Risk Measures and Decisions in Insurance

M. Goovaerts

[1] Catholic University of Leuven, Belgium

[2] University of Amsterdam, The Netherlands

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Agenda

Introduction

Definition

A risk measure κ *casu quo* decision principle ρ, π is a functional assigning a real number to any random variable defined on $(\Omega, \mathcal{F})$; that is, κ *casu quo* ρ, π are mappings from $\mathcal{X}$ to $\mathbb{R}$.

Difference between κ (Risk measure) and ρ, π (Decision principles)

Mathematically they are similar concepts. Justifications / derivations differ: Justifications of risk measures should be based on axiomatic characterizations. Derivations of decision principles should be based on an optimization procedure, e.g., by minimizing the total risk as measured by a risk measure, or on an equilibrium criterion.

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Mean value principle as risk measure

- Condition (c) (continuity condition)

$$\begin{cases} prob(X_{aq} = a) = q \\ prob(X_{aq} = 0) = 1 - q \end{cases}$$

For fixed $a > 0$, the premium $P_a(q) = \kappa(X_{qa})$ is strictly increasing $(0 \leq q \leq 1)$ with $P_a(0) = 0, P_a(1) = a$.

- **Theorem:** A premium principle satisfying condition (c) is iterative if and only if it is the mean value principle

$$v(\kappa(X)) = E(v(X)).$$

- Example: $v(x) = e^{\alpha x}$ (providing the same results as utility)

Application to Premium Calculation

- We use $v(x) = e^{\alpha x}$ and minimize the risk measure in an optimal premium problem that, for any random variable X, only allows premiums of the form $\mathbb{E}[\varphi(X)X]$, with $\varphi(\cdot)$ a real-valued, continuous and strictly increasing function satisfying $\mathbb{E}[\varphi(X)] = 1$. Then, we state the following problem:

$$\min_{\Psi} \; \alpha\mathbb{E}\left[\exp\left(-\alpha(\mathbb{E}[\varphi(X)X] - X)\right)\right], \qquad X \in \mathcal{X}_{[a,b]}$$

  where Ψ is the class of all functions φ that satisfy the aforementioned conditions and $\mathcal{X}_{[a,b]}$ is the class of all random variables with support $[a, b]$, $a < b$.
- The optimal premium can be expressed as $\mathbb{E}[\varphi(X)X] = \frac{\mathbb{E}[Xe^{\alpha X}]}{\mathbb{E}[e^{\alpha X}]}$.

Premium calculation top down

$$U_t = U_{t-1} + c - S_t, \quad t = 1, 2...$$

Criterion of ruin probability gives Lundberg upper bound $e^{-Ru} < \epsilon$

$$e^{Rc} = E(e^{RS_t})$$

In case one starts with $R = \frac{|\ln \epsilon|}{u}$

$$\pi(X) = \frac{u}{|\ln \epsilon|} \ln E(e^{|\ln \epsilon| \frac{S_t}{u}})$$

Interpretation Esscher premium with risk aversion $\alpha = \frac{|\ln \epsilon|}{u}$

Given two continuous and strictly increasing functions f, g in $[a, b]$, are $\pi(X, f)$ and $\pi(X, g)$ comparable, i.e. is there an inequality

$$\pi(X, f) \leq \pi(X, g) \text{ for } \forall X \in B$$

Theorem 3: Let f and g be two continuous and strictly increasing function in $\mathbb{R}$, then a necessary and sufficient condition that $\pi(X, f)$ and $\pi(X, g)$ should be comparable is that

$$h = gf^{-1}$$

should satisfy

$$h(E(X)) \leq E(h(X))$$

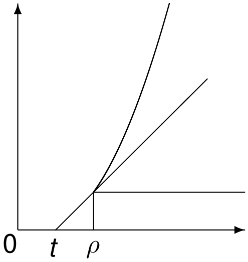
or the reversed inequality for $\forall X \in B$.

