Construction of the Itô Integral

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Some references

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Background and notation

Definition (Measurable space)

A *measurable space* is a pair $(\Omega, \mathcal{F})$, where Ω is a set and $\mathcal{F}$ is a collection of subsets of Ω with a σ-algebra structure, i.e.:

- $\emptyset \in \mathcal{F}$
- $\mathcal{F}$ is closed under complementation and countable unions.

Definition (Measure space and probability space)

A *measure space* is a triple $(\Omega, \mathcal{F}, \mu)$, where $(\Omega, \mathcal{F})$ is a measurable space and μ is a *measure* on $(\Omega, \mathcal{F})$, i.e.:

- $\mu(\emptyset) = 0$;
- $\mu(A) \geq 0$ for all $A \in \mathcal{F}$;
- if $\{A_i\}_{i \in I}$ is a countable collection of pairwise disjoint elements of $\mathcal{F}$ then $\mu(\cup_i A_i) = \sum_i \mu(A_i)$.

A *probability measure* is a measure with total measure one, i.e. $\mu(\Omega) = 1$.
A *probability space* is a measure space with a probability measure.

Background and notation

Definition (Stochastic process)

A *stochastic process* is a collection of random variables $X = \{X_t; 0 \leq t < \infty\}$ on a measurable space $(\Omega, \mathcal{F})$, which takes values on a second measurable space $(\Pi, \mathcal{G})$.

- $(\Omega, \mathcal{F})$ is called the *sample space*.
- $(\Pi, \mathcal{G})$ is called the *state space*. We take it to be $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.
- For a fixed sample point $\omega \in \Omega$, the function $t \rightarrow X_t(\omega); t \geq 0$ is the *sample path* of the process X associated with ω.

Definition (Measurable stochastic process)

The stochastic process X is called *measurable* if for every $A \in \mathcal{B}(\mathbb{R}^d)$ the set $\{(t, \omega) : X_t(\omega) \in A\}$ belongs to $\mathcal{B}([0, \infty)) \otimes \mathcal{F}$, i.e.

$$X_t(\omega) : ([0, \infty] \times \Omega, \mathcal{B}([0, \infty)) \otimes \mathcal{F}) \rightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$$

is measurable.

Background and notation

- We assume that our sample space $(\Omega, \mathcal{F})$ is equipped with a filtration.

Definition (Filtration)

A *filtration* on a measurable space $(\Omega, \mathcal{F})$ is a nondecreasing family $\{\mathcal{F}_t; t \geq 0\}$ of sub-σ-algebras of $\mathcal{F}$, i.e. $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $0 \leq s < t < \infty$.

- For a given stochastic process, the simplest choice of a filtration is the one generated by the process itself, $\mathcal{F}_t^X = \sigma(X_s; 0 \leq s \leq t)$, the smallest σ-algebra with respect to which X_s is measurable for every $s \in [0, t]$.

Definition (Adapted stochastic process)

The stochastic process X is *adapted* to the filtration $\{\mathcal{F}_t\}$ if, for every $t \geq 0$, X_t is an $\mathcal{F}_t$-measurable random variable.

- Every stochastic process X is adapted to $\{\mathcal{F}_t^X\}$.

Background and notation

Definition (standard, one-dimensional Brownian motion)

A *standard, one-dimensional Brownian motion* is a continuous, adapted process $B = \{B_t, \mathcal{F}_t, 0 \leq t < \infty\}$, defined on some probability space $(\Omega, \mathcal{F}, P)$, with the following properties:

- $B_0 = 0$ a.s.;
- for $0 \leq s < t$, the increment $B_t - B_s$ is independent of $\mathcal{F}_s$;
- for $0 \leq s < t$, the increment $B_t - B_s$ is normally distributed with mean zero and variance $t - s$.

Analogously, we can define a Brownian motion $B = \{B_t, \mathcal{F}_t, 0 \leq t < T\}$ on $[0, T]$, for some $T > 0$.

