Notes on Furstenberg's paper "Noncommuting Random products", Part I

Pedro Duarte

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Linear Cocycles

Let G be a semisimple group of $d \times d$ matrices, (X, μ) be a probability space, $T : X \to X$ a μ-preserving map, and $A : X \to G$ a measurable function.

Definition

We call *G-linear cocycle* to the skew-product map determined by T and A,

$$F : X \times \mathbb{R}^d \to X \times \mathbb{R}^d \qquad F(x, v) = (T(x), A(x)\, v)$$

whose iterates are given by $F^n(x, v) = (T^n(x), A^n(x)\, v)$ where $A^n(x) = A(T^{n-1}(x)) \cdots A(T(x))\, A(x)$

Cocycle Actions

Let $(T, *)$ be a group or semigroup.

Definition

We call *G-linear cocycle action of T* to an action of T on the trivial bundle $X \times \mathbb{R}^d$,

$$F : T \times (X \times \mathbb{R}^d) \to X \times \mathbb{R}^d \qquad F^t(x, v) = (T^t(x), A(t, x)\, v)$$

for which there is some measure μ preserved by every base map $T^t : X \to X$, and such that the action is linear on each fiber

- $A(t, x) \in G$, for $t \in T$, $x \in X$
- $A(1, x) = I$,
- $A(t' * t, x) = A(t', T^t(x))\, A(t, x)$, for $t', t \in T$, $x \in X$

Integrability

Definition

The cocycle $F = (\mu, T, A)$ is said to be *integrable* iff

$$\int_X \log^+ \|A(x)\| \; d\mu(x) < \infty \, .$$

where $\log^+(x) = \max\{\log x, 0\}$.

Oseledet's Theorem (Non-invertible case)

Assume G is any subgroup of $\mathrm{SL}(d, \mathbb{R})$.

Theorem (V. Oseledet)

If the cocycle (μ, T, A) is integrable then there are measurable functions $n(x) \in \mathbb{N}$ and $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_{n(x)}(x)$ and an F-invariant measurable filtration $\mathbb{R}^d = E_1(x) \supset E_2(x) \supset \ldots \supset E_{n(x)}$ such that for μ-almost every $x \in X$,

1. $\lambda_i(x) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\, v\|, \quad \forall\, v \in E_i(x) - E_{i+1}(x)$
2. $\sum_{i=1}^{n(x)} \lambda_i(x)\, (\dim E_i(x) - \dim E_{i+1}) = 0$

Lyapunov Exponents

The numbers $\lambda_i(x)$ are called the *Lyapunov exponents* of the linear cocycle $F = (\mu, T, A)$. These numbers are independent of x whenever T is ergodic w.r.t. μ.

The largest Lyapunov exponent is

$$\lambda_1(x) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\|$$

Oseledet's Theorem (Invertible case)

Assume T is invertible.

Theorem (V. Oseledet)

If the cocycle (μ, T, A) and its inverse (μ, A^{-1}, T^{-1}) are integrable then there are measurable functions $n(x) \in \mathbb{N}$ and $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_{n(x)}(x)$ and an F-invariant measurable decomposition $\mathbb{R}^d = \oplus_{i=1}^{n(x)} E_i(x)$ such that for μ-almost every $x \in X$,

1. $\lambda_i(x) = \lim_{n\to\pm\infty} \frac{1}{n} \log \|A^n(x)\, v\| \quad \forall\, v \in E_i(x)$
2. $\sum_{i=1}^{n(x)} \lambda_i(x) \dim E_i(x) = 0$

Furstenberg's perspective: Law of Large Numbers

Theorem

Given any sequence of i.i.d. real valued random variables $X_n(\omega)$ on some probability space (Ω, P), for P-almost every $\omega \in \Omega$,

$$\lim_{n\to\infty} \frac{1}{n} \left(X_0(\omega) + \cdots + X_{n-1}(\omega)\right) = \mu \ ,$$

where $\mu = \mathbb{E}(X_n)$ is the common expected value of the random variables $X_n(\omega)$.

The same theorem holds for i.i.d. processes valued in any commutative group.

Multiplicative L.L.N.