Application of mean value principle to solvency

$$E\left(\phi\left(\frac{(X-t)_+}{\rho - t}\right)\right) = \phi(\kappa(X)) = \phi(\alpha)$$

where ϕ is strictly increasing.
Interpretation 1): X claim size, t premium, $\rho - t$ solvency margin
Interpretation 2): $\phi(\alpha) = \alpha$ for $0 \le \alpha \le 1$, $\phi(\alpha) \nearrow \quad \alpha > 1$.
Then

$$E\left(\phi\left(\frac{(X-t)_+}{\rho - t}\right)\right) = \alpha_\phi(X, \rho, t)$$



Application of mean value principle to solvency

For $\alpha(X, \rho, t) = \alpha(Y, \rho, t)$, X and Y are "'equally solvent"'.
ϕ_1 and ϕ_2 have comparable tails in case

$$\alpha_{\phi_1}(X, \rho, t) < \alpha_{\phi_2}(X, \rho, t) \quad \forall X(\phi_2 \text{ convex in } \phi_1$$

Special case $\phi_1(x) = x$

$$\alpha_{\phi_1}(X, \rho, t) \leq \alpha_{\phi_2}(X, \rho, t)$$

Hence $E\left(\frac{(X-t)_+}{\rho-t}\right) = \alpha$

$$\rho = t + \frac{1}{\alpha}E((X-t)_+)$$

Application of mean value principle to solvency

Hence

$$E\left(\phi_2\left(\frac{(X-t)_+}{\rho-t}\right)\right) > \alpha$$

Such that

$$E\left(\phi_2\left(\frac{(X-t)_+}{\rho_2-t}\right)\right) = \alpha$$

resulting in $\rho_2(X,t,\alpha) \geq t + \frac{1}{\alpha}E((X-t)_+) \quad \forall t$.
Hence

$$\min_t \rho_\phi(X) \geq F_X^{-1}(1-\alpha) + \frac{1}{\alpha}E(X - F_X^{-1}(1-\alpha)).$$

Haezendonck Risk Measure

$$Pr[X > \rho] = Pr[X - t > \rho - t] \leq \mathbb{E}\left[\phi\left(\frac{(X-t)_+}{\rho - t}\right)\right] \qquad (1)$$

Lemma: Let X be a risk and let $\phi(\cdot)$ be a nonnegative, strictly increasing and continuous function on $[0, +\infty)$ with $\phi(0) = 0, \phi(1) = 1$ and $\phi(+\infty) = +\infty$. Then for any $-\infty < x < \max[X]$ and $0 < a < 1$, the right hand side of the equation (1) has a unique solution $\pi_\alpha[X, t]$ satisfying

$$\rho_\alpha[X, t] \geq F_X^{-1}(1 - \alpha) \text{ and } \rho_\alpha[X, t] > t$$

Definition 1: Let $\phi(\cdot)$ be as in Lemma and let $0 < \alpha < 1$ be arbitrarily fixed. We consider

$$\rho_\alpha[X] = \inf_{-\infty < t < \max[X]} \rho_\alpha[X, t]$$

as the risk measure of a risk X, where $\rho_\alpha[X, x]$ is the unique solution to the equation (1). In honor of the late J.Haezendonck we call it the Haezendonck risk measure, which is a minimal Orlicz norm risk measure.

Optimal value for ρ (depending on α and t)

$$\begin{cases} \int_t^{\infty} \phi\left(\frac{(x-t)_+}{\rho-t}\right) dF_X(x) = \alpha \\ \frac{\partial}{\partial t}\rho = \frac{E\left(\phi'\left(\frac{(x-t)_+}{\rho-t}\right)(x-\rho)\right)}{E\left((x-t)_+\phi'\left(\frac{(x-t)_+}{\rho-t}\right)\right)} = 0 \end{cases}$$

$$\rho_0 = t_0 + \frac{E\left(\phi'\left(\frac{(x-t_0)_+}{\rho_0-t_0}\right)(x-t_0)\right)}{E\left(\phi'\left(\frac{(x-t_0)_+}{\rho_0-t_0}\right)\right)}$$

