Notes on Furstenberg's paper "Noncommuting Random products", Part II

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Existence & Unicity of Invariant Measures

Theorem (1)

If M is a boundary of G, for every absolutely continuous measure, w.r.t. Haar, with compact support $\mu \in \mathcal{P}(G)$ there is a unique measure $\nu \in \mathcal{P}(M)$ such that $\mu * \nu = \nu$.

For the existence define $P_\mu : \mathcal{P}(M) \to \mathcal{P}(M), \quad P_\mu(\nu) = \mu * \nu$

$$\begin{aligned} M \text{ compact} \quad &\Rightarrow \quad \mathcal{P}(M) \text{ is weakly compact} \\ &\Rightarrow \quad P_\mu \text{ has fixed points} \end{aligned}$$

Stochastic Matrices

Let $P = [P_{i,j}] \in \mathrm{M}_{d\times d}(\mathbb{R})$.

Definition

We say that P is a *stochastic matrix* iff $P_{i,j} \geq 0$ and $\sum_{j=1}^{d} P_{i,j} = 1$.

We write $P^n = [P_{i,j}^{(n)}]$.

Definition

A state $i = 1, \ldots, d$ is called *transient* iff $P_{i,i}^{(n)} = 0 \quad \forall\, n \in \mathbb{N}$.
A stochastic matrix P is called *irreducible* iff $\forall\, i, j = 1, \ldots, d$ non-transient states, $\exists n \in \mathbb{N}$ such that $P_{i,j}^{(n)} > 0$.

Finite State Markov Chains

Definition

A stochastic process $\{X_n : \Omega \to \{1, \ldots, d\}\}_{n\geq 0}$ is called a *Markov process* iff $\exists\, P \in \mathrm{M}_{d\times d}(\mathbb{R})$ stochastic matrix such that $\forall\, i, j = 1, \ldots, d$, $\forall n \in \mathbb{N}$, $\mathbb{P}\,[\, X_{n+1} = j | X_n = i\,] = P_{i,j}$. P is called the *transition probability matrix of* $\{X_n\}_{n\geq 0}$.

Definition

A process $\{X_n\}_{n\geq 0}$ is called *stationary* iff all random variables X_n have the same distribution vector $q \in \Delta^{d-1} = \{\, (x_1, \ldots, x_d) \,:\, x_i \geq 0,\ \sum_{i=1}^d x_i = 1\,\}$.

Kolmogorov Extension Theorem

Assume $P \in \mathrm{M}_{d\times d}(\mathbb{R})$ is a stochastic matrix, $q \in \Delta^{d-1}$. Let $\sigma : \Omega(P) \to \Omega(P)$ be the shift on the Bernoulli space $\Omega(P) = \{\, \underline{\omega} \in \{1,\ldots,d\}^{\mathbb{N}} \,:\, \forall\, n \in \mathbb{N},\ P_{\omega_n,\omega_{n+1}} > 0 \,\}$.

Theorem (Kolmogorov)

The function $\mu_{P,q}$ defined on the cylinders of $\Omega(P)$ by

$$\mu_{P,q}\left(\{\omega_0\} \times \cdots \times \{\omega_n\} \times \{1,\ldots,d\} \times \cdots\right) = q_{\omega_0}\, P_{\omega_0,\omega_1} \, \cdots \, P_{\omega_{n-1},\omega_n}$$

extends uniquely to a measure on $\Omega(P)$, which is σ-invariant iff $q^t\, P = q$.

Let $X_0 : \Omega(P) \to \{1,\ldots,d\}$, $X_0(\underline{\omega}) = \omega_0$. Then $\{X_n = X_0 \circ \sigma^n\}_{n\geq 0}$ is a Markov process on the probability space $(\Omega(P), \mu_{P,q})$, with transition probability matrix P. Furthermore, $\{X_n\}_{n\geq 0}$ is stationary iff $q^t\, P = q$.

