Avila-Bochi formula for certain products of 2×2 matrices

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Outline

1. $\mathrm{SL}(2,\mathbb{R})$-cocycles
2. Herman's theorem
3. Avila-Bochi improvement

$\mathrm{SL}(2,\mathbb{R})$-cocycles

Let

- (X,μ) compact probability space
- $f\colon X \to X$ μ-preserving, **ergodic**
- $A\colon X \to \mathrm{SL}(2,\mathbb{R})$ measurable
- $\int \log \|A\| \, d\mu < +\infty$
- $A_n(x) = A(f^{n-1}(x)) \dots A(x)$

KAN Iwasawa decomposition

$$A(x) = R_\varphi(x)\, H(x)\, N(x) = R_{\varphi(x)}\, T(x)$$

where

$$R_x = \begin{bmatrix} \cos(2\pi x) & -\sin(2\pi x) \\ \sin(2\pi x) & \cos(2\pi x) \end{bmatrix}, \quad T(x) = \begin{bmatrix} c(x) & b(x) \\ 0 & c(x)^{-1} \end{bmatrix}$$

and $c(x) \geq 1$

Remark

For $A \in \mathrm{SL}(2,\mathbb{R})$*:*

- $\|A\| = \sup_{\|v\|_2=1} \|A\,v\|_2$
- $\|A\| = \sqrt{\rho(A^T A)} = \sqrt{\beta + \sqrt{\beta^2 - 1}}$ *where* $\beta = \frac{1}{2}\sum_{ij} A_{ij}^2$
- $\|A\| + \|A\|^{-1} = \sqrt{2(\beta+1)} = |c + c^{-1} + ib|$
- $\|A\| = \|A^{-1}\| \geq 1$

Outline

(Upper) (fiber) Lyapunov exponent

$$\lambda(A) = \lim_{n\to+\infty} \frac{1}{n} \int_X \log \|A_n(x)\| \, d\mu(x)$$

(generalizes to complex matrices)

Theorem (Herman)

$$\int_0^1 \lambda(R_\theta A)\, d\theta \geq \int_X \log \left(\frac{\|A(x)\| + \|A(x)\|^{-1}}{2} \right) d\mu(x)$$

Example (Herman's)

$\varphi(x) = x$, $c(x) = c$, $b(x) = 0$. Thus, $R_\theta A(x) = A(x+\theta)$ and

$$\lambda(A) = \lambda(R_\theta A) \geq \log\left(\frac{c+c^{-1}}{2}\right)$$

Proof

Let

$$S_z = z \begin{bmatrix} \frac{z+z^{-1}}{2} & -\frac{z-z^{-1}}{2i} \\ \frac{z-z^{-1}}{2i} & \frac{z+z^{-1}}{2} \end{bmatrix}$$

- $S_0 = \frac{1}{2}\left[\begin{smallmatrix} 1 & -i \\ i & 1 \end{smallmatrix}\right]$
- $S_{e^{2\pi it}} = e^{2\pi it} R_t$

Fix $x \in X$. Let

$$C_z = S_{ze^{2\pi i\varphi(x)}} T(x)$$

- $C_{e^{2\pi i\theta}} = e^{2\pi i(\theta+\varphi(x))} R_\theta A(x)$
- $\lambda(R_\theta A) = \lambda(C_{e^{2\pi i\theta}})$

Lemma

The map $\mathbb{C} \to \mathbb{R}_0^+$,

$$z \mapsto \lambda(C_z)$$

is subharmonic.