Special cases:
(1) $\phi(x) = x$

$$\rho_0 = t_0 + \frac{\int_{t_0}^{\infty}(x-t_0)dF_X(x)}{1-F_X(t_0)}$$

(2)$\phi(x) = e^{\alpha x}$

$$\rho_0 = t_0 + \frac{E\left((x-t_0)_+ e^{\frac{\alpha x}{\rho_0-t_0}}\right)}{E\left(e^{\frac{\alpha x}{\rho_0-t_0}}\right)}$$

Haezendonck Risk Measure

Theorem 1: Let $\phi(\cdot)$ be as in Lemma. The Haezendonck risk measure $\rho_\alpha[X]$ satisfies

$$F_X^{-1}(1-\alpha) \leq \rho_\alpha[X] \leq \max[X]$$

Example: Now we specify the risk in Definition 1 as B_q, as a Bernoulli variable with

$$Pr[B_q = 1] = 1 - Pr[B_q = 0] = q \in [0, 1].$$

Let $\phi(y)y$ for $y \geq 0$ and let $-\infty < t < 1$ and $0 < \alpha < 1$ be arbitrarily given. In case $-\infty < x < 0$ equation (1) leads to

$$(1-q)\frac{-t}{\rho - t} + q\frac{1-t}{\rho - t} = \alpha$$

whereas in case $0 \leq t < 1$ it leads to $q\frac{1-t}{\rho-t} = \alpha$.

Haezendonck Risk Measure

Theorem 2: Let $\rho_\alpha[\cdot]$ be the The Haezendonck risk measure with $\phi(\cdot)$ given in Lemma and let $0 < \alpha < 1$ be arbitrarily given. Then we have

B1. Monotonicity: If $X \leq_{st} Y$ then $\rho_\alpha[X] \leq \rho_\alpha[Y]$;

B2. Positive homogeneity: $\rho_\alpha[cX] = c\rho_\alpha[X]$ for any $c > 0$;

B3. Subadditivity: If $\phi(\cdot)$ is convex, then $\rho_\alpha[X + Y] \leq \rho_\alpha[X] + \rho_\alpha[Y]$ holds for any (X, Y) such that

$$\max[X + Y] = \max[X] + \max[Y];$$

B4. Translation invariance: $\rho_\alpha[X + a] = \rho_\alpha[X] + a$ for any a;

B5. Preservation of convex ordering: If $\phi(\cdot)$ is convex, then $X \leq_{cx} Y \Rightarrow \rho_\alpha(X) \leq \rho_\alpha(Y)$, where $X \leq_{cx} Y$; means that $\mathbb{E}\varphi(X) \leq \mathbb{E}\varphi(Y)$ holds for all convex functions $\varphi(\cdot)$ for which the expectations involved exist.

Definition 2: Let $\phi_1(\cdot)$ and $\phi_2(\cdot)$ be two real functions on $(0, +\infty)$. We say $\phi_2(\cdot)$ is convex (concave) in $\phi_1(\cdot)$ if and only if $\phi_2\phi_1^{-1}(\cdot)$ is convex (concave).

Solvency Capital Principles

- Several types of solvency capital need to be distinguished, namely regulatory capital, economic (or management) capital, rating capital and book capital. For further details we refer to Laeven & Goovaerts (2004), Goovaerts, Van den Borre & Laeven (2005) and Dhaene *et al.* (2008).
- Tradeoff between risk exposure on the one hand and the cost of economic capital on the other hand. (Compare in statistics type I and type II errors)

$$\min_k \; ik + E((X-k)_+) \Rightarrow k = F_X^{-1}(1-i)$$

Optimal Risk Sharing

A cooperating "pool" with n participating insurance companies wants to insure a risk X. The pool looks for

$$\min_{(X_1,\ldots,X_n)|X=X_1+\cdots+X_n} \rho[X] = \sum_{i=1}^{n} \frac{1}{\alpha_i} \log \mathbb{E}[e^{\alpha_i X_i}],$$

where we assume that participant i has an exponential(α_i) utility function and the claim amount this participant has to pay is denoted by X_i, hence $X = X_1 + \cdots + X_n$.