Finite State Irreducibility

Theorem

The following statements are equivalent:

1. $\sigma : \Omega(P) \to \Omega(P)$ *is ergodic w.r.t.* $\mu_{P,q}$,
2. P *is irreducible,*
3. $P_\infty = \lim_{n\to\infty} P^n$ *exists, and has constant columns,*
4. *there is a unique* $x \in \Delta^{d-1}$ *such that* $x^t\, P = x$

Absolutely Continuous Random Walk

$\mu \in \mathcal{P}(G)$ determines the *random walk*
$\hat{\mu} : M \to \mathcal{P}(M), \quad x \mapsto \hat{\mu}_x := \mu * \delta_x.$

$\mu \ll$ Haar measure on $G \;\Rightarrow\; \hat{\mu}_x \ll m$, the Riem. meas. on M

We write $x \overset{\hat{\mu}}{\rightsquigarrow} y \;\Leftrightarrow\; \exists n \in \mathbb{N}$ such that $\frac{d\mu_x^n}{dm}(y) > 0.$
$x \in M$ is called *transient* $\;\Leftrightarrow\; x \overset{\hat{\mu}}{\rightsquigarrow} x$ does not hold

Definition

The random walk $\hat{\mu} : M \to \mathcal{P}(M)$ is said to be *irreducible* iff there is $C \subseteq M$, measurable set, such that

1. $x \overset{\hat{\mu}}{\rightsquigarrow} y$ and $x \overset{\hat{\mu}}{\rightsquigarrow} y$, for every $x, y \in C$,
2. x is transient, for almost every $x \in M - C$.

Continuous State Irreducibility

Theorem (Doeblin)

If $\hat{\mu} : M \to \mathcal{P}(M)$ is an absolutely continuous random walk then the following statements are equivalent:

1. *$\hat{\mu}$ is irreducible,*
2. $\hat{\mu}_\infty(A) = \lim_{n\to\infty} \hat{\mu}^n_x(A)$ *exists,* $\forall\, x \in M$, $A \subset M$,
3. *there is a unique measure $\nu \in \mathcal{P}(M)$ such that $\nu = P_\mu(\nu)$*

Irreducibility of $\hat{\mu}$

Let $\mu \in \mathcal{P}(G)$, $\mu \ll$ Haar measure on G.

Definition

We say that $x \in M$ is a *μ-boundary point* iff $\exists$ $\underline{g} \in G^{\mathbb{N}}$ density point of $\mu^{\mathbb{N}}$ such that: $(*)$ $\delta_x = \lim_{n\to\infty} g_n \cdots g_1 \, g_0 \, m$.

Define $W^s(\underline{g}, x) = \{ \, z \in M \, : \, x = \lim_{n\to\infty} g_n \cdots g_1 \, g_0 \, z \, \}$

$$
\begin{aligned}
(*) \quad & \Rightarrow \quad m(W^s(\underline{g}, x)) = 1 \\
& \Rightarrow \quad W^s(\underline{g}, x) \subseteq \{ \, z \in M \, : \, z \overset{\hat{\mu}}{\rightsquigarrow} x \, \} \\
& \quad\quad \Downarrow \\
& \hat{\mu} \quad \text{is irreducible}
\end{aligned}
$$

A-cocycles finite dimension

Let M be a boundary of G.
An *A-cocycle* is any function $\rho : G \times M \to \mathbb{R}$ such that

1. $\rho(k, x) = 0$, for every rotation $k \in G$
2. $\rho(g\, g', x) = \rho(g, g'\, x) + \rho(g', x)$, for $g, g' \in G$, $x \in M$

$\mathcal{W}_G(M)$ denotes the space of all A-cocycles.

Theorem (2)

$\mathcal{W}_G(M)$ has finite dimension.
$\dim \mathcal{W}_G(B(G)) = \dim A$.

An isomorphism

$\mathrm{Hom}(S,\mathbb{R}) = \{\, \chi : S \to \mathbb{R} \,:\, \chi$ is a group character $\}$
Take $x_0 \in B(G)$, and let
$H = \{\, g \in G \,:\, g\, x_0 = x_0 \,\}$ be the isotropy group at x_0.

Theorem

There is an isomorphism
$\Phi : \mathcal{W}_G(B(G)) \to \mathrm{Hom}(S,\mathbb{R}) \simeq \mathrm{Hom}(A,\mathbb{R}),\ \ \rho \to \chi$ *such that*

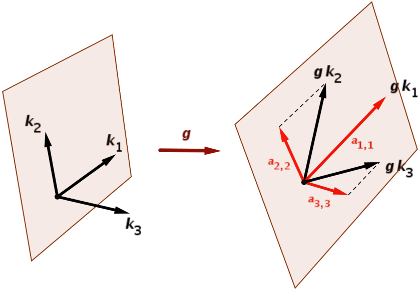
$$\chi(s) = \rho(s, x_0), \quad \forall\, s \in S$$
$$\Updownarrow$$
$$\chi(\Pi_S(g\,k)) = \rho(g, k\, x_0), \quad \forall\, g \in G,\ k \in K$$

where $\Pi_S : G = K\,S \to S$ *is the projection* $\Pi_S(k\,s) = s$.