Therefore,

$$\int_0^1 \lambda(C_{e^{2\pi i\theta}})\,d\theta \geq \lambda(C_0) = \lambda(S_0\,T)$$

Now,

- $$LS_0T(x)L^{-1} = \begin{bmatrix} \frac{c(x)+c(x)^{-1}+ib(x)}{2} & \frac{b(x)-ic(x)^{-1}}{2} \\ 0 & 0 \end{bmatrix}$$
  $$= \begin{bmatrix} \alpha(x) & \beta(x) \\ 0 & 0 \end{bmatrix}$$
  where $L = \left[\begin{smallmatrix} 1 & 0 \\ -i & 1 \end{smallmatrix}\right]$
- $\|(LS_0TL^{-1})_n(x)\| = \prod_{i=1}^{n-1} |\alpha(f^i x)| \left\| \left[\begin{smallmatrix} \alpha(x) & \beta(x) \\ 0 & 0 \end{smallmatrix}\right] \right\|$
- $$\lambda(S_0T) = \lambda(LS_0TL^{-1}) \geq \lim \frac{1}{n} \sum_{i=0}^{n-1} \log |\alpha(f^i x)|$$

Using Birkhoff's ergodic theorem

$$\begin{aligned}\int_0^1 \lambda(R_\theta A)\, d\theta &\geq \lambda(LS_0TL^{-1}) \\ &\geq \int_X \log|\alpha(x)|\, d\mu(x) \\ &= \int_X \log \frac{|c(x)+c(x)^{-1}+ib(x)|}{2}\, d\mu(x) \\ &= \int_X \log\left(\frac{\|A(x)\|+\|A(x)\|^{-1}}{2}\right)\, d\mu(x)\end{aligned}$$

End of the proof of theorem.

Outline

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Avila-Bochi improvement

Theorem (Avila-Bochi-Herman)

$$\int_0^1 \lambda(R_\theta A)\, d\theta = \int_X \log\left(\frac{\|A(x)\| + \|A(x)\|^{-1}}{2}\right)\, d\mu(x)$$

Example (Herman's revisited)

$A(x) = R_x \begin{bmatrix} c & 0 \\ 0 & \frac{1}{c} \end{bmatrix}$,

$$\lambda(A) = \lambda(R_\theta A) = \log\left(\frac{c + c^{-1}}{2}\right)$$

Fix $x \in X$. Let

$$B_{\theta,n} = (R_\theta A)_n(x) = R_\theta A(f^{n-1}x)\dots R_\theta A(x)$$
$$N(A) = \log\left(\frac{\|A\| + \|A\|^{-1}}{2}\right)$$

Theorem (Avila-Bochi formula)

$$\int_0^1 \log \rho(B_{\theta,n})\, d\theta = \sum_{i=0}^{n-1} N(A(f^i x)) = \int_0^1 N(B_{\theta,n})\, d\theta$$

Proof of Theorem A-B-H

$$\begin{aligned}
\int_0^1 \lambda(R_\theta A)\, d\theta &= \lim \frac{1}{n} \int_0^1 \log \|B_{\theta,n}\|\, d\theta \quad \text{(dominated conv thm)} \\
&= \lim \frac{1}{n} \int_0^1 N(B_{\theta,n})\, d\theta \quad (|\log A - \log N(A)| \leq \log 2) \\
&= \lim \frac{1}{n} \sum_{i=0}^{n-1} N(A(f^i x)) \quad \text{(A-B formula)} \\
&= \int_X N(A(x))\, d\mu(x) \quad \text{(Birkhoff)} \\
&= \int_X \log\left(\frac{\|A(x)\| + \|A(x)\|^{-1}}{2}\right) d\mu(x)
\end{aligned}$$

Proof of Avila-Bochi formula

Let for every $z \in \mathbb{C}$

$$S_z = z \begin{bmatrix} \frac{z+z^{-1}}{2} & -\frac{z-z^{-1}}{2i} \\ \frac{z-z^{-1}}{2i} & \frac{z+z^{-1}}{2} \end{bmatrix} \qquad C_z = S_z A(x)$$

- $S_{e^{2\pi i\theta}} = e^{2\pi i\theta} R_\theta$
- $(C_{e^{2\pi i\theta}})_n = e^{2\pi i n\theta} B_{\theta,n}$
- $\rho(B_{\theta,n}) = \rho((C_{e^{2\pi i\theta}})_n)$
- $\det((C_0)_n) = \det(S_0)^n = 0$, thus 0 is e-value of $(C_0)_n$

Proposition (A-B)

