Billiard dynamics

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Outline

Setting

- M is a smooth manifold ($M = I = [0, 1]$),
- $\mathcal{B}$ is a Borel σ-algebra of M,
- m is the Lebesgue measure on M,
- $f : M \to M$ is a measurable map,
- μ is an invariant probability on M.

Physical measure

Definition

The basin $B(\mu)$ of μ is the set of all $x \in M$ such that

$$\frac{1}{n}\sum_{k=0}^{n-1}\delta_{T^k(x)} \xrightarrow{\text{weak*}} \mu.$$

Definition

If $m(B(\mu)) > 0$, then μ is called a physical (or SRB) probability.

Example

Every ergodic and absolutely continuous μ is physical (Birkhoff Ergodic Theorem). **why?**

Piecewise expanding maps

Definition

Let $M = I = [0, 1]$. Then f is piecewise expanding if there exists $S = \{a_0 = 0 < a_1, \ldots, a_N = 1\} \subset I$ such that

1. for each i, $f_{(a_{i-1}, a_i)}$ admits an extension f_i to $[a_{i-1}, a_i]$ that is a C^1 diffeomorphism,
2. there exists $c > 1$ such that $|f_i'| \geq c$ for each i.

Example

- Uniform expanding maps,
- Markov maps,
- ....

Strategy

- Look for acip's of f,
- acip's are fixed points of the Perron-Frobenius P operator of f.

1. First strategy: P is a contraction, and so $\frac{1}{n}\sum_{k=0}^{n-1} P^k g$ may converge to a fixed point of f.
2. Second strategy: show that 1 is an eigenvalue for P, and check that its eigenvectors are non-negative. These vectors are acip's of f.
3. Third strategy: show that P restrcited to the cone of densities $\{g \geq 0\}$ is a strict contraction.

Perron-Frobenius operator

Definition

The Perron-Frobenius operator associated to a non-singular map $f : M \to M$ is the operator $P : L^1(m) \to L^1(m)$ given by

$$g \mapsto Pg = \frac{d(f_* gm)}{dm}, \qquad f \in L^1(m).$$

Properties:

1. P is linear, positive **why?** and bounded;
2. P is a contraction: $\|Pg\|_1 \leq \|g\|_1$;
3. $P^* : L^\infty \to L^\infty$ is the Koopman operator: $\int h(x)Pg(x)dm(x) = \int h(f(x))g(x)dm(x)$,
4. acip's of f are fixed points of P.

Mean ergodic theorem