# Constructions of Brownian Motion I

## Based on the book
## “Brownian motion and Stochastic Calculus”
## by I. Karatzas and S. Shreve

Diogo Pinheiro

CEMAPRE - ISEG - UTL
dpinheiro@iseg.utl.pt

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# Background

## Definition (Measurable space)

A *measurable space* a pair $(\Omega, \mathcal{F})$, where $\Omega$ is a set and $\mathcal{F}$ is a collection of subsets of $\Omega$ with a $\sigma$-algebra structure, i.e.:

- $\emptyset \in \mathcal{F}$
- $\mathcal{F}$ is closed under complementation and countable unions.

## Definition (Measure space and probability space)

A *measure space* is a triple $(\Omega, \mathcal{F}, \mu)$, where $(\Omega, \mathcal{F})$ is a measurable space and $\mu$ is a *measure* on $(\Omega, \mathcal{F})$, i.e.:

- $\mu(\emptyset) = 0$;
- $\mu(A) \geq 0$ for all $A \in \mathcal{F}$;
- if $\{A_i\}_{i \in I}$ is a countable collection of pairwise disjoint elements of $\mathcal{F}$ then $\mu(\cup_i A_i) = \sum_i \mu(A_i)$.

A *probability measure* is a measure with total measure one, i.e. $\mu(\Omega) = 1$.
A *probability space* is a measure space with a probability measure.

# Background

## Definition (Stochastic process)

A *stochastic process* is a collection of random variables $X = \{X_t; 0 \leq t < \infty\}$ on a measurable space $(\Omega, \mathcal{F})$, which takes values on a second measurable space $(\Pi, \mathcal{G})$.

- $(\Omega, \mathcal{F})$ is called the *sample space*.
- $(\Pi, \mathcal{G})$ is called the *state space*. We will take it to be $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.
- For a fixed sample point $\omega \in \Omega$, the function $t \to X_t(\omega); t \geq 0$ is the *sample path* of the process $X$ associated with $\omega$.

- One can think of the index $t \in [0, \infty]$ of the random variables $X_t$ as time.

# Background

- Implicit in the statement that a random process $X = \{X_t; 0 \leq t < \infty\}$ is a collection of $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$-valued random variables on $(\Omega, \mathcal{F})$, is the assumption that each $X_t$ is $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$ measurable.
- However, since $X$ is a function of the pair of variables $(t, \omega)$, it is convenient to have joint measurability properties.

## Definition (Measurable stochastic process)

The stochastic process $X$ is called *measurable* if for every $A \in \mathcal{B}(\mathbb{R}^d)$ the set $\{(t, \omega) : X_t(\omega) \in A\}$ belongs to $\mathcal{B}([0, \infty]) \otimes \mathcal{F}$, i.e.

$$X_t(\omega) : ([0, \infty] \times \Omega, \mathcal{B}([0, \infty]) \otimes \mathcal{F}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$$

is measurable

# Background

- The temporal feature of a stochastic process suggests a flow of time, in which, at every moment $t \geq 0$, we can:
  - talk about past, present and future.
  - ask how much an observer of the process knows about it at present, as compared to how much he knew at some point in the past or will know at some point in the future.
- $\sigma$-algebras are used in the study of stochastic processes to keep track of information.
- From now on, we assume that our sample space $(\Omega, \mathcal{F})$ is equipped with a filtration.

## Definition (Filtration)

A *filtration* on a measurable space $(\Omega, \mathcal{F})$ is a nondecreasing family $\{\mathcal{F}_t; t \geq 0\}$ of sub-$\sigma$-algebras of $\mathcal{F}$, i.e. $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $0 \leq s < t < \infty$.
We set $\mathcal{F}_\infty = \sigma(\cup_{t \geq 0} \mathcal{F}_t)$, the smallest $\sigma$-algebra containing $\cup_{t \geq 0} \mathcal{F}_t$.

