Constructions of Brownian Motion II

Based on the book
"Brownian motion and Stochastic Calculus"
by I. Karatzas and S. Shreve

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Third construction of Brownian motion

- The sample spaces for the Brownian motions in the two previous constructions were, respectively:
  - the space $\mathbb{R}^{[0,\infty)}$ of all real-valued functions on the half-line;
  - a space Ω rich enough to carry a countable collection of independent standard normal random variables.
- The "canonical" space for Brownian motion is $C[0,\infty)$, the space of all continuous real-valued functions on the half-line with metric

$$\rho(\omega_1,\omega_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0\le t\le n} \{|\omega_1(t) - \omega_2(t)| \wedge 1\} \ ,$$

where $a \wedge b$ is used to denote $\min\{a, b\}$.

  - Under the metric ρ, $C[0,\infty)$ is a complete, separable metric space.

Third construction of Brownian motion

- Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with values in a measurable space $(S, \mathcal{B}(S))$, i.e. $X : \Omega \to S$ is $\mathcal{F}/\mathcal{B}(S)$-measurable.
  - Then X induces a probability measure PX^{-1} on $(S, \mathcal{B}(S))$ by

$$PX^{-1}(B) = P[\omega \in \Omega : X(\omega) \in B] \ , \ B \in \mathcal{B}(S) \ .$$

- When $X = \{X_t : 0 \leq t < \infty\}$ is a continuous stochastic process on $(\Omega, \mathcal{F}, P)$, we can think of X as a random variable $(\Omega, \mathcal{F}, P)$ with values in $(C[0, \infty), \mathcal{B}(C[0, \infty)))$
  - PX^{-1} is called the *law of* X.

Third construction of Brownian motion

Definition (weak convergence)

Let (S, ρ) be a metric space with Borel σ-algebra $\mathcal{B}(S)$.
Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of probability measures on $(S, \mathcal{B}(S))$, and let P be another measure on this space.
We say that $\{P_n\}_{n=1}^{\infty}$ *converges weakly* to P and write $P_n \to P$ if and only if

$$\lim_{n\to\infty} \int_S f(s)\mathrm{d}P_n(s) = \int_S f(s)\mathrm{d}P(s)$$

for every bounded, continuous real-valued function f on S.

- The weak limit P is a probability measure and it is unique.

Third construction of Brownian motion

Definition (convergence in distribution)

Let $\{(\Omega_n, \mathcal{F}_n, P_n)\}_{n=1}^{\infty}$ be a sequence of probability spaces, and on each of them consider a random variable X_n with values in the metric space (S, ρ).
Let $(\Omega, \mathcal{F}, P)$ be another probability space, on which a random variable X with values in (S, ρ) is given.

We say that $\{X_n\}_{n=1}^{\infty}$ *converges to X in distribution*, and write $X_n \xrightarrow{\mathcal{D}} X$, if the sequence of measures $\{P_n X_n^{-1}\}_{n=1}^{\infty}$ converges weakly to the measure PX^{-1}.

- Equivalently, $X_n \xrightarrow{\mathcal{D}} X$ if and only if

$$\lim_{n\to\infty} E_n f(X_n) = Ef(x)$$

  for every bounded, continuous real-valued function f on S, where E_n and E denote expectations with respect to P_n and P, respectively.
- If $S = \mathbb{R}^d$, then $X_n \xrightarrow{\mathcal{D}} X$ if and only if the sequence of characteristic functions $\phi_n(u) = E_n[\exp\{\mathrm{i}(u, X_n)\}]$ converges to $\phi(u) = E[\exp\{\mathrm{i}(u, X)\}]$ for every $u \in \mathbb{R}^d$.

Third construction of Brownian motion

- The most important example of convergence in distribution is provided by the central limit theorem:
  - if $\{\xi_n\}_{n=1}^{\infty}$ is a sequence of iid random variables with mean zero and variance σ^2, then $\{S_n\}$ defined by
$$S_n = \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{n} \xi_k$$
    converges in distribution to a standard normal random variable.
  - It is this fact that dictates that a properly normalized sequence of random walks will converge in distribution to Brownian motion.

Third construction of Brownian motion

Definition (relatively compact and tight family of probability measures)

Let (S, ρ) be a metric space and let Π be a family of probability measures on $(S, \mathcal{B}(S))$.

- We say that Π is *relatively compact* if every sequence of elements of Π contains a weakly convergent subsequence.
- We say that Π is *tight* if for every $\epsilon > 0$, there exists a compact set $K \subseteq S$ such that $P(K) \geq 1 - \epsilon$, for every $P \in \Pi$.

- Let $\{X_\alpha\}_{\alpha \in A}$ be a family of random variables, each one defined on a probability space $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$ and taking values in S.
  We say that this family is *relatively compact* or *tight* if the family of induced measures $P_\alpha X_\alpha^{-1}{}_{\alpha \in A}$ has the appropriate property.

Third construction of Brownian motion

Theorem (Prohorov (1956))

Let Π be a family of probability measures on a complete, separable metric space S. This family is relatively compact if and only if is tight.

- We are interested in the case $S = C[0, \infty)$, for which we have another characterization of tightness.
- The special case $S = \mathbb{R}$ can be used to prove the central limit theorem.

