Some properties and extensions of the Itô integral

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A stochastic integral for smooth functions

- Let $g : [0,1] \to \mathbb{R}$ be a continuously differentiable (deterministic) function with $g(0) = g(1) = 0$.
- Define (integration by parts analogue):

$$\int_0^1 g(t)\mathrm{d}B_t(\omega) = \int_0^1 g'(t)B_t(\omega)\mathrm{d}t \ .$$

- The resulting integral has the following nice properties:
  (i) $E\left[\int_0^1 g\mathrm{d}B\right] = 0$.
  (ii) $E\left[\left(\int_0^1 g\mathrm{d}B\right)^2\right] = \int_0^1 g^2\mathrm{d}t$.

A stochastic integral for smooth functions

- Suppose now that $g \in L^2(0,1)$.
  - Then, we can take a sequence of C^1 function $\{g_n\}$, as above, such that

$$\int_0^1 (g - g_n)^2 \mathrm{d}t \to 0 \qquad \text{as } n \to \infty.$$

  - By property (ii) above, we get that

$$E\left[\left(\int_0^1 g_n \mathrm{d}B - \int_0^1 g_m \mathrm{d}B\right)^2\right] = \int_0^1 (g_n - g_m)^2 \mathrm{d}t \ ,$$

    and therefore, $\{\int_0^1 g_n \mathrm{d}B\}$ is a Cauchy sequence in $L^2(\Omega)$ and we can define

$$\int_0^1 g \mathrm{d}B = \lim_{n\to\infty} \int_0^1 g_n \mathrm{d}B \qquad \text{limit in } L^2(\Omega).$$

  - The extended definition still satisfies properties (i) and (ii) above.
  - This is a reasonable definition for $\int_0^1 g \mathrm{d}B$, except that this only makes sense for functions $g \in L^2(0,1)$, and not for stochastic processes.

The Itô Integral

Definition

Denote by $\{\mathcal{F}_t\}_t$ the filtration generated by the one-dimensional Brownian motion B_t and by $\mathcal{B}$ the Borel σ-algebra on $[0,\infty)$.
Let $\mathcal{V} = \mathcal{V}(S,T)$ be the class of functions $f : [0,\infty) \times \Omega \to \mathbb{R}$ such that

(i) $(t,\omega) \to f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$ measurable.

(ii) $f(t,\omega)$ is $\mathcal{F}_t$ adapted.

(iii) $E\left[\int_S^T (f(t,\omega))^2 \mathrm{d}t\right] < \infty$.

- For functions $f \in \mathcal{V}$ we have defined the Itô integral

$$\mathcal{I}[f](\omega) = \int_S^T f(t,\omega)\mathrm{d}B_t(\omega) \ ,$$

where B_t is a one-dimensional Brownian motion, as the limit of the integrals of a sequence of elementary functions converging to f.

The Itô Integral

Definition (Elementary function)

A function $\phi \in \mathcal{V}$ is called elementary if it has the form

$$\phi(t,\omega) = \sum_j e_j(\omega) I_{[t_j,t_{j+1})}(t) \ .$$

- For elementary functions $\phi(t,\omega)$ we define the stochastic integral as

$$\int_S^T \phi(t,\omega)\mathrm{d}B_t(\omega) = \sum_{j\geq 0} e_j(\omega)[B_{t_{j+1}} - B_{t_j}](\omega) \ ,$$

The Itô Integral

Definition (The Itô integral)

Let $f \in \mathcal{V}(S,T)$. Then the *Itô integral* of f (from S to T) is defined by

$$\int_S^T f(t,\omega)\mathrm{d}B_t(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega)\mathrm{d}B_t(\omega) \quad \text{limit in } L^2(P) \;, \tag{1}$$

where $\{\phi_n\}$ is a sequence of elementary functions such that

$$E\left[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2 \mathrm{d}t\right] \to 0 \quad \text{as } n \to \infty \;. \tag{2}$$

- The limit in (1) exists and does not depend on the choice of $\{\phi_n\}$, as long as (2) holds.