Optimal Risk Sharing

We will find that the minimal premium $\rho^-[X]$ is obtained by choosing $X_i = \alpha X/\alpha_i$, where α is such that $\sum_{i=1}^n \alpha/\alpha_i = 1$. Hence, we get

$$\begin{aligned}\rho^-[X] &= \sum_{i=1}^n \frac{1}{\alpha_i} \log \mathbb{E}[e^{\alpha_i \frac{\alpha}{\alpha_i} X_i}] \\ &= \frac{1}{\alpha} \log \mathbb{E}[e^{\alpha X}].\end{aligned}$$

Optimal asset allocation in case of marginal information

A conglomerate or insurance regulator faces a total risk $S = S_1 + S_2 + \cdots + S_n$ and has an economic capital $u = u_1 + u_2 + \cdots + u_n$. The capital allocation problem can be formulated as

$$\text{Minimize} \sum_{i=1}^{n} \frac{u_i}{|\log \epsilon|} \log \mathbb{E}\left[exp\left(\frac{|\log \epsilon|}{u_i} X_i\right)\right]$$

over all u_i with $\sum u_i = u$.

The solution can be obtained by means of the Lagrange method,

$$\rho^i_{exp}(X_i) = \frac{u_i}{|\log \epsilon|} \log \mathbb{E}[e^{(|\log \epsilon|/u_i)X_i}];$$

$$\rho^i_{Ess}(X_i) = \frac{\mathbb{E}[X_i e^{(|\log \epsilon|/u_i)X_i}]}{\mathbb{E}[e^{(|\log \epsilon|/u_i)X_i}]}.$$

The optimal solution satisfies the following system of equations:

$$\frac{1}{u_j}(\rho^j_{exp}(X_j) - \rho^j_{Ess}(X_j)) = \frac{1}{u} \sum_i (\rho^i_{exp}(X_i) - \rho^i_{exp}(X_i)).$$

Optimal asset allocation in case of marginal information

For small values of the parameters $|\log\epsilon|/u_i$ in Esscher and the exponential premiums, the solution can be written in the following form:

$$\frac{u_j}{u} \approx \frac{Var[X_j]/(2u_j)}{\sum_i Var[X_i]/(2u_i)}.$$

(X, Y) is comonotonic, we have

$$\pi[X; u] + \pi[Y; u] \leq \pi[X + Y; u] \leq \pi[X; u_1] + \pi[Y; u_2].$$

Reinsurance Principles

The insurer might also consider a more general problem. Let risk X be decomposed as follows:

$$X = X_1 + X_2 + X_3 + X_4,$$

with

- $X_1 = X \cdot 1_{\{X \leq 0\}}$: the profit layer;
- $X_2 = \min(X \cdot 1_{\{X \leq 0\}}, c)$: the reinsurance layer with retention 0 and cap c;
- $X_3 = \min((X \cdot 1_{\{X \leq 0\}} - c)_+, \rho[X])$: the economic capital layer;
- $X_4 = (X - \rho[X])_+$: the residual risk layer.

Applications and Interpretations

Economic Capital Allocation Derived from Risk Measures

a.
- Two sided risk measure (TRM)
- One-sided risk measure (ORM)

b.
- $Var[\sum X_i^{\perp}] = nVar[X_i]$
- $Var[\sum X_i^c] = n^2 Var[X_i]$

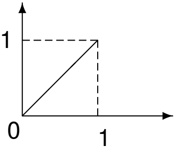
c. Minimize $\sum_j \rho_j(X_j, \pi_j, u_j, \epsilon)$ over $u_1, \cdots, u_n$ with $\sum u_i = u$. This result should be compared with $\rho_m(X_1 + \cdots + X_n, \pi_1 + \cdots + \pi_n, u_1 + \cdots + u_n, \epsilon)$, the risk measure for the parent company.

