Herman's subharmonic trick

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Outline

$\mathrm{SL}(2,\mathbb{R})$-cocycles

Uniformly hyperbolic cocycles

Herman's examples

Plurisubharmonic functions

$\mathrm{SL}(2,\mathbb{R})$-cocycles

Let

- (X,μ) compact probability space
- $f\colon X \to X$ μ-preserving
- $A\colon X \to \mathrm{SL}(2,\mathbb{R})$ measurable
- $\int \log \|A\| \, d\mu < +\infty$

Cocycle:

$$\begin{aligned} F = (f, A)\colon X \times \mathbb{R}^2 &\to X \times \mathbb{R}^2 \\ (x, v) &\mapsto (f(x), A(x)\, v) \end{aligned}$$

n-th iteration:

$$F^n(x,v) = (f^n(x), \underbrace{A(f^{n-1}(x)) \dots A(x)}_{=A_n(x)}\, v)$$

Cocycle identity:

$$A_{n+m}(x) = A_m(f^n(x))\, A_n(x)$$

(Upper) (fiber) Lyapunov exponent of F at $x \in X$

$$\lambda(x) = \lim_{n \to +\infty} \frac{1}{n} \log \|A_n(x)\|$$

Remark

For $A \in \mathrm{SL}(2, \mathbb{R})$*:*

- $\|A\| = \sup_{\|v\|_2 = 1} \|A\,v\|_2$
- $\|A\| = \sqrt{\rho(A^T A)} = \sqrt{\beta + \sqrt{\beta^2 - 1}}$ *where* $\beta = \frac{1}{2} \sum_{ij} A_{ij}^2$
- $\|A\| = \|A^{-1}\| \geq 1$

Theorem (Kingman's subadditive ergodic theorem)

There is $L = X \bmod 0$ *such that* $\lambda\colon L \to [0, +\infty[$ *is*

- f*-invariant*
- μ*-integrable and*

$$\int \lambda \, d\mu = \lim \frac{1}{n} \int \log \|A_n\| \, d\mu = \inf_{n \geq 1} \frac{1}{n} \int \log \|A_n\| \, d\mu$$

Remark

If f *is ergodic, then* $\lambda(x) = \lambda = \int \lambda \, d\mu$ *a.e.*
(if $\mu(\Omega) > 0$*, then* $\bigcup_n f^n(\Omega)$ *is* f*-invariant and full measure with constant* λ*)*

Let

- $\mathbb{P}^1 = \{(\cos\theta, \sin\theta) \colon \theta \in [0, 2\pi[\}/_{x\sim -x} \simeq \mathbb{T}^1$ projective space
- $\mathrm{SL}(2,\mathbb{R})$-action $A \cdot z = \frac{A\,z}{\|A\,z\|}, \qquad z \in \mathbb{P}^1$

Theorem (Oseledets)

Let $v \in \mathbb{R}^2$.

1. *If $x \in \lambda^{-1}(\{0\})$, then*

$$\lim \frac{1}{n} \log \|A_n(x)\, v\| = 0$$

2. *There is $E^s \colon \lambda^{-1}(\mathbb{R}^+) \to \mathbb{P}^1$ measurable such that*
   - $A(x) \cdot E^s(x) = E^s(f(x))$
   - $$\lim \frac{1}{n} \log \|A_n(x)\, v\| = \begin{cases} -\lambda(x), & \frac{v}{\|v\|} = E^s(x) \\ \lambda(x), & \text{o.c.} \end{cases}$$

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The cocycle $F = (f, A)$ is **uniformly hyperbolic** iff there is $E^s \in C^0(X, \mathbb{P}^1)$ st $A(x) \cdot E^s(x) = E^s(f(x))$ and attracting.

Theorem (Yoccoz)

Let $A \in C^0$.