Theorem

Given any sequence of i.i.d. random variables $X_n(\omega)$ valued in the multiplicative group of positive real numbers $(\mathbb{R}_+, \cdot)$, for almost every $\omega \in \Omega$,

$$\lim_{n\to\infty} \frac{1}{n} \log(X_{n-1} \cdots X_1\, X_0)(\omega) = \mathbb{E}(\log X_n) \; .$$

Does such a theorem holds for i.i.d. processes valued in a noncommutative group G?

Difficulties with a noncommutative L.L.N.

For any non-commutative semisimple Lie group G,

1. There is no continuous globally defined logarithm function $\log : G \to \mathfrak{g}$, characterized by the functional equation

$$\log(a\, b) = \log(a) + \log(b) \ .$$

2. There is no continuous non-vanishing group homomorphism $f : G \to (\mathbb{R}, +)$.

Furstenberg's A-cocycles

Assume $G \times M \to M$ is some action on M.

Definition

We call *A-cocycle* to any function $\rho : G \times M \to \mathbb{R}$ such that

1. $\rho(k, x) = 0$, for every rotation $k \in G$
2. $\rho(g\, g', x) = \rho(g, g'\, x) + \rho(g', x)$, for $g, g' \in G$, $x \in M$

Each A-cocycle determines a 1-*dimensional linear cocycle action of* G on the trival bundle $M \times \mathbb{R}$,

$$F : G \times (M \times \mathbb{R}) \to M \times \mathbb{R}, \qquad F^g(x, v) = (g\, x, e^{\rho(g,x)}\, v)\ .$$

Examples of A-cocycles

The *log-norm* $\rho : G \times \mathbb{R}^d - \{0\} \to \mathbb{R}^d - \{0\}$,
$\rho(g, x) = \log \|g\,x\|/\|x\|$ is an A-cocycle which induces another A-cocycle, still referred as the *norm-logarithm cocycle*, on the real projective space $\rho : G \times \mathbb{P}^{d-1} \to \mathbb{P}^{d-1}$.

The *Jacobian cocycle* $\rho : G \times \mathbb{P}^{d-1} \to \mathbb{P}^{d-1}$, is defined by
$\rho(g, x) = \log \det [(D\varphi_g)_x]$ where $\varphi_g : \mathbb{P}^{d-1} \to \mathbb{P}^{d-1}$, $\varphi_g(x) = g\,x$.

Let G be the Möbius group (linear fractional transformations), and $\mathbb{D}$ be the Poincaré disk. Then $\rho : G \times \mathbb{D} \to \mathbb{R}$, $\rho(g, z) = \log |g'(z)|$ is an A-cocycle, called the log-derivative cocycle.

Furstenberg's Theorems (I)

Let G be a semisimple group with finite center. Denote by $\mathcal{P}(M)$ the compact convex space of probability measures on M, and by $\mathcal{W}_G(M)$ the space of A-cocycles $\rho : G \times M \to \mathbb{R}$.

Theorem

If M is a boundary of G, for every absolutely continuous measure, w.r.t. Haar, with compact support $\mu \in \mathcal{P}(G)$ there is a unique measure $\nu \in \mathcal{P}(M)$ such that $\mu * \nu = \nu$.

Theorem

If M is a boundary of G then $\mathcal{W}_G(M)$ is a finite dimensional vector space.

A Functional on A-cocycles

A measure $\mu \in \mathcal{P}(G)$ is said to be of *class* B_1 iff it has compact support, it is absolutely continuous w.r.t. Haar, and every A-cocycle ρ on the maximal boundary M of G is integrable,

$$\int_G \sup_{x \in M} |\rho(g,x)| \, d\mu(g) < \infty \, .$$

Given a probability of class B_1, $\mu \in \mathcal{P}(G)$, we can define a linear functional

$$\alpha_\mu : \mathcal{W}_G(M) \to \mathbb{R}, \quad \alpha_\mu(\rho) = \int_G \int_M \rho(g,x) \, d\nu(x) \, d\mu(g) \, ,$$

where $\nu \in \mathcal{P}(M)$ is the unique μ-stationary measure $(\mu * \nu = \nu)$.

