_0^t \phi_n \mathrm{d}B_s = \lim_{\Delta t_j \to 0} \sum_j B_j \Delta B_j \ .$$

We now note that

$$\Delta(B_j^2) = B_{j+1}^2 - B_j^2 = (B_{j+1} - B_j)^2 + 2B_j(B_{j+1} - B_j) = (\Delta B_j)^2 + 2B_j \Delta B_j \ ,$$

and therefore

$$B_t^2 = \sum_j \Delta(B_j^2) = \sum_j (\Delta B_j)^2 + 2\sum_j B_j \Delta B_j \ ,$$

that is

$$\sum_j B_j \Delta B_j = \frac{1}{2} B_t^2 - \frac{1}{2} \sum_j (\Delta B_j)^2 \ .$$

Noting that $\sum_j (\Delta B_j)^2 \to t$ in $L^2(P)$ as $\Delta t_j \to 0$, we obtain the result.

# Motivation for the Itô formula

## Example

$$\int_0^t B_s \mathrm{d}B_s = \frac{1}{2}B_t^2 - \frac{1}{2}t \ .$$

- This example illustrates that the definition of Itô integral is not very useful when we try to evaluate a given integral.
  - We face the same kind of problems posed by the computation of ordinary Riemann integrals using its definition.
    - But to compute Riemann integrals we do not usually use its definition, but rather a combination of the fundamental theorem of calculus and the chain rule.
  - However, in the context of stochastic calculus, we have no differentiation theory, only integration theory.
  - Nevertheless, it is possible to establish an Itô integral version of the chain rule, called the Itô formula.
  - The Itô formula turns out to be extremely useful for evaluating Itô integrals.

# Itô processes

- From the identity
$$\int_0^t B_s \mathrm{d}B_s = \frac{1}{2}B_t^2 - \frac{1}{2}t \ .$$
we get
$$\frac{1}{2}B_t^2 = \frac{1}{2}t + \int_0^t B_s \mathrm{d}B_s \ .$$
- Note that the image of the Itô integral $B_t = \int_0^t \mathrm{d}B_s$ by the map $g(x) = \frac{1}{2}x^2$ is not again an Itô integral of the form
$$\int_0^t f(s,\omega)\mathrm{d}B_s(\omega)$$
but rather a combination of two integrals.
$$\frac{1}{2}B_t^2 = \int_0^t \frac{1}{2}\mathrm{d}s + \int_0^t B_s \mathrm{d}B_s \ .$$
- If we introduce Itô processes (also called stochastic integrals) as sums of a Itô integral and a Riemann integral then this family of integrals is stable under smooth maps.

# Itô processes

- Before giving a precise definition of Itô processes, let us recall the definition of the class of functions $\mathcal{W}_{\mathcal{H}}$.

## Definition (Class of functions $\mathcal{W}_{\mathcal{H}}(S, T)$)

We denote by $\mathcal{W}_{\mathcal{H}}(S, T)$ the class of functions $f : [0, \infty) \times \Omega \to \mathbb{R}$ satisfying the following three conditions

(i) $(t, \omega) \to f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable.

(ii)' There exists an increasing family of $\sigma$-algebras $\mathcal{H}_t, \; t \geq 0$ such that:

- a) $B_t$ is a martingale with respect to $\mathcal{H}_t$
- b) $f(t, \omega)$ is $\mathcal{H}_t$-adapted.

(iii)' $P\left[\int_S^T f(s, \omega)^2 \mathrm{d}s < \infty\right] = 1$.

We denote by $\mathcal{W}_{\mathcal{H}}$ the set defined by $\mathcal{W}_{\mathcal{H}} = \cap_{T>0} \mathcal{W}_{\mathcal{H}}(0, T)$.
We denote by $\mathcal{W}_{\mathcal{H}}^{m \times n}(S, T)$ the set of $m \times n$ matrices with entries on $\mathcal{W}_{\mathcal{H}}(S, T)$.