- *(f, A) is uniformly hyperbolic iff $\exists c, \tau > 0$ st*

$$\|A_n(x)\| \geq c e^{\tau n}, \qquad n \in \mathbb{N}, \, x \in X$$

- *Spp $f \in \mathrm{Homeo}(X)$ minimal. (f, A) is C^0-conjugated to a $\mathrm{SO}(2, \mathbb{R})$-cocycle iff $\exists x_0 \in X$ st*

$$\|A_n(x_0)\| \leq cst, \quad n \in \mathbb{N}$$

Proposition

Let

- $f(x) = x + \alpha \in \mathbb{T}^1$
- $A \in C^0(\mathbb{T}^1, \mathrm{SL}(2,\mathbb{R}))$ *not homotopic to* I

Then (f, A) *is not uniformly hyperbolic*

Proof.

Representatives of homotopy classes of (f, A, E^s)

$$f_0(x) = mx \quad A_0(x) = R_{nx} \quad E_0^s(x) = \frac{r}{2}x$$

Uniform hyperbolicity
$\Rightarrow$ invariance of a C^0-section $A_0(x) \cdot E_0^s(x) = E_0^s(f_0(x))$
$\Rightarrow$

$$n + \frac{r}{2} = \frac{mr}{2}$$

With $m = 1$ get $n = 0$ and A is homotopic to I □

Theorem (Bochi)

Let $f \in \mathrm{Homeo}(X)$*. Then, for a* C^0*-generic* A*,* (f, A) *is either uniformly hyperbolic or* $\lambda = 0$*.*

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Let

- $D_r = \{z \in \mathbb{C}^d \colon |z_i| < r\}$ polydisk
- $\mathbb{T}^d = \{z \in \mathbb{C}^d \colon |z_i| = r\} \simeq \mathbb{R}^d/\mathbb{Z}^d$ d-torus
- μ Haar measure on $\mathbb{T}^d$
- $f \colon D_r \to \mathbb{C}^d$ holomorphic, $r > 1$, such that
  - $f(D_r) \subset D_r$
  - $f(0) = 0$
  - $f_* \mu = \mu$

Example (Base maps)

1. $f(z) = e^{iA}z$ where $A = \mathrm{diag}(\alpha_1, \ldots, \alpha_d)$, $z \in \mathbb{C}^d$

$$f(e^{ix_1}, \ldots, e^{ix_d}) = (e^{i(x_1+\alpha_1)}, \ldots, e^{i(x_d+\alpha_d)})$$

2. $f(z_1, z_2) = (z_1^2 z_2, z_1 z_2)$ (Anosov)

$$f(e^{ix}, e^{iy}) = (e^{i(2x+y)}, e^{i(x+y)})$$

Let

- $(\mathcal{B}, \|\cdot\|)$ Banach algebra over $\mathbb{C}$
- $\rho(A) = \lim \|A^n\|^{1/n}$ spectral radius of $A \in \mathcal{B}$

Theorem (Herman)

If $A \in C^\omega(D_r, \mathcal{B})$ *and*

$$\begin{aligned} F \colon \mathbb{T}^d \times \mathcal{B} &\to \mathbb{T}^d \times \mathcal{B} \\ (z, v) &\mapsto (f(z), A(z)\, v) \end{aligned}$$

Then

$$\int_{\mathbb{T}^d} \lambda \, d\mu \geq \log \rho(A(0))$$

Proof.

Lemma

If $\varphi\colon \mathbb{C}^d \to \mathbb{R}$ is plurisubharmonic, then

$$\varphi(0) \leq \int_{\mathbb{T}^d} \varphi \, d\mu$$

- $z \mapsto \log \|A_n(z)\|$ is plurisubharmonic
- $\int \log \|A_n\| \, d\mu \geq \log \|A_n(0)\| = \log \|A(0)^n\|$ (since $f(0) = 0$)
- $\lim \frac{1}{n} \int \log \|A_n\| \, d\mu \geq \lim \log \|A(0)^n\|^{1/n} = \log \rho(A(0))$

□

Corollary

Let

- $X = \mathbb{T}^1$
- $f(x) = x + \alpha$ *ergodic*
- $c > 1$
- $A(x) = R_x H \in \mathrm{SL}(2, \mathbb{R})$ *where*

$$R_x = \begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix} \quad \textit{and} \quad H = \begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix}$$

Then

$$\lambda \geq \log\left(\frac{c + c^{-1}}{2}\right)$$

Proof.