A Basis of cocycles

$$\rho_i \quad \leftrightarrow \quad \chi_i, \qquad \chi_i(a) = \log a_{i,i}$$

Consider the flag $k\, x_0 = (V_1, \ldots, V_i)$ where $V_j = \langle k_1, \ldots, k_j \rangle$



$$a_{i,i} = \text{ norm of } g : V_i/V_{i-1} \to gV_i/gV_{i-1}$$

$$\sim i^{\text{th}} \text{ most largest expansion of } g$$

A-Cocycles' Pullbacks

Assume M, M' are boundaries of G, and $f : M \to M'$ is a G-equivariant map.

Definition

The *pullback* of $\rho \in \mathcal{W}_G(M')$ is the function $f^*\rho : G \times M \to \mathbb{R}$ defined by $(f^*\rho)(g,x) := \rho(g, f(x))$.

Theorem

The pullback is an injective linear map $f^* : \mathcal{W}_G(M') \to \mathcal{W}_G(M)$.

The Rank of a Boundary

Assume M is a boundary of G.

Definition

We call *rank* of M to the dimension of $\mathcal{W}_G(M)$.

Theorem

For every $1 \leq i < d$, the boundary $\mathcal{F}_{d,i}$ has rank i.

$$\mathbb{P}^{d-1} = \mathcal{F}_{d,1} \preceq \mathcal{F}_{d,2} \preceq \cdots \preceq \mathcal{F}_{d,d-1} = \mathcal{F}_{d,d} = B(G).$$

Some A-cocycles in Flag manifolds

$\rho_1 =$ logarithm-norm in $\mathbb{P}^{d-1}$
$\rho_i =$ logarithm of i^{th} largest expansion in $\mathcal{F}_{d,i}$
$\rho_{V,i} = \rho_1 + \ldots + \rho_i$ logarithm of i-volume in $\mathcal{G}_{d,i} \preceq \mathcal{F}_{d,i}$
$\rho_{J,i} =$ logarithm of jacobian in $\mathcal{F}_{d,i}$

These A-cocycles are related as follows

$$\rho_{J,i} = -(d+i-1)\,\rho_1 - (d+i-3)\,\rho_2 - \cdots - (d+1-i)\,\rho_i$$
$$\rho_{J,d} = -(2d-1)\,\rho_1 - (2d-3)\,\rho_2 - \cdots - 3\,\rho_{d-1} - \rho_d$$
$$\rho_{J,1} = -d\,\rho_1$$

Spherical Functions

Definition

A function $\psi : G \to \mathbb{R}$ is called *left uniformely continuous (l.u.c.)* iff $\lim_{g' \to e} \max_{g \in G} |f(g'\, g) - f(g)| = 0$.

Definition

A function $\psi : G \to \mathbb{R}$ is called *spherical* iff ψ is l.u.c. and for ever $g_1, g_2 \in G$, $\int_K \psi(g_1\, k\, g_2)\, dk = \psi(g_1) + \psi(g_2)$.

Each spherical function $\psi : G \to \mathbb{R}$ satisfies
$\psi(e) = 0$ and $\psi(k_1\, g\, k_2) = \psi(g)$, for $k_1, k_2 \in K$, $g \in G$.

$\mathcal{V}_G$ shall denote the space of all spherical functions $\psi : G \to \mathbb{R}$.

A-cocycles & Spherical Functions

Assume M is a boundary of G, and let m denote the normalized G-invariant Riemannian measure in M.
Define a linear map $\Psi : \mathcal{W}_G(M) \to \mathcal{V}_G$,

$$\Psi : \rho \mapsto \psi_\rho \,, \quad \psi_\rho(g) = \int_M \rho(g,x)\, dm(x) \,.$$

Theorem (3)

Let $M = B(G)$ be the maximal boundary. Then $\Psi : \mathcal{W}_G(M) \to \mathcal{V}_G$ is a linear isomorphism.