1. $z \mapsto \log \rho((C_z)_n)$ *harmonic on* $\mathbb{D}$, *continuous on* $\overline{\mathbb{D}}$
2. $\log \rho((C_0)_n) = \sum_{i=0}^{n-1} N(A(f^i x))$

Remark

Herman proved that $z \mapsto \lambda(C_z)$ *is subharmonic*

First equality of Avila-Bochi formula

$$\int_0^1 \log \rho(B_{\theta,n})\, d\theta = \int_0^1 \log \rho((C_{e^{2\pi i\theta}})_n)\, d\theta = \log \rho((C_0)_n)$$

Remark

- $\int_0^1 \log \rho(R_{\theta'} B_{\theta,n})\, d\theta = \int_0^1 \log \rho(B_{\theta,n})\, d\theta$
- $\int_0^1 \log \rho(R_{\theta'} B_{\theta,n})\, d\theta' = N(B_{0,n})$

Second equality of Avila-Bochi formula

$$\begin{aligned}\sum_{i=0}^{n-1} N(A(f^i x)) &= \int_0^1 \log \rho(B_{\theta,n})\, d\theta \\ &= \int_0^1 \int_0^1 \log \rho(R_{\theta'} B_{\theta,n})\, d\theta\, d\theta' \\ &= \int_0^1 N(B_{\theta,n})\, d\theta\end{aligned}$$

Proof of Prop A-B

Want to show that $(C_z)_n = (S_z A)_n(x)$ and its e-values $\lambda_1(z), \lambda_2(z)$, (holomorphic maps on $\mathbb{C}$) verify:

- $$z \mapsto \log \rho((C_z)_n) = \log \max_i |\lambda_i(z)|$$
  is harmonic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$
- $\rho((C_0)_n) = \max_i |\lambda_i(0)| = \prod_{i=0}^{n-1} \frac{\|A(f^i x)\| + \|A(f^i x)\|^{-1}}{2}$

Finally

Enough to prove that $|\lambda_1(z)| \neq |\lambda_2(z)|$, $z \in \mathbb{D}$, and compute $\max_i |\lambda_i(0)|$

Remark

$$|\lambda_1| = |\lambda_2| \quad \textit{iff} \quad \frac{(\operatorname{tr} C)^2}{4 \det C} = \frac{(\lambda_1 + \lambda_2)^2}{4\lambda_1 \lambda_2} \in [0,1]$$

- $\det(C_z)_n = z^{2n}$

Let the rational function $Q\colon \mathbb{C}_* \to \mathbb{C}$, holomorphic, be

$$Q(z) = \frac{\operatorname{tr} C_z}{2z^n}$$

$\deg(Q) \leq 2n$ and $\deg(Q') \leq 2n$

Thus

$$|\lambda_1(z)| = |\lambda_2(z)| \quad \text{iff} \quad Q(z) \in [-1,1]$$

So

Want to show that $Q(\mathbb{D}) \cap [-1,1] = \emptyset$

- $Q(e^{2\pi i\theta}) = \frac{1}{2}\operatorname{tr}(B_{\theta,n}) \in [-1,1]$ iff $B_{\theta,n}$ is non-hyperbolic ($|\operatorname{tr}| \leq 2$)
- $Q(\partial\mathbb{D}) \subset \mathbb{R}$

Lemma (A-B)

If $\partial\mathbb{D} \cap Q^{-1}([-1,1])$ has at least $2n$ connected components, then

$$Q(\mathbb{D}) \cap [-1,1] = \emptyset$$

Proof.

- $\deg(Q') \leq 2n$ hence $\leq 2n$ zeros of Q'
- Each component of the complement in $\partial\mathbb{D}$ has a zero of Q'
- Thus, there are no zeros of Q' in $Q^{-1}([-1,1])$
- So, each component is diffeo to $[-1,1]$

□

Lemma (A-B)

$\partial\mathbb{D} \cap Q^{-1}([-1,1])$ *has at least* $2n$ *connected components*

Proof.

Uses a topological argument... □