# Itô processes

## Definition (1-dimensional Itô process)

Let $B_t$ be 1-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$.
A *(1-dimensional) Itô process* is a stochastic process $X_t$ on $(\Omega, \mathcal{F}, P)$ of the form

$$X_t = X_0 + \int_0^t u(s,\omega)\mathrm{d}s + \int_0^t v(s,\omega)\mathrm{d}B_s \ ,$$

where $v \in \mathcal{W}_{\mathcal{H}}$ so that

$$P\left[\int_0^t v(s,\omega)^2 \mathrm{d}s < \infty \text{ for all } t \geq 0\right] = 1$$

and $u$ is $\mathcal{H}_t$-adapted and such that

$$P\left[\int_0^t |u(s,\omega)| \mathrm{d}s < \infty \text{ for all } t \geq 0\right] = 1 \ .$$

# Itô processes

- An Itô process $X_t$ of the form

$$X_t = X_0 + \int_0^t u(s,\omega)\mathrm{d}s + \int_0^t v(s,\omega)\mathrm{d}B_s \ ,$$

can be written in the shorter differential form

$$\mathrm{d}X_t = u\mathrm{d}t + v\mathrm{d}B_t \ .$$

- Recalling the example above, we note that the Itô process

$$\frac{1}{2}B_t^2 = \int_0^t \frac{1}{2}\mathrm{d}s + \int_0^t B_s\mathrm{d}B_s \ .$$

can be rewritten as

$$\mathrm{d}\left(\frac{1}{2}B_t^2\right) = \frac{1}{2}\mathrm{d}t + B_t\mathrm{d}B_t \ .$$

# The 1-dimensional Itô formula

## Theorem (The 1-dimensional Itô formula)

*Let $X_t$ be an Itô process given by*

$$\mathrm{d}X_t = u\mathrm{d}t + v\mathrm{d}B_t$$

*and let $g(t,x) \in C^2([0,\infty) \times \mathbb{R})$. Then*

$$Y_t = g(t, X_t)$$

*is again an Itô process, and*

$$\mathrm{d}Y_t = \frac{\partial g}{\partial t}(t, X_t)\mathrm{d}t + \frac{\partial g}{\partial x}(t, X_t)\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)(\mathrm{d}X_t)^2 \ ,$$

*where $(\mathrm{d}X_t)^2 = (\mathrm{d}X_t).(\mathrm{d}X_t)$ is computed according to the "multiplication table"*

$$\mathrm{d}t.\mathrm{d}t = \mathrm{d}t.\mathrm{d}B_t = \mathrm{d}B_t.\mathrm{d}t = 0 \ , \qquad \mathrm{d}B_t.\mathrm{d}B_t = \mathrm{d}t \ .$$

# The 1-dimensional Itô formula

## Proof.

Observe that if we substitute

$$\mathrm{d}X_t = u\mathrm{d}t + v\mathrm{d}B_t$$

in

$$\mathrm{d}Y_t = \frac{\partial g}{\partial t}(t, X_t)\mathrm{d}t + \frac{\partial g}{\partial x}(t, X_t)\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)(\mathrm{d}X_t)^2 \ ,$$

and use the "multiplication table"

$$\mathrm{d}t.\mathrm{d}t = \mathrm{d}t.\mathrm{d}B_t = \mathrm{d}B_t.\mathrm{d}t = 0 \ , \qquad \mathrm{d}B_t.\mathrm{d}B_t = \mathrm{d}t \ .$$

we get the equivalent expression

$$\begin{aligned} g(t, X_t) &= g(0, X_0) + \int_0^t \frac{\partial g}{\partial t}(s, X_s) + u\frac{\partial g}{\partial x}(s, X_s) + \frac{1}{2}v^2\frac{\partial^2 g}{\partial x^2}(s, X_s)\mathrm{d}s \\ &\quad + \int_0^t v\frac{\partial g}{\partial x}(s, X_s)\mathrm{d}B_s \ , \end{aligned}$$

which is still an Itô process.