Furstenberg's Theorems (II)

Theorem (Non-commutative Law of Large Numbers)

If M is a boundary of G, for every measure $\mu \in \mathcal{P}(G)$ of class B_1, every i.i.d. G-valued process $\{X_n\}$ with distribution μ, every cocycle $\rho \in \mathcal{W}_G(M)$, and every $x \in M$, with probability 1,

$$\lim_{n\to\infty} \frac{1}{n}\,\rho(X_{n-1}\cdots X_1\,X_0, x) = \alpha_\mu(\rho)\ .$$

Furstenberg's Context in our Linear Cocycles' Setting

Let G be a subgroup of $\mathrm{SL}(d,\mathbb{R})$, and $\mu \in \mathcal{P}(G)$ a measure of class B_1. Consider also $X = G^{\mathbb{N}}$ and $\mu^{\mathbb{N}}$ product measure in X,
$T : G^{\mathbb{N}} \to G^{\mathbb{N}}$ the shift map $T(g_n)_{n\geq 0} = (g_{n+1})_{n\geq 0}$,
$A : G^{\mathbb{N}} \to G$ which "observes" the first matrix $A(g_n)_{n\geq 0} = g_0$,
$F = (\mu, T, A)$ a G-linear cocycle,
$\mathbb{P}^{d-1}$, which is a boundary of the group $\mathrm{SL}(d,\mathbb{R})$,
$\rho \in \mathcal{W}_{\mathrm{SL}(d,\mathbb{R})}(\mathbb{P}^{d-1})$ the norm-logarithm A-cocycle.
Then $\alpha_\mu(\rho)$ = largest Lyapunov exponent of $F = (\mu, T, A)$.

Irreducibility

Let G be a subgroup of $\mathrm{SL}(d, \mathbb{R})$.

Definition

A subspace $V \subset \mathbb{R}^d$ is said to be *G-invariant* iff $g\,V = V$ for every $g \in G$.

Definition

The group G is said to be *irreducible* iff $\{0\}$ and $\mathbb{R}^d$ are the only G-invariant subspaces of $\mathbb{R}^d$.

Furstenberg's Theorems (III)

Let G be a subgroup of $\mathrm{SL}(d, \mathbb{R})$.

Theorem

Given $\mu \in \mathcal{P}(\mathrm{SL}(d, \mathbb{R}))$ with compact support such that $\int \log \|g\| \, d\mu(g) < \infty$, let G be the closed subgroup generated by the support of μ. If G is irreducible then for every, every i.i.d. $\mathrm{SL}(d, \mathbb{R})$-valued process $\{X_n\}$ with distribution μ, and every $x \in \mathbb{P}^{d-1}$, with probability 1,

$$\lim_{n\to\infty} \frac{1}{n} \log \|X_{n-1} \cdots X_1 \, X_0 \, x\| = \alpha(\mu) \; ,$$

where $\alpha(\mu) = \int_G \int_{\mathbb{P}^{d-1}} \log \|g\, x\| \; d\nu(x) \, d\mu(g)$ for every measure $\nu \in \mathcal{P}(\mathbb{P}^{d-1})$ such that $\mu * \nu = \nu$.

A Trivial Example

Consider $g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ and $\mu = \delta_g$.

The subgroup generated by $\mathrm{supp}(\mu)$, $G = \{ g^n : n \in \mathbb{Z} \}$, is reducible.

If $p_1, p_2 \in \mathbb{P}^1$ correspond to the two eigen-directions of g then δ_{p_1} and δ_{p_2} are the only two μ-stationary measures.
$\int_G \int_{\mathbb{P}^{d-1}} \log \|g\, x\| \; d\nu(x)\, d\mu(g)$ takes the values $\log\lambda$ and $-\log\lambda$ for $\nu = \delta_{p_1}$ and $\nu = \delta_{p_2}$.

Furstenberg's Example

Consider the matrices $g_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in $\mathrm{SL}(2, \mathbb{R})$ and let $\mu = \frac{1}{2}\,\delta_{g_1} + \frac{1}{2}\,\delta_{g_2}$.

The subgroup G generated by $\mathrm{supp}(\mu)$ is irreducible, but it contains the cyclic subgroup $H \subset G$ generated by g_1 with index $[G : H] = 2$, which is reducible.

If $p_1, p_2 \in \mathbb{P}^1$ correspond to the eigen-directions of g_1 then $\nu = \frac{1}{2}\,\delta_{p_1} + \frac{1}{2}\,\delta_{p_2}$ is the only μ-stationary measure.

