Numerical Approximation of Partial Differential Equations Arising in Financial Option Pricing

Fernando Gonçalves

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(revised version)

To Sílvia, my wife.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Fernando Gonçalves*)

Abstract

We consider the Cauchy problem for a second-order parabolic PDE in half spaces, arising from the stochastic modelling of a multidimensional European financial option. To improve generality, the asset price drift and volatility in the underlying stochastic model are taken time and space-dependent and the payoff function is not specified.

The numerical methods and possible approximation results are strongly linked to the theory on the solvability of the PDE. We make use of two theories: the theory of linear PDE in Hölder spaces and the theory of linear PDE in Sobolev spaces.

First, instead of the problem in half spaces, we consider the corresponding problem in domains. This localized PDE problem is solvable in Hölder spaces. The solution is numerically approximated, using finite differences (with both the explicit and implicit schemes) and the rate of convergence of the time-space finite differences scheme is estimated. Finally, we estimate the localization error.

Then, using the L^2 theory of solvability in Sobolev spaces and in weighted Sobolev spaces, the solution of the PDE problem is approximated in space, also using finite differences. The approximation in time is considered in abstract spaces for evolution equations (making use of both the explicit and implicit schemes) and then specified to the second-order parabolic PDE problem. The rates of convergence are estimated for the approximation in space and in time.

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Chapter 1

Introduction

Financial options or derivatives are contingent financial claims and their modelling is made in a stochastic framework (according with the Financial Mathematics theory initiated by the works of Fisher Black and Myron Scholes (1973) and Robert Merton (1973)). We are interested, in particular, in one basic type of financial option: the European option, in its general multidimensional version (the option on a basket of assets).

The European option modelling lies on the stochastic equation describing the dynamic of the underlying asset prices. It is well known that pricing an option can be reduced, with the use of Feynman-Kac formula, to solving the Cauchy problem with a final condition for a second-order parabolic PDE in half spaces, where the parabolic operator's coefficients associated with the first and second-order partial derivatives are unbounded.

The topic of this research is the numerical approximation of the PDE arising from the stochastic financial problem, in this general multidimensional version.

In the available numerical analysis literature, several numerical schemes can be found for the European option price approximation. However, we could not find a systematic approach to the subject, namely considering the PDE problem in its general form (with time and space dependent coefficients and non specified independent term and final condition) and simultaneously producing the rates of convergence for the corresponding approximation schemes. The aim of the present study is to contribute to this systematic approach.

We make some comments on the choice of the European option (in the general multidimensional form) as the derivative type motivating this research. This choice seemed to be appropriate as its general modelling can be applied or be adapted, more or less easily, to the other several types of options with no early exercise. At the same time, the particularities of the study of each of the multiple different types of options are avoided in this first stage. We expect that our numerical approximation study can be used beyond the particular derivative type motivating it.

Finally, we mention that, in this research, we will not put the emphasis in the numerical methods sophistication: the basic finite differences explicit and implicit schemes will be used.

We summarize the chapters' content.

Chapter 2 - European financial options After briefly reviewing the stochastic background for the European option modelling, we consider the simple unidimensional Black-Scholes model and a few of its immediate generalizations. Then, we outline the way the parabolic PDE Cauchy problem arises from the stochastic problem.

Chapter 3 - Parabolic PDE in Hölder spaces: space and time discretization In this chapter, we follow the approach by N. V. Krylov (in Krylov [29]). We approximate the parabolic PDE Cauchy problem in Hölder spaces (imposing that the operator is non-degenerate elliptic in space and its coefficients are bounded). We localize the problem on a bounded domain and study the approximation for this localized problem, using both the implicit and the explicit schemes. Then we estimate the localization error, i.e. the error due to considering the Cauchy problem on a bounded domain instead of the whole space. The main content of the chapter is:

- Existence and uniqueness result for the solution of the discrete problem corresponding to the continuous initial-boundary value problem - this is a result stated in Krylov [29], but proved only for an elliptic problem.

- Estimate for the convergence rate of the discrete problem solution to the corresponding continuous problem solution - this result is also stated in Krylov [29], but proved only for an elliptic problem. We also estimate the rate of convergence for a case where weaker conditions are imposed over the initial data.

- Construction of discrete operators approximating the corresponding continuous operator, using the explicit and implicit schemes - these operators are considered in Krylov [29], but for a more specific example of the equation.

- Stochastic representation of the solutions of the Cauchy and the initial-boundary value problems for a parabolic PDE, under milder conditions and capturing wider situations than we could find in the literature. Estimate of the localization error. These results are obtained for the cases where strong and weak solutions of the corresponding stochastic equation are considered.

Chapter 4 - Parabolic PDE in Sobolev and weighted Sobolev spaces: space discretization We consider the Cauchy problem in Sobolev spaces (assuming that the operator is non-degenerate elliptic in space but imposing less regularity from the data) and study its space-discretized version in discrete Sobolev spaces. Next, in order to consider PDE with unbounded coefficients, we take the problem in weighted Sobolev spaces and study its space-discretization in discrete weighted Sobolev spaces. The main results we obtain are:

- Existence and uniqueness of the discretized problem solution in discrete Sobolev spaces.
- Estimate for the discrete problem solution rate of convergence to the corresponding continuous problem solution in Sobolev spaces. Stronger estimate for the particular unidimensional (in space) case.
- Existence and uniqueness result for the discrete problem solution in discrete weighted Sobolev spaces.
- Estimate for the discrete problem solution rate of convergence to the continuous problem solution in weighted Sobolev spaces.

Chapter 5 - Evolution equations in abstract spaces: time discretization We consider the approximation in time in abstract spaces for evolution equations, using both the implicit and the explicit schemes. The particular second-order parabolic PDE problem approximation is given as an example. We prove the following main results for each of the approximation schemes:

- Existence and uniqueness result for the solution of the discrete problem.
- Estimate for the solution of the discrete problem.
- Estimate for the discrete problem solution rate of convergence to the corresponding continuous problem solution.

Chapter 6 - Conclusion and further research We discuss some of the results obtained in the previous chapters and outline further research directions.

Appendix A - Notation The notation is mostly introduced in the text. For the convenience of the reader, we list the basic notation symbols used.

Appendix B - Useful results We list some basic inequalities and convergence theorems we use.

Chapter 2

European financial options

We will introduce the European financial option. Basically, this derivative is a contract giving its owner the right (and not the obligation) to trade (either to buy or to sell) a stock (or a commodity, an index or a currency) for a fixed price at a fixed future date.

We will sketch the stochastic model for the pricing of a European option and the way this problem can be reduced to solve the Cauchy problem for a second-order parabolic PDE. Finally, we will discuss the potentiality of the modelling for application to other types of options.

2.1 Stochastic processes background

In this section we summarize the basic stochastic processes concepts and results (see e.g. Lamberton et all [34], pp. 29-56, Friedman [18], ch. 5).

Stochastic processes.

Definition 2.1.1. A continuous-time stochastic process in a space E endowed with a $\sigma-$algebra $\mathcal{E}$ is a family $(X_t)_{t\in\mathbb{R}^+}$ of random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ with values in a measurable space $(E, \mathcal{E})$.

We introduce the concept of *filtration*, which represents the information available at time t.

Definition 2.1.2. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space. A filtration $(\mathcal{F}_t)_{t\geq 0}$ is an increasing family of $\sigma-$algebras included in $\mathcal{A}$.

A process $(X_t)_{t\geq 0}$ is said to be *adapted* to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if, for any t, X_t is $\mathcal{F}_t-$measurable. We say that the filtration $\mathcal{F}_t = \sigma(X_s,\, s \leq t)$ is *generated* by the process $(X_t)_{t\geq 0}$. We will work with filtrations which contain all the $\mathbf{P}-$null sets of $\mathcal{A}$. The completion of $(\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by both

$\sigma(X_s,\ s \leq t)$ and $\mathcal{N}$ (the $\sigma-$algebra generated by all the $\mathbf{P}-$null sets of $\mathcal{A}$) and is called the *natural filtration* of the process $(X_t)_{t\geq 0}$.

A *stopping time* is a random time that depends on the process (X_t) in a non-anticipative way.

Definition 2.1.3. τ is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if τ is a mapping $\Omega \to [0, +\infty]$ such that, for any $t \geq 0$, $\{\tau \leq t\} \in \mathcal{F}_t$. The $\sigma-$algebra associated with τ is $\mathcal{F}_\tau = \{A \in \mathcal{A} :$ for any t $\geq 0,\ A \cap \{\tau \leq t\} \in \mathcal{F}_t\}$, and represents the information available until the random time τ.

Next we state some stopping time properties (see Lamberton et all [34], p. 31).

Proposition 2.1.4. *The following hold*

1. *If S is a stopping time then S is $\mathcal{F}_S$ measurable;*
2. *If S is a stopping time, finite almost surely, and $(X_t)_{t\geq 0}$ is a continuous adapted process then X_S is $\mathcal{F}_S$ measurable;*
3. *If S and T are two stopping times such that $S \leq T$ $\mathbf{P}$ a.s. then $\mathcal{F}_S \subset \mathcal{F}_T$;*
4. *If S and T are two stopping times then $S \wedge T = \inf(S, T)$ is a stopping time. In particular, if S is a stopping time and t is a deterministic time then $S \wedge t$ is a stopping time.*

Brownian motion.
An important example of stochastic process is the *Brownian motion* (or *Wiener process*). This process is central in the financial option modelling.

Definition 2.1.5. A Brownian motion is a real-valued, continuous stochastic process $(X_t)_{t\geq 0}$, with independent and stationary increments. That is

1. Continuity: $\mathbf{P}$ *a.s.* the map $s \to X_s(\omega)$ is continuous;
2. Independent increments: If $s \leq t$ then $X_t - X_s$ is independent of $\mathcal{F}_s = \sigma(X_u, u \leq s)$;
3. Stationary increments: If $s \leq t$ then $X_t - X_s$ and $X_{t-s} - X_0$ have the same probability law.

We state the Gaussian property of a Brownian motion (see Lamberton et all [34], p. 31).

Theorem 2.1.6. *If $(X_t)_{t\geq 0}$ is a Brownian motion then $X_t - X_0$ is a normal random variable with mean rt and variance $\sigma^2 t$, where r and σ are constant real numbers.*

Definition 2.1.7. A Brownian motion is standard if

1. $X_0 = 0$ $\mathbf{P}$ *a.s.*;

2. $\mathbf{E}(X_t) = 0$;

3. $\mathbf{E}(X_t^2) = t$.

In the sequel text, if we do not state differently, a Brownian motion is assumed to be standard. A stronger result for the Gaussian property holds (see Lamberton et all [34], p. 32).

Theorem 2.1.8. *If $(X_t)_{t\geq 0}$ is a Brownian motion and if $0 \leq t_1 < \cdots < t_d$ then $(X_{t_1}, \ldots, X_{t_d})$ is a Gaussian vector.*

We define the Brownian motion with respect to a filtration.

Definition 2.1.9. A real-valued continuous stochastic process is a $(\mathcal{F}_t)-$Brownian motion if it satisfies

1. For any $t \geq 0$, X_t is $\mathcal{F}_t-$measurable;

2. If $s \leq t$ then $X_t - X_s$ is independent of the $\sigma-$algebra $\mathcal{F}_s$;

3. If $s \leq t$ then $X_t - X_s$ and $X_{t-s} - X_0$ have the same probability law.

Martingales.

The financial notion of *arbitrage*, to be introduced in the next section, is explained with the concept of martingale.

Definition 2.1.10. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and $(\mathcal{F}_t)_{t\geq 0}$ a filtration on this space. An adapted family $(M_t)_{t\geq 0}$ of integrable random variables, i.e. $\mathbf{E}(|M_t|) < \infty$ for any t, is a martingale if, for any $s \leq t$, $\mathbf{E}(M_t|\mathcal{F}_s) = M_s$.

We give some examples of martingales (see Lamberton et all [34], p. 32).

Proposition 2.1.11. *If $(X_t)_{t\geq 0}$ is a standard $\mathcal{F}_t-$Brownian motion then*

1. *X_t is a $\mathcal{F}_t-$martingale;*

2. *$X_t^2 - t$ is a $\mathcal{F}_t-$martingale;*

3. *$\exp(\sigma X_t - (\sigma^2/2)t)$ is a $\mathcal{F}_t-$martingale.*

The martingale property $\mathbf{E}(M_t|\mathcal{F}_s) = M_s$ still holds when t and s are bounded stopping times (see Lamberton et all [34], p. 34).

Theorem 2.1.12. (Optional sampling Theorem). *If* $(M_t)_{t\geq 0}$ *is a continuous martingale with respect to the filtration* $(\mathcal{F}_t)_{t\geq 0}$*, and if* τ_1 *and* τ_2 *are two stopping times such that* $\tau_1 \leq \tau_2 \leq K$*, where* K *is a finite real number, then* M_{τ_2} *is integrable and* $\mathbf{E}(M_{\tau_2}|\mathcal{F}_{\tau_1}) = M_{\tau_1}$ $\mathbf{P}$ *a.s.*

We state a property of the *hitting time* of a point a by a Brownian motion (see Lamberton et all [34], p. 34). If a is a real number, we define $T_a := \inf\{s \geq 0, X_s = a\}$ or $+\infty$ if that set is empty.

Proposition 2.1.13. *Let* $(X_t)_{t\geq 0}$ *be an* $\mathcal{F}_t-$*Brownian motion and* a *a real number. Then* T_a *is a stopping time, finite almost surely, and its distribution is characterized by its Laplace transform* $\mathbf{E}(e^{-\lambda T_a}) = e^{-\sqrt{2\lambda}|a|}$.

Next result gives an estimate for the second-order moment of $\sup_{0\leq t\leq T}|M_t|$, where M_t is a square integrable martingale (see Lamberton et all [34], p. 35).

Theorem 2.1.14. (Doob inequality). *If* $(M_t)_{0\leq t\leq T}$ *is a continuous martingale then* $\mathbf{E}\left(\sup_{0\leq t\leq T}|M_t|^2\right) \leq 4\mathbf{E}(|M_T|^2)$.

Stochastic integral.
In the financial option modelling, we will deal with expressions of the type $(\int_0^t H_s dW_s)_{0\leq t\leq T}$, where $(W_t)_{t\geq 0}$ is a $\mathcal{F}_t-$Brownian motion and $(H_t)_{0\leq t\leq T}$ is a $\mathcal{F}_t-$adapted process. As Brownian motion paths are, almost surely, not differentiable at any point, this integral with respect to a Brownian motion (the *stochastic integral*) needs to be defined.

Let $(W_t)_{t\geq 0}$ be a standard $\mathcal{F}_t-$Brownian motion defined on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. Take T a strictly positive, finite real number. We will begin by considering a set of processes called *simple processes.*

Definition 2.1.15. $(H_t)_{0\leq t\leq T}$ is a simple process if it can be written as

$$(H_t)(\omega) = \sum_{i=1}^{p} \phi_i(\omega)\mathbf{1}_{]\,t_{i-1},t_i]}(t),$$

where $0 = t_0 < t_1 < \cdots < t_p = T$ and ϕ_i is $\mathcal{F}_{t_{i-1}}-$measurable and bounded.

By definition, the stochastic integral of a simple process is the continuous process $(I(H)_t)_{0\leq t\leq T}$ defined for any $t \in]t_k, t_{k+1}]$ as

$$I(H)_t = \sum_{1\leq i\leq k} \phi_i(W_{t_i} - W_{t_{i-1}}) + \phi_{k+1}(W_t - W_{t_k}).$$

We write $\int_0^t H_s dW_s = I(H)_t$.

We next state some fundamental properties of the stochastic integral of a simple process (see Lamberton et all [34], p. 36).

Proposition 2.1.16. *If* $(H_t)_{0\leq t\leq T}$ *is a simple process then*

1. $\left(\int_0^t H_s dW_s\right)_{0\leq t\leq T}$ *is a continuous* $\mathcal{F}_t-$*martingale;*
2. $\mathbf{E}\left(\left(\int_0^t H_s dW_s\right)^2\right) = \mathbf{E}\left(\int_0^t H_s^2 ds\right)$*;*
3. $\mathbf{E}\left(\sup_{t\leq T}\left|\int_0^t H_s dW_s\right|^2\right) \leq 4\mathbf{E}\left(\int_0^T H_s^2 ds\right)$.