Let

$$\widetilde{A}(z) = \begin{bmatrix} \frac{z^2+1}{2} & -\frac{z^2-1}{2i} \\ \frac{z^2-1}{2i} & \frac{z^2+1}{2} \end{bmatrix} H = z \begin{bmatrix} \frac{z+z^{-1}}{2} & -\frac{z-z^{-1}}{2i} \\ \frac{z-z^{-1}}{2i} & \frac{z+z^{-1}}{2} \end{bmatrix} H$$

By Theorem, $\lambda \geq \log \rho(\widetilde{A}(0)) = \log(\frac{c+c^{-1}}{2})$ for cocycle $(f, \widetilde{A})$.
As $\widetilde{A}(e^{ix}) = e^{ix} R_x H = e^{ix} A(x)$ and

$$\|\widetilde{A}_n(e^{ix})\| = \|A_n(x)\|$$

same λ for cocycle (f, A). □

Remark

This example is non-uniformly hyperbolic (since A is not homotopic to the identity).

Corollary (Almost-Mathieu)

Let

$$A(x) = \begin{bmatrix} a\cos x + b & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

Then $\int \lambda \, d\mu \geq \max\{\log |\frac{a}{2}|, 0\}$

Proof.

Let

$$\widetilde{A}(z) = \begin{bmatrix} a\frac{z^2+1}{2} + bz & -z \\ z & 0 \end{bmatrix} = z \begin{bmatrix} a\frac{z+z^{-1}}{2} + b & -1 \\ 1 & 0 \end{bmatrix}$$

So, $\int \lambda \, d\mu \geq \log \rho(\widetilde{A}(0))$ for cocycle $(f, \widetilde{A})$.
Hence $\widetilde{A}(e^{ix}) = e^{ix} A(x)$ and $\|\widetilde{A}_n(e^{ix})\| = \|A_n(x)\|$, and same bound for λ of cocycle (f, A). □

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Subharmonic functions

Let $\Omega \subset \mathbb{C}$ open and connected (region).
$f : \Omega \to \mathbb{R} \cup \{-\infty\}$ is **subharmonic** ($f \in \mathrm{SH}(\Omega)$) iff

- f is upper semicontinuous ($\limsup_{z \to a} f(z) \leq f(a)$)
- $f(a) \leq \int_0^1 f(a + re^{i\theta})\, d\theta$ for any $\overline{B_r(a)} \subset \Omega$

Example

Harmonic functions ($\nabla^2 f = 0$, $f \in C^2$)

Remark

Let $f \in C^\omega(\Omega)$

- $f(a) = \int_0^1 f(a + re^{i\theta})\, d\theta$ *(Cauchy formula).*
- $\Re f$ *and* $\Im f$ *are harmonic thus subharmonic*

Theorem

If $f \in C^{\omega}(\Omega)$ *and* $\neq 0$*, then* $\log|f| \in \mathrm{SH}(\Omega)$

Proof.

□

Plurisubharmonic functions

Let $\Omega \subset \mathbb{C}^d$ open.
$u\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ is **plurisubharmonic** ($u \in P(\Omega)$) iff

- u is upper semicontinuous
- $\forall_{z,w \in \mathbb{C}^d} t \mapsto u(z + tw)$ is subharmonic (where defined $t \in \mathbb{C}$)

Example

$f \in C^\omega(\Omega) \Rightarrow \log|f| \in P(\Omega)$

Theorem

Let $u \in C^2(\Omega)$*.*

$u \in P(\Omega)$ *iff* $[\bar{\partial}_i \partial_j u]_{i,j}$ *positive semidefinite*