The largest Lyapunov exponent is zero,

$$\begin{aligned} \alpha(\mu) &= \int_G \frac{1}{2}\,\log \|g\, p_1\| + \frac{1}{2}\,\log \|g\, p_2\| \; d\mu(g) \\ &= \frac{1}{4}\left(\log \|g_1\, p_1\| + \log \|g_1\, p_2\| + \log \|g_2\, p_1\| + \log \|g_2\, p_2\|\right) = 0 \end{aligned}$$

Furstenberg's Theorems (IV)

In the next theorems $G \subset \mathrm{SL}(d, \mathbb{R})$ is assumed to be a non-compact closed subgroup such that every subgroup of G with finite index is irreducible.

Theorem

There is no measure $\nu \in \mathcal{P}(\mathbb{P}^{d-1})$ which is G-invariant, meaning $g\,\nu = \nu$ for every $g \in G$.

Theorem

Given $\mu \in \mathcal{P}(\mathrm{SL}(d, \mathbb{R}))$ with compact support such that $\int \log \|g\| \, d\mu(g) < \infty$, if G is the closed subgroup generated by the support of μ then $\alpha(\mu) > 0$.

QR Decomposition

Given $g \in \mathrm{GL}(d, \mathbb{R})$, there are unique matrices: k orthogonal and u upper triangular with positive diagonal, such that $g = k\, u$. This decomposition is obtained applying the *Gram-Schmidt orthogonalization process* to the columns of g.

The matrix u can be factored as $u = a\, n$, where a is a diagonal matrix and n an upper triangular matrix with 1's on the diagonal.

The resulting decomposition, $g = k\, \overbrace{a\, n}^{u}$, was generalized to semisimple Lie groups by Kenkichi Iwasawa.

Iwasawa Decomposition

Let G be a connected semisimple real Lie group.

Theorem

Then G has subgoups: K maximal compact , A abelian and N nilpotent such that $G = K \cdot A \cdot N$. For each $g \in G$, there is a unique decomposition $g = k\, a\, n$ with $k \in K$, $a \in A$ and $n \in N$.

The Iwasawa decomposition $G = K \cdot A \cdot N$ is not unique.

The subgroup $S = A \cdot N$ is *solvable* .

Iwasawa Decomposition of $\mathrm{SL}(d, \mathbb{R})$

$\mathrm{SL}(d, \mathbb{R})$ is a connected semisimple real Lie group.

$K = \mathrm{O}(d, \mathbb{R})$	orthogonal matrices
$A = \mathrm{Diag}_+(d, \mathbb{R})$	positive diagonal matrices
$N = \mathrm{UT}_1(d, \mathbb{R})$	upper triangular matrices with 1's on the diagonal
$S = \mathrm{UT}_+(d, \mathbb{R})$	upper triangular matrices with positive diagonal

Iwasawa Decomposition of $\mathrm{Sp}(d, \mathbb{R})$

$\mathrm{Sp}(d, \mathbb{R})$ is a connected semisimple real Lie group.

$K = \mathrm{U}(d, \mathbb{R})$	unitary (sympl. orthog.) matrices
$A = \mathrm{Diag}_+^{\mathrm{sp}}(d, \mathbb{R})$	symplectic positive diagonal matrices
$N = \mathrm{UT}_1^{\mathrm{sp}}(d, \mathbb{R})$	"upper triangular" matrices $\begin{pmatrix} u & * \\ 0 & u^{-T} \end{pmatrix}$ with $u \in \mathrm{UT}_1(d, \mathbb{R})$
$S = \mathrm{UT}_+^{\mathrm{sp}}(d, \mathbb{R})$	"upper triangular" matrices $\begin{pmatrix} u & * \\ 0 & u^{-T} \end{pmatrix}$ with $u \in \mathrm{UT}_+(d, \mathbb{R})$

G-spaces

Definition

We call *G-space* to any manifold M equipped with a transitive action $G \times M \to M$.

For any subgroup $H \subset G$, the quotient $G/H = \{ g\, H \, : \, g \in G \}$ is a G-space with the left multiplication action of G. In particular, G is itself a G-space.

$\mathcal{P}(M)$ denotes the space of probability measures in M with compact support.