# Background

- For a given stochastic process, the simplest choice of a filtration is the one generated by the process itself:

$$\mathcal{F}_t^X = \sigma(X_s; 0 \leq s \leq t) ,$$

  the smallest $\sigma$-algebra with respect to which $X_s$ is measurable for every $s \in [0, t]$.
  - We can interpret $A \in \mathcal{F}_t^X$ to mean that by time $t$, an observer of $X$ knows whether or not $A$ has occured.
- The concept of measurability for a stochastic process introduced before is still rather weak.
- The introduction of a filtration $\{\mathcal{F}_t\}$ enables us to use more interesting and useful concepts.

### Definition (Adapted stochastic process)

The stochastic process $X$ is *adapted* to the filtration $\{\mathcal{F}_t\}$ if, for every $t \geq 0$, $X_t$ is an $\mathcal{F}_t$-measurable random variable.

- Every stochastic process $X$ is adapted to $\{\mathcal{F}_t^X\}$.

# Background

## Definition (Independent $\sigma$-algebras)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_n$ be sub-$\sigma$-algebras of $\mathcal{F}$. A finite set of sub-$\sigma$-algebras $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_n$ is *independent* if for any set of events $A_i \in \mathcal{F}_i$, $i = 1, ..., n$, we have that

$$P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1)P(A_2)...P(A_n) \ .$$

An arbitrary set $\mathcal{S}$ of $\sigma$-algebras is *mutually independent* if any finite subset of $\mathcal{S}$ is independent.

- The above definitions are generalizations of the notions of independence for events and for random variables:
  - Events $B_1, ..., B_n \in \mathcal{F}$ are mutually independent if the sub-$\sigma$-algebras $\sigma(B_i) := \{\emptyset, B_i, \Omega - B_i, \Omega\}$ are mutually independent.
  - Random variables $X_1, ..., X_n$ defined on $(\Omega, \mathcal{F}, P)$ are mutually independent if the sub-$\sigma$-algebras $\sigma(X_i) = \{X_i^{-1}(B) : B \in \mathcal{B}(\mathbb{R}^d)\}$ are mutually independent.
  - In general, mutual independence among events $B_i$, random variables $X_j$ and $\sigma$-algebras $\mathcal{F}_k$ means the mutual independence among $\sigma(B_i)$, $\sigma(X_j)$ and $\mathcal{F}_k$.

# Brownian motion

## Definition (standard, one-dimensional Brownian motion)

A *standard, one-dimensional Brownian motion* is a continuous, adapted process $B = \{B_t, \mathcal{F}_t, 0 \leq t < \infty\}$, defined on some probability space $(\Omega, \mathcal{F}, P)$, with the following properties:

- $B_0 = 0$ a.s.;
- for $0 \leq s < t$, the increment $B_t - B_s$ is independent of $\mathcal{F}_s$;
- for $0 \leq s < t$, the increment $B_t - B_s$ is normally distributed with mean zero and variance $t - s$.

Analogously, we can define a Brownian motion $B = \{B_t, \mathcal{F}_t, 0 \leq t < T\}$ on $[0, T]$, for some $T > 0$.

- If $B$ is a Brownian motion and $0 = t_0 < t_1 < \ldots < t_n < \infty$, then the increments $\{B_{t_j} - B_{t_{j-1}}\}_{j=1}^{n}$ are independent and the distribution of $B_{t_j} - B_{t_{j-1}}$ depends on $t_j$ and $t_{j-1}$ only through the difference $t_j - t_{j-1}$: it is normal with mean zero and variance $t_j - t_{j-1}$.
  - We say that $B$ has stationary, independent increments.

# Brownian motion

- The filtration $\{\mathcal{F}_t\}$ is a key part in the definition of Brownian motion.
- However, if we are given $\{B_t; 0 \leq t < \infty\}$ but no filtration, and if we know that $B$ has stationary independent increments and that $B_t - B_0$ is normal with mean zero and variance $t$, then $\{B_t, \mathcal{F}_t^B; 0 \leq t < \infty\}$ is a Brownian motion.
- If $\{\mathcal{F}_t\}$ is "larger" than $\{\mathcal{F}_t^B\}$ (in the sense that $\mathcal{F}_t^B \subset \mathcal{F}_t$ for all $t \geq 0$) and if $B_t - B_s$ is independent of $\mathcal{F}_s$ whenever $0 \leq s < t$, then $\{B_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is still a Brownian motion.