Law of Large Numbers

Theorem (4)

If M is a boundary of G, for every measure $\mu \in \mathcal{P}(G)$ of class B_1, every i.i.d. G-valued process $\{X_n\}$ with distribution μ, every cocycle $\rho \in \mathcal{W}_G(M)$, and every $x \in M$, with probability 1,

$$\lim_{n\to\infty} \frac{1}{n}\, \rho(X_{n-1} \cdots X_1\, X_0, x) = \alpha_\mu(\rho)\ .$$

For $G = \mathrm{SL}(d,\mathbb{R})$ theorem (4) reduces to Oseledet's theorem. This is so for the log-norm cocycle ρ_1 on $\mathbb{P}^{d-1}$, but also for $\rho_{V,i}(g,x) = \log \left\| (\wedge^i g)\, x \right\|$ the logarithm of i-volume cocycle, on the Grassman boundary $\mathcal{G}_{d,i} \subset \mathbb{P}(\wedge^i \mathbb{R}^d)$.

Proof of Thm (4)

Denote by $\mathrm{LLN}(\mathrm{M}, \rho)$ the statement of theorem (4) for the boundary M and the A-cocycle ρ. Then

$$\mathrm{LLN}(\mathrm{M}, \rho_{\mathrm{i}}), \quad \forall\, i \quad \Rightarrow \quad \mathrm{LLN}(\mathrm{M}, \textstyle\sum_{\mathrm{i}} \rho_{\mathrm{i}})$$

$f : M \to M'$ is G-equivariant and $\rho = f^* \rho'$

$$\Rightarrow \quad \mathrm{LLN}(\mathrm{M}, \rho) \quad \Leftrightarrow \quad \mathrm{LLN}(\mathrm{M}', \rho')$$

The general case follows because any cocycle ρ on a boundary M is a linear combination of pullbacks of the volume cocycles $\rho_{V,i}$ on the Grassman boundaries $\mathcal{G}_{d,i}$, for $i = 1, \ldots, d$.

Formula for the Largest Lyapunov Exponent

Theorem (5)

Given $\mu \in \mathcal{P}(\mathrm{SL}(d,\mathbb{R}))$ *with compact support such that* $\int \log \|g\| \, d\mu(g) < \infty$, *let* G *be the closed subgroup generated by the support of* μ. *If* G *is irreducible then for every, every i.i.d.* $\mathrm{SL}(d,\mathbb{R})$*-valued process* $\{X_n\}$ *with distribution* μ, *and every* $x \in \mathbb{P}^{d-1}$, *with probability* 1,

$$\lim_{n\to\infty} \frac{1}{n} \log \|X_{n-1} \cdots X_1 \, X_0 \, x\| = \alpha(\mu) \, ,$$

where $\alpha(\mu) = \int_G \int_{\mathbb{P}^{d-1}} \log \|g\, x\| \; d\nu(x) \, d\mu(g)$ *for every measure* $\nu \in \mathcal{P}(\mathbb{P}^{d-1})$ *such that* $\mu * \nu = \nu$.

Largest Lyapunov Exponent: Proof of Thm (5)

$\mathcal{P}_\mu(\mathbb{P}^{d-1}) = \{\, \pi \in \mathcal{P}(\mathbb{P}^{d-1}) \;:\; \mu * \pi = \pi \,\}$
Define $F : G^{\mathbb{N}} \times \mathbb{P}^{d-1} \to G^{\mathbb{N}} \times \mathbb{P}^{d-1}$ by $F(\underline{g}, x) = (\sigma(\underline{g}), g_0\, x)$

$\pi \in \mathcal{P}_\mu(\mathbb{P}^{d-1}) \;\Rightarrow\; \mu \times \pi$ is F-invariant
π is an extremal point of $\mathcal{P}_\mu(\mathbb{P}^{d-1}) \;\Rightarrow\; \mu \times \pi$ is ergodic w.r.t. F
In this case we shall say that π is *ergodic*.