The Itô Integral

Corollary (The Itô isometry)

$$E\left[\left(\int_S^T f(t,\omega)\mathrm{d}B_t(\omega)\right)^2\right] = E\left[\int_S^T (f(t,\omega))^2\,\mathrm{d}t\right] \quad \textit{for all } f \in \mathcal{V}(S,T).$$

Corollary

If $f(t,\omega) \in \mathcal{V}(S,T)$, $f_n(t,\omega) \in \mathcal{V}(t,\omega)$ *for* $n = 1, 2, \ldots$ *and*

$$E\left[\int_S^T (f_n(t,\omega) - f(t,\omega))^2 \mathrm{d}t\right] \to 0 \quad \textit{as } n \to \infty\ ,$$

then

$$\int_S^T f_n(t,\omega)\mathrm{d}B_t(\omega) \to \int_S^T f(t,\omega)\mathrm{d}B_t(\omega) \quad \textit{in } L^2(P) \textit{ as } n \to \infty.$$

Properties of the Itô Integral

Theorem

Let $f, g \in \mathcal{V}(0, T)$ and let $0 \leq S < U < T$. Then:

(i) $\int_S^T f \mathrm{d}B_t = \int_S^U f \mathrm{d}B_t + \int_U^T f \mathrm{d}B_t$ *for a.e.* $\omega \in \Omega$.

(ii) $\int_S^T af + bg \mathrm{d}B_t = c \int_S^T f \mathrm{d}B_t + b \int_S^T g \mathrm{d}B_t$ *for a.e.* $\omega \in \Omega$ *and all* $a, b \in \mathbb{R}$.

(iii) $E\left[\int_S^T f \mathrm{d}B_t\right] = 0$.

(iv) $\int_S^T f \mathrm{d}B_t$ *is* $\mathcal{F}_T$*-measurable.*

Proof.

The statements above clearly hold for all elementary functions.
The results then follow trivially by taking limits. □

Properties of the Itô Integral

Definition (Martingale)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X = \{X_t : t \geq 0\}$ a stochastic process on it.

We say that X_t is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ if

(i) X is adapted to $\mathcal{F}_t$.

(ii) $E[|X_t|] < \infty$ for all t.

(iii) $E[X_s|\mathcal{F}_t] = X_t$ for all $s \geq t$.

We say that X_t is a submartingale with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ if (i) and (ii) above hold and condition (iii) is replaced by

(iii)' $E[X_s|\mathcal{F}_t] \geq X_t$ for all $s \geq t$.

We define a supermartingale in an analogous way.

Properties of the Itô Integral

Lemma

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X = \{X_t : t \geq 0\}$ a real-valued martingale with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is a convex function, i.e

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad \text{for all } x, y \in \mathbb{R} \text{ and all } t \in [0,1].$$

Then if $E[\phi(X_t)] < \infty$ for all $t \geq 0$, $\phi(X_t)$ is a submartingale.

Proof.

The result follows from Jensen's inequality:
If $\phi : \mathbb{R} \to \mathbb{R}$ is a convex function and $E[|\phi(X)|] < \infty$ then

$$\phi(E[X|\mathcal{F}]) \leq E[\phi(X)|\mathcal{F}] .$$

□

Properties of the Itô Integral

Example

Brownian motion B_t in $\mathbb{R}^n$ is a martingale with respect to the filtration $\mathcal{F}_t$ generated by $\{B_t : t \geq 0\}$:

(i) B_t is clearly adapted to $\mathcal{F}_t$.

(ii) We note that for all t

$$\begin{aligned} (E[|B_t|])^2 &\leq E\left[|B_t|^2\right] \\ &= E\left[|B_t - B_0|^2\right] + 2E[B_t.B_0] - E\left[|B_0|^2\right] = nt + E\left[|B_0|^2\right] \end{aligned}$$

(iii) Assume $s \geq t$. Using Brownian motion properties, we get

$$\begin{aligned} E[B_s|\mathcal{F}_t] &= E[B_s - B_t + B_t|\mathcal{F}_t] \\ &= E[B_s - B_t + B_t|\mathcal{F}_t] + E[B_t|\mathcal{F}_t] \\ &= B_t \ . \end{aligned}$$

Properties of the Itô Integral

Example

The stochastic process $B_t^2 - t$ in $\mathbb{R}$ is a martingale with respect to the filtration $\mathcal{F}_t$ generated by $\{B_t : t \geq 0\}$:

(i) B_t is clearly adapted to $\mathcal{F}_t$.