# Brownian motion

- The first problem one encounters with Brownian motion is its existence.
- We will go through the ideas in three distinct approaches to the proof of existence of Browinian motion.
  - A first approach is to write down what the finite-dimensional distributions of this process must be (based on the stationarity, independence and normality of its increments), and then construct a probability measure on an appropriate measurable space in such a way that we obtain the prescribed finite-dimensional distributions.
  - A second construction exploits the Gaussian property of this process and is closely related to Wiener's original construction.
  - The third proof is based on the idea of the weak limit of a sequence of random walks.

# First construction of Brownian motion

- Let $\mathbb{R}^{[0,\infty)}$ denote the set of all real-valued functions on the half-line $[0,\infty)$.

**Definition ($n$-dimensional cylinder set)**

An *n-dimensional cylinder set* in $\mathbb{R}^{[0,\infty)}$is a set of the form

$$C = \{\omega \in \mathbb{R}^{[0,\infty)} : (\omega(t_1), \ldots, \omega(t_n)) \in A\} ,$$

where $t_i \in [0,\infty)$, $i = 1, \ldots, n$, and $A \in \mathcal{B}(R^n)$.

- Let:
  - $\mathcal{C}$ denote the algebra of all cylinder sets (of all finite dimensions) in $\mathbb{R}^{[0,\infty)}$
  - $\mathcal{B}(\mathbb{R}^{[0,\infty)})$ denote the smallest $\sigma$-algebra containing $\mathcal{C}$.

# First construction of Brownian motion

## Definition (Family of finite-dimensional distributions)

Let $T$ be the set of finite sequences $t = (t_1, ..., t_n)$ of distinct, nonnegative numbers, where the length $n$ of these sequences ranges over the set of positive integers.
Suppose that for each $t$ of length $n$, we have a probability measure $Q_t$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$.
Then the collection $\{Q_t\}_{t \in T}$ is called a *family of finite-dimensional distributions*.

# First construction of Brownian motion

## Definition (Consistent family of finite-dimensional distributions)

A family of finite-dimensional distributions $\{Q_t\}_{t\in T}$ is *consistent* if the following two conditions are satisfied:

- if $s = (t_{i_1}, t_{i_2}, ..., t_{i_n})$ is a permutation of $t = (t_1, t_2, ..., t_n)$ then for any $A_i \in \mathcal{B}(\mathbb{R})$, $i = 1, ..., n$, we have

$$Q_t(A_1 \times A_2 \times ... \times A_n) = Q_s(A_{i_1} \times A_{i_2} \times ... \times A_{i_n}) \ ;$$

- if $t = (t_1, t_2, ..., t_n)$ with $n \geq 1$, $s = (t_1, t_2, ..., t_{n-1})$ and $A \in \mathcal{B}(\mathbb{R}^{n-1})$, then

$$Q_t(A \times \mathbb{R}) = Q_s(A) \ .$$

# First construction of Brownian motion

- If we have a probability measure $P$ on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$, then we can define a consistent family of finite-dimensional distributions by

$$Q_t(A) = P[\omega \in \mathbb{R}^{[0,\infty)} : (\omega(t_1), ..., \omega(t_n)) \in A] \ ,$$

where $A \in \mathcal{B}(\mathbb{R}^n)$ and $t = (t_1, ..., t_n) \in T$.

- We are interested in the converse of this fact, since it will enable us to construct a probability measure $P$ from the finite-dimensional distributions of Brownian motion.