Third construction of Brownian motion

Definition (modulus of continuity)

For each $\omega \in C[0,\infty)$, $T > 0$ and $\delta > 0$ the *modulus of continuity* on $[0, T]$:

$$m^T(\omega,\delta) = \max_{|s-t|\leq\delta,\ 0\leq s,t\leq T} |\omega(s) - \omega(t)| \ .$$

Theorem (Theorem A)

A sequence $\{P_n\}_{n=1}^{\infty}$ *of probability measures on* $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ *is tight if and only if*

$$\begin{aligned} \lim_{\lambda\uparrow\infty} \sup_{n\geq1} P_n[\omega : |\omega(0)| > \lambda] &= 0 \\ \lim_{\delta\downarrow 0} \sup_{n\geq1} P_n[\omega : m^T(\omega,\delta) > \epsilon] &= 0 \ , \forall T > 0, \epsilon > 0 \ . \end{aligned}$$

Third construction of Brownian motion

- Let X be a continuous process on some probability space $(\Omega, \mathcal{F}, P)$.
  - For each ω, the function $t \to X_t(\omega)$ is a member of $C[0,\infty)$, which we denote by $X(\omega)$.
  - We can then consider the random function $X : \Omega \to C[0,\infty)$.
- Let $\{X^{(n)}\}_{n=1}^{\infty}$ be a sequence of continuous processes (with each $X^{(n)}$ defined on a perhaps distinct probability space $(\Omega_n, \mathcal{F}_n, P_n)$.
  - We can ask whether $X^{(n)} \xrightarrow{\mathcal{D}} X$.
  - We can also ask whether the finite-dimensional distributions of $\{X^{(n)}\}_{n=1}^{\infty}$ converge to those of X, i.e. whether

$$(X_{t_1}^{(n)}, X_{t_2}^{(n)}, ..., X_{t_d}^{(n)}) \xrightarrow{\mathcal{D}} (X_{t_1}, X_{t_2}, ..., X_{t_d}) \ .$$

    - ⋆ This latter question is easier to answer than the former, since the convergence in distribution of finite-dimensional random vectors can be resolved by studying characteristic functions.

Third construction of Brownian motion

- For any finite subset $\{t_1, ..., t_d\}$ of $[0, \infty)$, define the *projection mapping* $\pi_{t_1,...,t_d} : C[0, \infty) \to \mathbb{R}^d$ as

$$\pi_{t_1,...,t_d}(\omega) = (\omega(t_1), ..., \omega(t_d)) \ .$$

  - If the function $f : \mathbb{R}^d \to \mathbb{R}$ is bounded and continuous, then the composite mapping $f \circ \pi_{t_1,...,t_d} : C[0, \infty) \to \mathbb{R}$ enjoys the same properties.
  - Thus, $X^{(n)} \xrightarrow{\mathcal{D}} X$ implies

$$\begin{aligned} \lim_{n\to\infty} E_n f(X_{t_1}^{(n)}, ..., X_{t_d}^{(n)}) &= \lim_{n\to\infty} E_n f \circ \pi_{t_1,...,t_d}(X^{(n)}) \\ &= Ef \circ \pi_{t_1,...,t_d}(X) = Ef(X_{t_1}, ..., X_{t_d}) \ . \end{aligned}$$

  - If the sequence of processes $\{X^{(n)}\}_{n=1}^{\infty}$ converges in distribution to the process X, then all finite-dimensional distributions converge as well.
  - The converse holds in the presence tightness.

Third construction of Brownian motion

Theorem (Theorem B)

Let $\{X^{(n)}\}_{n=1}^{\infty}$ *be a tight sequence of continuous processes with the property that, whenever* $0 \leq t_1 < \ldots < t_d < \infty$, *then the sequence of random vectors* $\{(X_{t_1}^{(n)}, \ldots, X_{t_d}^{(n)})\}_{n=1}^{\infty}$ *converges in distribution.*
Let P_n *be the measure induced on* $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ *by* $X^{(n)}$.
Then $\{P_n\}_{n=1}^{\infty}$ *converges weakly to a measure* P, *under which the coordinate mapping process* $W_t(\omega) = \omega(t)$ *on* $C[0,\infty)$ *satisfies*

$$(X_{t_1}^{(n)}, \ldots, X_{t_d}^{(n)}) \xrightarrow{\mathcal{D}} (W_{t_1}, \ldots, W_{t_d}) \ , \ 0 \leq t_1 < \ldots < t_d < \infty \ , \ d \geq 1 \ .$$

Third construction of Brownian motion

- Consider now:
  - a sequence $\{\xi_j\}_{j=1}^{\infty}$ of independent, identically distributed random variables with mean zero and variance σ^2 $(0 < \sigma^2 < \infty)$.
  - a sequence of partial sums $S_0 = 0$, $S_k = \sum_{j=0}^{k} \xi_j$, $k \geq 1$.
- We can obtain a continuous process $Y = \{Y_t : t \geq 0\}$ from the sequence $\{S_k\}_{k=0}^{\infty}$ by linear interpolation:

$$Y_t = S_{[t]} + (t - [t])\xi_{[t]+1} \ , \ t \geq 0 \ ,$$

  where $[t]$ denotes the greatest integer less than or equal to t.
- Scaling appropriately both time and space, we obtain from Y a sequence of processes $\{X^{(n)}\}$:

$$X_t^{(n)} = \frac{1}{\sigma\sqrt{n}} Y_{nt} \ , \ t \geq 0 \ .$$

Third construction of Brownian motion

- Let $s = k/n$ and $t = (k+1)/n$:
  - the increment $X_t^{(n)} - X_s^{(n)} = (1/\sigma\sqrt{n})\xi_{k+1}$ is independent of $\mathcal{F}_s^{X^{(n)}} = \sigma(\xi_1, ..., \xi_k)$.
  - $X_t^{(n)} - X_s^{(n)}$ has zero mean and variance $t - s$.
- This suggests that $\{X_t^{(n)} : t \geq 0\}$ is approximately a Brownian motion.
- In the next theorem we prove that even though the random variables ξ_j are not necessarily normal, the central limit theorem dictates that the limiting distributions of the increments of $X^{(n)}$ are normal.