$\hat{\rho} : G^{\mathbb{N}} \times \mathbb{P}^{d-1} \to \mathbb{R}$, $\quad \hat{\rho}(\underline{g}, x) = \log \|g_0\, x\|$ is integrable w.r.t. $\mu \times \pi$, for any measure $\pi \in \mathcal{P}_\mu(\mathbb{P}^{d-1})$

Largest Lyapunov Exponent: Proof of Thm (5)

If $\pi \in \mathcal{P}_\mu(\mathbb{P}^{d-1})$ is ergodic, by Birkhoff's Theorem, for $(\mu^{\mathbb{N}} \times \pi)$-almost every $(\underline{g}, x) \in G^{\mathbb{N}} \times \mathbb{P}^{d-1}$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \log \|X_{n-1} \cdots X_0\, x\| &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \hat{\rho} \circ F^i(\underline{g}, x) \\
&= \int_{G^{\mathbb{N}}} \int_{\mathbb{P}^{d-1}} \hat{\rho}(\underline{g}, x)\, d\pi(x)\, d\mu^{\mathbb{N}}(\underline{g}) \\
&= \int_G \int_{\mathbb{P}^{d-1}} \hat{\rho}(g, x)\, d\pi(x)\, d\mu(g) = \alpha_\pi
\end{aligned}$$

Largest Lyapunov Exponent: Proof of Thm (5)

G irreducible $\Rightarrow \exists\ v_1, \ldots, v_d \in \mathbb{R}^d - \{0\}$ linearly independent for which the limit above holds.

$\Rightarrow$ for $\mu^{\mathbb{N}}$-almost every $\underline{g} \in G^{\mathbb{N}}$ and every $x \in \mathbb{P}^{d-1}$,
$\lim_{n\to\infty} \frac{1}{n} \log \|X_{n-1} \cdots X_0\, x\| \leq \alpha_\pi$.
for every $x \in \mathbb{P}^{d-1}$ corresponds to a vector $v \in \mathbb{R}^d - \{0\}$ which can be written as a linear combination of $v_1, \ldots, v_d$.

$\Rightarrow\ \alpha_\pi \leq \alpha_{\pi'},\ \forall\ \pi, \pi' \in \mathcal{P}_\mu(\mathbb{P}^{d-1})$ both ergodic

$\Rightarrow\ \alpha_\pi = \alpha_{\pi'},\ \forall\ \pi, \pi' \in \mathcal{P}_\mu(\mathbb{P}^{d-1})$ both ergodic

$\Rightarrow\ \alpha_\pi = \alpha_{\pi'},\ \forall\ \pi, \pi' \in \mathcal{P}_\mu(\mathbb{P}^{d-1})$ □

Positive Lyapunov Exponents

Assume $G \subset \mathrm{SL}(d, \mathbb{R})$ is a non-compact closed subgroup such that every subgroup of G with finite index is irreducible.

Theorem (6)

Given $\mu \in \mathcal{P}(\mathrm{SL}(d, \mathbb{R}))$ with compact support such that $\int \log \|g\| \, d\mu(g) < \infty$, if G is the closed subgroup generated by the support of μ then $\alpha(\mu) > 0$.

Follows from theorems (7) and (8) bellow

Zero Lyapunov Exponents

Definition

We say that a measure $\nu \in \mathcal{P}(\mathbb{P}^{d-1})$ is *G-invariant* iff $g\,\nu = \nu$ for every $g \in G$.

Theorem (7)

Assume $\mu \in \mathcal{P}(\mathrm{SL}(d,\mathbb{R}))$ has compact support, $\int \log \|g\|\, d\mu(g) < \infty$, and the closed subgroup $G \subset \mathrm{SL}(d,\mathbb{R})$ generated by supp(μ) is non-compact. Then

$$\alpha(\mu) = 0 \quad \Rightarrow \quad \exists\ \nu \in \mathcal{P}(\mathbb{P}^{d-1})\ G\text{-invariant.}$$

No G-invariant measures

Assume $G \subset \mathrm{SL}(d, \mathbb{R})$ is a non-compact closed subgroup such that every subgroup of G with finite index is irreducible.

Theorem (8)

There is no measure $\nu \in \mathcal{P}(\mathbb{P}^{d-1})$ which is G-invariant.

Assume $\pi \in \mathcal{P}(\mathbb{P}^{d-1})$ is G-invariant.
$\exists \{g_n\} \subset G, \quad \exists \; V_1, V_2 \subset \mathbb{P}^{d-1}$ linear subvarieties s.t.
$\forall x \notin V_1, \quad g_n x \to y \in V_2.$
Because G is non-compact $\exists \; g_n \in G$ s.t. $g_n \to \infty$.
Let $g_n = k_n' \, a_n \, k_n =$ singular value decomp., $a_n \in A$, $k_n, k_n' \in K$.
Assume $k_n \to k$, $k_n' \to k'$ and let $W_1 \subset \mathbb{P}^{d-1}$ be linear subvariety with all directions where the entries of a_n stay bounded, while $W_2 \subset \mathbb{P}^{d-1}$ is the linear subvariety with all directions where the entries of a_n tend to ∞. $W_1 \neq \emptyset$ because $\det g_n = 1$.
Set $V_1 = k^{-1} \, W_1$ and $V_2 = k' \, W_2$.