# The 1-dimensional Itô formula

## Proof.

We can assume that:

1) the functions $g$, $\frac{\partial g}{\partial t}$, $\frac{\partial g}{\partial x}$ and $\frac{\partial^2 g}{\partial x^2}$ are bounded.
   - If this is the case, we obtain the general case by approximating by $C^2$ functions $g_n$ such that $g_n$, $\frac{\partial g_n}{\partial t}$, $\frac{\partial g_n}{\partial x}$ and $\frac{\partial^2 g_n}{\partial x^2}$ are bounded for each $n$ and converge uniformly on compact subsets of $[0, +\infty) \times \mathbb{R}$ to $g$, $\frac{\partial g}{\partial t}$, $\frac{\partial g}{\partial x}$ and $\frac{\partial^2 g}{\partial x^2}$, respectively.

2) $u(t, \omega)$ and $v(t, \omega)$ are elementary functions.
   - Because if $f \in \mathcal{W}_{\mathcal{H}}$ then for all $t$ there exist step functions $f_n \in \mathcal{W}_{\mathcal{H}}$ such that

$$\int_0^t |f - f_n| \mathrm{d}s \to 0 \qquad \text{(in probability)}$$

and, therefore,

$$\int_0^t f(s, \omega) \mathrm{d}B_s(\omega) = \lim_{n \to \infty} \int_0^t f_n(s, \omega) \mathrm{d}B_s(\omega) \qquad \text{(in probability)} .$$

# The 1-dimensional Itô formula

## Proof.

Using Taylor's theorem we get

$$
\begin{aligned}
g(t, X_t) &= g(0, X_0) + \sum_j \Delta g(t_j, X_j) \\
&= g(0, X_0) + \sum_j \frac{\partial g}{\partial t} \Delta t_j + \sum_j \frac{\partial g}{\partial x} \Delta X_j \\
&\quad + \frac{1}{2} \sum_j \frac{\partial^2 g}{\partial t^2} (\Delta t_j)^2 + \frac{1}{2} \sum_j \frac{\partial^2 g}{\partial t \partial x} \Delta t_j \Delta X_j + \frac{1}{2} \sum_j \frac{\partial^2 g}{\partial x^2} (\Delta X_j)^2 \\
&\quad + \sum_j R_j \ ,
\end{aligned}
$$

where

- the functions $g$, $\frac{\partial g}{\partial t}$, $\frac{\partial g}{\partial x}$, $\frac{\partial^2 g}{\partial t^2}$, $\frac{\partial^2 g}{\partial t \partial x}$ and $\frac{\partial^2 g}{\partial x^2}$ are all evaluated at $(t_j, X_{t_j})$.
- $\Delta t_j = t_{j+1} - t_j$, $\Delta X_j = X_{j+1} - X_j$, $\Delta g(t_j, X_j) = g(t_{j+1}, X_{j+1}) - g(t_j, X_j)$.
- $R_j = o\left(|\Delta t_j|^2 + |\Delta X_j|^2\right)$ for all j.

# The 1-dimensional Itô formula

## Proof.

If $\Delta t_j \to 0$ then

$$\sum_j \frac{\partial g}{\partial t} \Delta t_j = \sum_j \frac{\partial g}{\partial t}(t_j, X_{t_j}) \Delta t_j \to \int_0^t \frac{\partial g}{\partial s}(s, X_s) \mathrm{d}s$$

and

$$\sum_j \frac{\partial g}{\partial x} \Delta X_j = \sum_j \frac{\partial g}{\partial x}(t_j, X_{t_j}) \Delta X_j \to \int_0^t \frac{\partial g}{\partial x}(s, X_s) \mathrm{d}X_s \ .$$