# First construction of Brownian motion

**Theorem (Daniell (1918), Kolmogorov (1933))**

*Let $\{Q_t\}$ be a consistent family of finite-dimensional distributions. Then there is a probability measure $P$ on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$ such that the equality*

$$Q_t(A) = P[\omega \in \mathbb{R}^{[0,\infty)} : (\omega(t_1), ..., \omega(t_n)) \in A] \ ,$$

*holds for every $t \in T$.*

# First construction of Brownian motion

- We want to construct a probability measure $P$ on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$ so that the process $B = \{B_t, \mathcal{F}_t^B : 0 \leq t < \infty\}$ defined by $B_t(\omega) = \omega(t)$, the coordinate mapping process, is "almost'" a standard, one-dimensional Brownian motion under $P$.
- We say that the process is "almost" a Brownian motion because we are neglecting the requirement of sample path continuity.
- We concentrate on the finite-dimensional distributions now and will deal with continuity of the process later.

# First construction of Brownian motion

- Let $0 = s_0 < s_1 < s_2 < ... < s_n$.
- The cumulative distribution function for $(B_{s_1}, ..., B_{s_n})$ must be

$$F_{(s_1,s_2,\ldots,s_n)}(x_1,...,x_n) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}...\int_{-\infty}^{x_n} p(s_1;0,y_1)p(s_2;y_1,y_2)... \\ ...p(s_n - s_{n-1}, y_{n-1}, y_n)\mathrm{d}y_n\mathrm{d}y_{n-1}...\mathrm{d}y_1 \ ,$$

for $(x_1, ..., x_n) \in \mathbb{R}^n$, where $p$ is the Gaussian kernel

$$p(t;x,y) = \frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-(x-y)^2/2t} \ , \qquad t > 0, \qquad x, y \in \mathbb{R} \ .$$

# First construction of Brownian motion

- Let $t = (t_1, t_2, ..., t_n)$, where the $t_j$ are not ncessarily ordered but are distinct.
- Let the random vector $(B_{t_1}, B_{t_2}, ..., B_{t_n})$ have the distribution determined in the previous slide (where the $t_j$ must be ordered from smallest to largest to obtain $s = (s_1, ..., s_n)$.
- For $A \in \mathcal{B}(\mathbb{R}^n)$, let $Q_t(A)$ be the probability under this distribution that $(B_{t_1}, B_{t_2}, ..., B_{t_n})$ is in $A$.
- This defines a family of finite-dimensional distributions $\{Q_t\}_{t \in T}$.

# First construction of Brownian motion

**Lemma**

*The family of finite-dimensional distributions $\{Q_t\}_{t \in T}$ defined above is consistent.*

- Fix $t = (t_1, t_2, ..., t_n)$ and let $s = (t_{i_1}, t_{i_2}, ..., t_{i_n})$ be a permutation of $t$.
- We have a distribution for the random vector $(B_{t_1}, B_{t_2}, ..., B_{t_n})$ under which

$$\begin{aligned} Q_t(A_1 \times A_2 \times ... \times A_n) &= P[(B_{t_1}, B_{t_2}, ..., B_{t_n}) \in A_1 \times A_2 \times ... \times A_n] \\ &= P[(B_{t_{i_1}}, B_{t_{i_2}}, ..., B_{t_{i_n}}) \in A_{i_1} \times A_{i_2} \times ... \times A_{i_n}] \\ &= Q_s(A_{i_1} \times A_{i_2} \times ... \times A_{i_n}) \ . \end{aligned}$$

- Furthermore, for $A \in \mathcal{B}(\mathbb{R}^{n-1})$ and $s' = (t_{i_1}, t_{i_2}, ..., t_{i_{n-1}})$,

$$\begin{aligned} Q_t(A \in \mathbb{R}) &= P[(B_{t_1}, B_{t_2}, ..., B_{t_{n-1}}) \in A] \\ &= Q_{s'}(A) \ . \end{aligned}$$

# First construction of Brownian motion

- Combining the contruction of the consistent family of finite-dimensional distributions above with the Daniell-Kolmogorov theorem we obtain the following result

## Corollary

*There is a probability measure $P$ on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$, under which the coordinate mapping process*

$$B_t(\omega) = \omega(t) \ , \qquad \omega \in \mathbb{R}^{[0,\infty)} \ , \qquad t \geq 0 \ ,$$

*has stationary, independent increments. An increment $B_t - B_s$, where $0 \leq s < t$, is normally distributed with mean zero and variance $t - s$.*