No G-invariant measures: Proof of Thm (8)

Then $\mathrm{supp}(\pi) \subseteq V_1 \cup V_2$. Take $\{W_1, \ldots, W_k\}$ to be a minimal collection of linear subvarieties of $\mathbb{P}^{d-1}$ such that

$$\mathrm{supp}(\pi) \subseteq W_1 \cup \ldots \cup W_k \ .$$

minimality $\Rightarrow$ $\{W_1, \ldots, W_k\}$ is G-invariant
Denote by $\mathcal{C}(\pi)$ the set of all such collections $\{W_i\}_i$.
$\{W_i\}_i \in \mathcal{C}(\pi) \Rightarrow \{g\, W_i\}_i \in \mathcal{C}(\pi)$, $\forall\, g \in G$, because π is G-invariant. $\{W_i\}_i, \{W'_j\}_j \in \mathcal{C}(\pi) \Rightarrow \{W_i \cap W'_j\}_{i,j} \in \mathcal{C}(\pi)$, and this intersection is strictly smaller unless $\{W_i\}_i = \{W'_j\}_j$.

Whence, $H = \{\, g \in G \,:\, g\, W_i = W_i,\ \forall\, i = 1, \ldots, k \,\}$ is a reducible subgroup of G with finite index, a contradiction. $\square$

Positive Lyapunov Exponents

Denote by m the Riemannian (K-invariant) measure in $\mathbb{P}^{d-1}$.

$$\log \|g\, x\| = -\frac{1}{d} \log\det(D\varphi_g)_x = -\frac{1}{d}\, \log \frac{dg^{-1}m}{dm}(x)\ .$$

Theorem (9 Another Formula)

Given $\pi \in \mathcal{P}(\mathbb{P}^{d-1})$ *such that* $\mu * \pi = \pi$, $\pi \ll m$ *and* $m \ll \pi$,

$$\alpha(\mu) = -\frac{1}{d} \int_G \int_{\mathbb{P}^{d-1}} \log \frac{dg^{-1}\pi}{d\pi}(\xi)\; d\pi(\xi)\, d\mu(g)\ .$$

Positive Lyapunov Exponents: Proof of Thm 9

We have

$$\alpha = \alpha(\mu) = \int\int \log \|g\,x\|\; d\pi(x)\, d\mu(g)$$
$$= -\frac{1}{d} \int\int \log \frac{dg^{-1}m}{dm}(x)\, d\pi(x)\, d\mu(g)$$

Define now

$$\beta = \beta(\mu) = -\frac{1}{d} \int\int \log \frac{dg^{-1}\pi}{d\pi}(x)\, d\pi(x)\, d\mu(g)$$

Then, because $\mu * \pi = \pi$,

Positive Lyapunov Exponents: Proof of Thm 9

$$
\begin{aligned}
d(\alpha-\beta) &= \int\int \log\left(\frac{dg^{-1}m}{dm}\Big/\frac{dg^{-1}\pi}{d\pi}\right)\, d\pi(x)\, d\mu(g)\\
&= \int\int \log\left(\frac{dg^{-1}m}{dg^{-1}\pi}\Big/\frac{dm}{d\pi}\right)\, d\pi(x)\, d\mu(g)\\
&= \int\int \log\frac{dg^{-1}m}{dg^{-1}\pi} - \log\frac{dm}{d\pi}\, d\pi(x)\, d\mu(g)\\
&= \int\int \log\frac{dg^{-1}m}{dg^{-1}\pi}(x)\, d\pi(x)\, d\mu(g) - \int \log\frac{dm}{d\pi}\, d\pi(x)\\
&= \int\int \log\frac{dm}{d\pi}(g\,x)\, d\pi(x)\, d\mu(g) - \int \log\frac{dm}{d\pi}\, d\pi(x)\\
&= \int \log\frac{dm}{d\pi}(y)\, d(\mu*\pi)(y)\, d\mu(g) - \int \log\frac{dm}{d\pi}\, d\pi(x) = 0
\end{aligned}
$$

Positive Lyapunov Exponents: Proof of Thm (10)

Let $\mu \in \mathcal{P}(\mathrm{SL}(d, \mathbb{R}))$ and assume the closed subgroup G generated by $\mathrm{supp}(\mu)$ is non-compact and such that every subgroup of G with finite index is irreducible.