Third construction of Brownian motion

Theorem (Theorem C)

For $0 \leq t_1 < \ldots < t_d < \infty$*, we have that as* $n \to \infty$

$$(X_{t_1}^{(n)}, \ldots, X_{t_d}^{(n)}) \xrightarrow{\mathcal{D}} (B_{t_1}, \ldots, B_{t_d}) \ ,$$

where $\{B_t, \mathcal{F}_t^B : t \geq 0\}$ *is a standard, one-dimensional Brownian motion.*

- We prove the result for the case $d = 2$, the general case being analogous.
- Set $s = t_1$ and $t = t_2$. We want to show that

$$(X_s^{(n)}, X_t^{(n)}) \xrightarrow{\mathcal{D}} (B_t, B_t) \ .$$

  - Since

$$\left| X_t^{(n)} - \frac{1}{\sigma\sqrt{n}} S_{[tn]} \right| \leq \frac{1}{\sigma\sqrt{n}} \left| \xi_{[tn]+1} \right| \ ,$$

    we obtain by Chebyshev inequality

$$P\left[\left| X_t^{(n)} - \frac{1}{\sigma\sqrt{n}} S_{[tn]} \right| > \epsilon \right] \leq \frac{1}{\epsilon^2 n} \ ,$$

    as $n \to \infty$.

Third construction of Brownian motion

- Therefore,

$$\left\| (X_s^{(n)}, X_t^{(n)}) - \frac{1}{\sigma\sqrt{n}}(S_{[sn]}, S_{[tn]}) \right\| \to 0$$

in probability.

Lemma (Auxiliary lemma 1)

Let $\{X^{(n)}\}_{n=1}^{\infty}$, $\{Y^{(n)}\}_{n=1}^{\infty}$, and X be random variables with values in a separable metric space (S, ρ).
Assume also that for each $n \geq 1$, $X^{(n)}$ and $Y^{(n)}$ are defined on the same probability space.
If $X^{(n)} \xrightarrow{\mathcal{D}} X$ and $\rho(X^{(n)}, Y^{(n)}) \to 0$ in probability as $n \to \infty$ then $Y^{(n)} \xrightarrow{\mathcal{D}} X$ as $n \to \infty$.

- Therefore, we need only to show that

$$\frac{1}{\sigma\sqrt{n}}(S_{[sn]}, S_{[tn]}) \xrightarrow{\mathcal{D}} (B_t, B_t) \ .$$

Third construction of Brownian motion

- But proving the convergence

$$\frac{1}{\sigma\sqrt{n}}(S_{[sn]}, S_{[tn]}) \xrightarrow{\mathcal{D}} (B_t, B_t) \ .$$

is equivalent to proving

$$\frac{1}{\sigma\sqrt{n}}\left(\sum_{j=1}^{[sn]}\xi_j, \sum_{j=[sn]+1}^{[tn]}\xi_j\right) \xrightarrow{\mathcal{D}} (B_s, B_t - B_s)$$

by the auxiliary lemma below.

Lemma (Auxiliary lemma 2)

Let $\{X^{(n)}\}_{n=1}^{\infty}$ be a sequence of random variables taking values in a metric space (S_1, ρ_1) and converging in distribution to X.
Suppose that (S_1, ρ_1) is also a metric space and let $\phi : S_1 \to S_2$ be a continuous map. Then $Y^{(n)} = \phi(X^{(n)})$ converges in distribution to $Y = \phi(X)$.

Third construction of Brownian motion

- Independence of the random variables $\{\xi_j\}_{j=1}^{\infty}$ implies

$$\lim_{n\to\infty} E\left[\exp\left\{\frac{\mathrm{i}u}{\sigma\sqrt{n}}\sum_{j=1}^{[sn]}\xi_j + \frac{\mathrm{i}v}{\sigma\sqrt{n}}\sum_{j=[sn]+1}^{[tn]}\xi_j\right\}\right]$$

$$= \lim_{n\to\infty} E\left[\exp\left\{\frac{\mathrm{i}u}{\sigma\sqrt{n}}\sum_{j=1}^{[sn]}\xi_j\right\}\right]\lim_{n\to\infty} E\left[\exp\left\{\frac{\mathrm{i}v}{\sigma\sqrt{n}}\sum_{j=[sn]+1}^{[tn]}\xi_j\right\}\right]$$

  provided both limits on the right hand side exist.

- We deal with the first limit on the right hand side, the other being similar.