Random Walks

Definition

A *random walk* on M is any weakly continuous map $\mu : M \to \mathcal{P}(M)$, $x \mapsto \mu_x$.

The continuity means that $P_\mu^*(\phi)(x) := \int_M \phi(y)\, d\mu_x(y)$ is a continuous function on M whenever $\phi(x)$ is. The operator on continuous functions $P_\mu^* : \mathcal{C}(M) \to \mathcal{C}(M)$ is the adjoint of the Perron operator on measures, $P_\mu : \mathcal{P}(M) \to \mathcal{P}(M)$ defined by $P_\mu(\nu) = \int_G \mu_x\, d\nu(x)$.

A *μ-process* is any M-valued process $\{X_n\}$ such that $\mu_x(A) = \mathbb{P}(X_n \in A | X_{n-1} = x)$, for every $n \geq 1$.

An M-valued process $\{X_n\}$ is called *stationary* iff all X_n have the same distribution probability $\nu \in \mathcal{P}(M)$.

Convolution of Measures

Let M be a compact G-space.
The convolution of measures is the operation
$* : \mathcal{P}(G) \times \mathcal{P}(M) \to \mathcal{P}(M)$ defined by

$$\int_M \phi(x)\, d(\mu * \nu)(x) = \int_G \int_M \phi(g\,x)\, d\mu(g)\, d\nu(x)\,.$$

Given measures $\mu \in \mathcal{P}(G)$, $\nu \in \mathcal{P}(M)$, and points $g \in G$ and $x \in M$, we shall write $g\,\nu = \delta_g * \nu$ and $\mu\,x = \mu * \delta_x$

Convolution and Stationary Measures

Assume M is a compact G-space and $\mu \in \mathcal{P}(G)$.
μ determines the random walk $x \mapsto \mu\, x = \mu * \delta_x$.

Theorem

Consider an i.i.d. process $\{X_n\}$ where each X_n takes values in G with distribution μ, and let W_0 be an M-valued random variable with distribution $\nu \in \mathcal{P}(M)$. Then $W_n = X_n \cdots X_2\, X_1\, W_0$ is a μ-process, and $\{W_n\}$ is stationary iff $\mu * \nu = \nu$.

Boundaries of a Lie Group

Definition

A *boundary of G* is any compact G-space with the property that for every $\pi \in \mathcal{P}(M)$ there is a sequence $g_n \in G$ such that $g_n \pi$ converges weakly to a point mass δ_p, with $p \in M$.

Definition

Given boundaries M and M' of G, $M \preceq M'$ iff M is the epimorphic image of M' by some G-equivariant epimorphism.

The relation $\preceq$ is a partial order on the set of G-equivariant equivalence classes of boundaries of G.

Up to G-equivariant equivalence, there is a unique *maximal boundary of G*, w.r.t. $\preceq$, which is denoted by $B = B(G)$.

Geometry of a Boundary

Let M be a boundary of G.
The group K acts transitively on M, and there is a unique K-invariant Riemannian structure on M, whose associated normalized measure we denote by m.

Definition

We say that *$p \in M$ is an attractive boundary point for $g \in G$* iff the diffeo $\varphi_g : M \to M$, $\varphi_g(x) = g\,x$, has p as its unique attractive fixed point $p = g\,p$, and its basin of attraction has full Riemannian measure, $m(W^s(p)) = 1$.

Geometry of a Boundary

Assume $p \in M$ is an attractive boundary point for $g \in G$, and $\pi \in \mathcal{P}(M)$ is a probability measure such that $\pi(W^s(p)) = 1$. Then $g^n \pi$ converges weakly to the point mass δ_p.

If M is a boundary of G then every point $p \in M$ is an attractive boundary point for some $g \in G$.

B-sugroups of G

Definition

A subgroup $H \subset G$ is called a *B-subgroup* iff G/H is a boundary of G. Given B-subgroups H and H', $H \preceq H'$ iff H is conjugate to a subgroup of H'.

The relation $\preceq$ is a partial order on the set of conjugation classes of B-subgroups

Up to conjugation, there is a unique *minimal B-subgroup of G*, w.r.t. $\preceq$, which is denoted by $H = H(G)$.
Off course $B(G) = G/H(G)$.