We extend the concept of stochastic integral to a larger class of adapted processes $\mathcal{H}$

$$\mathcal{H} = \left\{(H_t)_{0\leq t\leq T},\ (\mathcal{F}_t)_{t\geq 0} - \text{adapted process}:\ \mathbf{E}\left(\int_0^T H_s^2 ds\right) < +\infty\right\}.$$

We define the extension (see Lamberton et all [34], p. 38).

Proposition 2.1.17. *Let* $(W_t)_{t\geq 0}$ *be an* $\mathcal{F}_t-$*Brownian motion. There exists a unique linear mapping* J *from* $\mathcal{H}$ *to the space of the continuous* $\mathcal{F}_t-$*martingales defined on* $[0,T]$*, such that*

1. *If* $(H_t)_{t\leq T}$ *is a simple process then* $\mathbf{P}$ *a.s. for any* $0 \leq t \leq T$, $J(H)_t = I(H)_t$*;*
2. *If* $t \leq T$ *then* $\mathbf{E}(J(H)_t^2) = \mathbf{E}\left(\int_0^t H_s^2 ds\right)$.

This linear mapping is unique in the sense that if both J *and* J' *satisfy the previous properties then* $\mathbf{P}$ *a.s.* $\forall\ 0 \leq t \leq T,\ J(H)_t = J'(H)_t$. *We denote, for* $H \in \mathcal{H}$, $\int_0^t H_s dW_s = J(H)_t$.

We note that the condition $\mathbf{E}(\int_0^T H_s^2 ds) < +\infty$ in the definition of $\mathcal{H}$ is satisfied if and only if $\mathbf{E}(\sup_{0\leq t\leq T}(\int_0^t H_s dWs)^2) < +\infty$.

The following properties hold (see Lamberton et all [34], p. 38).

Proposition 2.1.18. *If* $(H_t)_{0\leq t\leq T}$ *belongs to* $\mathcal{H}$ *then*

1. $\mathbf{E}\left(\sup_{t\leq T}\left|\int_0^t H_s dW_s\right|^2\right) \leq 4\mathbf{E}\left(\int_0^T H_s^2 ds\right)$;
2. *If* τ *is a* $\mathcal{F}_t-$*stopping time then* $\mathbf{P}$ *a.s.* $\int_0^\tau H_s dW_s = \int_0^T \mathbf{1}_{\{s\leq\tau\}} H_s dW_s$.

We extend the stochastic integral to a class of processes satisfying a weaker integrability condition. Let

$$\tilde{\mathcal{H}} = \left\{ (H_s)_{0\le s\le T},\ (\mathcal{F}_t)_{t\ge 0} - \text{adapted process} : \int_0^T H_s^2 ds < +\infty \ \ \mathbf{P}\ a.s. \right\}.$$

We define the extension to $\tilde{\mathcal{H}}$ (see Lamberton et all [34], p. 40).

Proposition 2.1.19. *There exists a unique linear mapping $\tilde{J}$ from $\tilde{\mathcal{H}}$ into the vector space of continuous processes defined on $[0,T]$, such that*

1. *If $(H_t)_{0\le t\le T}$ is a simple process then $\mathbf{P}\ a.s. \ \forall\ 0\le t\le T,\ \tilde{J}(H)_t = I(H)_t$;*
2. *If $(H^i)_{i\ge 0}$ is a sequence of processes in $\tilde{\mathcal{H}}$ such that $\int_0^T (H_s^i)^2 ds$ converges to 0 in probability then $\sup_{t\le T} |\tilde{J}(H^i)_t|$ converges to 0 in probability.*

We write, for $H \in \tilde{\mathcal{H}}$, $\int_0^t H_s dW_s = \tilde{J}(H)_t$.

In this case the integral is not necessarily a martingale.

We introduce next some basic concepts of *Itô calculus*. Let us define a *Itô process*.

Definition 2.1.20. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbf{P})$ be a filtered probability space and $(W_t)_{t\ge 0}$ an $\mathcal{F}_t-$Brownian motion. $(X_t)_{0\le t\le T}$ is an $\mathbb{R}-$valued Itô process if it can be written as

$$\mathbf{P}\ a.s. \ \ \forall\ t\le T,\ \ X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s,$$

where X_0 is $\mathcal{F}_0-$measurable, $(K_t)_{0\le t\le T}$ and $(H_t)_{0\le t\le T}$ are $\mathcal{F}_t-$adapted processes, $\int_0^T |K_s| ds < +\infty$ $\mathbf{P}\ a.s.$ and $\int_0^T |H_s|^2 ds < +\infty$ $\mathbf{P}\ a.s.$

The previous decomposition is unique (see Lamberton et all [34], p. 43).

Proposition 2.1.21. *If $(M_t)_{0\le t\le T}$ is a continuous martingale such that $M_t = \int_0^t K_s ds$, with $\mathbf{P}$ a.s. $\int_0^T |K_s| ds < +\infty$ then $\mathbf{P}$ a.s. $\forall t \le T$, $M_t = 0$. This implies that*

1. *An Itô process decomposition is unique. That means that if*

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s = X_0' + \int_0^t K_s' ds + \int_0^t H_s' dW_s$$

   then $X_0 = X_0'$ $d\mathbf{P}$ a.s. $H_s = H_s'$ $ds \times d\mathbf{P}$ a.e. $K_s = K_s'$ $ds \times d\mathbf{P}$ a.e.;

2. *If $(X_t)_{0\le t\le T}$ is a martingale of the form $X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s$ then $K_t = 0$ $\ dt \times d\mathbf{P}$ a.e.*

In next result the stochastic integral is defined in the interval $[0, \tau_0]$, with τ_0 a stopping time (the stochastic integral is interpreted as a random variable) (see Friedman [18], p. 72).

Theorem 2.1.22. *Let f a process such that $\mathbf{E} \int_0^\tau |f(t)|^2 dt < \infty$ and τ a stopping time with respect to $\mathcal{F}_t$, $0 \le \tau \le T$. Then the process $\int_0^{\tau \wedge t} f(s) dW(s)$, $0 \le t \le T$, is a martingale and $\mathbf{E} \int_0^{\tau \wedge t} f(s) dW(s) = 0$.*

We state *Itô* formula (see Lamberton et all [34], p. 44).

Theorem 2.1.23. (Itô formula). *Let $(X_t)_{0 \le t \le T}$ be an Itô process*

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s,$$

and f be a twice continuously differentiable function. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d\langle X, X \rangle_s,$$

where $\langle X, X \rangle_t := \int_0^t H_s^2 ds$ and $\int_0^t f'(X_s) dX_s := \int_0^t f'(X_s) K_s ds + \int_0^t f'(X_s) H_s dW_s$.

Also, if f is a function twice differentiable with respect to x and once differentiable with respect to t, with continuous partial derivatives in (t, x), then

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t f'_s(s, X_s) ds \\ &\quad + \int_0^t f'_x(s, X_s) dX_s + \frac{1}{2} \int_0^t f''_{xx}(s, X_s) d\langle X, X \rangle_s. \end{aligned}$$

We give the integration by parts formula (see Lamberton et all [34], p. 46).

Proposition 2.1.24. (Integration by parts formula). *Let (X_t) and Y_t be two Itô processes, $X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s$ and $Y_t = Y_0 + \int_0^t K'_s ds + \int_0^t H'_s dW_s$. Then*

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y \rangle_t,$$

with $\langle X, Y \rangle_t := \int_0^t H_s H'_s ds$.

We have a multidimensional version of Itô formula to be applied when f is a function of several Itô processes, each of them function of several Brownian motions.

Definition 2.1.25. A $p-$dimensional $\mathcal{F}_t-$Brownian motion is an $\mathbb{R}^p-$valued $\mathcal{F}_t-$adapted process $(W_t = (W_t^1, \ldots, W_t^p))_{t \ge 0}$, where all the $(W_t^i)_{t \ge 0}$ are independent standard $\mathcal{F}_t-$Brownian motions.

We define the multidimensional Itô process.

Definition 2.1.26. $(X_t)_{0\leq t\leq T}$ is a (multidimensional) Itô process if

$$X_t = X_0 + \int_0^t K_s ds + \sum_{i=1}^{p} \int_0^t H_s^i dW_s^i,$$

where K_t and all the processes (H_t^i) are adapted to $\mathcal{F}_t$, $\int_0^T |K_s| ds < +\infty$ $\mathbf{P}$ *a.s.* and $\int_0^T (H_s^i)^2 ds < +\infty$ $\mathbf{P}$ *a.s.*

We state the multidimensional Itô formula (see Lamberton et all [34], p. 48).

Theorem 2.1.27. (Multidimensional Itô formula). *Let $(X_t^1, \ldots, X_t^d)$ be d Itô processes*

$$X_t^i = X_0^i + \int_0^t K_s^i ds + \sum_{j=1}^{p} \int_0^t H_s^{i,j} dW_s^j,$$

and f a function twice differentiable with respect to x and once differentiable with respect to t, with continuous partial derivatives in (t, x). Then

$$\begin{aligned} f(t, X_t^1, \ldots, X_t^d) &= f(0, X_0^1, \ldots, X_0^d) + \int_0^t \frac{\partial f}{\partial s}(s, X_s^1, \ldots, X_s^d) ds \\ &+ \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x^i}(s, X_s^1, \ldots, X_s^d) dX_s^i \\ &+ \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 f}{\partial x^i x^j}(s, X_s^1, \ldots, X_s^d) d\langle X^i, X^j\rangle_s, \end{aligned}$$

with $dX_s^i = K_s^i ds + \sum_{j=1}^p H_s^{ij} dW_s^j$ *and* $d\langle X^i, X^j\rangle_s = \sum_{m=1}^p H_s^{im} H_s^{jm} ds.$

Stochastic differential equations.

We begin by considering a type of process that, as it will be mentioned later, models the behaviour of certain financial assets. Let

$$S_t = x_0 + \int_0^t S_s(\mu ds + \sigma dW_s), \tag{2.1}$$

where σ and μ are real numbers and $(W_t)_{t\geq 0}$ is a Brownian motion.

We show next that the process $S_t = x_0 \exp((\mu - \sigma^2/2)t + \sigma W_t)$ solves (2.1). Let $f(t, x) = x_0 \exp((\mu - \sigma^2/2)t + \sigma x)$ so that we can write $S_t = f(t, W_t)$. As $(W_t)_{t\geq 0}$ is an Itô process (identifying $K_s = 0$ and $H_s = 1$) we apply Itô formula and obtain

$$\begin{aligned} S_t = f(t, W_t) &= f(0, W_0) + \int_0^t f_s'(s, W_s) ds + \int_0^t f_x'(s, W_s) dW_s \\ &+ \frac{1}{2} \int_0^t f_{xx}''(s, W_s) d\langle W, W\rangle_s. \end{aligned}$$

As $d\langle W, W\rangle_t = dt$,

$$\begin{aligned} S_t &= x_0 + \int_0^t S_s(\mu - \sigma^2/2)ds + \int_0^t S_s \sigma dW_s + \frac{1}{2}\int_0^t S_s\sigma^2 ds \\ &= x_0 + \int_0^t S_s \mu ds + \int_0^t S_s \sigma dW_s. \end{aligned}$$

The uniqueness of this solution can obtained using Proposition 2.1.24. We have the following theorem (see Lamberton et all [34], p. 47):

Theorem 2.1.28. *Let σ, μ be two real numbers, T a strictly positive constant and $(W_t)_{t\geq 0}$ a Brownian motion. There exists a unique Itô process $(S_t)_{0\leq t\leq T}$ which satisfies, for any $t \leq T$, equation (2.1). This process is given by $S_t = x_0 \exp((\mu - \sigma^2/2)t + \sigma W_t)$.*

We consider now the equation

$$X_t = Z + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \tag{2.2}$$

a more general version of equation (2.1). Equation (2.2) is also written:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = Z.$$

Equations of this type are called *stochastic differential equations* and their solutions are called *diffusions*. Most financial assets are modelled using these equations. We define the solution of equation (2.2).

Definition 2.1.29. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$. Let b and σ be functions such that $b: \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $\sigma: \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, Z a $\mathcal{F}_0-$measurable random variable and $(W_t)_{t\geq 0}$ a $\mathcal{F}_t-$Brownian motion. A solution to the equation (2.2) is an $\mathcal{F}_t-$adapted stochastic process $(X_t)_{t\geq 0}$ such that

1. For any $t \geq 0$, the integrals $\int_0^t b(s, X_s)ds$ and $\int_0^t \sigma(s, X_s)dW_s$ exist, i.e.

$$\int_0^t |b(s, X_s)|ds < +\infty \quad \text{and} \quad \int_0^t |\sigma(s, X_s)|^2 ds < +\infty \quad \mathbf{P}\ a.s.;$$

2. $(X_t)_{t\geq 0}$ satisfies (2.2), i.e.

$$\forall t \geq 0 \quad \mathbf{P}\ a.s.\ \ X_t = Z + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s.$$

We state the existence and uniqueness of the solution of equation (2.2) (see Lamberton et all [34], pp. 49-50).

Theorem 2.1.30. *If b and σ are continuous functions and if there exist constants $K, L < +\infty$ such that*

1. $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K|x-y|$,
2. $|b(t,x)| + |\sigma(t,x)| \leq L(1+|x|)$,
3. $\mathbf{E}(|Z|^2) < +\infty$,

for all $t \in \mathbb{R}^+$, for all $x, y \in \mathbb{R}$. Then there exists a unique solution of (2.2) in $[0,T]$, $T \geq 0$. Moreover, this solution satisfies $\mathbf{E}(\sup_{0\leq t\leq T}|X_t|^2) < +\infty$. The uniqueness means that if $(X_t)_{0\leq t\leq T}$ and $(Y_t)_{0\leq t\leq T}$ are two solutions of (2.2) then $\mathbf{P}$ a.s. $\forall 0 \leq t \leq T,\ \ X_t = Y_t$.

We extend the stochastic differential equation analysis to the multidimensional case. Let

$$\begin{aligned}
&W_t = (W_t^1, \dots, W_t^p) \ \text{ an } \ \mathbb{R}^p - \text{valued} \ \ \mathcal{F}_t - \text{Brownian motion;}\\
&b : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d,\ b(s,x) = (b^1(s,x), \dots, b^d(s,x));\\
&\sigma : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{d\times p},\ \sigma(s,x) = (\sigma^{ij}(s,x))_{1\leq i\leq d, 1\leq j\leq p};\\
&Z = (Z^1, \dots, Z^d) \ \text{ an } \ \mathcal{F}_0 - \text{measurable random variable in } \mathbb{R}^d.
\end{aligned}$$

Consider the multidimensional equation

$$X_t^i = Z^i + \int_0^t b^i(s, X_s)ds + \sum_{j=1}^p \int_0^t \sigma^{ij}(s, X_s)dW_s^j, \ \text{ for } \ i = 1, \dots, d,$$

which can be written

$$X_t = Z + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s. \tag{2.3}$$

We state the existence and uniqueness of a solution of (2.3) (see Lamberton et all [34], p. 53). If $x \in \mathbb{R}^d$, denote by $|x|$ the Euclidean norm of x and if $\sigma \in \mathbb{R}^{d\times p}$ denote $|\sigma|^2 = \sum_{1\leq i\leq d,\ 1\leq j\leq p}(\sigma^{ij})^2$.

Theorem 2.1.31. *Assume that b and σ are continuous functions and that there exist constants $K, L < +\infty$ such that*

1. $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K|x-y|$,
2. $|b(t,x)| + |\sigma(t,x)| \leq L(1+|x|)$,
3. $\mathbf{E}(|Z|^2) < +\infty$,

for all $t \in \mathbb{R}^+$, for all $x, y \in \mathbb{R}^d$. Then there exists a unique solution of (2.3) in $[0,T]$, $T \geq 0$. Moreover, this solution satisfies $\mathbf{E}(\sup_{0\leq t\leq T}|X_t|^2) < +\infty$.

We next state the flow and Markov properties for the solution of equation (2.2).