Third construction of Brownian motion

- Since

$$\left| \frac{1}{\sigma\sqrt{n}} \sum_{j=1}^{[sn]} \xi_j - \frac{\sqrt{s}}{\sigma\sqrt{[sn]}} \sum_{j=1}^{[sn]} \xi_j \right| \to 0$$

  in probability and, by the central limit theorem $(\sqrt{s}/\sigma\sqrt{[sn]}) \sum_{j=1}^{[sn]} \xi_j$ converges in distribution to a normal variable with mean zero and variance s, we have

$$\lim_{n\to\infty} E\left[\exp\left\{\frac{\mathrm{i}v}{\sigma\sqrt{n}} \sum_{j=[sn]+1}^{[tn]} \xi_j\right\}\right] = \mathrm{e}^{-u^2 s/2} \ .$$

- Similarly

$$\lim_{n\to\infty} E\left[\exp\left\{\frac{\mathrm{i}u}{\sigma\sqrt{n}} \sum_{j=1}^{[sn]} \xi_j\right\}\right] = \mathrm{e}^{-v^2(t-s)/2} \ .$$

- Substitution in the equality in the previous slide completes the proof.

Third construction of Brownian motion

- In fact, the sequence $\{X^{(n)}\}$ of linearly interpolated and normalized random walks converges to Brownian motion in distribution.
- For the tightness required to carry out such an extension (see Theorem B), we need one more auxiliary result.

Lemma

Set $S_k = \sum_{j=1}^{k} \xi_j$, where $\{\xi_j\}_{j=1}^{\infty}$ is a sequence of independent, identically distributed random variables, with mean zero and finite variance $\sigma^2 > 0$. Then, for any $\epsilon \geq 0$

$$\lim_{\delta \downarrow 0} \overline{\lim}_{n\to\infty} \frac{1}{\delta} P\left[\max_{1\leq j\leq [n\delta]+1} |S_j| > \epsilon\sigma\sqrt{n}\right] = 0 \ .$$

Furthermore, for any $T > 0$

$$\lim_{\delta \downarrow 0} \overline{\lim}_{n\to\infty} P\left[\max_{1\leq j\leq [n\delta]+1,\ 1\leq k\leq [nT]+1} |S_{j+k} - S_k| > \epsilon\sigma\sqrt{n}\right] = 0 \ .$$

Third construction of Brownian motion

Theorem (The Invariance Principle of Donsker (1951))

Let $(\Omega, \mathcal{F}, P)$ be a probability space on which is given a sequence $\{\xi_j\}_{j=1}^{\infty}$ of independent, identically distributed random variables, with mean zero and finite variance $\sigma^2 > 0$.
Let $X^{(n)} = \{X_t^{(n)}\}$ be defined by

$$X_t^{(n)} = \frac{1}{\sigma\sqrt{n}} Y_{nt} \ , \ t \geq 0 \ ,$$

where

$$Y_t = S_{[t]} + (t - [t])\xi_{[t]+1} \ , \ t \geq 0 \ ,$$

$S_0 = 0$ and $S_k = \sum_{j=0}^{k} \xi_j$. Furthermore, let P_n be the measure induced by $X^{(n)}$ on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$.
Then $\{P_n\}_{n=1}^{\infty}$ converges weakly to a measure P_* under which the coordinate mapping process $W_t(\omega) = \omega(t)$ on $C[0,\infty)$ is a standard, one dimensional Brownian motion.

Third construction of Brownian motion

- Due to Theorems B and C, we only need to show that $\{X^{(n)}\}$ is tight.
  - To prove this, we use Theorem A:
    - the first condition is trivially satisfied since $X_0^{(n)} = 0$ a.s. for every n;
    - thus, we only need to establish that for arbitrary $\epsilon > 0$ and $T > 0$

$$\lim_{\delta \downarrow 0} \sup_{n \geq 1} P\left[\max_{|s-t| \leq \delta,\ 0 \leq s,t \leq T} |X_s^{(n)} - X_t^{(n)}| > \epsilon \right] = 0 \ .$$

    - We can replace $\sup_{n \geq 1}$ in the expression above by $\overline{\lim}_{n \to \infty}$ since for a finite number of integers n we can make the probability above as small as we choose by reducing δ.
    - But note that

$$P\left[\max_{|s-t| \leq \delta,\ 0 \leq s,t \leq T} |X_s^{(n)} - X_t^{(n)}| > \epsilon \right] = P\left[\max_{|s-t| \leq n\delta,\ 0 \leq s,t \leq nT} |Y_s - Y_t| > \epsilon \right]$$

and

$$\begin{aligned} \max_{|s-t| \leq n\delta,\ 0 \leq s,t \leq nT} |Y_s - Y_t| &\leq \max_{|s-t| \leq [n\delta]+1,\ 0 \leq s,t \leq [nT]+1} |Y_s - Y_t| \\ &\leq \max_{1 \leq j \leq [n\delta]+1,\ 0 \leq k \leq [nT]+1} |S_{j+k} - S_k| \ , \end{aligned}$$

where the last inequality follows from the fact that Y is piecewise linear and changes slope only at integer values of t.
    - The result now follows from the previous lemma.

Third construction of Brownian motion

Definition (Wiener measure)

The probability measure P_* on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ under which the coordinate mapping process $W_t(\omega) = \omega(t)$, $0 \le t < \infty$, is a standard, one dimensional Brownian motion is called *Wiener measure*.

- A standard, one-dimensional Brownian motion defined on any probability space can be thought of as a random variable with values in $C[0,\infty)$.
  - Regarded this way, Brownian motion induces the Wiener measure on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$.
  - For this reason, $(C[0,\infty), \mathcal{B}([0,\infty)), P_*)$, where P_* is the Wiener measure, is called the canonical probability space for Brownian motion.