Since $u$ and $v$ are elementary, we get

$$\begin{aligned} \sum_j \frac{\partial^2 g}{\partial x^2} (\Delta X_j)^2 &= \sum_j \frac{\partial^2 g}{\partial x^2}(t_j, X_{t_j}) u_j^2 (\Delta t_j)^2 + \sum_j \frac{\partial^2 g}{\partial x^2}(t_j, X_{t_j}) u_j v_j \Delta t_j \Delta B_j \\ &\quad + \sum_j \frac{\partial^2 g}{\partial x^2}(t_j, X_{t_j}) v_j^2 (\Delta B_j)^2 \ . \end{aligned}$$

where $u_j = u_j(t_j, \omega)$ and $v_j = v_j(t_j, \omega)$.

# The 1-dimensional Itô formula

## Proof.

The first two terms in the previous equality tend to zero as $\Delta t_j \to 0$. To see this, note that, for instance

$$E\left[\left(\sum_j \frac{\partial^2 g}{\partial x^2} u_j v_j \Delta t_j \Delta B_j\right)^2\right] = \sum_j E\left[\left(\frac{\partial^2 g}{\partial x^2} u_j v_j\right)^2\right] (\Delta t_j)^3 \to 0 \text{ as } \Delta t_j \to 0.$$

In a similar fashion the terms

$$\frac{1}{2} \sum_j \frac{\partial^2 g}{\partial t^2} (\Delta t_j)^2$$

and

$$\frac{1}{2} \sum_j \frac{\partial^2 g}{\partial t \partial x} \Delta t_j \Delta X_j$$

also tend to zero as $\Delta t_j \to 0$.

# The 1-dimensional Itô formula

## Proof.

It remains to be proved that

$$\sum_j \frac{\partial^2 g}{\partial x^2}(t_j, X_{t_j}) v_j^2 (\Delta B_j)^2 \to \int_0^t v^2 \frac{\partial^2 g}{\partial x^2} \mathrm{d}s \qquad \text{in } L^2(P) \text{ as } \Delta t_j \to 0.$$

To prove it, we introduce the notation $a(t) = v^2(t,\omega)\frac{\partial^2 g}{\partial x^2}(t, X_t)$, $a_j = a(t_j)$ and consider

$$E\left[\left(\sum_j a_j (\Delta B_j)^2 - \sum_j a_j \Delta t_j\right)^2\right] = \sum_{i,j} E\left[a_i a_j ((\Delta B_i)^2 - \Delta t_i)((\Delta B_j)^2 - \Delta t_j)\right].$$

If $i < j$ then the terms $a_i a_j ((\Delta B_i)^2 - \Delta t_i)$ and $((\Delta B_j)^2 - \Delta t_j)$ are independent and so the corresponding terms in the sum vanish in this case and similarly for $i > j$.

# The 1-dimensional Itô formula

## Proof.

Therefore, we are left with

$$\begin{aligned}\sum_j E\left[a_j^2((\Delta B_j)^2 - \Delta t_j)^2\right] &= \sum_j E\left[a_j^2\right] E\left[(\Delta B_j)^4 - 2(\Delta B_j)^2(\Delta t_j) + (\Delta t_j)^2\right] \\ &= \sum_j E\left[a_j^2\right] E\left[3(\Delta t_j)^2 - 2(\Delta t_j)^2 + (\Delta t_j)^2\right] \\ &= 2\sum_j E\left[a_j^2\right](\Delta t_j)^2 \qquad \text{as } \Delta t_j \to 0.\end{aligned}$$

Therefore, we have established that

$$\sum_j a_j(\Delta B_j)^2 \to \int_0^t a(s)\mathrm{d}s \qquad \text{in } L^2(P) \text{ as } \Delta t_j \to 0,$$

which is also often expressed as $(\mathrm{d}B_t)^2 = \mathrm{d}t$
The arguments above also prove that $\sum_j R_j \to 0$ as $\Delta t_j \to 0$, which completes the proof of the Itô formula. □

# The 1-dimensional Itô formula

- Note that:
  - it is enough that $g(t, x)$ is $C^2$ on $[0, \infty) \times U$, if $U \subset \mathbb{R}$ is an open set such that $X_t(\omega) \in U$ for all $t \geq 0$ and $\omega \in \Omega$.
  - it is enough to assume that $g(t, x)$ is $C^1$ in $t$ and $C^2$ in $x$.