We say that an $\mathcal{F}-$adapted process $(X_t)_{t\geq 0}$ satisfies the Markov property if, for any bounded Borel function f and for any s and t such that $s \leq t$,

$$\mathbf{E}(f(X_t)|\mathcal{F}_s) = \mathbf{E}(f(X_t)|X_s).$$

Intuitively, this means that the future behaviour of $(X_t)_{t\geq 0}$ depends only on the value X_t and not on any other previous information. We will see that this property will play an important role in the financial option pricing.

Let us denote by $X_s^{t,x}$, for $s \geq t$, the solution of equation (2.2) starting from x at time t. For $s \geq t$, $X_s^{t,x}$ satisfies

$$X_s^{t,x} = x + \int_t^s b(u, X_u^{t,x})du + \int_t^s \sigma(u, X_u^{t,x})dW_u.$$

We state the flow property of X_t (see Lamberton et all [34], p. 54).

Lemma 2.1.32. *Under the assumptions of Theorem 2.1.30, if $s \geq t$ then*

$$X_s^{0,x} = X_s^{t,X_t^x} \;\; \mathbf{P} \; a.s.$$

For the Markov property of X_t, we have the following result (see Lamberton et all [34], p. 55):

Theorem 2.1.33. *Let $(X_t)_{t\geq 0}$ be a solution of (2.2). Then $(X_t)_{t\geq 0}$ is a Markov process with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Furthermore, for any bounded Borel function f, we have $\mathbf{P}$ a.s. $\mathbf{E}(f(X_t)|\mathcal{F}_s) = \phi(X_s)$, with $\phi(x) = \mathbf{E}(f(X_t^{s,x}))$.*

We state an extension of Theorem 2.1.33, result useful when interest rate models are considered (see Lamberton et all [34], p. 55).

Theorem 2.1.34. *Let $(X_t)_{t\geq 0}$ be a solution of (2.2) and $r(s,x)$ be a non-negative measurable function. Then, for $t > s$,*

$$\mathbf{P} \; a.s. \;\; \mathbf{E}\left(e^{-\int_s^t r(u,X_u)du} f(X_t)|\mathcal{F}_s\right) = \phi(X_s),$$

with

$$\phi(x) = \mathbf{E}\left(e^{-\int_s^t r(u,X_u^{s,x})du} f(X_t^{s,x})\right).$$

It is also written as

$$\mathbf{E}\left(e^{-\int_s^t r(u,X_u)du} f(X_t)|\mathcal{F}_s\right) = \mathbf{E}\left(e^{-\int_s^t r(u,X_u^{s,x})du} f(X_t^{s,x})|_{x=X_s}\right).$$

2.2 European option stochastic modelling

In this section we briefly present the Black-Scholes model (see e.g. Lamberton et all [34], pp. 63-93).

Statement of the problem.
An *European option* on a *stock* S is a contract giving its owner the right to trade the stock (to buy it in the case of a *call option* or to sell it in the case of a *put option*) for a fixed price K (the *strike price)* at a future date T (the option *maturity* or *expiry)*. If, at time T, the option's owner opts to trade the stock the option is said to be *exercised* .

In the most simple case, the *payoff* of an option is

$$C_T = (S_T - K)_+ = \max(S_T - K, 0),$$

for a call option and

$$P_T = (K - S_T)_+ = \max(K - S_T, 0),$$

for a put option.

The model we will outline enables us to determine the price for this type of security, that is, what is the value at time t of an option worth C_T (for a call) or P_T (for a put) at time T.

As consequence of a model's assumption (the absence of *arbitrage opportunity* to be mentioned later), we have the *put-call parity* equation

$$C_t - P_t = S_t - Ke^{-r(T-t)},$$

which holds for all $t < T$. Then it suffices to consider one of the two cases: we will approach the call option case.

Remark 2.2.1. In the model we are presenting we assume, for simplification, that the stock does not pay dividends until the expiration date T.

Remark 2.2.2. We have defined a European option on a stock. It can be defined in the same way on a commodity, an index or a currency.

Behaviour of prices.
We will consider a model with two assets: a *riskless* asset S^0 and a *risky* asset S. Their price behaviour is described as follows.

For S^0 we have the ordinary differential equation

$$dS^0_t = rS^0_t dt,$$

where S_t^0 is the price of the asset at time t and r is a non-negative constant representing the riskless rate of interest. Assuming an initial condition $S_0^0 = 1$, we have

$$S_t^0 = e^{rt}, \quad t \geq 0.$$

For S we have the stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dB_t), \tag{2.4}$$

where μ and $\sigma > 0$ are constants representing the expected return or average growth rate of the asset (*drift rate*) and the standard deviation of returns (*volatility*), respectively, and (B_t) is a standard Brownian motion. The model is valid on $[0, T]$.

As we saw in the previous section (Theorem 2.1.28), a closed-form unique solution for the stochastic differential equation can be determined

$$S_t = S_0 \exp((\mu - \sigma^2/2)\, t + \sigma B_t),$$

where S_0 is the stock price observed at time 0. The process $(\log(S_t))$ is a (non necessarily standard) Brownian motion. We then have the following properties for the process (S_t):

1. Continuity of the sample paths;
2. Independent of the relative increments: If $u \leq t$ then $(S_t - S_u)/S_u$ is independent of $\sigma(S_v, v \leq u)$;
3. Stationarity of the relative increments: If $u \leq t$ then $(S_t - S_u)/S_u$ and $(S_{t-u} - S_0)/S_0$ have the same probability law.

These properties characterize the stock price behaviour assumed in Black-Scholes model.

Strategies.

A *strategy* is defined as a process

$$\phi = (\phi)_{0 \leq t \leq T} = ((H_t^0, H_t)),$$

with values in $\mathbb{R}^2$, adapted to the natural filtration $(\mathcal{F}_t)$ of the Brownian motion. The components H_t^0 and H_t of the *portfolio* (H_t^0, H_t) are the quantities of riskless asset and risky asset, respectively, held at time t. The value of the portfolio at time t is

$$V_t(\phi) = H_t^0 S_t^0 + H_t S_t.$$

We define strategies in which the decisions made on the composition of the portfolio do not affect its value, that is, changes in the portfolio value would only be brought by price moves.

Definition 2.2.3. A self-financing strategy is a pair ϕ of adapted processes $(H_t^0)_{0\leq t\leq T}$ and $(H_t)_{0\leq t\leq T}$ satisfying

1. $\int_0^T |H_t^0|dt + \int_0^T (H_t)^2 dt < +\infty$ a.s.;
2. $H_t^0 S_t^0 + H_t S_t = H_0^0 S_0^0 + H_0 S_0 + \int_0^t H_u^0 dS_u^0 + \int_0^t H_u dS_u$ a.s., for all $t \in [0,T]$.

Denote the discounted price of the risky asset by $\tilde{S}_t = e^{-rt} S_t$. We have the following result (see Lamberton et all [34], p. 65):

Proposition 2.2.4. *Let $\phi = (\phi)_{0\leq t\leq T} = ((H_t^0, H_t))$ be an adapted process with values in $\mathbb{R}^2$, satisfying $\int_0^T |H_t^0|dt + \int_0^T (H_t)^2 dt < +\infty$ a.s. Let $V_t(\phi) = H_t^0 S_t^0 + H_t S_t$ and $\tilde{V}_t(\phi) = e^{-rt} V_t(\phi)$. Then ϕ defines a self-financing strategy if and only if*

$$\tilde{V}_t(\phi) = V_0(\phi) + \int_0^t H_u d\tilde{S}_u \quad a.s.,$$

for all $t \in [0,T]$.

Remark 2.2.5. The model we are presenting assumes that the(continuous) changes in the portfolio composition are made with no cost (the model is called with no *transaction costs*).

Girsanov's Theorem. Martingale representation.
In order to price an option, we will construct self-financing strategies replicating the option. We need first to consider an equivalent probability measure under which discounted prices of assets are martingales.

We define equivalent probabilities (see Lamberton et all [34], p. 66).

Definition 2.2.6. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space. A probability measure $\mathbf{Q}$ on $(\Omega, \mathcal{A})$ is absolutely continuous with respect to $\mathbf{P}$ if $\forall A \in \mathcal{A}$ $\mathbf{P}(A) = 0 \Rightarrow \mathbf{Q}(A) = 0$.

Theorem 2.2.7. $\mathbf{Q}$ *is absolutely continuous relative to* $\mathbf{P}$ *if and only if there exists a non-negative random variable Z on $(\Omega, \mathcal{A})$ such that $\forall A \in \mathcal{A}$ $\mathbf{Q}(A) = \int_A Z(\omega) d\mathbf{P}(\omega)$. Z is called density of* $\mathbf{Q}$ *relative to* $\mathbf{P}$ *and denoted* $d\mathbf{Q}/d\mathbf{P}$.

Definition 2.2.8. Let $\mathbf{Q}$ and $\mathbf{P}$ be two probability measures on $(\Omega, \mathcal{A})$. $\mathbf{P}$ and $\mathbf{Q}$ are equivalent if each one is absolutely continuous relative to the other.

With next result, a probability measure $\mathbf{Q}$ equivalent to a given probability measure $\mathbf{P}$ is constructed (see Lamberton et all [34], p. 66).

Theorem 2.2.9. (Girsanov's Theorem). *Let* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbf{P})$ *be a probability space with* $(\mathcal{F}_t)_{0\le t\le T}$ *the natural filtration of the standard Brownian motion* $(B_t)_{0\le t\le T}$. *Let* $(\theta_t)_{0\le t\le T}$ *be an adapted process satisfying* $\int_0^T \theta_s^2 ds < +\infty$ *a.s. and such that the process* $(L_t)_{0\le t\le T}$ *defined by* $L_t = \exp\left(-\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right)$ *is a martingale. Then, under probability* $\mathbf{P}^{(L)}$ *with density* L_T *relative to* $\mathbf{P}$, *the process* $(W_t)_{0\le t\le T}$ *defined by* $W_t = B_t + \int_0^t \theta_s ds$ *is a standard Brownian motion.*

The stochastic integral is invariant by change of equivalent probability (see Lamberton et all [34], p. 79).

Proposition 2.2.10. *Assume that the hypothesis of Theorem 2.2.9 are satisfied. Let* $(H_t)_{0\le t\le T}$ *be an adapted process such that* $\int_0^T H_s^2 ds < \infty$ $\mathbf{P}$ *a.s. Let the processes*

$$X_t = \int_0^t H_s dB_s + \int_0^t H_s\theta_s ds, \quad \text{under } \mathbf{P}$$

and

$$Y_t = \int_0^t H_s dW_s, \quad \text{under } \mathbf{P}^{(L)},$$

with $W_t = B_t + \int_0^t \theta_s ds$ *and* $\mathbf{P}^{(L)}$ *the probability measure defined in Theorem 2.2.9. Then* $X_t = Y_t$.

We state next a result on the representation of a Brownian martingale in terms of a stochastic integral (see Lamberton et all [34], p. 67).

Theorem 2.2.11. *Let* $(B_t)_{0\le t\le T}$ *be a standard Brownian motion on a probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ *and let* $(\mathcal{F}_t)_{0\le t\le T}$ *be its natural filtration. Let* $(M_t)_{0\le t\le T}$ *be a square-integrable martingale, with respect to* $(\mathcal{F}_t)_{0\le t\le T}$. *There exists an adapted process* $(H_t)_{0\le t\le T}$ *such that* $\mathbf{E}(\int_0^T H_s^2 ds) < +\infty$ *and*

$$\forall t \in [0, T] \quad M_t = M_0 + \int_0^t H_s dB_s \ \ a.s.$$

Option pricing.

We consider now the problem of determining the price of an option.

First, we show that there exists a probability $\mathbf{P}^*$ equivalent to $\mathbf{P}$ under which the discounted risky asset price $\tilde{S}_t = e^{-rt}S_t$ is a martingale. From equation (2.4), we have

$$d\tilde{S}_t = -re^{-rt}S_t dt + e^{-rt}dS_t = \tilde{S}_t((\mu - r)dt + \sigma dB_t). \tag{2.5}$$

Setting $W_t = B_t + (\mu - r)t/\sigma$, we obtain

$$d\tilde{S}_t = \tilde{S}_t \sigma dW_t. \tag{2.6}$$

Owing to Theorem 2.2.9, with $\theta_t = (\mu - r)/\sigma$, there exists a probability measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$ under which $(W_t)_{0\leq t\leq T}$ is a standard Brownian motion. As, from Proposition 2.2.10, the stochastic integral is invariant by change of equivalent probability, under $\mathbf{P}^*$ we have

$$\tilde{S}_t = \tilde{S}_0 \exp(\sigma W_t - \sigma^2 t/2),$$

and, by Proposition 2.1.11, $\tilde{S}_t$ is a martingale.

Remark 2.2.12. The term $(\mu - r)$ in (2.5) is called the *risk premium.*

Remark 2.2.13. If we apply the transformation $W_t = B_t + (\mu - r)t/\sigma$ to S_t instead of to $\tilde{S}_t$, from

$$dS_t = S_t(\mu dt + \sigma dB_t)$$

we obtain

$$dS_t = S_t(r dt + \sigma dW_t),$$

and, for the same reasons, $(W_t)_{0\leq t\leq T}$ is a standard Brownian motion under the equivalent probability measure $\mathbf{P}^*$. Note that the drift μ is replaced by the riskless interest rate r, so that, under $\mathbf{P}^*$ the risk premium for S_t is null. This is why the probability measure $\mathbf{P}^*$ is sometimes called *risk-neutral.*

We will restrict the study to the class of admissible strategies.

Definition 2.2.14. A strategy $\phi = ((H_t^0, H_t))_{0\leq t\leq T}$ is admissible if it is self-financing and if the discounted value $\widetilde{V}_t(\phi) = H_t^0 + H_t\widetilde{S}_t$ of the corresponding portfolio is, for all t, non-negative and such that $\sup_{t\in[0,T]} \widetilde{V}_t$ is square integrable under $\mathbf{P}^*$.

For a self-financing strategy ϕ, from Proposition 2.2.4 and equation (2.6) we have

$$\tilde{V}_t = V_0 + \int_0^t H_u \sigma \tilde{S}_u dW_u.$$

If, additionally, ϕ is admissible, from Proposition 2.1.17 we have that $(\tilde{V}_t)$ is a square-integrable martingale under $\mathbf{P}^*$. Then, under $\mathbf{P}^*$, for any admissible strategy ϕ, $\widetilde{V}_0(\phi) = 0 \Rightarrow \widetilde{V}_T(\phi) = 0$ $\mathbf{P}^*$ a.s. This expresses the *no arbitrage opportunity* hypothesis of the model.

We define a call option by a non-negative, $\mathcal{F}_T-$measurable, random variable h (the option payoff).

Definition 2.2.15. An option is replicable if there is an admissible strategy $\phi = ((H_t^0, H_t))_{0\leq t\leq T}$ such that at time T its value equals the option payoff $V_T(\phi) = h$.

Note that for an option to be replicable h has to be square integrable under $\mathbf{P}^*$. This necessary condition is satisfied when h is written as $h = g(S_T)$, with $g(x) = (x - K)_+$.

We saw above that, for an admissible strategy ϕ, $(\tilde{V}_t)$ is a square-integrable martingale under $\mathbf{P}^*$. If ϕ replicates the option, from $\tilde{V}_t = \mathbf{E}^* \left(\tilde{V}_T | \mathcal{F}_t\right)$, we have

$$V_t = \mathbf{E}^* \left(e^{-r(T-t)} h | \mathcal{F}_t\right).$$

It could also be shown that if h is square integrable under $\mathbf{P}^*$ then there is an admissible strategy replicating the option.

We have the following main result which defines the option price (see Lamberton et all [34], p. 69):

Theorem 2.2.16. *In the Black-Scholes model, any option defined by a non-negative $\mathcal{F}_T-$measurable random variable h, which is square-integrable under the probability $\mathbf{P}^*$, is replicable and the value at time t of any replicating portfolio is given by*

$$V_t = \mathbf{E}^* \left(e^{-r(T-t)} h | \mathcal{F}_t\right). \tag{2.7}$$

The expression $\mathbf{E}^* \left(e^{-r(T-t)} h | \mathcal{F}_t\right)$ defines the option value at time t.