# First construction of Brownian motion

- The construction of Brownian motion would be over were not for the fact that we have built the process on the sample space $\mathbb{R}^{[0,\infty)}$ rather than on the space $C[0,\infty)$ of continuous functions on the half-line.
  - One could try to overcome this difficulty by showing that the probability measure $P$ of the previous corollary assigns measure one to $C[0,\infty]$.
  - However, $C[0,\infty)$ is not in the $\sigma$-algebra $\mathcal{B}(\mathbb{R}^{[0,\infty)})$ and, therefore, $P[C[0,\infty)]$ is not defined.
  - This failure is due to the fact that the $\sigma$-algebra $\mathcal{B}(\mathbb{R}^{[0,\infty)})$ is "too small" for a space as big as $\mathbb{R}^{[0,\infty)}$; no set in $\mathcal{B}(\mathbb{R}^{[0,\infty)})$ can have restrictions on uncountably many coordinates.
  - In constrast to the space $C[0,\infty)$, it is not possible to determine a function in $\mathbb{R}^{[0,\infty)}$ by specifying its values at only countably many coordinates.

# First construction of Brownian motion

- We avoid the problems described in the slide above by constructing a continuous modification of the coordinate mapping process in the previous corollary.

**Definition**

Let $X$ and $Y$ be stochastic processes on a probability space $(\Omega, \mathcal{F}, P)$.
The process $Y$ is a *modification* of $X$ if, for every $t \geq 0$, we have $P[X_t = Y_t] = 1$.

# First construction of Brownian motion

**Theorem (Kolmogorov, Čentsov (1956))**

*Suppose that a process $X = \{X_t : 0 \le t \le T\}$ on a probability space $(\Omega, \mathcal{F}, P)$ satisfies the condition*

$$E|X_t - X_s|^\alpha \le C|t-s|^{1+\beta} \ , \qquad 0 \le s,t \le T \ ,$$

*for some positive constants $\alpha$, $\beta$ and $C$. Then there exists a continuous modification $\tilde{X} = \{\tilde{X}_t : 0 \le t \le T\}$ of $X$, which is locally Hölder-continuous with exponent $\gamma$ for every $\gamma \in (0, \beta/\alpha)$, i.e.*

$$P\left[\omega : \sup_{0<t-s<h(\omega),\ s,t\in[0,T]} \frac{|\tilde{X}_t(\omega) - \tilde{X}_s(\omega)|}{|t-s|^\gamma} \le \delta\right] = 1 \ ,$$

*where $h(\omega)$ is an a.s. positive random variable and $\delta > 0$ is an appropriate constant.*

# First construction of Brownian motion

## Lemma

*Let $B_t - B_s$, $0 \leq s < t$, be normally distributed with mean zero and variance $t - s$. Then for each positive integer $n$, there is a positive constant $C_n$ for which*

$$E|B_t - B_s|^{2n} = C_n|t - s|^n \ .$$

- Let $I_n = E|B_t - B_s|^{2n}$. Then

$$I_n = \frac{1}{\sqrt{2\pi(t-s)}} \int_{\mathbb{R}} x^{2n} \mathrm{e}^{-x^2/2(t-s)} \mathrm{d}x \ .$$

- Clearly $I_1 = t - s = |t - s|$.
- Integration by parts gives $I_{n+1} = (2n+1)|t - s| I_n$.

# First construction of Brownian motion

## Corollary

*There is a probability measure $P$ on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$, and a stochastic process $W = \{W_t, \mathcal{F}_t^W : t \geq 0\}$ on the same space such that under $P$, $W$ is a Brownian motion.*