# The 1-dimensional Itô formula

## Example

To compute the integral

$$\int_0^t B_s \mathrm{d}B_s$$

one can choose $X_t = B_t$ and $g(t,x) = \frac{1}{2}x^2$ and apply Itô formula to

$$Y_t = g(t, B_t) = \frac{1}{2}B_t^2$$

to get

$$\begin{aligned} \mathrm{d}Y_t = g(t, B_t) &= \frac{\partial g}{\partial t}(t, B_t)\mathrm{d}t + \frac{\partial g}{\partial x}(t, B_t)\mathrm{d}B_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, B_t)(\mathrm{d}B_t)^2 \\ &= B_t \mathrm{d}B_t + \frac{1}{2}\mathrm{d}t \\ &= \frac{1}{2}\mathrm{d}t + B_t \mathrm{d}B_t \ . \end{aligned}$$

# The 1-dimensional Itô formula

## Example

Therefore,

$$\mathrm{d}\left(\frac{1}{2}B_t^2\right) = \frac{1}{2}\mathrm{d}t + B_t \mathrm{d}B_t$$

and thus

$$\frac{1}{2}B_t^2 = \frac{t}{2} + \int_0^t B_s \mathrm{d}B_s$$

or

$$\int_0^t B_s \mathrm{d}B_s = \frac{1}{2}B_t^2 - \frac{t}{2}$$

# The 1-dimensional Itô formula

## Example

Let us assume that $f : \mathbb{R} \to \mathbb{R}$ is $C^1$. To compute the integral

$$\int_0^t f(s)\mathrm{d}B_s \ .$$

we can use some real variable calculus intuition, i.e. it seem reasonable that a term of the form $f(t)B_t$ should appear when computing the integral above. Therefore, we choose $X_t = B_t$ and $g(t,x) = f(t)x$ and apply Itô formula to

$$Y_t = g(t, B_t) = f(t)B_t$$

to get

$$\begin{aligned} \mathrm{d}Y_t = g(t,B_t) &= \frac{\partial g}{\partial t}(t,B_t)\mathrm{d}t + \frac{\partial g}{\partial x}(t,B_t)\mathrm{d}B_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,B_t)(\mathrm{d}B_t)^2 \\ &= f'(t)B_t\mathrm{d}t + f(t)\mathrm{d}B_t \ . \end{aligned}$$

# The 1-dimensional Itô formula

**Example**

Therefore,

$$\mathrm{d}\left(f(t)B_t\right) = f'(t)B_t\mathrm{d}t + f(t)\mathrm{d}B_t$$

and thus

$$f(t)B_t = \int_0^t f'(s)B_s\mathrm{d}s + \int_0^t f(s)\mathrm{d}B_s$$

or

$$\int_0^t f(s)\mathrm{d}B_s = f(t)B_t - \int_0^t f'(s)B_s\mathrm{d}s$$

# The 1-dimensional Itô formula

- We can extend the previous reasoning to obtain the following analogue of the integration by parts formula.

## Theorem (Integration by parts)

*Suppose the $f(s,\omega) = f(s)$ only depends on $s$ and that $f$ is continuous and of bounded variation in $[0,t]$. Then*

$$\int_0^t f(s)\mathrm{d}B_s = f(t)B_t - \int_0^t B_s \mathrm{d}f(s) \ .$$

# The Multi-dimensional Itô formula

## Definition (Multi-dimensional Itô process)

Let $B(t,\omega) = (B_1(t,\omega), ..., B_m(t,\omega))$ denote $m$-dimensional Brownian motion. Assume that for each $i \in \{1, ..., n\}$ and each $j \in \{1, ..., m\}$ we have that the processes $u_i = u_i(t,\omega)$ and $v_{ij} = v_{ij}(t,\omega)$ are such that $v_{ij} \in \mathcal{W}_{\mathcal{H}}$ and