Remark 2.2.17. If the option value is written $h = g(S_T)$, under strong hypothesis over g it would be possible to determine explicitly the replicating portfolio, that is the composition of the portfolio (H_t^0, H_t) satisfying (2.7).

We make a final comment. Recall that in the modelling we assumed that there were no dividend payments and no transaction costs.

The inclusion of continuously payed dividends in the model is immediate. Unfortunately, this is not consistent with the discrete (usual annual) dividend payment in finance world. This points to the need to combine the continuous modelling we have presented with discrete modelling for the dividend payment. The same idea applies to the inclusion of transaction costs: the changes in the portfolio composition should rather be considered discrete.

Several models for these purposes are available in the Financial Mathematics literature (see e.g. Wilmott [47])

2.3 European option pricing and parabolic PDE

We will show the way the problem of pricing an European option is related to a parabolic PDE Cauchy problem (see e.g. Lamberton et all [34], pp. 95-101).

We will consider a more general version of the problem we have presented.

Let $(X_t)_{t\geq 0}$ be a diffusion in $\mathbb{R}$, solution of the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{2.8}$$

where b and σ satisfy the assumptions of Theorem 2.1.30. Let also $r(t,x)$ be a bounded continuous real-valued function defined on $\mathbb{R}^+ \times \mathbb{R}$, modelling the riskless interest rate. We write the payoff function h as $h = g(X_T)$.

We want to compute

$$V_t = \mathbf{E}\big(e^{-\int_t^T r(s,X_s)ds} g(X_T)|\mathcal{F}_t\big).$$

As a consequence of Theorem 2.1.34, V_t can be written

$$V_t = G(t, X_t),$$

where $G(t,x) = \mathbf{E}\big(e^{-\int_t^T r(s,X_s^{t,x})ds} g(X_T^{t,x})\big)$, and $X_s^{t,x}$ denotes the solution of (2.8) starting from x at time t.

First we state some results relating the infinitesimal generator of a diffusion.

Infinitesimal generator of a diffusion.

Let b and σ satisfy the assumptions of Theorem 2.1.30. We state the following result (see Lamberton et all [34], p. 98):

Proposition 2.3.1. *For any time t let $A_t(x)$ be the differential operator that maps a C^2 function v from $\mathbb{R}$ to $\mathbb{R}$ to a function $A_t v$ such that*

$$(A_t v)(x) = \frac{\sigma^2(t,x)}{2}\frac{\partial^2 v}{\partial x^2}(x) + b(t,x)\frac{\partial v}{\partial x}(x).$$

Let $u(t,x)$ be a $C^{1,2}$ real-valued function defined on $\mathbb{R}^+ \times \mathbb{R}$ with bounded derivatives in x. Let X_t be a solution of (2.8). Then the process

$$M_t = u(t, X_t) - \int_0^t \big(A_s u + \frac{\partial u}{\partial t}\big)(s, X_s)ds$$

is a martingale.

The differential operator A is called the *infinitesimal generator* of the diffusion (X_t).

We state a more general result where discounted prices are considered (see Lamberton et all [34], p. 98).

Proposition 2.3.2. *Let the assumptions of Proposition 2.3.1 be satisfied. Let $r(t,x)$ be a bounded continuous real-valued function defined on $\mathbb{R}^+ \times \mathbb{R}$. Then the process*

$$M_t = e^{-\int_0^t r(s,X_s)ds} u(t, X_t) - \int_0^t e^{-\int_0^s r(v,X_v)dv}\big(A_s u - ru + \frac{\partial u}{\partial t}\big)(s, X_s)ds$$

is a martingale.

This result still holds in the multidimensional case. Let

$$\begin{aligned} &W_t = (W_t^1, \dots, W_t^p) \ \text{ an } \ \mathbb{R}^p - \text{valued } \ \mathcal{F}_t - \text{Brownian motion;} \\ &b : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d, \ b(s,x) = (b^1(s,x), \dots, b^d(s,x)); \\ &\sigma : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{d\times p}, \ \sigma(s,x) = (\sigma^{ij}(s,x))_{1\le i\le d, 1\le j\le p}. \end{aligned}$$

Consider the multidimensional stochastic differential equation

$$dX_t^i = b^i(t,X_t)dt + \sum_{j=1}^{p} \sigma^{ij}(t,X_t)dW_t^j, \ \text{ for } \ i = 1, \dots, d,$$

which can be written

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t. \tag{2.9}$$

We assume that the assumptions of Theorem 2.1.31 are satisfied. For any time t we define the differential operator A_t which maps a C^2 function v from $\mathbb{R}^d$ to $\mathbb{R}$ to the function

$$(A_t v)(x) = \frac{1}{2} a^{ij}(t,x)\frac{\partial^2 v}{\partial x^i \partial x^j}(x) + b^i(t,x)\frac{\partial v}{\partial x^i}(x), \tag{2.10}$$

where $(a^{ij}(t,x))$ is the matrix with components

$$a^{ij}(t,x) = \sum_{k=1}^{p} \sigma^{ik}(t,x)\sigma^{jk}(t,x).$$

We have the following result (see Lamberton et all [34], p. 99):

Proposition 2.3.3. *Let $u(t,x)$ be a $C^{1,2}$ real-valued function defined on $\mathbb{R}^+\times\mathbb{R}^d$ with bounded derivatives in x and (X_t) a solution of system (2.9). Let $r(t,x)$ be a bounded continuous real-valued function defined on $\mathbb{R}^+ \times \mathbb{R}^d$. Then the process*

$$M_t = e^{-\int_0^t r(s,X_s)ds}u(t,X_t) - \int_0^t e^{-\int_0^s r(v,X_v)dv}\left(A_s u - ru + \frac{\partial u}{\partial t}\right)(s,X_s)ds$$

is a martingale.

Option pricing and solving a PDE.
We will now establish the connection between pricing an option and solving a parabolic PDE problem.

We consider the multidimensional stochastic differential equation (2.9). Let $(X_t)_{t\ge0}$ be the solution of (2.9), $g(x)$ a function from $\mathbb{R}^d$ to $\mathbb{R}$ and $r(t,x)$ a bounded continuous real-valued function defined on $\mathbb{R}^+ \times \mathbb{R}^d$.

We want to compute

$$V_t = \mathbf{E}\left(e^{-\int_t^T r(s,X_s)ds}g(X_T)|\mathcal{F}_t\right).$$

As in the unidimensional case, it can be proved that

$$V_t = G(t, X_t), \quad \text{with} \quad G(t,x) = \mathbf{E}\left(e^{-\int_t^T r(s,X_s^{t,x})ds} g(X_T^{t,x})\right),$$

where $X_s^{t,x}$ denotes the solution of (2.9) starting from x at time t.

The following main result is obtained owing to Proposition 2.3.3 and characterizes the function G as a solution of a parabolic partial differential equation (see Lamberton et all [34], p. 99).

Theorem 2.3.4. *Let $u(t,x)$ be a $C^{1,2}$ real-valued function defined on $[0,T]\times\mathbb{R}^d$ with bounded derivatives in x and (X_t) a solution of system (2.9). Let A_t be the operator defined by (2.10) and $r(t,x)$ a bounded continuous real-valued function defined on $\mathbb{R}^+ \times \mathbb{R}^d$. If u satisfies*

$$\left(A_t u - ru + \frac{\partial u}{\partial t}\right)(t,x) = 0 \quad \forall (t,x) \in [0,T]\times\mathbb{R}^d, \quad u(T,x) = g(x) \quad \forall x \in \mathbb{R}^d$$

then

$$\forall (t,x) \in [0,T]\times\mathbb{R}^d \quad u(t,x) = G(t,x) = \mathbf{E}\left(e^{-\int_t^T r(s,X_s^{t,x})ds} g(X_T^{t,x})\right).$$

This result offers a method to determine the price of an European option which consists in solving the corresponding PDE problem. To compute

$$G(t,x) = \mathbf{E}\left(e^{-\int_t^T r(s,X_s^{t,x})ds} g(X_T^{t,x})\right),$$

we have to solve

$$A_t u - ru + \frac{\partial u}{\partial t} = 0 \quad \text{in } [0,T]\times\mathbb{R}^d, \quad u(T,x) = g(x) \quad \text{for } x \in \mathbb{R}^d. \tag{2.11}$$

Equation (2.11) characterizes a parabolic PDE problem with a final condition.

We need to consider the proper function spaces for this problem to be well defined. We note that to have $u = G$, the solution u of (2.11) has to satisfy the smoothness assumptions in Theorem 2.3.4. In general, some regularity assumptions have to be made on the coefficients b and σ and the operator A_t have to satisfy the ellipticity condition

$$\exists \lambda > 0,\ \forall (t,x) \in [0,T]\times\mathbb{R}^d,\ \forall \xi \in \mathbb{R}^d \quad \sum_{i,j=1}^{d} a^{ij}(t,x)\xi^i\xi^j \geq \lambda \sum_{i=1}^{d} |\xi^i|^2.$$

Let us exemplify the method for the simple unidimensional Black-Scholes model (see Lamberton et all [34], pp. 100-101).

We consider the stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dB_t)$$

where μ and $\sigma > 0$ are constants and $(B_t)_{t\geq 0}$ is a standard $\mathcal{F}_t-$Brownian motion.

We have that (see Remark 2.2.13), under the risk-neutral probability measure $\mathbf{P}^*$, the asset price S_t satisfies

$$dS_t = S_t(rdt + \sigma dW_t),$$

where $r \geq 0$ is a constant and $(W_t)_{t\geq 0}$ is a standard $\mathcal{F}_t-$Brownian motion. The operator A is now independent of time t and is given by

$$A = \frac{\sigma^2}{2} x^2 \frac{\partial^2}{\partial x^2} + rx\frac{\partial}{\partial x}.$$

This operator is not elliptic.

We consider the diffusion $X_t = \log(S_t)$. Since $S_t = S_0\, e^{(r-\sigma^2/2)t+\sigma W_t}$, we have that $(X_t)_{t\geq 0}$ is solution of

$$dX_t = (r - \sigma^2/2)dt + \sigma dW_t.$$

The infinitesimal generator of this diffusion

$$A^{\log} = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + (r - \sigma^2/2)\frac{\partial}{\partial x}$$

has constant coefficients and the ellipticity condition is satisfied.

If we want to compute the option price $G(t,x)$, we then have to find a solution $v \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$, with bounded derivatives in x, of the problem

$$A^{\log}v - rv + \frac{\partial v}{\partial t} = 0 \quad \text{in } [0,T]\times\mathbb{R}, \qquad v(T,x) = g(e^x) \quad \text{for } x \in \mathbb{R}.$$

Finally,

$$G(t,x) = v(t, \log(x)).$$

The above example presented in Lamberton et all [34], can be generalized to the multidimensional version of Black-Scholes model (also with constant coefficients and interest rate). Let

$$\begin{aligned}
&B_t = (B_t^1,\dots,B_t^d) \ \text{ an } \mathbb{R}^d - \text{valued } \mathcal{F}_t - \text{Brownian motion;}\\
&\mu = (\mu^1,\dots,\mu^d) \ \text{ a constant vector;}\\
&\sigma = (\sigma^{ij})_{1\leq i,j\leq d} \ \text{ a constant matrix.}
\end{aligned}$$

The stochastic differential equation modelling the asset prices is

$$dS_t^i = S_t^i\left(\mu^i dt + \sum_{j=1}^d \sigma^{ij} dB_t^j\right), \quad \text{for } i = 1,\dots,d,$$

and can be written

$$dS_t = \hat{S}_t(\mu dt + \sigma dB_t),$$

where $\hat{S}_t$ denotes de diagonal matrix with diagonal elements $S_t^i, \;\; i = 1, \ldots, d$.

We assume that matrix σ is positive definite and define $\rho := (r, \ldots, r)$, with $r \geq 0$ a constant. Owing to Theorem 2.2.9, with $\theta = \sigma^{-1}(\mu - \rho)$, there exists a probability measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$ under which

$$W_t = B_t + \sigma^{-1}(\mu - \rho)t$$

is a $\mathbb{R}^d-$valued standard Brownian motion (see Elliot et all [14], p. 168).

We obtain

$$dS_t = \hat{S}_t(\rho dt + \sigma dW_t), \tag{2.12}$$

The infinitesimal generator of the diffusion S_t is

$$A = \frac{1}{2}\,(\sigma\sigma')^{ij}\, x^i x^j\, \frac{\partial^2}{\partial x^i \partial x^j} + r x^i \frac{\partial}{\partial x^i},$$

and it is not elliptic.

In the same way as for the unidimensional case, it could be checked that the stochastic differential equation (2.12) has the unique solution

$$S_t^i = S_0^i \,\exp\left(\left(r - \frac{1}{2}\sum_{j=1}^{d}(\sigma^{ij})^2\right) t + \sum_{j=1}^{d} \sigma^{ij} W_t^j\right), \;\; \text{for} \;\; i = 1, \ldots, d.$$

We use the logarithmic transformation $X_t^i = \log(S_t^i), \;\; i = 1, \ldots, d$, and denote it $X_t = \log(S_t)$. We have that $(X_t)_{t \geq 0}$ is solution of

$$dX_t^i = \left(r - \frac{1}{2}\sum_{j=1}^{d}(\sigma^{ij})^2\right) dt + \sum_{j=1}^{d} \sigma^{ij} dW_t^j, \;\; \text{for} \;\; i = 1, \ldots, d,$$

and its infinitesimal generator is

$$A^{\log} = \frac{1}{2}\,(\sigma\sigma')^{ij}\,\frac{\partial^2}{\partial x^i \partial x^j} + \left(r - \frac{1}{2}\sum_{j=1}^{d}(\sigma^{ij})^2\right)\frac{\partial}{\partial x^i}.$$

The coefficients in $A^{\log}$ are constant and, as σ is a positive definite matrix, the ellipticity condition is satisfied.

To compute the option price $G(t,x)$, we have to find a solution $v \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$, with bounded derivatives in x, of the problem

$$A^{\log} v - rv + \frac{\partial v}{\partial t} = 0 \quad \text{in } [0,T] \times \mathbb{R}^d, \quad v(T,x) = g(e^x) \quad \text{for } x \in \mathbb{R}^d,$$

and then obtain

$$G(t,x) = v(t, \log(x)),$$

where $e^x := (e^{x^1}, \ldots, e^{x^d})$ and $\log(x) := (\log(x^1), \ldots, \log(x^d))$.

In these two simple examples, the drift μ and the volatility σ were considered constant. Therefore we have a closed-form solution for the stochastic differential equation modelling the asset prices and, with the help of a logarithmic transformation, we could offset the linear growth of the equation coefficients and obtain a differential operator A with constant coefficients.

A more difficult situation occurs when μ and σ are not constant. In this case there does not exist in general a closed-form solution for the stochastic equation. We will approach this problem in Chapter 4, considering the appropriate function spaces in order to obtain the (uniform) ellipticity in space of the operator A.

We make a final comment on the application potentiality of the (multidimensional) European option modelling we have considered. We see that it extends Black-Scholes model in several ways:

- The option depends on several underlying assets;
- The payoff function is not specified;
- The coefficients of the stochastic equation modelling the stock prices are assumed to be time and space-dependent.

The model applies directly to options on a basket of assets (*basket options* or *rainbow options*).

The higher dimensionality together with the non-specification of the payoff function allows the model to be adapted to other types of options with no early exercise (that is, for which the exercise can only occur at a fixed time T) (see e.g. Lamberton et all [34], Wilmott [47]). For instance, to:

- European options on future contracts and foreign-exchange;
- *Compound options*: this type of option is an option on another option;
- *Exchange options*: in this case the option gives the right to exchange an asset for another;
- Some path-dependent types of options as *Asian options.*

The time and space-dependency of the stochastic equation's coefficients confers flexibility to the model: the assumption that the coefficients are constant would be restrictive, mainly for options with distant expiration dates.

Chapter 3

Parabolic PDE in Hölder spaces: space and time discretization

We have to consider the proper function spaces for the parabolic PDE problem we study to be well defined. In the present chapter, we will consider the solvability of the PDE in Hölder spaces, following the presentation of Krylov [29]. To approximate the solution of the Cauchy problem in half spaces, we first study the approximation of the solution of the corresponding (localized) problem in domains. Then we estimate the error due to the problem localization.