Characterization of Boundaries and B-subgroups

Let G be a semisimple Lie group with finite center,
and $L = \{\, k \in K \,:\, k\, a = a\, k,\ \forall\, a \in A \,\} =$ Centralizer of A in K.

Theorem

$H = L \cdot S$ is a minimal B-subroup and
$B = G/H \simeq K/L$ is a maximal boundary of G.

Theorem

Up to conjugacy, every B-subgroup of G has the form $H' = L' \cdot S$, and up to equivariant equivalence every boundary of G has the form $M' = K/L'$, for some subgroup $L \subset L' \subset K$.

Flag Manifolds

Take $1 \leq k \leq d$. The space of k-flags is defined as
$\mathfrak{F}_{d,k} = \{ V_* = (V_1, \ldots, V_k) \,:\, V_1 \subset \ldots \subset V_k \subset \mathbb{R}^d, \text{ dim } V_i = i \}$.
$\mathrm{SL}(d, \mathbb{R})$ acts transitively on k-flags: $\mathrm{SL}(d, \mathbb{R}) \times \mathfrak{F}_{d,k} \to \mathfrak{F}_{d,k}$ by
$g\, V_* = (g\, V_1, \ldots, g\, V_k)$.

Theorem

Each $\mathfrak{F}_{d,k}$ is a boundary of $\mathrm{SL}(d, \mathbb{R})$.
$\mathfrak{F}_{d,d}$ is the maximal boundary of $\mathrm{SL}(d, \mathbb{R})$.

For $k = 1$, $\mathfrak{F}_{d,1} = \mathbb{P}^{d-1}$.

Isotropic Flag Manifolds

Fix a linear symplectic structure on $\mathbb{R}^{2d}$. A k-flag $V_* \in \mathcal{F}_{2d,k}$ is called *isotropic* iff the subspace V_k is isotropic.

Let $\mathcal{F}^{\mathrm{sp}}_{d,k}$ = submanifold of isotropic k-flags in $\mathcal{F}_{2d,k}$.

$\mathcal{F}^{\mathrm{sp}}_{d,k}$ is invariant under $\mathrm{Sp}(d,\mathbb{R})$, and the symplectic group acts transitively there.

Theorem

Each $\mathcal{F}^{\mathrm{sp}}_{d,k}$ is a boundary of $\mathrm{Sp}(d,\mathbb{R})$.
$\mathcal{F}^{\mathrm{sp}}_{d,d}$ is the maximal boundary of $\mathrm{Sp}(d,\mathbb{R})$.

For $k = 1$, $\mathcal{F}^{\mathrm{sp}}_{d,1} = \mathbb{P}^{2d-1}$.

Isotropy Groups of Flag Manifolds

From now on we assume $G = \mathrm{SL}(d, \mathbb{R})$.

Define $\ell_i :=$ i^{th} axis of $\mathbb{R}^d$,
$V_*^{(i)} := (\ell_1, \ell_1 \oplus \ell_2, \ldots, \ell_1 \oplus \cdots \oplus \ell_i) \in \mathcal{F}_{d,i}$,
$H_i = \{ g \in G \, : \, g\, V_*^{(i)} = V_*^{(i)} \} =$ isotropy group of $\mathcal{F}_{d,i}$ at $V_*^{(i)}$.

Then $\quad H_k \subset \ldots \subset H_1 \quad$ and $\quad \mathcal{F}_{d,i} = G/H_i, \quad$ for $i = 1, \ldots, d$.

Orthogonal Isotropy Groups of Flag Manifolds

Let $L_i = \{ k \in K \,:\, k\,\ell_j = \ell_j, \text{ for } j = 1, \ldots, i \}$.
L_d is the group of order 2^{d-1} consisting of diagonal matrices with units ± 1 on the diagonal. It is also the centralizer of A in K, previously denoted by L. These groups satisfy $L_d \subset \ldots \subset L_1$.
Furthermore, $H_i = L_i\, S$, for every $i = 1, \ldots, k$.

Thus $K/L_i = K\,S/(L_i\,S) = G/H_i = \mathcal{F}_{d,i}$.

In particular, K acts transitively on $\mathcal{F}_{d,i}$, and
$L_i =$ isotropy group of this action at $V_*^{(i)}$.