Brownian motion in several dimensions

Definition (d-dimensional Brownian motion with initial distribution μ)

Let d be a positive integer and μ a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.
Let $B = \{B_t, \mathcal{F}_t : t \geq 0\}$ be a continuous, adapted process with values in $\mathbb{R}^d$, defined on some probability space $(\Omega, \mathcal{F}, P)$. This process is called a *d-dimensional Brownian motion with initial distribution μ*, if

- $P[B_0 \in \Gamma] = \mu(\Gamma)$, for all $\Gamma \in \mathcal{B}(\mathbb{R}^d)$,
- for $0 \leq s < t$, the increment $B_t - B_s$ is independent of $\mathcal{F}_s$,
- for $0 \leq s < t$, the increment $B_t - B_s$ is normally distributed with mean zero and covariance matrix equal to $(t-s)I_d$, where I_d denotes the $d \times d$ identity matrix.

If μ assigns measure one to some singleton $\{x\}$, we say that B is a *d-dimensional Brownian motion starting at x*.

Brownian motion in several dimensions

- Let us see how to construct a d-dimensional Brownian motion with initial distribution μ:
  - Let $X(\omega_0) = \omega_0$ be the identity random variable on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu)$.
  - For each $i = 1, ..., d$, let $\tilde{B}^{(i)} = \{\tilde{B}_t^{(i)}, \tilde{\mathcal{F}}_t^{\tilde{B}^{(i)}} : t \geq 0\}$ be a standard one-dimensional Brownian motion on some $(\Omega^{(i)}, \mathcal{F}^{(i)}, P^{(i)})$.
  - On the product space

    $$(\mathbb{R}^d \times \Omega^{(1)} \times ... \times \Omega^{(d)}, \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}^{(1)} \otimes ... \otimes \mathcal{F}^{(d)}, \mu \times P^{(1)} \times ... \times P^{(d)})$$

    define

    $$B_t(\omega) = X(\omega_0) + (\tilde{B}_t^{(1)}(\omega_1), ..., \tilde{B}_t^{(d)}(\omega_d))$$

    and set $\mathcal{F}_t = \mathcal{F}_t^B$
- $B = \{B_t, \mathcal{F}_t : t \geq 0\}$ is a d-dimensional Brownian motion with initial distribution μ

Brownian motion in several dimensions

- Another construction for a d-dimensional Brownian motion with initial distribution μ:
  - Let $P^{(i)}$, $i = 1, ..., d$, be d copies of the Wiener measure on $(C[0,\infty), \mathcal{B}([0,\infty)))$.
  - Then $P^0 = P^{(1)} \times ... \times P^{(d)}$ is a measure, called the *d-dimensional Wiener measure*, on $(C[0,\infty)^d, \mathcal{B}(C[0,\infty)^d))$.
  - Under P^0, the coordinate mapping process $B_t(\omega) = \omega(t)$ together with the filtration $\{\mathcal{F}_t^B\}$ is a d-dimensional Brownian motion starting at the origin.

Brownian motion in several dimensions

- For $x \in \mathbb{R}^d$, define the probability measure P^x on $\mathcal{B}(C[0,\infty)^d)$ by

$$P^x(F) = P^0(F - x) \ , \qquad F \in \mathcal{B}(C[0,\infty)^d) \ ,$$

  where $F - x = \{\omega \in C[0,\infty)^d : \omega(.) + x \in F\}$.
- Under P^x, $B = \{B_t, \mathcal{F}_t^B : t \geq 0\}$ is a d-dimensional Brownian motion starting at x.
- Finally, for a probability measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, we define P^μ on $\mathcal{B}(C[0,\infty)^d$ by

$$P^\mu(F) = \int_{\mathbb{R}^d} P^x(F)\mu(\mathrm{d}x) \ .$$

- The coordinate mapping process $B = \{B_t, \mathcal{F}_t^B : t \geq 0\}$ on $(C[0,\infty)^d, \mathcal{B}(C[0,\infty)^d), P^\mu)$ is a d-dimensional Brownian motion with initial distribution μ.

Brownian motion in several dimensions

Definition (Universally measurable function)

Given a metric space (S, ρ), we denote by $\overline{\mathcal{B}(S)}^{\mu}$ the completion of the Borel σ-field $\mathcal{B}(S)$ with respect to the finite measure μ on $(S, \mathcal{B}(S))$.
The *universal σ-field* is $\mathcal{U}(S) = \cap_{\mu} \overline{\mathcal{B}(S)}^{\mu}$, where the intersection is over all finite measures.
A $\mathcal{U}(S)/\mathcal{B}(\mathbb{R})$-measurable real-valued function is said to be *universally measurable*.

Brownian motion in several dimensions

Definition (d-dimensional Brownian family)

A *d-dimensional Brownian family* is an adapted, d-dimensional process $B = \{B_t, \mathcal{F}_t : t \geq 0\}$ on a measurable space $(\Omega, \mathcal{F})$, and a family of probability measures $\{P^x\}_{x \in \mathbb{R}^d}$, such that

- for each $F \in \mathcal{F}$, the mapping $x \mapsto P^x(F)$ is universally measurable,
- for each $x \in \mathbb{R}^d$, $P^x[B_0 = x] = 1$,
- under each P^x, the process B is a d-dimensional Brownian motion starting at x.

- The d-dimensional Brownian motion constructed above, together with the family of probability measures $\{P^x\}$, is an example of a d-dimensional Brownian family.