In the previous chapter, arising from the stochastic modelling of the stock price, we considered a parabolic problem (with final condition and null term)

$$Au + cu + u_t = 0 \ \text{ in } \ [0,T] \times \mathbb{R}^d, \quad u(T,x) = g(x) \ \text{ in } \mathbb{R}^d,$$

where

$$A(t,x) = \frac{1}{2} a^{ij}(t,x) \frac{\partial^2}{\partial x^i \partial x^j} + b^i(t,x) \frac{\partial}{\partial x^i}$$

and g is a given function.

In this chapter and in the following chapters (except for Section 3.3 where the stochastic representation of the PDE problem is needed) we will consider the more standard form of the PDE problem (with initial condition)

$$Lu - u_t + f = 0 \ \text{ in } \ [0,T] \times \mathbb{R}^d, \quad u(0,x) = g(x) \ \text{ in } \mathbb{R}^d, \tag{3.1}$$

where

$$L(t,x) = a^{ij}(t,x) \frac{\partial^2}{\partial x^i \partial x^j} + b^i(t,x) \frac{\partial}{\partial x^i} + c\,(t,x),$$

and f and g are given functions (with f not necessarily null).

Note that problem (3.1) (with the initial condition $u(0,x) = g(x)$), using the change of variable $(t,x) \mapsto (T-t,x)$, is obviously equivalent to the problem with final condition $u(T,x) = g(x)$

$$Lu + u_t + f = 0 \ \text{ in } \ [0,T] \times \mathbb{R}^d, \quad u(T,x) = g(x) \ \text{ in } \mathbb{R}^d.$$

3.1 Classical results

We introduce the Hölder spaces (see Krylov [29], pp. 33-34 and 117-118).

Let U be a domain in $\mathbb{R}^d$, meaning an open subset of $\mathbb{R}^d$. For $k = 0, 1, 2, \ldots$ we denote $C^k_{loc}(U)$ the set of all functions $u : U \to \mathbb{R}$ whose derivatives $D^\alpha u$ for $|\alpha| \leq k$ are continuous in every bounded subset V of U. We define

$$|u|_{0;U} := [u]_{0;U} := \sup_U |u|, \quad [u]_{k;U} := \max_{|\alpha|=k} |D^\alpha u|_{0;U}.$$

Definition 3.1.1. For $k = 0, 1, 2, \ldots$, the space $C^k(U)$ is the Banach space of all functions $u \in C^k_{loc}(U)$ for which the norm

$$|u|_{k;U} = \sum_{j=0}^{k} [u]_{j;U}$$

is finite. If $0 < \delta < 1$, we call u *Hölder continuous with exponent δ in U* if the seminorm

$$[u]_{\delta;U} = \sup_{x,y \in U,\, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\delta}$$

is finite. The seminorm is called *Hölder's constant* of u of order δ.

We define

$$[u]_{k+\delta;U} := \max_{|\alpha|=k} [D^\alpha u]_{\delta;U}.$$

Definition 3.1.2. For $0 < \delta < 1$ and $k = 0, 1, 2, \ldots$, the Hölder space $C^{k+\delta}(U)$ is the Banach space of all functions $u \in C^k(U)$ for which the norm

$$|u|_{k+\delta;U} = |u|_{k;U} + [u]_{k+\delta;U}$$

is finite.

Now denote $\mathbb{R}^{d+1} = \{(t, x) : t \in \mathbb{R},\ x \in \mathbb{R}^d\}$. In $\mathbb{R}^{d+1}$ define the parabolic distance between the points $z_1 = (t_1, x_1)$, $z_2 = (t_2, x_2)$ as

$$\rho(z_1, z_2) := |x_1 - x_2| + |t_1 - t_2|^{1/2}.$$

We fix a constant $\delta \in (0, 1)$. If u is a real-valued function defined in $Q \subset \mathbb{R}^{d+1}$, we denote

$$[u]_{\delta/2,\delta;Q} := \sup_{z_1 \neq z_2,\, z_i \in Q} \frac{|u(z_1) - u(z_2)|}{\rho^\delta(z_1, z_2)}, \quad |u|_{\delta/2,\delta;Q} := |u|_{0;Q} + [u]_{\delta/2,\delta;Q}.$$

Definition 3.1.3. For $0 < \delta < 1$, $C^{\delta/2,\delta}(Q)$ is the Banach space of all functions u defined in Q for which $|u|_{\delta/2,\delta;Q} < \infty$.

We introduce the parabolic Hölder spaces.

Definition 3.1.4. For $0 < \delta < 1$, the parabolic Hölder space $C^{1+\delta/2,2+\delta}(Q)$ is the Banach space of all real-valued functions $u(z)$ defined in Q for which both

1. $[u]_{1+\delta/2,2+\delta;Q} := [u_t]_{\delta/2,\delta;Q} + \sum_{i,j=1}^{d} [u_{x^i x^j}]_{\delta/2,\delta;Q}$
2. $|u|_{1+\delta/2,2+\delta;Q} := |u|_{0;Q} + |u_x|_{0;Q} + |u_t|_{0;Q} + \sum_{i,j=1}^{d} |u_{x^i x^j}|_{0;Q} + [u]_{1+\delta/2,2+\delta;Q}$

are finite.

We now summarize some classical results on solvability of parabolic PDE in Hölder spaces.

Consider the elliptic and parabolic operators of order m.

Definition 3.1.5. Let $m \geq 1$ be an integer and $a^\alpha(x)$ be some real-valued functions in $\mathbb{R}^d$, given for any multi-index α with $|\alpha| \leq m$. The operator $L = \sum_{|\alpha|\leq m} a^\alpha(x) D^\alpha$ is called mth order (uniformly) *elliptic* if there exists a constant $\lambda > 0$ called the constant of ellipticity, such that

$$\sum_{|\alpha|\leq m} a^\alpha(x)\xi^\alpha \geq \lambda|\xi|^m \quad \forall x, \xi \in \mathbb{R}^d.$$

Definition 3.1.6. Let $m \geq 1$ be an integer and $a^\alpha(t,x)$ be some given real-valued functions in $\mathbb{R}^{d+1}$, with $|\alpha| \leq m$ a multi-index. The operator $L - \partial/\partial t$, with $L = \sum_{|\alpha|\leq m} a^\alpha(t,x) D^\alpha$ is called mth order (uniformly) *parabolic* if there exists a constant $\lambda > 0$ such that

$$\sum_{|\alpha|\leq m} a^\alpha(t,x)\xi^\alpha \geq \lambda|\xi|^m \quad \forall (t,x) \in \mathbb{R}^{d+1},\ \forall \xi \in \mathbb{R}^d.$$

Consider the second-order operator (in the non-divergence form)

$$L(t,x) = a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t,x)\frac{\partial}{\partial x^i} + c\,(t,x), \tag{3.2}$$

with real coefficients. We assume that, for some $\lambda > 0$ and for each $t > 0$, the operator satisfies $a^{ij}(t,x)\xi^i\xi^j \geq \lambda|\xi|^2$, for all $x, \xi \in \mathbb{R}^d$, so that L is uniformly elliptic with respect to the space variables, with constant of ellipticity λ. Then, for each t, the symmetric matrix $(a^{ij}(x,t))$ is positive definite for any $x \in \mathbb{R}^d$. We also assume that there exists a constant K such that $|a|_{\delta/2,\delta} \le K$, $|b|_{\delta/2,\delta} \le K$, $|c|_{\delta/2,\delta} \le K$, where $\delta \in (0,1)$ is fixed.

We consider first the Cauchy problem for second-order parabolic equations in half spaces. Let $T \in (0,\infty)$, $Q = [0,T] \times \mathbb{R}^d$. The problem to be solved is

$$Lu - u_t + f = 0 \text{ in } Q, \quad u(0,x) = g(x) \text{ in } \mathbb{R}^d, \tag{3.3}$$

where f and g are given functions.

Remark 3.1.7. In the presentation of Krylov [29], the parabolic equation is defined for the time variable t taking values in $(0,T)$, with $T \in (0,\infty]$.

As, for any constant μ, the function $v(t,x) = u(t,x)e^{-\mu t}$ satisfies $Lv - \mu v - v_t + fe^{-\mu t} = 0$ if and only if u satisfies $Lu - u_t + f = 0$, we set $c \le 0$ without loss of generality.

We have the following existence and uniqueness result for the solution of (3.3) (see Krylov [29], p. 140).

Theorem 3.1.8. *Assume that $c \le -\mu$ for a constant $\mu > 0$. Let $g \in C^{2+\delta}(\mathbb{R}^d)$ and $f \in C^{\delta/2,\delta}(Q)$. Then there exists a unique function $u \in C^{1+\delta/2,2+\delta}(Q)$ such that it satisfies (3.3). Moreover, there is a constant N depending only on d, λ, δ, K and μ such that $|u|_{1+\delta/2,2+\delta;Q} \le N(|f|_{\delta/2,\delta;Q} + |g|_{2+\delta})$.*

We consider now the initial-boundary value problem in $Q = [0,T] \times U$, with $U \subset \mathbb{R}^d$ a bounded domain. For this, we give a preliminary definition (see Krylov [29], p. 78). Denote $B_R(x_0) \subset \mathbb{R}^d$ the open ball in $\mathbb{R}^d$ with center x_0 and radius R. For any $U \subset \mathbb{R}^d$, denote ∂U the boundary of U. Denote also

$$\mathbb{R}^d_+ = \{(x', x^d) : x' = (x^1, \dots, x^{d-1}) \in \mathbb{R}^{d-1},\ x^d > 0\}.$$

Definition 3.1.9. Let $r > 0$ and U be a bounded domain in $\mathbb{R}^d$. We write $U \in C^r$ (or $\partial U \in C^r$) and say that the domain U is of class C^r if there are numbers $\rho_0, K_0 > 0$ such that for any point $x_0 \in \partial U$ there exists a one-to-one mapping ψ of $B_{\rho_0}(x_0)$ onto a domain $D \subset \mathbb{R}^d$ such that

1. $D_+ := \psi(B_{\rho_0}(x_0) \cap U) \subset \mathbb{R}^d_+$ and $\psi(x_0) = 0$;
2. $\psi(B_{\rho_0}(x_0) \cap \partial U) = D \cap \{y \in \mathbb{R}^d : y^d = 0\}$;
3. $[\psi]_{s;B_{\rho_0}(x_0)} + [\psi^{-1}]_{s;D} \le K_0$ for any $s \in [0,r]$, and $|\psi^{-1}(y_1) - \psi^{-1}(y_2)| \le K_0\,|y_1 - y_2|$ for any $y_i \in D$.

We say that the diffeomorphism ψ straightens the boundary near x_0.

We consider the initial-boundary value problem

$$Lu - u_t + f = 0 \text{ in } Q, \quad u(0,x) = g(x) \text{ for } x \in \bar{U}, \quad u = \bar{g} \text{ on } \partial_x Q, \tag{3.4}$$

where $Q = [0,T] \times U$, with $T \in (0,\infty)$, the domain $U \subset \mathbb{R}^d$ is of class $C^{2+\delta}$, $\partial_x Q := [0,T] \times \partial U$ and f, g and $\bar{g}$ are given functions.

Remark 3.1.10. We denote $\partial_t Q := \{0\} \times \bar{U}$ and $\partial Q := \partial_x Q \cup \partial_t Q$.

Assumption 3.1.11. We assume the consistency conditions:

1. $\bar{g}(0,x) = g(x)$ for $x \in \partial U$;

2. $L(0,x)g(x) - \bar{g}_t(0,x) + f(0,x) = 0$ for $x \in \partial U$.

The following result states the solvability of the problem in Hölder spaces (see Krylov [29], p. 153). Denote $\mathbb{R}^{d+1}_+ = \{(t,x) : t \geq 0,\ x \in \mathbb{R}^d\}$.

Theorem 3.1.12. *Let $f \in C^{\delta/2,\delta}(\mathbb{R}^{d+1}_+)$, $g \in C^{2+\delta}(\mathbb{R}^d)$, $\bar{g} \in C^{1+\delta/2,2+\delta}(Q)$, with $Q = [0,\infty) \times U$. Let (1)–(2) in Assumption 3.1.11 be satisfied. Then there exists a unique function $u \in C^{1+\delta/2,2+\delta}(Q)$ satisfying (3.4). Moreover*

$$|u|_{1+\delta/2,2+\delta;Q} \leq N \left(|f|_{\delta/2,\delta;\mathbb{R}^{d+1}_+} + |g|_{2+\delta;\mathbb{R}^d} + |\bar{g}|_{1+\delta/2,2+\delta;Q} \right),$$

where N is a constant depending on d, λ, δ, K, ρ_0, K_0 and the diameter of U.

Further results under weaker conditions.
We consider the Cauchy problem in half spaces under weaker smoothness conditions imposed over the initial data.

Let Z be the fundamental solution for the parabolic operator $L - \partial/\partial t$. We have estimates for the derivatives of Z (see Ladyženskaja et all [33], pp. 376-377).

Proposition 3.1.13. *The following inequalities hold:*

1. $|D_t^\alpha D_x^\beta Z(t,\tau,x,y)| \leq K(t-\tau)^{-\frac{d+2|\alpha|+|\beta|}{2}} e^{-M\frac{|x-y|^2}{t-\tau}}$,
   where K, M constants, $2|\alpha| + |\beta| \leq 2$ and $\tau < t$;
2. $|D_t^\alpha D_x^\beta Z(t,\tau,x,y) - D_{t'}^\alpha D_x^\beta Z(t',\tau,x,y)|$
   $\leq K[(t-t')(t'-\tau)^{-\frac{d+2|\alpha|+|\beta|+2}{2}} + (t-t')^{\frac{\delta-2|\alpha|-|\beta|+2}{2}}(t'-\tau)^{-\frac{d+2}{2}}]e^{-M\frac{|x-y|^2}{t-\tau}}$,
   where K, M constants, $2|\alpha| + |\beta| = 1, 2$ and $\tau < t' < t$;
3. $|D_t^\alpha D_x^\beta Z(t,\tau,x,y) - D_t^\alpha D_{x'}^\beta Z(t,\tau,x',y)|$
   $\leq K[|x-x'|^\gamma (t-\tau)^{-\frac{\gamma+d+2}{2}} + |x-x'|^\zeta (t-\tau)^{-\frac{\delta-\zeta+d+2}{2}}]e^{-M\frac{|x''-y|^2}{t-\tau}}$,
   where K, M constants, $2|\alpha|+|\beta| = 2$, $0 \leq \gamma \leq 1$, $0 \leq \zeta \leq \delta$, $\tau < t$ and x'' is the one of the points x and x' which is closest to y.

We state an existence and uniqueness result for the solution of problem (3.3), when the initial data g is continuous and f and g are allowed polynomial growth (see Ladyženskaja et all [33], pp. 389-390, where weaker hypothesis over the growth of f and g are assumed):

Theorem 3.1.14. *Let f be a function in $Q = [0,T] \times \mathbb{R}^d$, for $T \in (0,\infty)$, such that $[f]_{\delta/2,\delta;Q} < \infty$, and $g \in C(\mathbb{R}^d)$. Let f, g satisfy $|f(t,x)| \leq K(1+|x|^m)$ in Q and $|g(x)| \leq K(1+|x|^m)$ in $\mathbb{R}^d$, respectively, with K, m positive constants. Then problem (3.3) has a unique solution $u(t,x)$ in Q. Moreover*

$$u(t,x) = \int_0^t d\tau \int_{\mathbb{R}^d} Z(t,\tau,x,y) f(\tau,y) dy + \int_{\mathbb{R}^d} Z(t,0,x,y) g(y) dy,$$

where Z is the fundamental solution for the parabolic operator $L - \partial/\partial t$.

From the estimates in Proposition 3.1.13, it can be shown that the solution u in Theorem 3.1.14 is in $C^{1,2}(Q)$ and satisfies $|D_x^\beta u(t,x)| \leq N(1+|x|^m)$, $\beta = 0, 1$, in Q, with N, m positive constants (m the constant in Theorem 3.1.14) (see e.g. Friedman [18], pp. 141 and 148).