- According to Kolmogorov-Čentsov theorem and the previous lemma, there is for each $T > 0$ a modification $W^T$ of the process $B$ in the previous corollary such that $W^T$ is continuous on $[0, T]$.
  - Let
    $$\Omega_T = \{\omega : W_t^T(\omega) = B_t(\omega) \text{ for every rational } t \in [0, T]\} ,$$
    so $P(\Omega_T) = 1$.
  - On $\tilde{\Omega} = \cap_{T=1}^{\infty} \Omega_T$, we have for positive integers $T_1$ and $T_2$, $W_t^{T_1}(\omega) = W_t^{T_2}(\omega)$, for every rational $t \in [0, \min\{T_1, T_2\}]$.
  - Since both processes are continuous on $t \in [0, \min\{T_1, T_2\}]$, we must have $W_t^{T_1}(\omega) = W_t^{T_2}(\omega)$ for every $t \in [0, \min\{T_1, T_2\}]$, $\omega \in \tilde{\Omega}$.
  - Define $W_t(\omega)$ to be this common value.
  - For $\omega \notin \tilde{\Omega}$, set $W_t(\omega) = 0$ for all $t \geq 0$.

# First construction of Brownian motion

- Actually, Kolmogorov-Čentsov theorem gives a bit more than what is stated in the previous result:
  - For $P$ a.e. $\omega \in \mathbb{R}^{[0,\infty)}$, the Brownian sample path $\{W_t(\omega) : 0 \leq t < \infty\}$ is locally Hölder-continuous with exponent $\gamma$, for every $\gamma \in (0, 1/2)$.

# Second construction of Brownian motion

- Suppose that $\{B_t, \mathcal{F}_t : t \geq 0\}$ is a Brownian motion, fix $0 \leq s < t < \infty$ and set $\theta = (t+s)/2$.
- Conditioned on $B_s = x$ and $B_t = z$, the random variable $B_\theta$ is normal with mean $\mu = (x+z)/2$ and variance $\sigma^2 = (t-s)/4$.
  - Knowing the distribution and independence of the increments $B_s$, $B_\theta - B_s$ and $B_t - B_\theta$ leads to the joint density

$$\begin{aligned} & P[B_s \in \mathrm{d}x, B_\theta \in \mathrm{d}y, B_t \in \mathrm{d}z] = \\ & \quad = \quad p(s,0,x)p\left(\frac{t-s}{2}, x, y\right) p\left(\frac{t-s}{2}, y, z\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\ & \quad = \quad p(s,0,x)p\left(t-s, x, z\right) \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-(y-\mu)^2/2\sigma^2} \mathrm{d}x\mathrm{d}y\mathrm{d}z \end{aligned}$$

  - Dividing by $P[B_s \in \mathrm{d}x, B_t \in \mathrm{d}z] = p(s,0,x)p\left(t-s,x,z\right)\mathrm{d}x\mathrm{d}z$ we get

$$P[B_\theta \in \mathrm{d}y | B_s \in \mathrm{d}x, B_t \in \mathrm{d}z] = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-(y-\mu)^2/2\sigma^2} \mathrm{d}y \ .$$

# Second construction of Brownian motion

- The simple form of the conditional distribution for $B_{(t+s)/2}$ suggests that we can construct Brownian motion on some finite time-interval, say $[0,1]$, by interpolation.
- Once we have completed the construction on $[0,1]$, patching together a sequence of such Brownian motions will result in a Brownian motion defined for all $t \geq 0$.

# Second construction of Brownian motion

- Let $(\Omega, \mathcal{F}, P)$ be a probability space.
- Let $I(n)$ be the set of odd integers between 0 and $2^n$
  $I(0) = \{1\}$, $I(1) = \{1\}$, $I(2) = \{1, 3\}$, etc. ...
  - Consider a countable collection $\{\xi_k^{(n)} : k \in I(n), n = 0, 1, ...\}$ of independent, standard normal variables on $(\Omega, \mathcal{F}, P)$.
  - For each $n \geq 0$, define a process $B^{(n)} = \{B_t^{(n)} : 0 \leq t \leq 1\}$ by recursion and linear interpolation, as follows:
    - For $n \geq 1$, $B^{(n)}_{k/2^{n-1}} = B^{(n-1)}_{k/2^{n-1}}$, for $k = 0, 1, ..., 2^{n-1}$.
    - Thus, for each value of $n$, we only need to specify $B^{(n)}_{k/2^n}$ for $k \in I(n)$.
    - Set $B_0^{(0)} = 0$, $B_1^{(0)} = \xi_1^{(0)}$.
    - If the values of $B^{(n-1)}_{k/2^{n-1}}$, $k = 0, 1, ..., 2^{n-1}$ have been specified (and $B_t^{(n-1)}$ defined for $0 \leq t \leq 1$ be piecewise-linear interpolation) and $k \in I(n)$, we denote