(ii) We note that for all t

$$\begin{aligned} E\left[B_t^2 - t\right] &= E\left[B_t^2\right] - t \\ &= t + E\left[B_0^2\right] - t = E\left[B_0^2\right] \ . \end{aligned}$$

(iii) Assume $s \geq t$. Using Brownian motion properties, we get

$$\begin{aligned} E\left[B_s^2 - s|\mathcal{F}_t\right] &= E\left[(B_s - B_t)^2 + 2B_sB_t - B_t^2|\mathcal{F}_t\right] - s \\ &= E\left[(B_s - B_t)^2|\mathcal{F}_t\right] + 2E\left[B_sB_t|\mathcal{F}_t\right] - E\left[B_t^2|\mathcal{F}_t\right] - s \\ &= s - t + 2B_tE\left[B_s|\mathcal{F}_t\right] - B_t^2 - s \\ &= B_t^2 - t \ . \end{aligned}$$

Properties of the Itô Integral

- For the proof of the next result we will need two auxiliary results: Doob's martingale inequality and the Borel–Cantelli lemma.

Theorem (Doob's martingale inequality)

If X_t is a martingale such that $t \to X_t(\omega)$ is continuous a.s., then for all $p \geq 1$, $T \geq 0$ and all $\lambda > 0$

$$P\left[\sup_{0 \leq t \leq T} |X_t| \geq \lambda\right] \leq \frac{1}{\lambda^p} E\left[|X_T|^p\right] .$$

- This is a generalization of the *Chebychev's inequality*:
  Let $X : \Omega \to \mathbb{R}^n$ be a random variable such that $E[|X|^p] < \infty$ for some p, $0 < p < \infty$. Then

$$P\{|X| \geq \lambda\} \leq \frac{1}{\lambda^p} E[|X|^p] \quad \text{for all } \lambda > 0.$$

Properties of the Itô Integral

- We will start by proving a discrete version of Doob's martingale inequality.

Theorem (Discrete martingale inequality)

If $\{X_n\}_{n\in\mathbb{N}}$ *is a submartingale then*

$$P\left[\max_{1\leq k\leq N} X_k \geq \lambda\right] \leq \frac{1}{\lambda} E\left[X_N^+\right] \ ,$$

where $X^+ = \max\{0, X\}$.

Properties of the Itô Integral

Proof.

Let us define the sets

$$A_k = \bigcap_{j=1}^{k-1} \{\omega \in \Omega : X_j \leq \lambda\} \cap \{\omega \in \Omega : X_k > \lambda\} \ , \quad k = 1, 2, ..., N \ .$$

Then the set

$$A = \left\{ \omega \in \Omega : \max_{1 \leq k \leq N} \{X_k > \lambda\} \right\}$$

can be written as the disjoint union of the sets $A_1, A_2, ..., A_N$, i.e.

$$A = \dot{\bigcup}_{k=1}^{N} A_k \ .$$

Properties of the Itô Integral

Proof.

Noting that

$$\int_{A_k} X_k \mathrm{d}P \geq \lambda P(A_k)$$

we have

$$\lambda P(A) = \lambda \sum_{k=1}^{N} P(A_k) \leq \sum_{k=1}^{N} E\left[I_{A_k} X_k\right] \ .$$

Then

$$\begin{aligned} E[X_N^+] &\geq \sum_{k=1}^{N} E\left[I_{A_k} X_N^+\right] = \sum_{k=1}^{N} E[E[I_{A_k} X_N^+ | X_1, \ldots X_k]] \\ &= \sum_{k=1}^{N} E[I_{A_k} E[X_N^+ | X_1, \ldots X_k]] \geq \sum_{k=1}^{N} E[I_{A_k} E[X_N | X_1, \ldots X_k]] \\ &\geq \sum_{k=1}^{N} E[I_{A_k} X_k] \geq \lambda P(A) \ . \end{aligned}$$

Properties of the Itô Integral

Proof.

Thus, we have proved that

$$\lambda P\left[\max_{1\leq k\leq N} X_k \geq \lambda\right] \leq \int_{\{\omega\in\Omega:\max_{1\leq k\leq N} X_k>\lambda\}} X_n^+ \mathrm{d}P \ .$$

□

Corollary

If $\{X_n\}_{n\in\mathbb{N}}$ is a martingale and $E\left[|X_n|^p\right] < \infty$ for some $p \geq 1$ and all $n \in \mathbb{N}$ then

$$P\left[\max_{1\leq k\leq N} |X_k| \geq \lambda\right] \leq \frac{1}{\lambda^p} E\left[|X_N|^p\right] \quad \textit{for any } \lambda > 0 \textit{ and } N \in \mathbb{N}.$$

Proof.