The smoothness of the solution u can be improved stepping away from the time origin in problem (3.3). We will see that, in this case, and if f and g are bounded, we obtain a $C^{1+\delta/2,2+\delta}$ solution.

Theorem 3.1.15. *Let the hypothesis of Theorem 3.1.14 be satisfied, f and g bounded functions in Q and $\mathbb{R}^d$, respectively, and u the corresponding solution of problem (3.3). Let $c \leq -\mu$ for a constant $\mu > 0$ and define the set $Q_\varepsilon = [\varepsilon, T] \times \mathbb{R}^d$, where ε is a positive constant. Then $u \in C^{1+\delta/2,2+\delta}(Q_\varepsilon)$.*

Proof Denote

$$u_1(t,x) = \int_0^t d\tau \int_{\mathbb{R}^d} Z(t,\tau,x,y) f(\tau,y) dy \quad \text{and} \quad u_2(t,x) = \int_{\mathbb{R}^d} Z(t,0,x,y) g(y) dy,$$

so that $u(t,x) = u_1(t,x) + u_2(t,x)$.

We have that $u_1(t,x)$ solves the problem

$$Lu - u_t + f = 0 \text{ in } Q, \quad u(0,x) = 0 \text{ for } x \in \mathbb{R}^d,$$

and that $u_2(t,x)$ solves the problem

$$Lu - u_t = 0 \text{ in } Q, \quad u(0,x) = g(x) \text{ for } x \in \mathbb{R}^d.$$

From Theorem 3.1.8 we have that $u_1 \in C^{1+\delta/2,2+\delta}(Q)$ and, therefore, $u_1 \in C^{1+\delta/2,2+\delta}(Q_\varepsilon)$. We will show that $u_2 \in C^{1+\delta/2,2+\delta}(Q_\varepsilon)$.

We see from estimate (2) in Proposition 3.1.13, with $|\alpha| = 1$, $|\beta| = 0$ and $0 < \varepsilon < t' < t$,

$$\begin{aligned} |D_t Z(t,0,x,y) - D_{t'} Z(t',0,x,y)| &\leq K[(t-t')t'^{-\frac{d+4}{2}} + (t-t')^{\frac{\delta}{2}}\, t'^{-\frac{d+2}{2}}] \\ &\quad \times \exp(-M\frac{|x-y|^2}{t}) \\ &\leq K[(t-t')\varepsilon^{-\frac{d+4}{2}} + (t-t')^{\frac{\delta}{2}}\, \varepsilon^{-\frac{d+2}{2}}] \\ &\quad \times \exp(-M|x-y|^2) \\ &\leq N(t-t')^{\frac{\delta}{2}}[(t-t')^{1-\frac{\delta}{2}}+1]\exp(-M|x-y|^2) \\ &\leq N(t-t')^{\frac{\delta}{2}}\exp(-M|x-y|^2), \qquad (3.5) \end{aligned}$$

with N a constant depending on ε.

From estimate (3) in Proposition 3.1.13, with $|\alpha| = 0$, $|\beta| = 2$, $\gamma = \zeta = \delta$ and $0 < \varepsilon < t$, we have

$$\begin{aligned} |D_x^\beta Z(t,0,x,y) - D_{x'}^\beta Z(t,0,x',y)| &\leq K[|x-x'|^\delta t^{-\frac{\delta+d+2}{2}} + |x-x'|^\delta t^{-\frac{d+2}{2}}] \\ &\quad \times \exp(-M\frac{|x''-y|^2}{t}) \\ &\leq K[|x-x'|^\delta \varepsilon^{-\frac{\delta+d+2}{2}} + |x-x'|^\delta \varepsilon^{-\frac{d+2}{2}}] \\ &\quad \times \exp(-M|x-y|^2) \\ &\leq N|x-x'|^\delta \exp(-M|x-y|^2), \qquad (3.6) \end{aligned}$$

with N a constant depending on ε.

From estimate (1) in Proposition 3.1.13, with $|\alpha| = |\beta| = 0$ and $0 < \varepsilon < t$, we have

$$|Z(t,0,x,y)| \leq Kt^{-\frac{d}{2}}\exp(-M\frac{|x-y|^2}{t}) \leq N\exp(-M|x-y|^2), \qquad (3.7)$$

with N a constant depending on ε.

Similarly, with $|\alpha| = 1$, $|\beta| = 0$ and $0 < \varepsilon < t$ we obtain

$$|D_t Z(t,0,x,y)| \leq N\exp(-M|x-y|^2), \qquad (3.8)$$

and with $|\alpha| = 0$, $|\beta| = 1$ or $|\alpha| = 0$, $|\beta| = 2$ and $0 < \varepsilon < t$

$$|D_x^\beta Z(t,0,x,y)| \leq N\exp(-M|x-y|^2), \qquad (3.9)$$

with N a constant depending on ε.

As g is a bounded function in $\mathbb{R}^d$, from (3.5) and (3.6) we conclude that $[u_2]_{1+\delta/2,2+\delta;Q_\varepsilon} < \infty$ and from (3.7), (3.8) and (3.9) that

$$|u_2|_{0;Q_\varepsilon} + |D_x u_2|_{0;Q_\varepsilon} + |D_t u_2|_{0;Q_\varepsilon} + \sum_{i,j=1}^{d} |D_{x^j} D_{x^i} u_2|_{0;Q_\varepsilon} < \infty.$$

Note that the factor $\exp(-M|x-y|^2)$ in the above estimates guarantees the convergence of the integral in u_2.

We have that $u_2 \in C^{1+\delta/2,2+\delta}(Q_\varepsilon)$ and, finally, $u \in C^{1+\delta/2,2+\delta}(Q_\varepsilon)$. □

Consider now the particular case of the initial-boundary value problem (3.4) where $\bar{g}=0$, under weaker smoothness imposed over the initial data g.

We have the following main result for the existence and uniqueness of the solution of (3.4) (proved in Ladyženskaja et all [33], pp. 412-413, for interior and exterior domains).

Theorem 3.1.16. *Let $f \in C^{\delta/2,\delta}(Q)$, $g \in C(\bar{U})$, with $Q=[0,T]\times U$, $T\in(0,\infty)$. Assume that (1) in Assumption 3.1.11 is satisfied. Then problem (3.4) with $\bar{g}=0$ has a unique solution $u(t,x)$ in Q. Moreover*

$$u(t,x)=\int_0^t d\tau \int_U G(t,\tau,x,y)f(\tau,y)dy+\int_U G(t,0,x,y)g(y)dy,$$

where G is the Green's function for problem (3.4).

Note that function u is not defined for $t=0$. The initial condition is satisfied by u in limit.

We have estimates for the derivatives of the Green's function (see Ladyženskaja et all [33], pp. 412-414).

Proposition 3.1.17. *Let G be the Green's function considered in Theorem 3.1.16. The following inequalities hold:*

1. $|D_t^\alpha D_x^\beta G(t,\tau,x,y)| \le K(t-\tau)^{-\frac{d+2|\alpha|+|\beta|}{2}} \exp\left(-M\frac{|x-y|^2}{t-\tau}\right)$,
   where K, M constants, $2|\alpha|+|\beta|\le 2$ and $\tau<t$;
2. $|D_t^\alpha D_x^\beta G(t,\tau,x,y)-D_{t'}^\alpha D_x^\beta G(t',\tau,x,y)|$
   $\le K(t-t')^{\frac{\delta-2|\alpha|-|\beta|+2}{2}}(t'-\tau)^{-\frac{\delta+d+2}{2}} \exp\left(-M\frac{|x-y|^2}{t-\tau}\right)$,
   where K, M constants, $2|\alpha|+|\beta|=1,2$ and $\tau<t'<t$;
3. $|D_t^\alpha D_x^\beta G(t,\tau,x,y)-D_t^\alpha D_{x'}^\beta G(t,\tau,x',y)|$
   $\le K|x-x'|^\delta (t-\tau)^{-\frac{\delta+d+2}{2}} \exp\left(-M\frac{|x''-y|^2}{t-\tau}\right)$,
   where K, M constants, $2|\alpha|+|\beta|=2$, $\tau<t$ and x'' is the one of the points x and x' which is closest to y.

As for the Cauchy problem (3.3), it can be shown from estimate (1) in Proposition 3.1.17 that the solution u of problem (3.4) in Theorem 3.1.16 is in $C^{1,2}(Q)$.

Also, if we step away from the time origin we obtain a $C^{1+\delta/2,2+\delta}$ solution.

Theorem 3.1.18. *Assume that the hypothesis of Theorem 3.1.16 are satisfied and denote by u the corresponding solution of problem (3.4) with* $\bar{g} = 0$*. Let the set* $Q_\varepsilon = [\varepsilon, T] \times U$*, where* ε *is a positive constant. Then* $u \in C^{1+\delta/2, 2+\delta}(Q_\varepsilon)$*.*

Proof The result is obtained following the same steps as in the proof of Theorem 3.1.15.

Denote

$$u_1(t,x) = \int_0^t d\tau \int_U G(t,\tau,x,y) f(\tau,y) dy \quad \text{and} \quad u_2(t,x) = \int_U G(t,0,x,y) g(y) dy,$$

so that $u(t,x) = u_1(t,x) + u_2(t,x)$.

We note that $u_1(t,x)$ solves the problem

$$Lu - u_t + f = 0 \text{ in } Q, \quad u(0,x) = 0 \text{ for } x \in \bar{U}, \quad u = 0 \text{ on } \partial_x Q,$$

and that $u_2(t,x)$ solves the problem

$$Lu - u_t = 0 \text{ in } Q, \quad u(0,x) = g(x) \text{ for } x \in \bar{U}, \quad u = 0 \text{ on } \partial_x Q.$$

From Theorem 3.1.12 we have that $u_1 \in C^{1+\delta/2, 2+\delta}(Q)$ and then $u_1 \in C^{1+\delta/2, 2+\delta}(Q_\varepsilon)$. It remains to prove that $u_2 \in C^{1+\delta/2, 2+\delta}(Q_\varepsilon)$.

From estimate (2) in Proposition 3.1.17, with $|\alpha| = 1$, $|\beta| = 0$ and $0 < \varepsilon < t' < t$, we have

$$\begin{aligned} |D_t G(t,0,x,y) - D_{t'} G(t',0,x,y)| &\leq K(t-t')^{\frac{\delta}{2}}\, t'^{-\frac{d+2+\delta}{2}} \exp\left(-M\frac{|x-y|^2}{t}\right) \\ &\leq N(t-t')^{\frac{\delta}{2}}, \end{aligned} \tag{3.10}$$

with N a constant depending on ε.

From estimate (3) in Proposition 3.1.17, with $|\alpha| = 0$, $|\beta| = 2$ and $0 < \varepsilon < t$, we have

$$\begin{aligned} |D_x^\beta G(t,0,x,y) - D_{x'}^\beta G(t,0,x',y)| &\leq K|x-x'|^\delta\, t^{-\frac{d+2+\delta}{2}} \exp\left(-M\frac{|x''-y|^2}{t}\right) \\ &\leq N|x-x'|^\delta, \end{aligned} \tag{3.11}$$

with N a constant depending on ε.

From estimate (1) in Proposition 3.1.17, with $0 < \varepsilon < t$ and $|\alpha|$, $|\beta|$ taking the appropriate values we have

$$|G(t,0,x,y)| \leq N, \quad |D_t G(t,0,x,y)| \leq N, \quad |D_x^\beta G(t,0,x,y)| \leq N, \tag{3.12}$$

with N a constant depending on ε.

As g is a bounded function in $\bar{U}$, from (3.10) and (3.11) we have that $[u_2]_{1+\delta/2,2+\delta;Q_\varepsilon} < \infty$ and from (3.12) that

$$|u_2|_{0;Q_\varepsilon} + |D_x u_2|_{0;Q_\varepsilon} + |D_t u_2|_{0;Q_\varepsilon} + \sum_{i,j=1}^{d} |D_{x^j} D_{x^i} u_2|_{0;Q_\varepsilon} < \infty.$$

Then $u_2 \in C^{1+\delta/2,2+\delta}(Q_\varepsilon)$. Finally, we have $u \in C^{1+\delta/2,2+\delta}(Q_\varepsilon)$ and the result is proved. □

3.2 Numerical approximation

We want to discretize problem (3.4). For the discretization, we set the framework in Krylov [29] (p. 155).

Take a number $T \in (0, \infty)$ and denote $Q = [0, T] \times U$. Let $l(h)$ be a function on $(0, 1]$ such that $l(h) > 0$ and $l(h) \to 0$ as $h \downarrow 0$. For $h \in (0, 1]$ define the $(l(h), h)-$grid on $\mathbb{R}^{d+1}_+$

$$Z_h^{d+1} = \{(t, x) : t = l(h)k, \;\; x = h \sum_{i=1}^{d} e_i n_i, \;\; k = 0, 1, 2, \dots \;, \; n_i = 0, \pm 1, \pm 2, \dots\}. \tag{3.13}$$

Let $Q(h) = Q \cap Z_h^{d+1}$ and $Q^0(h) = \{(t, x) \in Q(h) : \mathrm{dist}(x, \partial U) \geq h \text{ and } t \geq l(h)\}$. Denote $\partial' Q(h) = Q(h) \setminus Q^0(h) = \partial'_x Q(h) \cup \partial'_t Q(h)$, with

$$\partial'_x Q(h) = \{(t, x) \in Q(h) : \mathrm{dist}(x, \partial U) < h\}, \quad \partial'_t Q(h) = \{(t, x) \in Q(h) : t < l(h)\}.$$

For any $h \in (0, 1]$, $z \in Q^0(h)$, $z_1 \in Q(h)$ denote

$$\mathcal{L}_h u(z) = \sum_{z_1 \in Q(h)} p_h(z, z_1) u(z_1), \tag{3.14}$$

where $p_h(z, z_1)$ are some given numbers.

We make assumptions on the behaviour of the discrete operator $\mathcal{L}_h$.

Assumption 3.2.1. (Maximum principle). If u is a function defined on $Q(h)$ and for a point $z_0 \in Q^0(h)$ we have $u(z_0) = \max_{Q(h)} u(z) > 0$, then $\mathcal{L}_h u(z_0) \leq 0$.

Assumption 3.2.2. The operators $\mathcal{L}_h$ approximate $L - \partial/\partial t$. More precisely, for any $u \in C^{1+\delta/2,2+\delta}(Q)$ and any $z \in Q^0(h)$ we have

$$|Lu(z) - u_t(z) - \mathcal{L}_h u(z)| \leq K h^\delta |u|_{1+\delta/2,2+\delta;Q},$$

with K a constant.

We state a lemma (see Krylov [29], p. 77).

Lemma 3.2.3. *For any $R > 0$ there exists a function $v_0 \in C^\infty(\bar{B}_R)$, with $B_R \subset \mathbb{R}^d$, such that $Lv_0 \leq -1$ in B_R. Moreover, $0 < v_0 \leq N_0 = N_0(\lambda, K, R, d)$ in B_R and $v_0 = 0$ on ∂B_R.*

Next, we prove a result on the uniqueness of the solution for the discretized problem (stated in Krylov [29], p. 154, but only proved for the elliptic problem).

Theorem 3.2.4. *There is a constant $h_0 > 0$ depending only on κ, K, δ, d and the diameter of U such that for $h \in (0, h_0]$ for any bounded functions f, $\bar{g}$ the system of linear equations*

$$\mathcal{L}_h u(z) + f(z) = 0 \ \ \forall z \in Q^0(h), \quad u(z) = \bar{g}(z) \ \ \forall z \in \partial' Q(h), \tag{3.15}$$

has a unique solution $u_h(z)$, $z \in Q(h)$. In addition

$$\begin{aligned} \max\nolimits_{Q(h)} (u_h(z))_+ &\leq N \max\nolimits_{Q^0(h)} f_+(z) + \max\nolimits_{\partial' Q(h)} \bar{g}_+(z), \\ \max\nolimits_{Q(h)} (u_h(z))_- &\leq N \max\nolimits_{Q^0(h)} f_-(z) + \max\nolimits_{\partial' Q(h)} \bar{g}_-(z), \\ \max\nolimits_{Q(h)} |u_h(z)| &\leq N \max\nolimits_{Q^0(h)} |f(z)| + \max\nolimits_{\partial' Q(h)} |\bar{g}(z)|, \end{aligned}$$

where the constant N depends only on λ, K, d and the diameter of U.