Brownian motion in several dimensions

Definition (d-dimensional Brownian motion with drift μ and dispersion coefficient σ)

Let $B = \{B_t, \mathcal{F}_t : t \geq 0\}$, $(\Omega, \mathcal{F})$, $\{P^x\}_{x \in \mathbb{R}^d}$ be a d-dimensional Brownian family. If $\mu \in \mathbb{R}^d$ and $\sigma \in L(\mathbb{R}^d, \mathbb{R}^d)$ are constant and σ is nonsingular, then with

$$Y_t = \mu t + \sigma B_t \ ,$$

we say that $Y = \{Y_t, \mathcal{F}_t : t \geq 0\}$, $(\Omega, \mathcal{F})$, $\{P^{\sigma^{-1}x}\}_{x \in \mathbb{R}^d}$ is a *d-dimensional Brownian motion with drift μ and dispersion coefficient σ*.

Background

Definition (Conditional expectation with respect to a σ-algebra)

Let $(\Omega, \mathcal{F}, P)$ be a probability space, X a $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$-valued random variable on $(\Omega, \mathcal{F})$ and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$.
The *conditional expectation of* X *given* $\mathcal{G}$ is denoted by $E[X|\mathcal{G}]$ and defined as the function from Ω to $\mathbb{R}^d$ satisfying

- $E[X|\mathcal{G}]$ is $\mathcal{G}$-measurable
- $\int_A E[X|\mathcal{G}]\mathrm{d}P = \int_A X\mathrm{d}P$, for all $A \in \mathcal{G}$.

- It can be shown, via Radon-Nikodym Theorem, that $E[X|\mathcal{G}]$ always exists and is unique almost everywhere:
  - any two $\mathcal{G}$-measurable random variables Y and Z with

$$\int_A Y\mathrm{d}P = \int_A Z\mathrm{d}P = \int_A X\mathrm{d}P$$

for every $A \in \mathcal{G}$, differ by a null event in $\mathcal{G}$.

Background

- Let X and Y be $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$-valued random variables on $(\Omega, \mathcal{F})$ with $E[|X|] < \infty$ and $E[|Y|] < \infty$. Let also $\alpha, \beta \in \mathbb{R}$.
- The conditional expecta
  - $E[\alpha X + \beta Y|\mathcal{G}] = \alpha E[X|\mathcal{G}] + \beta E[X|\mathcal{G}]$.
  - $E[E[X|\mathcal{G}]] = E[X]$.
  - $E[X|\mathcal{G}] = X$ if X is $\mathcal{G}$-measurable.
  - $E[X|\mathcal{G}] = E[X]$ if X is independent of $\mathcal{G}$.
  - $E[YX|\mathcal{G}] = YE[X|\mathcal{G}]$ if Y is $\mathcal{G}$-measurable.

Background

Definition (conditional probability)

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, $A \in \mathcal{F}$ an event and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. The *conditional probability of A given* $\mathcal{G}$ is the conditional expectation of I_A given $\mathcal{G}$, i.e.:

$$P[A|\mathcal{G}] = E[I_A|\mathcal{G}].$$

Similarly, we can define the *conditional probability of A given a random variable X* on $(\Omega, \mathcal{F})$ as

$$P[A|X] = E[I_A|\mathcal{F}^X].$$

Background

Definition (Martingale)

Let $X = \{X_t : 0 \leq t < \infty\}$ be a real-valued process defined on a probability space $(\Omega, \mathcal{F}, P)$, adapted to a given filtration $\{\mathcal{F}_t\}$.
Furthermore, assume that $E|X_t| < \infty$ for all $t \geq 0$.
The process X is a *martingale* if, for every $0 \leq s < t < \infty$, we have, a.s. P:

$$E[X_t|\mathcal{F}_s] = X_s \ .$$

Markov Processes and Markov families

- Suppose that we observe a Brownian motion with initial distribution μ up to time s, $0 \leq s < t$.
- In particular, assume we see the value of B_s, to which we call y.
  - Conditioned on these observations, what is the probability that B_t is in some set $\Gamma \in \mathcal{B}(\mathbb{R}^d)$?
    - ★ $B_t = (B_t - B_s) + B_s$ and the increment $B_t - B_s$ is independent of the observations up to time s and is distributed just as B_{t-s} is under P^0.
    - ★ B_s depends on the observations: we are conditioning on $B_s = y$.
    - ★ We get that $B_t = (B_t - B_s) + B_s$ is distributed as B_{t-s} is under P^y
  - Summarizing:
    (i) Knowledge of the whole past up to time s provides as much information about B_t as knowledge of the value of B_s:

$$P^\mu[B_t \in \Gamma | \mathcal{F}_s] = P^\mu[B_t \in \Gamma | B_s] \ , \qquad 0 \leq s < t \ , \qquad \Gamma \in \mathcal{B}(\mathbb{R}^d) \ .$$

    (ii) Conditioned on $B_s = y$, B_t is distributed as B_{t-s} is under P^y:

$$P^\mu[B_t \in \Gamma | B_s = y] = P^y[B_{t-s} \in \Gamma] \ , \qquad 0 \leq s < t \ , \qquad \Gamma \in \mathcal{B}(\mathbb{R}^d) \ .$$

Markov Processes and Markov families

Definition (Markov Process with initial distribution μ)