Theorem (10)

$\pi \in \mathcal{P}(\mathbb{P}^{d-1})$, $\mu * \pi = \pi$, $\pi \ll m$ *and* $m \ll \pi \quad \Rightarrow \quad \alpha(\mu) > 0$.

Assume $\alpha(\mu) = 0$. By Jensen's inequality, for $g \in G$,

$$0 = \int \log \frac{dg^{-1}\pi}{d\pi}(x)\, d\pi(x) \leq \log \int \frac{dg^{-1}\pi}{d\pi}(x)\, d\pi(\xi) = \log 1 = 0$$

$$\Rightarrow \quad \frac{dg^{-1}\pi}{d\pi}(x) \text{ is constant for } \pi\text{-a.e. } x \in \mathbb{P}^{d-1}$$

Positive Lyapunov Exponents: Proof of Thm (10)

$\Rightarrow \frac{dg^{-1}\pi}{d\pi}(x) = 1$ for π-a.e. $x \in \mathbb{P}^{d-1}$

for $\frac{dg^{-1}\pi}{d\pi}(x)$ is a probability density.

$\Rightarrow g\,\pi = \pi$

$\Rightarrow$ Contradicts Thm (8).

□

Positive Lyapunov Exponents: Proof of Thm (7)

Take $\mu \in \mathcal{P}(\mathrm{SL}(d,\mathbb{R}))$ and assume $\alpha(\mu) = 0$.

$\mu = \lim_{n\to\infty} \mu_n$ with $\mu_n \ll$ Haar measure on G.

$\pi = \lim_{n\to\infty} \pi_n$ with $\pi_n \ll m \ll \pi_n$, $\mu_n * \pi_n = \pi_n$

Take for instance μ_n with supported on a compact neighbourhood of the orthogonal group K.

Then $\mu * \pi = \pi$, and

$$\alpha(\mu) = \int_G \int_{\mathbb{P}^{d-1}} \log \|g\, x\|\ d\pi(x)\, d\mu(g)$$

$$= \lim_{n\to\infty} \int_G \int_{\mathbb{P}^{d-1}} \log \|g\, x\|\ d\pi_n(x)\, d\mu_n(g)$$

$$0 = \lim_{n\to\infty} -\frac{1}{d} \int_G \int_{\mathbb{P}^{d-1}} \log \frac{dg^{-1}\pi_n}{d\pi_n}(x)\ d\pi_n(x)\, d\mu(g)\ . \qquad (1)$$

Positive Lyapunov Exponents: Lemma (1)

Lemma (1)

Given $\pi_1, \pi_2 \in \mathcal{P}(\mathbb{P}^{d-1})$, *such that* $\pi_1 \ll \pi_2$,

$$-\int_{\mathbb{P}^{d-1}} \log \frac{d\pi_1}{d\pi_2}\, d\pi_1 \geq 4^{-1} \, \|\pi_1 - \pi_2\|^2$$

$\|\pi\| = \sup_{|\phi|\leq 1} \left|\int \phi d\pi\right|$ denotes the *total variation norm of a measure* π. By Jensen's Inequality

$$\int_{\mathbb{P}^{d-1}} \log \frac{d\pi_1}{d\pi_2}\, d\pi_1 \leq \log \int_{\mathbb{P}^{d-1}} \frac{d\pi_1}{d\pi_2}\, d\pi_1 = \log 1 = 0$$

$$\int_{\mathbb{P}^{d-1}} \sqrt{\frac{d\pi_1}{d\pi_2}}\, d\pi_1 \leq \sqrt{\int_{\mathbb{P}^{d-1}} \frac{d\pi_1}{d\pi_2}\, d\pi_1} = 1$$