- If B is a Brownian motion and $0 = t_0 < t_1 < \ldots < t_n < \infty$, then the increments $\{B_{t_j} - B_{t_{j-1}}\}_{j=1}^n$ are independent and the distribution of $B_{t_j} - B_{t_{j-1}}$ depends on t_j and t_{j-1} only through the difference $t_j - t_{j-1}$: it is normal with mean zero and variance $t_j - t_{j-1}$.
    - We say that B has stationary, independent increments.

Background and notation

Definition (d-dimensional Brownian motion with initial distribution μ)

Let d be a positive integer and μ a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.
Let $B = \{B_t, \mathcal{F}_t : t \geq 0\}$ be a continuous, adapted process with values in $\mathbb{R}^d$, defined on some probability space $(\Omega, \mathcal{F}, P)$. This process is called a *d-dimensional Brownian motion with initial distribution μ*, if

- $P[B_0 \in \Gamma] = \mu(\Gamma)$, for all $\Gamma \in \mathcal{B}(\mathbb{R}^d)$,
- for $0 \leq s < t$, the increment $B_t - B_s$ is independent of $\mathcal{F}_s$,
- for $0 \leq s < t$, the increment $B_t - B_s$ is normally distributed with mean zero and covariance matrix equal to $(t-s)I_d$, where I_d denotes the $d \times d$ identity matrix.

If μ assigns measure one to some singleton $\{x\}$, we say that B is a *d-dimensional Brownian motion starting at x*.

Background and notation

- Some useful Brownian motion properties:
  - One-dimensional Brownian motion is a zero-mean Gaussian process with covariance function $\rho(s,t) = \min\{s,t\}$.
  - For almost every $\omega \in \Omega$, the sample path $W_{\cdot}(\omega)$ is of unbounded variation on every finite interval $[0,t]$.
  - For almost every $\omega \in \Omega$, the quadratic variation of the sample path $W_{\cdot}(\omega)$ on $[0,t]$ converges to t in L^2.
  - For almost every $\omega \in \Omega$, the Brownian sample path $W_{\cdot}(\omega)$ is nowhere differentiable.

Motivation

- Suppose we are given a differential equation of the form

$$\begin{cases} \mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t \\ X(0) = X_0 \end{cases} .$$

- We would like to give a precise meaning to the expression

$$X_t = X_0 + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}B_s \ .$$

  - Thus, we need to give a proper definition for the *stochastic integral*

$$\int_0^t f(s,\omega)\mathrm{d}B_s(\omega) \ ,$$

    where B_t is a one-dimensional Brownian motion and $f : [0,\infty) \times \Omega \to \mathbb{R}$ belongs to a wide class of functions.

Motivation

- Assume $0 \leq S < T$ and $f(t,\omega)$ are given. We want to define

$$\int_S^T f(t,\omega)\mathrm{d}B_t(\omega) \ .$$

- We start with a definition for a simple class of functions f and extend it by an approximation procedure.
  - Assume that f has the form

$$\phi(t,\omega) = \sum_{j\geq 0} e_j(\omega).I_{[j2^{-n},(j+1)2^{-n})}(t) \ ,$$

    where I denotes the characteristic function and n is a natural number.
  - For this class of functions we can define

$$\int_S^T f(t,\omega)\mathrm{d}B_t(\omega) = \sum_{j\geq 0} e_j(\omega)[B_{t_{j+1}} - B_{t_j}](\omega) \ ,$$

    where

$$t_k = t_k^{(n)} = \begin{cases} k2^{-n} & \text{if } S \leq k2^{-n} \leq T \\ S & \text{if } k2^{-n} < S \\ T & \text{if } k2^{-n} > T \end{cases}$$

Motivation

- However, without further assumptions on the functions $e_j(\omega)$ the definition above may lead to difficulties, as the next example shows.