Follows from the previous discrete martingale inequality and the fact that $\{|X_n|^p\}_{n\in\mathbb{N}}$ is a submartingale.

□

Properties of the Itô Integral

Proof of the Doob's martingale inequality.

Let X_t be a martingale such that $t \to X_t(\omega)$ is continuous a.s..
Choose $\lambda > 0$ and $t > 0$ and select a partition of $[0, t]$ such that
$0 = t_0 < t_1 < ... < t_n = t$.
Note that $X(t_i)_{i=0}^n$ is a discrete martingale and therefore the discrete martingale inequality applies.
The proof is completed by choosing smaller partitions and passing to the limit. □

Properties of the Itô Integral

Theorem (Borel–Cantelli lemma)

Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of events in some probability space $(\Omega, \mathcal{F}, P)$ such that

$$\sum_n P(E_n) < \infty \ .$$

Then

$$P\left(\limsup_{n\to\infty} E_n\right) = P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\right) = 0 \ ,$$

i.e., the probability that infinitely many events E_n's occur is zero.

Proof.

Note that

$$P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\right) = \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \limsup_{n\to\infty} \sum_{k=n}^{\infty} P(E_k) = 0$$

since $\sum_{k=n}^{\infty} P(E_k) < \infty$.

Properties of the Itô Integral

Theorem (Itô integral has a continuous version)

Let $f \in \mathcal{V}(0,T)$. Then there exists a t-continuous version of

$$\int_0^t f(s,\omega)\mathrm{d}B_s(\omega) \ , \qquad 0 \le t \le T \ ,$$

that is, there exists a t-continuous stochastic process J_t on $(\Omega, \mathcal{F}, P)$ such that

$$P\left[J_t = \int_0^t f \mathrm{d}B \right] = \qquad \text{for all } t,\ 0 \le t \le T .$$

Properties of the Itô Integral

Proof.

Define a sequence $\phi_n = \phi_n(t,\omega)$ of elementary functions

$$\phi_n(t,\omega) = \sum_j e_j^{(n)}(\omega) I_{[t_j^{(n)}, t_{j+1}^{(n)})}(t)$$

such that

$$E\left[\int_0^T (f-\phi_n)^2 \mathrm{d}t\right] \to 0 \qquad \text{when } n \to \infty.$$

Put

$$I_n = I_n(t,\omega) = \int_0^t \phi_n(s,\omega)\mathrm{d}B_s(\omega)$$

and

$$I_t = I_t(t,\omega) = \int_0^t f(s,\omega)\mathrm{d}B_s(\omega)\ , \qquad 0 \le t \le T\ .$$

Then I_n is clearly continuous for all $n \in \mathbb{N}$.
Furthermore, I_n is also a martingale with respect to the filtration $\mathcal{F}_t$ for all n.

Properties of the Itô Integral

Proof.

Let us prove that I_n is a martingale. Let $s > t$:

$$\begin{aligned}
E\left[I_n(s,\omega)|\mathcal{F}_t\right] &= E\left[\int_0^s \phi_n \mathrm{d}B|\mathcal{F}_t\right] \\
&= E\left[\int_0^t \phi_n \mathrm{d}B + \int_t^s \phi_n \mathrm{d}B|\mathcal{F}_t\right] \\
&= \int_0^t \phi_n \mathrm{d}B + E\left[\int_t^s \phi_n \mathrm{d}B|\mathcal{F}_t\right] \\
&= \int_0^t \phi_n \mathrm{d}B + E\left[\sum_{t \le t_j^{(n)} \le t_{j+1}^{(n)} \le s} e_j^{(n)} \Delta B_j|\mathcal{F}_t\right] \\
&= \int_0^t \phi_n \mathrm{d}B + \sum_{t \le t_j^{(n)} \le t_{j+1}^{(n)} \le s} E\left[e_j^{(n)} \Delta B_j|\mathcal{F}_t\right] .
\end{aligned}$$

We need to prove that the expected value on the right hand side is equal to zero.

Properties of the Itô Integral

Proof.