$$s = \frac{k-1}{2^n} \ , \ t = \frac{k+1}{2^n} \ , \ \mu = \frac{1}{2}(B_s^{(n-1)} + B_t^{(n-1)}) \ , \ \sigma^2 = \frac{1}{4}(t-s) = \frac{1}{2^{n+1}}$$

and set

$$B^{(n)}_{k/2^n} = B^{(n)}_{(t+s)/2} = \mu + \sigma \xi_k^{(n)} \ .$$

# Second construction of Brownian motion

- Our goal is show that, almost surely, $B_t^{(n)}$ converges uniformly in $t$ to a continuous function $B_t$, and $\{Bt, \mathcal{F}_t^B : 0 \leq t \leq 1\}$ is a Brownian motion.
- The first step is to give a more convenient representation for the processes $B^{(n)}$, $n = 0, 1, ....$
  - Define the *Haar functions* by:
    - $H_1^{(0)}(t) = 1$, $0 \leq t \leq 1$;
    - for $n \geq 1$, $k \in I(n)$, $H_k^{(n)}(t)$ is given by

$$H_k^{(n)}(t) = \begin{cases} 2^{(n-1)/2} \, , & \frac{k-1}{2^n} \leq t < \frac{k}{2^n} \\ -2^{(n-1)/2} \, , & \frac{k}{2^n} \leq t < \frac{k+1}{2^n} \\ 0 \, , & \text{otherwise} \end{cases} .$$

    - Define the *Schauder functions* by

$$S_k^{(n)}(t) = \int_0^t H_k^{(n)}(u) \mathrm{d}u \, , \; 0 \leq t \leq 1 \, , \; n \geq 0 \, , \; k \in I(n) \, .$$

    - Note that $S_1^{(0)}(t) = t$ and for $n \geq 1$ the graphs of $S_k^{(n)}$ are tents of height $2^{-(n-1)/2}$ centered at $k/2^n$ and nonoverlapping for different values of $k \in I(n)$.
    - Clearly $B_t^{(0)} = \xi_1^{(0)} S_1^{(0)}(t)$.
    - By induction on $n$, we get the following representation for the processes $B^{(n)}$

$$B_t^{(n)} = \sum_{m=0}^{n} \sum_{k \in I(m)} \xi_k^{(m)} S_k^{(m)}(t) \, , \; 0 \leq t \leq 1 \, , \; n \geq 0 \, .$$

# Second construction of Brownian motion

## Lemma

*As $n \to \infty$, the sequence of functions $\{B_t^{(n)} : 0 \leq t \leq 1\}$, $n \geq 0$, converges uniformly in $t$ to a continuous function $\{B_t(\omega) : 0 \leq t \leq 1\}$ for a.e. $\omega \in \Omega$.*

- Define $b_n = \max_{k \in I(n)} |\xi_k^{(n)}|$.
  - For $x > 0$

$$P[|\xi_k^{(n)}| > x] = \sqrt{\frac{2}{\pi}} \int_x^\infty \mathrm{e}^{-u^2/2} \mathrm{d}u \leq \sqrt{\frac{2}{\pi}} \int_x^\infty \frac{u}{x} \mathrm{e}^{-u^2/2} \mathrm{d}u = \sqrt{\frac{2}{\pi}} \frac{\mathrm{e}^{-x^2/2}}{x} .$$

which, for $n \geq 1$, gives

$$P[b_n > n] = P\left[\cup_{k \in I(n)} \{|\xi_k^{(n)}| > n\}\right] \leq 2^n P[|\xi_1^{(n)}| > n] \leq \sqrt{\frac{2}{\pi}} \frac{2^n \mathrm{e}^{-n^2/2}}{n} .$$

# Second construction of Brownian motion

- We now use Borel-Cantelli lemma.