Example

Take the following two approximations to $f(t,\omega) = B_t(\omega)$:

$$\begin{aligned}\phi_1(t,\omega) &= \sum_{j\geq 0} B_{j2^{-n}}(\omega).I_{[j2^{-n},(j+1)2^{-n})}(t) \\ \phi_2(t,\omega) &= \sum_{j\geq 0} B_{(j+1)2^{-n}}(\omega).I_{[j2^{-n},(j+1)2^{-n})}(t) \ .\end{aligned}$$

We would like both choices to provide a reasonable approximation for the stochastic integral

$$\int_S^T f(t,\omega)\mathrm{d}B_t(\omega) \ .$$

Motivation

Example

However, we have that

$$E\left[\int_0^T \phi_1(t,\omega)\mathrm{d}B_t(\omega)\right] = \sum_{j\geq 0} E\left[B_{t_j}(B_{t_{j+1}} - B_{t_j})\right] = 0$$

but

$$\begin{aligned} E\left[\int_0^T \phi_2(t,\omega)\mathrm{d}B_t(\omega)\right] &= \sum_{j\geq 0} E\left[B_{t_{j+1}}(B_{t_{j+1}} - B_{t_j})\right] \\ &= \sum_{j\geq 0} E\left[(B_{t_{j+1}} - B_{t_j})^2\right] = T \ . \end{aligned}$$

Motivation

- Despite the fact that both ϕ_1 and ϕ_2 seem to be good approximations to $B_t(\omega)$, their integrals as defined above are not close, no matter how large n is chosen.
  - This example reflects the fact that the variations of the paths of B_t are too big to enable us to define the integral in the Riemann-Stieltjes sense.
- In general it is natural to approximate a given function $f(t,\omega)$ by

$$\sum_j f(t_j^*,\omega) I_{[t_j,t_{j+1})}(t) ,$$

  where the points t_j^* belong to the intervals $[t_j, t_{j+1})$, and then define $\int_S^T f(t,\omega)\mathrm{d}B_t(\omega)$ as the limit of $\sum_j f(t_j^*,\omega)[B_{t_{j+1}} - B_{t_j}](\omega)$ as $n \to \infty$.
  - The previous example shows that, unlike the Riemann-Stieltjes integral, in our case it does make a difference what point t_j^* we choose:
    - ★ The choice $t_j^* = t_j$ (the left end point) leads to the Itô Integral.
    - ★ The choice $t_j^* = (t_j + t_{j+1})/2$ (the mid point) leads to the Stratonovich Integral.

Construction of the Itô Integral

- Therefore, we must restrict ourselves to a special class of functions $f(t,\omega)$ (even in the case where they are as simple as above) to obtain a reasonable definition of the integral.
  - The approximation procedure leading to the Itô Integral (left end point) will work out successfully provided f is such that each of the functions $\omega \to f(t_j,\omega)$ only depends on the behaviour of $B_s(\omega)$ up to time t_j.

Definition

Denote by $\{\mathcal{F}_t\}_t$ the filtration generated by the one-dimensional Brownian motion B_t and by $\mathcal{B}$ the Borel σ-algebra on $[0,\infty)$.
Let $\mathcal{V} = \mathcal{V}(S,T)$ be the class of functions $f : [0,\infty) \times \Omega \to \mathbb{R}$ such that

(i) $(t,\omega) \to f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$ measurable.

(ii) $f(t,\omega)$ is $\mathcal{F}_t$ adapted.

(iii) $E\left[\int_S^T (f(t,\omega))^2 \mathrm{d}t\right] < \infty$.

Construction of the Itô Integral

- For functions $f \in \mathcal{V}$ we will define the Itô integral

$$\mathcal{I}[f](\omega) = \int_S^T f(t, \omega) \mathrm{d}B_t(\omega) \ ,$$

  where B_t is a one-dimensional Brownian motion.
- We use the following approach:
  1) We define $\mathcal{I}[\phi]$ for a simple class of functions ϕ.
  2) We show that each $f \in \mathcal{V}$ can be approximated in an appropriate sense by some function ϕ.
  3) We use this approximation to define $\int f \mathrm{d}B$ as the limit of $\int \phi \mathrm{d}B$ as $\phi \to f$.