d. $X \leq_{st} Y$ if $F_X \geq F_Y$,
$X \leq_{st} Y \Rightarrow \rho(X) \leq \rho(Y)$,
$X \leq_{st} Y \Rightarrow Var[X] \leq Var[Y]$, nor $\sigma(X) \leq \sigma(Y)$.
$X \leq_{st} Y$ and $E[X] = E[Y] \Rightarrow X \sim Y$,
$E[(X - t)_+] \leq E[(Y - t)_+], E[X] = E[Y] \sim X \leq_{cx} Y$.

Examples

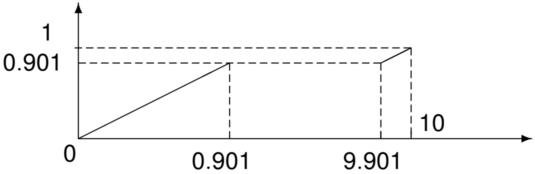
Ex 1: Earthquake risk insurance: exchange of portions of life portfolios between different continents. Splitting of risks.

Ex 2: a.



$F_u^{-1}(0.9) = 0.9$

b.



Examples

Ex 3: Allocation of economic capital
Economic capital: $u = u_1 + u_2 + \cdots + u_n$
$\rho_{congl}(X_1 + \cdots + X_n - u) \& \rho_1(X_1 - u_1) + \cdots + \rho_n(X_n - u_n)$
$\Rightarrow \rho_{congl}(X_1 + \cdots + X_n) - u \& \rho_1(X_1) + \cdots + \rho_n(X_n) - u$ in case of translation invariance.

Ex 4: Rational decision maker $\rho(\alpha X) \neq \alpha\rho(X)$.

Ex 5: Firewalls

Ex 6: Uniform risk X in the interval $(9, 10)$ and a risk Y that is 20 with certainty. Clearly, $Pr[X < Y] = 1$, but $X - E[X]$ is risky while $Y - E[Y]$ represents no risk at all.
$X = X_I + X_R$ where X_I is the retained risk while X_R is the reinsured part. In the case where $\rho(X_I + X_R) \geq \rho(X_I) + \rho(X_R)$ it is possible that $\rho(X_I + X_R) \geq \rho(X_I) + \rho_R(X_R)$, where $\rho_R(\cdot)$ is reinsurer's risk measure, and these are the reinsurance treaties that exist.

Examples

Ex 7: The condition of subadditivity, $\rho(X+Y) \leq \rho(X) + \rho(Y)$, for a translation invariant risk measure can be rewritten as $\rho(X + Y - \rho(Y)) \leq \rho(X)$.
Consider $0 \leq \rho(X) \leq 1$ for a Bernoulli(q) risk, add n comonotonic risks, then the new surplus equals $u + n\rho(X)$ with probability $1 - q$ and $u + n\rho(X) - n$ with probability q.
Note that translation invariance implies that $\rho(X - \rho(X)) = 0$.

Ex 8: $E[X]$ and $Max[X]$

Ex 9: $(1+\alpha)E[(X-K)_+] + i_D K$

Examples

Ex 10: Economic capital K, we have to minimize

$$E[(X-(1+r)K)_+]+(i-r)K.$$

Based on Yaari's dual theory, introducing a distortion function g with $g(0)=0, g(1)=1$, $g(x)$ increasing and $g(x)\geq x$, "cost of avoiding insolvency" can be calculated by

$$\int_{K(1+r)}^{\infty} g(1-F_X(x))dx.$$

Therefore, we just need to minimize

$$\int_{K(1+r)}^{\infty} g(1-F_X(x))dx+(i-r)K.$$

The optimal solution is given by

$$K=\frac{1}{1+r}F_X^{-1}\left(1-g^{-1}\left(\frac{i-r}{1+r}\right)\right).$$

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