Compact G-spaces

Theorem

If M is a compact G-space then K acts transitively on M.
In particular, $M = K/\Sigma = G/(\Sigma\, S)$ for some subgroup $\Sigma \subset K$.

Therefore, the maximal compact G-space is $K = G/S$.

Action of G on $K = G/S$
Given $g \in G$ and $k \in K$, $\;g * k := k'$, where $g\,k = k'\,u$ is the QR decomposition, with $k' \in K$ and $u \in S$.

K is not a boundary

Given $g \in G$, the diffeomorphism $\varphi_g : K \to K$ is L-equivariant.

Thus, every weak-$*$ limit measure $\nu = \lim_{n\to\infty} g_n\,\pi$, with $\pi \in \mathcal{P}(K)$, must be L-invariant. Since point masses δ_p are not L-invariant, K is not a boundary.

If $k \in K$ is an attractive fixed point of φ_g then all points in $L\,k$ are also fixed points of φ_g with the same character as k. In particular, the attractive fixed points of $\varphi_g : K \to K$ are never unique. The typical limit measures will be convex linear combinations of Dirac measures.

Maximal Boundary and Minimal B-subgroup

The previous argument shows that given a subgroup $\Sigma \subset K$, if $M = K/\Sigma$ is a boundary then $L \subset \Sigma$.

Therefore,
$\mathcal{F}_{d,d} = K/L_d = K/L$ is the maximal boundary of G.

and
H_d is the minimal B-subgroup of G.

Semisimple Groups

Definition

A non-commutative group G is called *semisimple* iff if it has no connected normal solvable subgroup $H \subset G$.

The special linear group $G = \mathrm{SL}(d, \mathbb{R})$, and the symplectic group $G = \mathrm{Sp}(d, \mathbb{R})$ are examples of semisimple groups.

Solvable Groups

Definition

A group G is called *solvable* iff if there is a finite series of normal subgroups $\{1\} = H_0 \subset H_1 \subset \cdots \subset H_m = G$ such that H_i/H_{i-1} is commutative for every $i = 1, \ldots, m$.

The group $G = \mathrm{UT}(d, \mathbb{R})$ of upper triangular matrices is solvable. It admits the series $\{1\} \subset H_1 \subset \cdots \subset H_{d-1} = G$, where H_i is the subgroup of upper triangular matrices whose restriction to the first $d - i + 1$ columns is diagonal.

Nilpotent Groups

The *lower central series* of a group G is the descending series of normal subgroups $G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_n \supseteq \ldots$, where $G_{n+1} = [G_n, G]$ is the subgroup of G generated by all commutators $[x, y] = x^{-1}\, y^{-1}\, x\, y$ with $x \in G_n$ and $y \in G$.

Definition

A group G is called *nilpotent* iff $G_m = \{1\}$ for some $m \geq 1$.

The group $G = \mathrm{UT}_1(d, \mathbb{R})$ of upper triangular matrices with 1's on the diagonal is nilpotent. Its lower central series terminates with $G_{d+1} = \{1\}$.

Center of a Group

Definition

The *center* of a group G is the normal subgroup $Z(G) = \{ g \in G \, : \, g\, h = h\, g, \; \forall\, h \in G \}$.

The center of $\mathrm{GL}(d, \mathbb{R})$ is the group $\{ a\, I \, : \, a \in \mathbb{R} \}$. The semisimple groups $G = \mathrm{SL}(d, \mathbb{R})$ and $G = \mathrm{Sp}(d, \mathbb{R})$ have centers equal to $\{-I, I\}$.

Maximal Compact subgroup

Definition

A *maximal compact subgroup* K of G is any compact subgroup of G which is maximal amongst such subgroups.

Semisimple groups always have maximal compact subgroups. Any compact subgroup $H \subset \mathrm{SL}(d, \mathbb{R})$ preserves some euclidean inner product on $\mathbb{R}^d$. Therefore, the maximal compact subgroups of $\mathrm{SL}(d, \mathbb{R})$ are the conjugates of the orthogonal group $\mathrm{O}(d, \mathbb{R})$.

Equivariant maps

Let M and M' be G-spaces.

Definition

A map $f : M \to M'$ is said to be *G-equivariant* iff $f(g\,x) = g\,f(x)$, for every $x \in M$ and $g \in G$.

A G-equivariant equivalence is any G-equivariant diffeomorphism.