Proof Let n be the number of points in $Q(h)$. Then the linear system (3.15) is a system of n equations about n variables $u_h(z), z \in Q(h)$. Therefore, to prove the first assertion we only need to prove uniqueness of the trivial solution for $f \equiv \bar{g} \equiv 0$. This uniqueness follows at once from the second assertion.

To prove the second assertion, it suffices only to prove the first estimate. In fact, if

$$\max_{Q(h)} (u_h(z))_+ \leq N \max_{Q^0(h)} f_+(z) + \max_{\partial' Q(h)} \bar{g}_+(z),$$

then

$$\begin{aligned} \max_{Q(h)} (u_h(z))_- = \max_{Q(h)} ((-u_h(z))_+ &\leq N \max_{Q^0(h)} (-f)_+(z) + \max_{\partial' Q(h)} (-\bar{g})_+(z) \\ &= N \max_{Q^0(h)} f_-(z) + \max_{\partial' Q(h)} \bar{g}_-(z). \end{aligned}$$

Note that if u_h is a solution of (3.15) then $-u_h$ is a solution of the system obtained from (3.15) taking $-f$ and $-\bar{g}$ instead of f and $\bar{g}$, respectively.

Also

$$\max_{Q(h)} |u_h(z)| = \max_{Q(h)} [(u_h(z))_+ + (u_h(z))_-] = \max_{Q(h)} (u_h(z))_+ + \max_{Q(h)} (u_h(z))_-,$$

and the third estimate follows.

In the proof of the first estimate we assume without loss of generality that $0 \in \bar{Q}$. We take the function v_0 from Lemma 3.2.3 with R defined as the diameter of U. Observe that $(L - \partial/\partial t)v_0 \le -1$ in Q, so that by Assumption 3.2.2 we can choose h_0 to have $\mathfrak{L}_h v_0 \le -\frac{1}{2}$ for any $h \in (0, h_0]$ and for any $z \in Q^0(h)$

$$|Lv_0(z) - \frac{\partial}{\partial t}v_0(z) - \mathfrak{L}_h v_0(z)| \le Kh^\delta |v_0|_{1+\delta/2,2+\delta;Q}$$
$$\Longrightarrow \mathfrak{L}_h v_0(z) \le Kh^\delta |v_0|_{1+\delta/2,2+\delta;Q} + Lv_0(z) - \frac{\partial}{\partial t}v_0(z),$$

and, as $(L - \partial/\partial t)v_0 \le -1$ in Q, then $\mathfrak{L}_h v_0(z) \le Kh^\delta |v_0|_{1+\delta/2,2+\delta;Q} - 1$. If $h \le ((2K|v_0|_{1+\delta/2,2+\delta;Q})^{-1})^{1/\delta}$ then

$$\mathfrak{L}_h v_0(z) \le -\frac{1}{2}, \ \forall z \in Q^0(h).$$

Now we take a solution u_h of (3.15) and consider $w = u_h - 2(F + \varepsilon)v_0 - \bar{G}$ where $F = \max_{Q^0(h)} f_+$, $\bar{G} = \max_{\partial' Q(h)} \bar{g}_+$ and ε is a positive constant.

If we prove that for any ε we have $w \le 0$ in $Q(h)$, the first estimate will obviously follow:

If $w \le 0$ in $Q(h)$ then

$$u_h \le 2(\max_{Q^0(h)} f_+ + \varepsilon)v_0 + \max_{\partial' Q(h)} \bar{g}_+$$

and

$$\max_{Q(h)}(u_h)_+ = \max_{Q(h)} u_h = \sup_{Q(h)} u_h \le 2v_0 \max_{Q^0(h)} f_+ + \max_{\partial' Q(h)} \bar{g}_+.$$

By Lemma 3.2.3, $v_0 \le N_0 = N_0(\lambda, K, R, d)$ in Q (with R = diameter of U) and we obtain

$$\max_{Q(h)}(u_h)_+ \le 2N_0 \max_{Q^0(h)} f_+ + \max_{\partial' Q(h)} \bar{g}_+.$$

Assume that $w > 0$ at some points and define z_0 as a point in $Q(h)$ where w takes its maximum value $w(z_0) > 0$. Since $u_h = \bar{g}$ and $v_0 \ge 0$ on $\partial' Q(h)$,

$$w = \bar{g} - \max_{\partial' Q(h)} \bar{g}_+ - 2v_0(\max_{Q^0(h)} f_+ + \varepsilon) \le 0, \ \text{ on } \partial' Q(h),$$

so that $z_0 \in Q^0(h)$.

By Assumption 3.2.1 we obtain $\mathfrak{L}_h \bar{G} \le 0$ and $\mathfrak{L}_h w(z_0) \le 0$. Note that if $\bar{G} = \max_{\partial' Q(h)} \bar{g}_+ = 0$ then $\mathfrak{L}_h \bar{G} = 0 \le 0$ trivially.

Then we have

$$\begin{aligned} 0 &\ge \mathfrak{L}_h w(z_0) = \mathfrak{L}_h u_h(z_0) - 2(F + \varepsilon)\mathfrak{L}_h v_0(z_0) - \mathfrak{L}_h \bar{G}(z_0) \\ &= -f(z_0) - 2(F + \varepsilon)\mathfrak{L}_h v_0(z_0) - \mathfrak{L}_h \bar{G}(v_0) \\ &\ge -f(z_0) + F + \varepsilon \ge \varepsilon > 0. \end{aligned}$$

We obtained a contradiction and the proposition is proved. □

Furthermore, a rate of convergence can be determined (also stated in Krylov [29], p. 155, but only proved for the elliptic problem).

Theorem 3.2.5. *Let* $f \in C^{\delta/2,\delta}(\mathbb{R}^{d+1}_+)$, $\bar{g} \in C^{1+\delta/2,2+\delta}(\mathbb{R}^{d+1}_+)$. *In Theorem 3.1.12 take* $g(x) = \bar{g}(0,x)$ *and assume that its hypotheses are satisfied. Let* $u \in C^{1+\delta/2,2+\delta}(Q)$ *be the solution of (3.4). Take a number* $h \in (0, h_0]$ *and denote by* u_h *the corresponding solution of (3.15). Then*

$$|u - u_h|_{0,Q(h)} \leq Nh^\delta \left(|f|_{\delta/2,\delta;\mathbb{R}^{d+1}_+} + |\bar{g}|_{1+\delta/2,2+\delta;\mathbb{R}^{d+1}_+}\right), \tag{3.16}$$

where the constant N *depends only on* d, K, δ, λ, ρ_0, K_0 *and the diameter of* U.

Proof For $z \in Q^0(h)$

$$\begin{aligned} |\mathfrak{L}_h(u_h - u)(z)| &= |-f(z) - \mathfrak{L}_h u(z)| = |Lu(z) - u_t(z) - \mathfrak{L}_h u(z)| \\ &\leq Kh^\delta |u|_{1+\delta/2,2+\delta;\mathbb{R}^{d+1}_+} \leq Nh^\delta \left(|f|_{\delta/2,2;\mathbb{R}^{d+1}_+} + |\bar{g}|_{1+\delta/2,2+\delta;\mathbb{R}^{d+1}_+}\right), \end{aligned}$$

owing to Theorem 3.1.12.

As $u_h - u$ is solution of the problem

$$\begin{cases} \mathfrak{L}_h(u_h - u)(z) = -f(z) - \mathfrak{L}_h u(z) & \forall z \in Q^0(h) \\ (u_h - u)(z) = 0 & \forall z \in \partial' Q(h) \cap \partial Q \\ (u_h - u)(z) = (\bar{g} - u)(z) & \forall z \in \partial' Q(h) \setminus \partial Q, \end{cases}$$

owing to Theorem 3.2.4 inequality (3.16) is obtained.

If $z \in \partial' Q(h)$, then the distance from z to ∂Q is less than h, so that there is a $y \in \partial Q$ satisfying $\rho(z,y) = |z - y| \leq h$. Note that $\partial'_t Q(h) \subset \partial_t Q$ so that if $z \in \partial'_t Q(h)$ then $\rho(z, \partial_t Q) = 0$ and the inequality is trivial.

We have

$$\begin{aligned} |(u_h - u)(z)| &= |\bar{g}(z) - u(z)| = |\bar{g}(z) - u(z) + \bar{g}(y) - \bar{g}(y)| \\ &= |\bar{g}(z) - u(z) + u(y) - \bar{g}(y)| \\ &\leq |\bar{g}(z) - \bar{g}(y)| + |u(z) - u(y)| \\ &\leq h(\frac{|\bar{g}(z) - \bar{g}(y)|}{|z - y|} + \frac{|u(z) - u(y)|}{|z - y|}) \\ &\leq h(\sup_{w \in [z,y]} |\nabla \bar{g}(w)| + \sup_{w \in [z,y]} |\nabla u(w)|) \\ &\leq h(\sup_{w \in [z,y]} (\sum_{i=1}^d |\bar{g}_{x^i}(w)| + |\bar{g}_t(w)|) + \sup_{w \in [z,y]} (\sum_{i=1}^d |u_{x^i}(w)| + |u_t(w)|)) \\ &\leq h(|\bar{g}|_{1+\delta/2,2+\delta;\mathbb{R}^{d+1}_+} + |u|_{1+\delta/2,2+\delta;\mathbb{R}^{d+1}_+}) \\ &\leq h(|\bar{g}|_{1+\delta/2,2+\delta;\mathbb{R}^{d+1}_+} + N(|f|_{\delta/2,\delta;\mathbb{R}^{d+1}_+} + |\bar{g}|_{1+\delta/2,2+\delta;\mathbb{R}^{d+1}_+})) \\ &\leq Nh^\delta (|f|_{\delta/2,2;\mathbb{R}^{d+1}_+} + |\bar{g}|_{1+\delta/2,2+\delta;\mathbb{R}^{d+1}_+}, \end{aligned}$$

using the mean-value theorem and Theorem 3.1.12. The result is proved. □

Discretization under weaker conditions.

In Section 3.1 we considered the case where weaker smoothness was imposed over the initial data g, when the boundary condition is $u = 0$. Under this framework, Theorem 3.2.4 holds for the same reasons.

For the rate of convergence we state a new proposition. Let $Q_\varepsilon = [\varepsilon, T] \times U$, with $\varepsilon > 0$ a constant and $Q_\varepsilon(h) = Q(h) \cap Q_\varepsilon$.

Theorem 3.2.6. *Let $f \in C^{\delta/2,\delta}(Q)$, $g \in C(\bar{U})$, with $Q = [0,T] \times U$ for $T \in (0, \infty)$. Define*

$$\bar{g}(t,x) = \begin{cases} 0, & x \in \partial U \\ g(x), & \text{otherwise} \end{cases}$$

and assume that the hypothesis in Theorem 3.1.16 are satisfied. Let u be the solution of (3.4) and u_ε its restriction to Q_ε. Take a number $h \in (0, h_0]$ and let u_h be the solution of (3.15) and $u_{h\varepsilon}$ its restriction to $Q_\varepsilon(h)$. Then

$$|u_\varepsilon - u_{h\varepsilon}|_{0,Q_\varepsilon(h)} \leq N h^\delta \left(|f|_{\delta/2,\delta;\bar{Q}_\varepsilon} + |\bar{g}|_{1+\delta/2,2+\delta;\bar{Q}_\varepsilon} \right),$$

where the constant N depends only on d, K, δ, λ, ρ_0, K_0, the diameter of U and ε.

Proof The proof is the same as for Theorem 3.2.5 taking, when needed, Q_ε and $Q_\varepsilon(h)$ in the place of Q and $Q(h)$, respectively. □

From what saw in the present section, to obtain an approximation for the solution of the continuous problem (3.4), with a known rate of convergence, it suffices to have an operator $\mathfrak{L}_h$ with the form of operator (3.14), verifying Assumptions 3.2.1 and 3.2.2.

We will now discretize the problem constructing particular operators, using both the explicit and implicit schemes.

In Krylov [29] (pp. 155-156) discrete operators are considered for the particular case where $L = \sum_{i=1}^{d} a^{ii}(t,x) D_i^2$ (in the elliptic case, discrete schemes for the operator $L = \sum_{i=1}^{d} a^{ii}(x) D_i^2 + \sum_{i=1}^{d} b^i(x) D_i$ are also introduced, [29] pp. 86-87). We will construct discrete operators for the more general case

$$L = \sum_{i,j=1}^{d} a^{ij}(t,x) D_i D_j + \sum_{i=1}^{d} b^i(t,x) D_i + c(t,x),$$

where coefficients $a^{ij}(t,x)$ satisfy $\sum_{j=1}^{d} a^{ij}(t,x) \geq 0$, for $i = 1, 2, \ldots, d$ and $a^{ij}(t,x) < 0$, for $i \neq j$, $i,j = 1, 2, \ldots, d$. Note that there is a large class of positive definite matrices $a^{ij}(t,x)$ satisfying the preceding conditions. The matrix defined by $a^{ii}(t,x) = d$ for $i = 1,2,\dots,d$ and $a^{ij}(t,x) = -1$ for $i \neq j$, $i,j = 1,2,\dots,d$, with eigen values 1 and $d+1$ with multiplicity 1 and $d-1$, respectively, is an example.

Explicit scheme.

For $(t,x) \in Q^0(h)$, we define the operator

$$\begin{aligned}
\mathfrak{L}_h u(t,x) &= -\varepsilon^{-1}h^{-2}[u(t,x) - u(t-\varepsilon h^2, x)] \\
&\quad + \sum_{i,j} a^{ij}(t-\varepsilon h^2, x) 2^{-1} h^{-2}[u(t-\varepsilon h^2, x+he_i) \\
&\quad + u(t-\varepsilon h^2, x-he_i) - u(t-\varepsilon h^2, x+h(e_i-e_j)) \\
&\quad - 2u(t-\varepsilon h^2, x) - u(t-\varepsilon h^2, x-h(e_i-e_j)) \\
&\quad + u(t-\varepsilon h^2, x-he_j) + u(t-\varepsilon h^2, x+he_j)] \\
&\quad + \sum_i |b^i(t-\varepsilon h^2, x)| h^{-1}[u(t-\varepsilon h^2, x+he_i \operatorname{sign} b^i(t-\varepsilon h^2,x)) \\
&\quad - u(t-\varepsilon h^2, x)] + c(t,x)u(t,x). \qquad (3.17)
\end{aligned}$$

Theorem 3.2.7. *Assume the coefficients $a^{ij}(t,x)$ are such that $\sum_{j=1}^d a^{ij}(t,x) \geq 0$, for $i = 1,2,\dots,d$ and $a^{ij}(t,x) < 0$ for $i \neq j$, $i,j = 1,2,\dots,d$. Let $l(h) = \varepsilon h^2$, where $\varepsilon^{-1} \geq \sup_z(2\sum_{i\leq j} a^{ij}(z) + \sum_i |b^i(z)|)$. Then the discrete operator $\mathfrak{L}_h$ (3.17) satisfies Assumptions 3.2.1 and 3.2.2.*

Proof To check Assumption 3.2.1, let $z_0 = (t_0,x_0) \in Q_h^0$ and $u(t_0,x_0) = M = \max_{Q(h)} u(z) > 0$. Denote $t_0^{'} = t_0 - \varepsilon h^2$. Then

$$\begin{aligned}
h^2\mathfrak{L}_h u(t_0,x_0) &= -M\varepsilon^{-1} + u(t_0^{'},x_0)[\varepsilon^{-1} - 2\sum_{i\leq j} a^{ij}(t_0^{'},x_0) - h\sum_i |b^i(t_0^{'},x_0)|] \\
&\quad + \sum_i\sum_j a^{ij}(t_0^{'},x_0)[u(t_0^{'},x_0+he_i) + u(t_0^{'},x_0-he_i)] \\
&\quad - \sum_{i<j} a^{ij}(t_0^{'},x_0)[u(t_0^{'},x_0+h(e_i-e_j)) + u(t_0^{'},x_0-h(e_i-e_j))] \\
&\quad + h\sum_i |b^i(t_0^{'},x_0)| u(t_0^{'}, x_0+he_i \operatorname{sign} b^i(t_0^{'},x_0)) + h^2Mc(t_0,x_0) \\
&\leq -M\varepsilon^{-1} + M[\varepsilon^{-1} - 2\sum_{i\leq j} a^{ij}(t_0^{'},x_0) - h\sum_i |b^i(t_0^{'},x_0)|] \\
&\quad + 2M\sum_i\sum_j a^{ij}(t_0^{'},x_0) - 2M\sum_{i<j} a^{ij}(t_0^{'},x_0) \\
&\quad + hM\sum_{i=1}^d |b^i(t_0^{'},x_0)| \\
&= 0.
\end{aligned}$$