Construction of the Itô Integral

Definition (Elementary function)

A function $\phi \in \mathcal{V}$ is called elementary if it has the form

$$\phi(t,\omega) = \sum_j e_j(\omega) I_{[t_j,t_{j+1})}(t) \ .$$

- Note that since $\phi \in \mathcal{V}$ each function e_j must be $\mathcal{F}_{t_j}$-measurable
  - Remember the example: ϕ_1 was elementary while ϕ_2 was not.
- For elementary functions $\phi(t,\omega)$ we define the integral as was done previously:

$$\int_S^T \phi(t,\omega) \mathrm{d}B_t(\omega) = \sum_{j \geq 0} e_j(\omega)[B_{t_{j+1}} - B_{t_j}](\omega) \ ,$$

Construction of the Itô Integral

Lemma (The Itô isometry)

If $\phi(t,\omega)$ is bounded and elementary then

$$E\left[\left(\int_S^T \phi(t,\omega)\mathrm{d}B_t(\omega)\right)^2\right] = E\left[\int_S^T (\phi(t,\omega))^2\,\mathrm{d}t\right] .$$

Proof.

Let $\Delta B_j = B_{t_{j+1}} - B_{t_j}$. Then

$$E[e_i e_j \Delta B_i \Delta B_j] = \begin{cases} 0 & \text{if } i \neq j \\ E[e_j^2](t_{j+1} - t_j) & \text{if } i = j \end{cases} .$$

using independence of $e_i e_j \Delta B_i$ and ΔB_j if $i < j$. Thus

$$E\left[\left(\int_S^T \phi \mathrm{d}B\right)^2\right] = \sum_{i,j} E[e_i e_j \Delta B_i \Delta B_j] = \sum_j E[e_j^2](t_{j+1} - t_j) = E\left[\int_S^T \phi^2 \mathrm{d}t\right] .$$

Construction of the Itô Integral

- We now use the Itô isometry to extend the definition from elementary functions to functions on $\mathcal{V}$.
- We need the following two results.

Lemma (Dominated convergence theorem)

Suppose $\{f_n\}_n$ is a sequence of measurable functions on a measure space $(\Omega, \mathcal{F}, \mu)$ such that

$$f(x) = \lim_{n \to \infty} f_n(x) \ , \quad \mu \text{ a.e. } x \in \Omega \ .$$

If there is an integrable function g, i.e. $\int_\Omega g \mathrm{d}\mu < \infty$, such that

$$|f_n(x)| \leq g(x) \ , \qquad \text{for } \mu \text{ a.e. } x \in \Omega \text{ and for all } n \in \mathbb{N}$$

then f is integrable and

$$\lim_{n \to \infty} \int_\Omega f_n \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\mu \ .$$

Construction of the Itô Integral

Lemma (Bounded convergence theorem)

Let $\{f_n\}_n$ be a sequence of uniformly bounded and measurable functions on a bounded measure space $(\Omega, \mathcal{F}, \mu)$ such that

$$f(x) = \lim_{n\to\infty} f_n(x) \ , \quad \mu \ a.e. \ x \in \Omega \ .$$

Then, f is integrable and

$$\lim_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\mu \ .$$

Construction of the Itô Integral

Lemma (Lemma 1)

Let $g \in \mathcal{V}$ be bounded and $g(\cdot, \omega)$ continuous for each ω. Then there exist elementary functions $\phi_n \in \mathcal{V}$ such that

$$E\left[\int_S^T (g - \phi_n)^2 \mathrm{d}t\right] \to 0 \quad \text{as } n \to \infty \, .$$

Proof.