We use the following property of conditional expectations:
"if $\mathcal{G}$ and $\mathcal{H}$ are σ-algebras such that $\mathcal{G} \subset \mathcal{H}$ then $E[X|\mathcal{G}] = E[E[X|\mathcal{H}]|\mathcal{G}]$"
to obtain

$$\begin{aligned} E\left[I_n(s,\omega)|\mathcal{F}_t\right] &= \int_0^t \phi_n \mathrm{d}B + \sum_{t \le t_j^{(n)} \le t_{j+1}^{(n)} \le s} E\left[e_j^{(n)} \Delta B_j | \mathcal{F}_t\right] \\ &= \int_0^t \phi_n \mathrm{d}B + \sum_{t \le t_j^{(n)} \le t_{j+1}^{(n)} \le s} E\left[E\left[e_j^{(n)} \Delta B_j | \mathcal{F}_{t_j^{(n)}}\right] |\mathcal{F}_t\right] \\ &= \int_0^t \phi_n \mathrm{d}B + \sum_{t \le t_j^{(n)} \le t_{j+1}^{(n)} \le s} E\left[e_j^{(n)} E\left[\Delta B_j | \mathcal{F}_{t_j^{(n)}}\right] |\mathcal{F}_t\right] \\ &= \int_0^t \phi_n \mathrm{d}B = I_n(t,\omega) \; . \end{aligned}$$

Properties of the Itô Integral

Proof.

Hence $I_n - I_m$ is also an $\mathcal{F}_t$-martingale, so by the martingale inequality and the Itô isometry it follows that

$$\begin{aligned} P\left[\sup_{0\le t\le T} |I_n(t,\omega) - I_m(t,\omega)| > \epsilon\right] &\le \frac{1}{\epsilon^2} E\left[|I_n(T,\omega) - I_m(T,\omega)|^2\right] \\ &= \frac{1}{\epsilon^2} E\left[\left|\int_0^T (\phi_n - \phi_m)^2 \mathrm{d}s\right|\right] \to 0 \end{aligned}$$

as $n, m \to \infty$.
Hence, we can choose a subsequence $n_k \to \infty$ such that

$$P\left[\sup_{0\le t\le T} |I_{n_{k+1}}(t,\omega) - I_{n_k}(t,\omega)| > 2^{-k}\right] < 2^{-k} \ .$$

By the Borel-Cantelli lemma

$$P\left[\sup_{0\le t\le T} |I_{n_{k+1}}(t,\omega) - I_{n_k}(t,\omega)| > 2^{-k} \text{ for infinitely many } k\right] = 0 \ .$$

Properties of the Itô Integral

Proof.

So for almost all $\omega \in \Omega$ there exists $k_1(\omega)$ such that

$$\sup_{0 \le t \le T} |I_{n_{k+1}}(t,\omega) - I_{n_k}(t,\omega)| \le 2^{-k} \text{ for } k \ge k_1(\omega) \, .$$

Therefore $I_{n_k}(t,\omega)$ is uniformly convergent for $t \in [0, T]$ for a.a. $\omega \in \Omega$. Thus, the limit denoted by $J_t(\omega)$ is t-continuous for for $t \in [0, T]$ a.s.. Since $I_{n_k}(t,\cdot) \to I(t,\cdot)$ in $L^2(P)$ for all t, we must have

$$I_t = J_t \quad \text{a.s.} \, , \qquad \text{for all } t \in [0, T].$$

□

- From now on, we will always assume that $\int_0^t f(s,\omega) \mathrm{d}B_s(\omega)$ means a t-continuous version of the integral.

Properties of the Itô Integral

Corollary (Itô integral is a martingale)

Let $f \in \mathcal{V}(0,T)$ for all T. Then

$$M_t(\omega) = \int_0^t f(s,\omega)\mathrm{d}B_s(\omega)$$

is a martingale with respect to $\mathcal{F}_t$ and for all $\lambda, T > 0$ we have

$$P\left[\sup_{0\leq t\leq T} |M_t| \geq \lambda\right] \leq \frac{1}{\lambda^2} E\left[\int_0^T f(s,\omega)^2 \mathrm{d}s\right] .$$

- The result above follows from:
  - the proof that I_n is a martingale for all $n \in \mathbb{N}$.
  - the a.s. continuity of M_t.
  - the Doob's martingale inequality.
  - the Itô isometry.