Let d be a positive integer and μ a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.
An adapted, d-dimensional process $X = \{X_t, \mathcal{F}_t : t \geq 0\}$ on some probability space $(\Omega, \mathcal{F}, P^\mu)$ is said to be a *Markov process with initial distribution* μ if

- $P^\mu[X_0 \in \Gamma] = \mu(\Gamma)$, for every $\Gamma \in \mathcal{B}(\mathbb{R}^d)$.
- for $s, t \geq 0$ and $\Gamma \in \mathcal{B}(\mathbb{R}^d)$,

$$P^\mu[X_{t+s} \in \Gamma | \mathcal{F}_s] = P^\mu[X_{t+s} \in \Gamma | X_s] \ , \qquad P^\mu \text{ a.s.}$$

Markov Processes and Markov families

Definition (Markov family)

Let d be a positive integer. A d-dimensional *Markov family* is an adapted process $X = \{X_t, \mathcal{F}_t : t \geq 0\}$ on some measurable space $(\Omega, \mathcal{F})$, together with a family of probability measures $\{P^x\}_{x \in \mathbb{R}^d}$ on $(\Omega, \mathcal{F})$, such that

- for each $F \in \mathcal{F}$, the mapping $x \mapsto P^x(F)$ is universally measurable,
- for each $x \in \mathbb{R}^d$, $P^x[X_0 = x] = 1$,
- for $x \in \mathbb{R}^d$, $s, t \geq 0$ and $\Gamma \in \mathcal{B}(\mathbb{R}^d)$,

$$P^x[X_{t+s} \in \Gamma | \mathcal{F}_s] = P^x[X_{t+s} \in \Gamma | X_s] \,, \qquad P^x \text{ a.s.}$$

- for $x \in \mathbb{R}^d$, $s, t \geq 0$ and $\Gamma \in \mathcal{B}(\mathbb{R}^d)$,

$$P^x[X_{t+s} \in \Gamma | X_s = y] = P^y[X_t \in \Gamma] \,, \qquad P^x X_s^{-1} \text{ a.e.} y$$

Markov Processes and Markov families

- The following properties hold:
  - A d-dimensional Brownian motion is a Markov process.
  - A d-dimensional Brownian family is a Markov family.
  - Standard, one dimensional Brownian motion is both a martingale and a Markov process.
  - Brownian motion with non-zero drift is a Markov process but is not a martingale.

Brownian sample paths

- We will state the basic absolute properties of Brownian motion, i.e. those which hold with probability one, also called sample path properties.
  - These include:
    - Bad behaviour: nondifferentiability and lack of points of increase.
    - Good behaviour: law of the iterated logarithm
- It is worth to remark that sample paths of any continuous martingale can be obtained by running those of a Brownian motion according to a different, path-dependent clock.
  - Therefore, the study of Brownian motion provides the sample path properties for a much more general class of processes, which includes continuous martingales and diffusions.

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Brownian sample paths

Definition (Gaussian process)

An $\mathbb{R}^d$-valued stochastic process $X = \{X_t : 0 \leq t < \infty\}$ is called *Gaussian* if, for any integer $k \geq 1$ and real numbers $0 \leq t_1 < t_2 < ... < t_k < \infty$, the random vector $(X_{t_1}, X_{t_2}, ..., X_{t_k})$ has a joint normal distribution.
If the distribution of $(X_{t+t_1}, X_{t+t_2}, ..., X_{t+t_k})$ does not depend on t, we say that the process is *stationary*.

Brownian sample paths

- The finite dimensional distributions of a Gaussian process X are determined by its expectation vector $m(t) = E[X_t],\ t \geq 0$, and its covariance matrix

$$\rho(s,t) = E[(X_s - m(s))(X_t - m(t))^T], \qquad s,t \geq 0 \ .$$

- If $m(t) = 0$ for all $t \geq 0$ we say that X is a zero-mean Gaussian process.
- One-dimensional Brownian motion is a zero-mean Gaussian process with covariance function $\rho(s,t) = \min\{s,t\}$
- Conversely, any zero-mean Gaussian process $X = \{X_t, \mathcal{F}_t^X : 0 \leq t < \infty\}$ with a.s. continuous paths and covariance function given by $\rho(s,t) = \min\{s,t\}$ is a one-dimensional Brownian motion.

Brownian sample paths

- From now on, let:
  - $W = \{W_t, \mathcal{F}_t : 0 \leq t < \infty\}$ be a standard one dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$.
  - $W_0 = 0$ a.s. P.
  - for fixed $\omega \in \Omega$, $W_\cdot(\omega)$ denotes the sample path $t \mapsto W_t(\omega)$.

- *Strong Law of Large Numbers*:

$$\lim_{t\to\infty} \frac{W_t}{t} = 0 \qquad \text{a.s.}$$

Brownian sample paths

- *Equivalence transformations*:
  - When $W = \{W_t, \mathcal{F}_t : 0 \leq t < \infty\}$ be a standard Brownian motion, so are the processes obtained from the following equivalence transformations:
    - ⋆ Scaling: $X = \{X_t, \mathcal{F}_{ct} : 0 \leq t < \infty\}$ defined for $c > 0$ by
$$X_t = \frac{1}{\sqrt{c}} W_{ct} \ , \qquad 0 \leq t < \infty \ .$$
    - ⋆ Time-inversion: $Y = \{Y_t, \mathcal{F}_t^Y : 0 \leq t < \infty\}$ defined by
$$Y_t = tW_{1/t} \ , \ 0 < t < \infty \ , \qquad Y_0 = 0 \ .$$
    - ⋆ Time-reversal: $Z = \{Z_t, \mathcal{F}_t^Z : 0 \leq t \leq T\}$ defined for $T > 0$ by
$$Z_t = W_T - W_{T-t} \ , \ 0 \leq t \leq T \ .$$
    - ⋆ Symmetry: $-W = \{-W_t, \mathcal{F}_t : 0 \leq t < \infty\}$.