Define $\phi_n(t, \omega) = \sum_j g(t_j, \omega) I_{[t_j, t_{j+1})}(t)$.
Then ϕ_n is elementary since $g \in \mathcal{V}$, and, for each ω

$$\int_S^T (g - \phi_n)^2 \mathrm{d}t \to 0 \quad \text{as } n \to \infty \, ,$$

since $g(\cdot, \omega)$ is continuous for each ω.
Hence $E\left[\int_S^T (g - \phi_n)^2 \mathrm{d}t\right] \to 0$ as $n \to \infty$ by bounded convergence. □

Construction of the Itô Integral

Lemma (Lemma 2)

Let $h \in \mathcal{V}$ be bounded.
Then there exist bounded functions $g_n \in \mathcal{V}$ such that $g_n(\cdot, \omega)$ is continuous for all ω and n, and

$$E\left[\int_S^T (h - g_n)^2 \mathrm{d}t\right] \to 0 \quad \textit{as } n \to \infty \ .$$

Proof.

Suppose $|h(t,\omega)| \leq M$ for all (t,ω).
For each n let ψ_n be a non-negative, continuous function on $\mathbb{R}$ such that

(i) $\psi_n(x) = 0$ for $x \leq -1/n$ and $x \geq 0$

(ii) $\int_{-\infty}^{+\infty} \psi_n(x) \mathrm{d}x = 1$.

Let us define

$$g_n(t,\omega) = \int_0^t \psi_n(s-t) h(s,\omega) \mathrm{d}s \ .$$

Clearly, $g_n(\cdot,\omega)$ is continuous for each ω and $|g_n(t,\omega)| \leq M$.

Construction of the Itô Integral

Proof.

Since $h \in \mathcal{V}$, $g_n(t, \cdot)$ is $\mathcal{F}_t$-measurable for all t.
Moreover,

$$\int_S^T (h(s,\omega) - g_n(s,\omega))^2 \mathrm{d}s \to 0 \quad \text{as } n \to \infty, \text{ for each } \omega \ ,$$

By bounded convergence we get

$$E\left[\int_S^T (h(s,\omega) - g_n(s,\omega))^2 \mathrm{d}s\right] \to 0 \ , \quad \text{as } n \to \infty \ .$$

□

Construction of the Itô Integral

Lemma (Lemma 3)

Let $f \in \mathcal{V}$. There exists a sequence of bounded functions $\{h_n\} \subset \mathcal{V}$ such that

$$E\left[\int_S^T (f - h_n)^2 \mathrm{d}t\right] \to 0 \quad \text{as } n \to \infty \ .$$

Proof.

Let us define

$$h_n(t,\omega) = \begin{cases} -n & \text{if } f(t,\omega) < -n \\ f(t,\omega) & \text{if } -n < f(t,\omega) < n \\ n & \text{if } f(t,\omega) > n \end{cases} .$$

Then h_n is bounded for each $n \in \mathbb{N}$ and

$$\int_S^T (f(s,\omega) - h_n(s,\omega))^2 \mathrm{d}s \to 0 \quad \text{as } n \to \infty, \text{ for each } \omega \ .$$

The result then follows by dominated convergence. □

Construction of the Itô Integral

- We can now complete the definition of the Itô integral

$$\int_S^T f(t,\omega)\mathrm{d}B_t(\omega) \ , \quad \text{for } f \in \mathcal{V} \ .$$

  - If $f \in \mathcal{V}$ by Lemmas 1 to 3 we can choose elementary functions $\phi_n \in \mathcal{V}$ such that

$$E\left[\int_S^T (f - \phi_n)^2 \mathrm{d}t\right] \to 0 \quad \text{as } n \to \infty \ .$$

  - We can then define

$$\mathcal{I}[f](\omega) = \int_S^T f(t,\omega)\mathrm{d}B_t(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega)\mathrm{d}B_t(\omega)$$

  - The limit above exists as an element of $L^2(P)$ since

$$\left\{\int_S^T \phi_n(t,\omega)\mathrm{d}B_t(\omega)\right\}$$

    is a Cauchy sequence in $L^2(P)$ by the Itô Isometry.