Extensions of the Itô Integral

- Recall that we have defined the Itô integral for a class $\mathcal{V}$ of functions $f : [0, \infty) \times \Omega \to \mathbb{R}$ satisfying the following conditions:
  (i) $(t, \omega) \to f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$ measurable.
  (ii) $f(t, \omega)$ is $\mathcal{F}_t$ adapted.
  (iii) $E\left[\int_S^T (f(t,\omega))^2 \mathrm{d}t\right] < \infty$.
- We will now discuss possible relaxations to conditions (ii) and (iii).

Extensions of the Itô Integral

- Condition (ii) in the definition of $\mathcal{V}$ can be relaxed to the following condition
  (ii)' There exists an increasing family of σ-algebras $\mathcal{H}_t,\ t \geq 0$ such that:
    a) B_t is a martingale with respect to $\mathcal{H}_t$
    b) $f(t, \omega)$ is $\mathcal{H}_t$ adapted.
- Note that:
  - condition a) implies that $\mathcal{F}_t \subset \mathcal{H}_t$.
  - this extension allows f to depend on more than $\mathcal{F}_t$ as long as B_t remains a martingale with respect to the "history" of $f_s,\ s \leq t$.
  - if condition (ii)' holds, then $E[B_s - B_t | \mathcal{H}_t] = 0$ for all $s > t$ and this condition is sufficient to carry out the construction of the Itô integral that we have seen previously.

Extensions of the Itô Integral

Example (One example where condition (ii)' applies but (ii) does not)

Denote by $B_k(t,\omega)$ the k-th coordinate of n-dimensional Brownian motion. Let $\mathcal{F}_t^{(n)}$ be the σ-algebra generated by

$$B_1(s_1,\cdot),...,B_n(s_n,\cdot)\ ,\ s_k \leq t \text{ for } k=1,...,n\ .$$

Then $B_k(t,\omega)$ is a martingale with respect to $\mathcal{F}_t^{(n)}$. If we take $\mathcal{H}_t = \mathcal{F}_t^{(n)}$ we are now able to define

$$\int_0^t f(s,\omega)\mathrm{d}B_k(s,\omega)$$

for $\mathcal{H}_t$ adapted integrands $f(t,\omega)$. This includes integrals like

$$\int B_2 \mathrm{d}B_1 \qquad \text{and} \quad \int \sin(B_1^2+B_2^2)\mathrm{d}B_2\ .$$

involving several components of n-dimensional Brownian motion.

Extensions of the Itô Integral

Definition (Multi-dimensional Itô integral)

Let $B = (B_1, ..., B_n)$ be the n-dimensional Brownian motion.
Denote by $\mathcal{V}_{\mathcal{H}}^{m\times n}(S,T)$ the set of $m \times n$ matrices $v = [v_{ij}(t,\omega)]$ where each entry $v_{ij}(t,\omega)$ satisfies conditions (i) and (iii) and (ii)' with respect to some filtration $\mathcal{H} = \{\mathcal{H}_t\}_{t\geq 0}$.
If $v \in \mathcal{V}_{\mathcal{H}}^{m\times n}(S,T)$ we define

$$\int_S^T v\mathrm{d}B = \int_S^T \begin{pmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & & \vdots \\ v_{m1} & \cdots & v_{mn} \end{pmatrix} \begin{pmatrix} \mathrm{d}B_1 \\ \vdots \\ \mathrm{d}B_n \end{pmatrix}$$

to be the $m \times 1$ matrix whose i-th component is the following sum of 1-dimensional Itô integrals:

$$\sum_{j=1}^{n} \int_S^T v_{ij}(s,\omega)\mathrm{d}B_j(s,\omega) \ .$$

Extensions of the Itô Integral

- A second extension of the Itô integral consists of weakening condition (iii) to the following:

(iii)' $P\left[\int_S^T f(s,\omega)^2 \mathrm{d}s < \infty\right] = 1.$

Definition

We denote by $\mathcal{W}_{\mathcal{H}}(S,T)$ the class of stochastic processes $f(t,\omega) \in \mathbb{R}$ satisfying conditions (i), (ii)' and (iii)' with respect to some filtration $\mathcal{H}$.
Similarly to the notation for $\mathcal{V}$, we write $\mathcal{W}_{\mathcal{H}}^{m\times n}(S,T)$ in the matrix case.