Brownian sample paths

- *Zero set*:
  - For fixed $\omega \in \Omega$, define the zero set of $W_{\cdot}(\omega)$ as

$$\mathcal{L}_\omega = \{0 \leq t < \infty : W_t(\omega) = 0\} .$$

  - For P-a.e. $\omega \in \Omega$, the zero set $\mathcal{L}_\omega$
    (i) has Lebesgue measure zero,
    (ii) is closed and unbounded,
    (iii) has an accumulation point at $t = 0$,
    (iv) has no isolated point in $(0, \infty)$, and therefore
    (v) is dense in itself.
  - With probability one, a standard, one-dimensional Brownian motion changes sign infinitely many times in any time-interval $[0, \epsilon]$, $\epsilon > 0$.
  - For every fixed $b \in \mathbb{R}$ and P-a.e. $\omega \in \Omega$, the level set

$$\mathcal{L}_\omega(b) = \{0 \leq t < \infty : W_t(\omega) = b\}$$

    is closed, unbounded, of Lebesgue measure zero, and dense in itself.

Brownian sample paths

- *Quadratic Variation*:
  - Let $\{\Pi_n\}_{n=1}^{\infty} = \{0 = t_0^{(n)}, t_1^{(n)}, \dots, t_{m_n}^{(n)} = t\}_{n=1}^{\infty}$ be a sequence of partitions of the interval $[0, t]$ with $\lim_{n\to\infty} \|\Pi_n\| = 0$.
  - Then the quadratic variations

$$V_t^{(2)}(\Pi_n) = \sum_{k=1}^{m_n} |W_{t_k^{(n)}} - W_{t_{k-1}^{(n)}}|^2$$

    of the Brownian motion W over these partitions converge to t in L^2 as $n \to \infty$.
  - Furthermore, if the partitions become so fine that $\sum_{n=1}^{\infty} \|\Pi_n\| < \infty$ holds, the convergence above takes place also with probability one.
  - As a consequence, for almost every $\omega \in \Omega$, the sample path $W_\cdot(\omega)$ is of unbounded variation on every finite interval $[0, t]$.

Brownian sample paths

- *Local maxima and points of increase*:
  - For almost every $\omega \in \Omega$, the sample path $W_{\cdot}(\omega)$ is monotone in no interval.
  - For almost every $\omega \in \Omega$, the set of points of local maximum for the Brownian path $W_{\cdot}(\omega)$ is countable and dense in $[0, \infty)$, and all local maxima are strict.
  - (Dvoretzky, Erdös, Kakutani (1961):
    Almost every Brownian sample path has no point of increase (or decrease).

Brownian sample paths

- *Nowhere differentiability*:
  - For a continuous function $f : [0, \infty) \to \mathbb{R}$, we denote by

    $$D^{\pm} f(t) = \overline{\lim}_{h \to 0^{\pm}} \frac{f(t+h) - f(t)}{h}$$

    the *upper (right and left) Dini derivatives* at t, and by

    $$D_{\pm} f(t) = \underline{\lim}_{h \to 0^{\pm}} \frac{f(t+h) - f(t)}{h}$$

    the *lower (right and left) Dini derivatives* at t.
  - The function f is said to be *differentiable at t from the right* (resp. *left*), if $D^{+} f(t)$ and $D_{+} f(t)$ (resp. $D^{-} f(t)$ and $D_{-} f(t)$) are finite and equal.
  - The function f is said to be *differentiable at* $t > 0$ if it is differentiable from both the right and the left and the four Dini derivatives agree.
  - At $t = 0$, differentiability is defined as differentiability from the right.

  - (Paley, Wiener, Zygmund (1933)):
    For almost every $\omega \in \Omega$, the Brownian sample path $W_{\cdot}(\omega)$ is nowhere differentiable. More precisely, the set

    $$\{\omega \in \Omega : \text{ for each } t \in [0, \infty), \text{ either } D^{+} W_t(\omega) = \infty \text{ or } D_{+} W_t(\omega) = -\infty\}$$

    contains an event $F \in \mathcal{F}$ with $P(F) = 1$.

Brownian sample paths

- *Law of the iterated Logarithm*:
  - (A. Hinčin (1933)):
    For almost every $\omega \in \Omega$, we have

$$
\begin{aligned}
&\text{(i)} \qquad \overline{\lim}_{t\downarrow 0} \frac{W_t(\omega)}{\sqrt{2t\log\log(1/t)}} = 1 \\
&\text{(ii)} \qquad \underline{\lim}_{t\downarrow 0} \frac{W_t(\omega)}{\sqrt{2t\log\log(1/t)}} = -1 \\
&\text{(iii)} \qquad \overline{\lim}_{t\to\infty} \frac{W_t(\omega)}{\sqrt{2t\log\log t}} = 1 \\
&\text{(iv)} \qquad \underline{\lim}_{t\to\infty} \frac{W_t(\omega)}{\sqrt{2t\log\log t}} = -1 \; .
\end{aligned}
$$