$$P\left[\int_0^t v_{ij}(s,\omega)^2 \mathrm{d}s < \infty \text{ for all } t \geq 0\right] = 1,$$

and $u_i$ is $\mathcal{H}_t$-adapted and such that

$$P\left[\int_0^t |u_i(s,\omega)| \mathrm{d}s < \infty \text{ for all } t \geq 0\right] = 1 \ .$$

A *(Multi-dimensional) Itô process* is a stochastic process of the form

$$\begin{cases} \mathrm{d}X_1 & = & u_1 \mathrm{d}t & + & v_{11}\mathrm{d}B_1 & + & \cdots & + & v_{1m}\mathrm{d}B_m \\ \vdots & & \vdots & & & & & & \vdots \\ \mathrm{d}X_n & = & u_n \mathrm{d}t & + & v_{n1}\mathrm{d}B_1 & + & \ldots & + & v_{nm}\mathrm{d}B_m \end{cases} .$$

# The Multi-dimensional Itô formula

- Note that one could also represent a Multi-dimensional Itô process on matrix form

$$\mathrm{d}X = u\mathrm{d}t + v\mathrm{d}B \ ,$$

where

$$X(t) = \begin{pmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{pmatrix} \ , \qquad u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

$$v = \begin{pmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & & \vdots \\ v_{n1} & \cdots & v_{nm} \end{pmatrix} \ , \qquad \mathrm{d}B(t) = \begin{pmatrix} \mathrm{d}B_1(t) \\ \vdots \\ \mathrm{d}B_m(t) \end{pmatrix} \ .$$

# The Multi-dimensional Itô formula

## Theorem (The Multi-dimensional Itô formula)

*Let*

$$\mathrm{d}X(t) = u\mathrm{d}t + v\mathrm{d}B(t)$$

*be an n-dimensional Itô process and let* $g(t,x) = (g_1(t,x), ..., g_p(t,x))$ *be a* $C^2$ *map from* $[0,\infty) \times \mathbb{R}^n$ *into* $\mathbb{R}^p$*.*
*Then the process*

$$Y(t) = g(t, X(t))$$

*is again an Itô process and its* $k$*th component* $Y_k$ *is given by*

$$\mathrm{d}Y_k = \frac{\partial g_k}{\partial t}(t,X)\mathrm{d}t + \sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i}(t,X)\mathrm{d}X_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(t,X)\mathrm{d}X_i\mathrm{d}X_j \ ,$$

*where the following "multiplication table" holds*

$$\mathrm{d}t.\mathrm{d}t = \mathrm{d}t.\mathrm{d}B_i = \mathrm{d}B_i.\mathrm{d}t = 0 \ , \qquad \mathrm{d}B_i.\mathrm{d}B_j = \delta_{ij}\mathrm{d}t \ .$$

- The proof is analogous to the 1-dimensional version.

# The Multi-dimensional Itô formula

## Example

Let $B = (B_1, ..., B_n)$ be a Brownian motion in $\mathbb{R}^n$ and $n \geq 2$. Consider the stochastic process given by

$$R(t, \omega) = |B(t, \omega)| = \left(B_1^2(t, \omega) + \cdots + B_n^2(t, \omega)\right)^{1/2} ,$$

i.e. the distance to the origin to the Brownian motion $B$.
Although $g(t, x) = |x|$ is not a $C^2$ function at the origin, Itô's formula still holds since $B$ never hits the origin a.s. when $n \geq 2$. We get

$$\mathrm{d}R = \sum_{i=1}^{n} \frac{B_i}{R} \mathrm{d}B_i + \frac{n-1}{2R} \mathrm{d}t .$$

The process $R$ above is called the *n-dimensional Bessel process*.