**Lemma (Borel-Cantelli lemma)**

*Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of events in some probability space $(\Omega, \mathcal{F}, P)$. If $\sum_n P(E_n) < \infty$ then the probability that infinitely many of them occur is zero, i.e. $P(\limsup_{n\to\infty} E_n) = 0$.*

- Since

$$\sum_{n=1}^{\infty} \frac{2^n \mathrm{e}^{-n^2/2}}{n} < \infty \ ,$$

Borel-Cantelli lemma implies that there is a set $\tilde{\Omega}$ with $P(\tilde{\Omega}) = 1$ such that for each $\omega \in \tilde{\Omega}$ there is an integer $n(\omega)$ satisfying $b_n(\omega) \leq n$ for all $n \geq n(\omega)$.

# Second construction of Brownian motion

- We get that

$$
\begin{aligned}
|B_t - B_t^{(n)}| &= \left| \sum_{n=n(\omega)}^{\infty} \sum_{k \in I(n)} \xi_k^{(n)} S_k^{(n)}(t) \right| \\
&\leq \sum_{n=n(\omega)}^{\infty} \sum_{k \in I(n)} |\xi_k^{(n)} S_k^{(n)}(t)| \\
&\leq \sum_{n=n(\omega)}^{\infty} \sum_{k \in I(n)} n 2^{-(n+1)/2} < \infty
\end{aligned}
$$

- Therefore, for $\omega \in \tilde{\Omega}$, $B_t^{(n)}$ converges uniformly in $t$ to a limit $B_t(\omega)$.
- Continuity of $\{B_t(\omega) : 0 \leq t \leq 1\}$ follows from the uniformity of the convergence.

# Second construction of Brownian motion

**Theorem**

*With $\{B_t^{(n)}\}_{n=1}^{\infty}$ as defined above and $B_t = \lim_{n\to\infty} B_t^{(n)}$, the process $\{B_t, \mathcal{F}_t^B : 0 \le t \le 1\}$ is a Brownian motion on $[0,1]$.*

- It is enough to prove that, for $0 = t_0 < t_1 < \ldots < t_n \le 1$, the increments $\{B_{t_j} - B_{t_{j-1}}\}_{j=1}^{n}$ are independent, normally distributed, with mean zero and variance $t_j - t_{j-1}$.
    - This follows by proving the following equality, for $\lambda_j \in \mathbb{R}$, $j = 1, \ldots, n$:

$$E\left[\exp\left\{\mathrm{i}\sum_{j=1}^{n}\lambda_j(B_{t_j} - B_{t_{j-1}})\right\}\right] = \prod_{j=1}^{n}\exp\left\{-\frac{1}{2}\lambda_j{}^2(t_j - t_{j-1})\right\} .$$

    - The equality is proved using the sequence $\{B_t^{(n)}\}_{n=1}^{\infty}$ and the independence and standard normality of the random variables $\{\xi_k^{(n)}\}$.

# Second construction of Brownian motion

**Corollary**

*There is a probability space* $(\Omega, \mathcal{F}, P)$ *and a stochastic process* $B = \{B_t, \mathcal{F}_t^B : 0 \leq t < \infty\}$ *on it, such that* $B$ *is a standard, one-dimensional Brownian motion.*

- According to the previous theorem, there is a sequence $(\Omega_n, \mathcal{F}_n, P_n)$, $n = 1, 2, \ldots$ of probability spaces together with a Brownian motion $\{X_t^{(n)} : 0 \leq t \leq 1\}$ on each space.
- Let $\Omega = \Omega_1 \times \Omega_2 \times \ldots$, $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \ldots$, and $P = P_1 \times P_2 \times \ldots$.
- Define $B$ on $\Omega$ recursively by

$$
\begin{aligned}
B_t &= X_t^{(1)} , \qquad 0 \leq t \leq 1 \\
B_t &= B_n + X_{t-n}^{(n+1)} , \qquad n \leq t \leq n+1 .
\end{aligned}
$$

- The process $B$ is clearly continuous and its increments are independent and normal with mean zero and the proper variances.