The Itô Integral

Definition (The Itô integral)

Let $f \in \mathcal{V}(S,T)$. Then the *Itô integral* of f (from S to T) is defined by

$$\int_S^T f(t,\omega)\mathrm{d}B_t(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega)\mathrm{d}B_t(\omega) \quad \text{limit in } L^2(P) \ , \tag{1}$$

where $\{\phi_n\}$ is a sequence of elementary functions such that

$$E\left[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2 \mathrm{d}t\right] \to 0 \quad \text{as } n \to \infty \ . \tag{2}$$

- Note that:
  - such a sequence $\{\phi_n\}$ exists by Lemmas 1 to 3.
  - the limit in (1) exists and does not depend on the choice of $\{\phi_n\}$, as long as (2) holds

The Itô Integral

We easily obtain the following two consequences from the construction above.

Corollary (The Itô isometry)

$$E\left[\left(\int_S^T f(t,\omega)\mathrm{d}B_t(\omega)\right)^2\right] = E\left[\int_S^T (f(t,\omega))^2 \,\mathrm{d}t\right] \quad \textit{for all } f \in \mathcal{V}(S,T).$$

Corollary

If $f(t,\omega) \in \mathcal{V}(S,T)$, $f_n(t,\omega) \in \mathcal{V}(t,\omega)$ *for* $n = 1, 2, \ldots$ *and*

$$E\left[\int_S^T (f_n(t,\omega) - f(t,\omega))^2 \mathrm{d}t\right] \to 0 \quad \textit{as } n \to \infty \ ,$$

then

$$\int_S^T f_n(t,\omega)\mathrm{d}B_t(\omega) \to \int_S^T f(t,\omega)\mathrm{d}B_t(\omega) \quad \textit{in } L^2(P) \textit{ as } n \to \infty.$$

The Itô Integral

Example

Assume $B_0 = 0$. Then

$$\int_0^t B_s \mathrm{d}B_s = \frac{1}{2}B_t^2 - \frac{1}{2}t \ .$$

To prove this we consider the sequence of elementary functions

$$\phi_n(t,\omega) = \sum_j B_j(\omega) I_{[t_j,t_{j+1})}(t),$$

where $B_j = B_{t_j}$. Then

$$\begin{aligned} E\left[\int_0^t (\phi_n - B_s)^2 \, \mathrm{d}s\right] &= E\left[\sum_j \int_{t_j}^{t_{j+1}} (B_j - B_s)^2 \, \mathrm{d}s\right] \\ &= \sum_j \int_{t_j}^{t_{j+1}} (s - t_j) \, \mathrm{d}s \\ &= \sum_j \frac{1}{2} (t_{j+1} - t_j)^2 \to 0 \quad \text{as } \Delta t_j \to 0. \end{aligned}$$

The Itô Integral

Example

By the previous corollary, we get that

$$\int_0^t B_s \mathrm{d}B_s = \lim_{\Delta t_j \to 0} \int_0^t \phi_n \mathrm{d}B_s = \lim_{\Delta t_j \to 0} \sum_j B_j \Delta B_j \ .$$

We now note that

$$\Delta(B_j^2) = B_{j+1}^2 - B_j^2 = (B_{j+1} - B_j)^2 + 2B_j(B_{j+1} - B_j) = (\Delta B_j)^2 + 2B_j \Delta B_j \ ,$$

and therefore

$$B_t^2 = \sum_j \Delta(B_j^2) = \sum_j (\Delta B_j)^2 + 2 \sum_j B_j \Delta B_j \ ,$$

that is

$$\sum_j B_j \Delta B_j = \frac{1}{2} B_t^2 - \frac{1}{2} \sum_j (\Delta B_j)^2 \ .$$

Noting that $\sum_j (\Delta B_j)^2 \to t$ in $L^2(P)$ as $\Delta t_j \to 0$, we obtain the result.

The Itô Integral

- Next time:
  - will state several properties of the Itô integral, namely the martingale property.
  - will study possible changes to the assumptions made in the set $\mathcal{V}$ in order to generalize the concept of Itô integral to a broader class.
  - will extend the notions above to multi-dimensional Itô integrals.
  - will (very briefly) discuss the Stratonovich integral.