Extensions of the Itô Integral

- Let B_t denote 1-dimensional Brownian motion:
  - If $f \in \mathcal{W}_{\mathcal{H}}$ we can show that there exist a sequence of step functions $\{f_n\} \in \mathcal{W}_{\mathcal{H}}$ such that

$$\int_0^t |f_n - f|^2 \mathrm{d}s \to 0 \quad \text{as } n \to \infty \text{ in probability.}$$

  - For such a sequence we get that $\int_0^t f_n(s,\omega)\mathrm{d}B_s$ converges in probability to some random variable and the limit depends only on f, not on $\{f_n\}$.
  - Thus, we can define

$$\int_0^t f(s,\omega)\mathrm{d}B_s(\omega) = \lim_{n\to\infty} \int_0^t f_n(s,\omega)\mathrm{d}B_s(\omega) \quad \text{in probability}$$

    for all $f \in \mathcal{W}_{\mathcal{H}}$.
  - There exists a t-continuous version of this integral.
  - This integral is not in general a martingale, but rather a local martingale.

Extensions of the Itô Integral

Let $(\Omega, \mathcal{F})$ be a measurable space equipped with a filtration $\{\mathcal{F}_t\}$.

Definition (Random time)

A *random time* T is an $\mathcal{F}$ measurable random variable with values in $[0, \infty]$.

Definition (Stopping time)

A random time T is a stopping time of the filtration $\mathcal{F}_t$ if the event $\{\omega : T(\omega) < t\}$ is in $\mathcal{F}_t$ for every $t \geq 0$.

Definition (Local martingale)

Let $X = \{X_t, t \geq 0\}$ be an adapted stochastic process with respect to the filtration $\mathcal{F}_t$.
We say that X is a *local martingale* with respect to the filtration $\mathcal{F}_t$ if there exists a non-decreasing sequence of $\mathcal{F}_t$-stopping times $\{T_n\}$ such that

$$T_n \to \infty \quad \text{a.s. as } n \to \infty$$

and $X_{t \wedge T_n}$ is an $\mathcal{F}_t$-martingale for all n, where $x \wedge y = \min\{x, y\}$.

Comparison with the Stratonovich integral

- Let us consider again the stochastic differential equation

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t \tag{3}$$

or, equivalently

$$X_t = X_0 + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}B_s \ . \tag{4}$$

- We have seen that the Itô integral is one of several reasonable choices to define the $\int_0^t \sigma(s, X_s)\mathrm{d}B_s$.
- One can raise the following question:
  "which interpretation of the stochastic integral makes (4) the right mathematical model for equation (3)?"

Comparison with the Stratonovich integral

- The Stratonovich interpretation may be the most appropriate in some situations:
  - Choose t-continuously differentiable processes $B_t^{(n)}$ such that for a.a. $\omega \in \Omega$

    $$B^{(n)}(t,\omega) \to B(t,\omega) \quad \text{as } n \to \infty$$

    uniformly in t in bounded intervals.
  - For each ω let $X_t^{(n)}(\omega)$ be the solution of the corresponding deterministic differential equation

    $$\frac{\mathrm{d}X_t}{\mathrm{d}t} = b(t,X_t) + \sigma(t,X_t)\frac{\mathrm{d}B_t^{(n)}}{\mathrm{d}t} \ .$$

  - Then $X_t^{(n)}$ converges to some function $X_t(\omega)$ uniformly in t in bounded intervals for a.a. ω.
  - It turns out that X_t coincides with the solutions of the stochastic differential equation obtained using the Stratonovich integral.

Comparison with the Stratonovich integral

- However, the specific feature of the Itô formulation of "not looking into the future" seems to be a reason for its choice in many cases such as applications to biology, finances and economics.
- In any case, there is an explicit connection between the two formulations:

$$X_t = X_0 + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}B_s \quad \text{(Stratonovich)}$$

  is equivalent to

$$X_t = X_0 + \int_0^t b(s, X_s)\mathrm{d}s + \frac{1}{2}\int_0^t \sigma'(s, X_s)\sigma(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}B_s \ ,$$

  where σ' denotes the derivative of $\sigma(t, x)$ with respect to x.
- Therefore, for many purposes it is enough to do the general mathematical treatment for one of the two types of integrals.

Comparison with the Stratonovich integral

- Chain rule:
  - Stratonovich integral leads to ordinary chain rules formulas under transformations.
  - Itô transformation formula has second order correction terms.
- Martingale property:
  - Stratonovich integrals are not martingales.
  - We have seen that Itô integrals are martingales – this property is an important computational advantage.