Positive Lyapunov Exponents:Proof of Lemma (1)

$$\|\pi_1 - \pi_2\| = \int \left| 1 - \frac{d\pi_2}{d\pi_1} \right| d\pi_1 = \int \left| 1 - \sqrt{\frac{d\pi_2}{d\pi_1}} \right| \left| 1 - \sqrt{\frac{d\pi_2}{d\pi_1}} \right| d\pi_1$$

$$\leq \left\{ \int \left(1 - \sqrt{\frac{d\pi_2}{d\pi_1}} \right)^2 d\pi_1 \int \left(1 + \sqrt{\frac{d\pi_2}{d\pi_1}} \right)^2 d\pi_1 \right\}^{1/2}$$

$$= \left\{ \left(2 - 2 \int \sqrt{\frac{d\pi_2}{d\pi_1}} d\pi_1 \right)^2 \left(2 + 2 \int \sqrt{\frac{d\pi_2}{d\pi_1}} d\pi_1 \right)^2 \right\}^{1/2}$$

$$= 2 \left\{ 1 - \left(\int \sqrt{\frac{d\pi_2}{d\pi_1}} d\pi_1 \right)^2 \right\}^{1/2}$$

Positive Lyapunov Exponents: Proof of Lemma (1)

$$\leq 2\left\{1-\left(\exp\frac{1}{2}\int\log\frac{d\pi_2}{d\pi_1}d\pi_1\right)^2\right\}^{1/2}$$

$$\leq 2\left\{-\int\log\frac{d\pi_2}{d\pi_1}d\pi_1\right\}^{1/2}$$

We have used Jensen's inequality

$$\int\sqrt{\frac{d\pi_2}{d\pi_1}}d\pi_1 \geq \exp\int\log\sqrt{\frac{d\pi_2}{d\pi_1}}d\pi_1$$

and

$$1-e^t \leq -t, \quad \forall\, t\in\mathbb{R}$$

Positive Lyapunov Exponents: Proof of Thm (7)

Thus, by (1) and lemma (1),

$$0 = \lim_{n\to\infty} \overbrace{\int_G \left\| g^{-1}\,\pi_n - \pi_n \right\|^2 \, d\mu_n(g)}^{a_n} \; .$$

$\Rightarrow \; \forall\, g$ density point of μ, $\exists \; g_n \to g$ in G s.t.
$\left\| g_n^{-1}\,\pi_n - \pi_n \right\| = \left\| g_n\,\pi_n - \pi_n \right\| \to 0.$

Take $\delta > 0$ and any U neighbourhood of g. There is $\epsilon > 0$ s.t. for all sufficientely large n, $\mu_n(U) \geq \epsilon$ and $a_n < \delta$. Consider any n large enouhg so that $a_n < \delta^2\,\epsilon$. Then there is $g \in U$ such that $\left\| g^{-1}\,\pi_n - \pi_n \right\| < \delta$, for otherwise we would get

$$a_n = \int_G \left\| g^{-1}\,\pi_n - \pi_n \right\|^2 \, d\mu_n(g) \geq \delta^2\,\epsilon > a_n .$$

Positive Lyapunov Exponents: Proof of Thm (7)

$\Rightarrow\ \forall\, g$ density point of μ, $g\,\pi_n - \pi_n \to 0$ weakly.

For any continuous function ϕ,

$$\begin{aligned}
|g\,\pi_n(\phi) - \pi_n(\phi)| &\leq |g\,\pi_n(\phi) - g_n\,\pi_n(\phi)| + \|g_n\,\pi_n(\phi) - \pi_n(\phi)\| \\
&\leq |\pi_n(L_g(\phi)) - \pi_n(L_{g_n}(\phi))| + \|g_n\,\pi_n - \pi_n\| \\
&\leq \underbrace{\|L_g(\phi) - L_{g_n}(\phi)\|_\infty}_{\to 0} + \underbrace{\|g_n\,\pi_n - \pi_n\|}_{\to 0} \to 0
\end{aligned}$$

where $L_g(\phi)(x) = \phi(g\,x)$.

Positive Lyapunov Exponents: Proof of Thm (7)

$\Rightarrow \ \forall g \in \text{supp}(\mu), \ \ g\,\pi = \pi$.

First we prove this for every g density point of μ, using the weak continuity of the action $G \times \mathcal{P}(\mathbb{P}^{d-1}) \to \mathcal{P}(\mathbb{P}^{d-1})$, and the fact that $g_n\,\pi_n - \pi_n \to 0$ weakly. But μ-density points are dense in $\text{supp}(\mu)$.

$\Rightarrow \ \pi$ is G-invariant.

Because the group G is generated by $\text{supp}(\mu)$.

□