Assumption 3.2.2 can be checked using Taylor's formula and the assumptions on the smoothness of the coefficients. Denote $t' = t - \varepsilon h^2$.

$$
\begin{aligned}
&|Lu(t,x) - u_t(t,x) - \mathfrak{L}_h u(t,x)| \\
&= |\sum_{i,j} a^{ij}(t,x) u_{x^i x^j}(t,x) + \sum_i b^i(t,x) u_{x^i}(t,x) + c(t,x)u(t,x) - u_t(t,x) \\
&\quad + \varepsilon^{-1} h^{-2}[u(t,x) - u(t',x)] - \sum_{i,j} a^{ij}(t',x) 2^{-1} h^{-2} [u(t', x + he_i) \\
&\quad + u(t', x - he_i) - u(t', x + h(e_i - e_j)) - 2u(t',x) - u(t', x - h(e_i - e_j)) \\
&\quad + u(t', x + he_j) + u(t', x - he_j)] - \sum_i |b^i(t',x)| h^{-1} [u(t', x + he_i \operatorname{sign} b^i(t',x)) \\
&\quad - u(t',x)] - c(t,x)u(t,x)| \\
&\le |\sum_{i,j} a^{ij}(t,x) u_{x^i x^j}(t,x) - \sum_{i,j} a^{ij}(t',x) 2^{-1} h^{-1} [h^{-1}(u(t', x + he_i) - u(t',x)) \\
&\quad - h^{-1}(u(t',x) - (u(t', x - he_i)) + h^{-1}(u(t', x + he_j) - u(t', x - h(e_i - e_j))) \\
&\quad - h^{-1}(u(t', x + h(e_i - e_j)) - u(t', x - he_j)))]| + |\sum_i b^i(t,x) u_{x^i}(t,x) \\
&\quad - \sum_i |b^i(t',x)| h^{-1} [u(t', x + he_i \operatorname{sign} b^i(t',x)) - u(t',x)]| \\
&\quad + |u_t(t,x) - \varepsilon^{-1} h^{-2} [u(t,x) - u(t',x)]|.
\end{aligned}
$$

Using the mean-value theorem repeatedly we obtain

$$
\begin{aligned}
&|Lu(t,x) - u_t(t,x) - \mathfrak{L}_h u(t,x)| \\
&\le |\sum_{i,j} a^{ij}(t,x)[u_{x^i x^j}(t,x) - u_{x^i x^j}(t', x + \theta_1 hej)]| \\
&\quad + |\sum_{i,j} [a^{ij}(t,x) - a^{ij}(t',x)] u_{x^i x^j}(t', x + \theta_2 he_j)| \\
&\quad + |\sum_{i,j} a^{ij}(t',x)[-u_{x^i x^j}(t', x + \theta_1 he_j) + u_{x^i x^i}(t', x + \theta_3 he_i) \\
&\qquad + u_{x^i x^j}(t', x + \theta_4 he_i + \theta_5 he_j) - u_{x^i x^i}(t', x + \theta_6 he_i + he_j)]| \\
&\quad + |\sum_i b^i(t,x)[u_{x^i}(t,x) - u_{x^i}(t', x + \theta_7 he_i)| \\
&\quad + |\sum_i [b^i(t,x) - b^i(t',x)] u_{x^i}(t', x + \theta_7 he_i)| \\
&\quad + |u_t(t,x) - \varepsilon^{-1} h^{-2}[u(t,x) - u(t',x)]|,
\end{aligned}
$$

with $\theta_k,\ k = 1, \dots, 7$ constants such that $|\theta_k| < 1,$ for all k.

Finally, we have

$$|Lu(t,x) - u_t(t,x) - \mathfrak{L}_h u(t,x)| \le K h^{\delta} |u|_{1+\delta/2, 2+\delta; Q},$$

and the result is proved. □

The operator we have constructed furnishes an explicit scheme for approximation. It allows the computation of $u_h(t,x)$ on $Q(h)$, starting from $u(0,x)$ which is given and then finding $u_h(\varepsilon h^2,x)$, $u_h(2\varepsilon h^2,x)$ and so on consecutively from the explicit formula

$$\begin{aligned} u_h(t,x) &= c'\varepsilon h^2 f(t,x) - c'u(t-\varepsilon h^2,x) \\ &\quad -2^{-1}c'\varepsilon \sum_{i,j} a^{ij}(t-\varepsilon h^2,x)[u(t-\varepsilon h^2,x+he_i)+u(t-\varepsilon h^2,x-he_i) \\ &\quad -u(t-\varepsilon h^2,x+h(e_i-e_j))-2u(t-\varepsilon h^2,x) \\ &\quad -u(t-\varepsilon h^2,x-h(e_i-e_j))+u(t-\varepsilon h^2,x-he_j)+u(t-\varepsilon h^2,x+he_j)] \\ &\quad -c'\varepsilon h\sum_i |b^i(t-\varepsilon h^2,x)|[u(t-\varepsilon h^2,x+he_i \operatorname{sign} b^i(t-\varepsilon h^2,x)) \\ &\quad -u(t-\varepsilon h^2,x)], \end{aligned}$$

where $c' = (\varepsilon h^2 c(t,x)-1)^{-1}$.

We note that the restrictions over ε in the sub-cases where a^{ij} is diagonal or where there are no first-order partial derivatives can be obtained immediately from the more general condition we presented.

Implicit scheme.

For the same particular case of the continuous operator L, we define, for $(t,x)\in Q^0(h)$, the discrete operator

$$\begin{aligned} \mathfrak{L}_h u(t,x) &= -\varepsilon^{-1}h^{-2}[u(t,x)-u(t-\varepsilon h^2,x)] \\ &\quad +\sum_{i,j} a^{ij}(t,x)2^{-1}h^{-2}[u(t,x+he_i)+u(t,x-he_i) \\ &\quad -u(t,x+h(e_i-e_j))-2u(t,x)-u(t,x-h(e_i-e_j)) \\ &\quad +u(t,x-he_j)+u(t,x+he_j)] \\ &\quad +\sum_i |b^i(t,x)|h^{-1}[u(t,x+he_i \operatorname{sign} b^i(t,x))-u(t,x)] \\ &\quad +c(t,x)u(t,x). \end{aligned} \tag{3.18}$$

Theorem 3.2.8. *Let the coefficients $a^{ij}(t,x)$ and the discrete function $l(h)$ satisfy the hypothesis in Theorem 3.2.7. Then the discrete operator (3.18) satisfies Assumptions 3.2.1 and 3.2.2.*

Proof The operator $\mathfrak{L}_h$ satisfies Assumption 3.2.2 for the same reasons as in Theorem 3.2.7 and Assumption 3.2.1 with no restrictions on ε. □

The method of computation of $u_h(t,x)$ on $Q(h)$ is implicit: in order to find $u_h((k+1)\varepsilon h^2,x)$ from $u_h(k\varepsilon h^2,x)$ a system of linear equations has to be solved.

3.3 Localization error estimate

Finally, we should estimate the error in approximating the solution of the Cauchy problem

$$Lu - u_t + f = 0 \text{ in } [0,T] \times \mathbb{R}^d, \quad u(0,x) = g(x) \text{ in } \mathbb{R}^d. \tag{3.19}$$

where $T \in (0, \infty)$, by the solution of the initial-boundary value problem

$$Lu - u_t + f = 0 \text{ in } Q, \quad u(0,x) = g(x) \text{ for } x \in \bar{U}, \quad u = \bar{g} \text{ on } \partial_x Q, \tag{3.20}$$

where $Q = [0,T] \times U$, U is a bounded domain in $\mathbb{R}^d$ and $\partial_x Q = [0,T] \times \partial U$.

In fact, in Section 3.2 we have produced an estimate for the second term of the right hand of the inequality

$$|v(t,x) - u_h(t,x)| \leq |v(t,x) - u(t,x)| + |u(t,x) - u_h(t,x)|,$$

where $v(t,x)$ and $u(t,x)$ represent, respectively, the solutions of (3.19) and (3.20), and $u_h(t,x)$ is the solution of the discretized problem (3.15). It remains to estimate the localization error:

$$|v(t,x) - u(t,x)|.$$

Localize problem (3.19), considering the particular case of (3.20) where Dirichlet boundary conditions are imposed:

$$Lu - u_t + f = 0 \text{ in } Q, \quad u(0,x) = g(x) \text{ for } x \in \bar{U}, \quad u = 0 \text{ on } \partial_x Q. \tag{3.21}$$

Remark 3.3.1. We recall that in Section 3.1 we studied the solvability of problems (3.19) and (3.21) when weaker smoothness is imposed over the initial data g (Theorems 3.1.14 and 3.1.16). We saw also that the restrictions v_ε and u_ε of the unique solutions v and u of problems (3.19) and (3.21) to the sets $[\varepsilon, T] \times \mathbb{R}^d$ and $[\varepsilon, T] \times U$, respectively, are of class $C^{1+\delta/2, 2+\delta}$ (Theorems 3.1.15 and 3.1.18). Finally, in Section 3.2, we studied the numerical approximation for the restriction $u_\varepsilon \in C^{1+\delta/2, 2+\delta}([\varepsilon, T] \times U)$ of u (Theorems 3.2.4 and 3.2.6).

In order to estimate the localization error, we will consider the stochastic representation of problems (3.19) and (3.21), written for t replaced by $T - t$, respectively

$$Lu + u_t + f = 0 \text{ in } [0,T] \times \mathbb{R}^d, \quad u(T,x) = g(x) \text{ in } \mathbb{R}^d \tag{3.22}$$

and

$$Lu + u_t + f = 0 \text{ in } Q, \quad u(T,x) = g(x) \text{ for } x \in \bar{U}, \quad u = 0 \text{ on } \partial_x Q. \tag{3.23}$$

Let the multidimensional stochastic problem

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \tag{3.24}$$

$$X_0 = x_0, \tag{3.25}$$

where

$$W_t = (W_t^1, \ldots, W_t^p) \text{ an } \mathbb{R}^p - \text{valued } \mathcal{F}_t - \text{Brownian motion;}$$
$$b : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d,\ b(s,x) = (b^1(s,x), \ldots, b^d(s,x));$$
$$\sigma : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{d\times p},\ \sigma(s,x) = (\sigma^{ij}(s,x))_{1\leq i\leq d, 1\leq j\leq p}.$$

Let σ be such that $1/2(\sigma\sigma')^{ij} = a^{ij}$, for $i = 1, \ldots, d$, $j = 1, \ldots, p$, where a^{ij} is the coefficient associated with the second-order derivatives in the operator L. We assume that b and σ satisfy the proper integrability conditions so that the process

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \tag{3.26}$$

is an Itô process.

We will obtain the stochastic representation of problems (3.22) and (3.23) (and then approximate the localization error), assuming that these problems' solutions exist, as well as the solution of the stochastic equation. Under these assumptions, the deduction will be made imposing weaker conditions over the operator L and the data f and g.

Assumption 3.3.2. Let coefficients in the operator L, defined by (3.2), satisfy:

1. a, b and c are continuous functions in $[0,T] \times \mathbb{R}^d$;
2. $c \leq 0$ in $[0,T] \times \mathbb{R}^d$;
3. $|a(t,x)| \leq C(1 + |x|^2)$, for all $t \in [0,T]$, $x \in \mathbb{R}^d$, $C > 0$ a constant;
4. $|b(t,x)| \leq C(1 + |x|)$, for all $t \in [0,T]$, $x \in \mathbb{R}^d$, $C > 0$ a constant.

We state a preliminary result which gives a moment estimate of the solution of a stochastic equation (see Krylov [31], p. 85, where estimates are also given for a more general case).

Consider the more general multidimensional stochastic equation

$$X_t = Z_t + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \tag{3.27}$$

where W_t, b, σ are as defined for equation (3.26) and Z_t is a $d-$dimensional random vector. Let us assume that the coefficients b, σ in equation (3.27) satisfy the condition:

Assumption 3.3.3. There exists a constant $M > 0$ and nonnegative functions $r(t)$, $h(t)$ such that

1. $|\sigma(t,x)|^2 \le 2r^2(t) + 2M^2|x|^2$;
2. $|b(t,x)| \le h(t) + M^2|x|$.

for all $t \in [0,T]$, $x, y \in \mathbb{R}^d$, where $|\sigma|^2 = \sum_{1\le i\le d,\, 1\le j\le p}(\sigma^{ij})^2$.

Proposition 3.3.4. *Under Assumption 3.3.3, if Z_t is a separable process then a solution X_t of (3.27) satisfies, for all $q > 1$, $t \in [0,T]$,*

$$\mathbf{E}(\sup_{s\le t}|X_s|^{2q}) \le N\,\mathbf{E}(\sup_{s\le t}|Z_s|^{2q}) + Nt^{q-1}e^{Nt}\,\mathbf{E}(\int_0^t(|Z_s|^{2q} + (h(s))^{2q} + (r(s))^{2q})ds),$$

where N is a constant depending on q and M.

Remark 3.3.5. In Assumption 3.3.2 we assumed some growth conditions for the coefficients $a = 1/2\,\sigma\sigma'$ and b. Under this assumption, Assumption 3.3.3 is not restrictive and is met with $r^2 = h = M$. Also, in the framework we are considering in the present section, $Z_t = x_0$ is a $d-$dimensional non-random vector. Under these conditions, the estimate in Proposition 3.3.4, written for $X_s^{t,x}$, is

$$\begin{aligned}
\mathbf{E}(\sup_{t\le s\le T}|X_s^{t,x}|^{2q}) &\le N\,\mathbf{E}(\sup_{s\le T}|x|^{2q}) + NT^{q-1}e^{NT}\,\mathbf{E}\left(\int_t^T(|x|^{2q} + M^{2q} + M^q)ds\right)\\
&\le N\,|x|^{2q} + NT^{q-1}e^{NT}\int_t^T(|x|^{2q} + M^{2q} + M^q)ds\\
&\le N\,\left(|x|^{2q} + T^q e^{NT}\,(|x|^{2q} + M^{2q} + M^q)\right)\\
&\le N\,(|x|^{2q} + 1),
\end{aligned} \tag{3.28}$$

where N is a constant depending on T, q, and M.

Next two theorems give the stochastic representation of the two parabolic problems.

Theorem 3.3.6. *Let (1)–(4) in Assumption 3.3.2 be satisfied. Let functions f and g in (3.22) be such that*

$$|f(t,x)| \leq K(1+|x|^m) \;\; in \;\; [0,T]\times\mathbb{R}^d \quad and \quad |g(x)| \leq K(1+|x|^m) \;\; in \;\; \mathbb{R}^d,$$

respectively, with K, m positive constants. Assume that parabolic problem (3.22) has a unique solution v in $C^{1,2}([0,T]\times\mathbb{R}^d)$ and this solution satisfies

$$|D_x^\beta v(t,x)| \leq N(1+|x|^p), \;\; \beta = 0,1, \;\; in \;\; Q,$$

with N, p positive constants. Assume also that there exists a unique solution X_t of the stochastic problem (3.24)–(3.25) in $[0,T]$. Denote $X_s^{t,x}$, with $s \geq t$, the solution of equation (3.24) starting from x at time t. Then v is given by

$$v(t,x) = \mathbf{E}\left(e^{\int_t^T c(s,X_s^{t,x})\,ds} g(X_T^{t,x})\right) + \mathbf{E}\left(\int_t^T e^{\int_t^s c(r,X_r^{t,x})dr} f(s,X_s^{t,x})ds\right).$$

for all $t \in [0,T]$, $x \in \mathbb{R}^d$.

Proof Consider the Itô processes $Y_s = e^{\int_t^s c(r,X_r^{t,x})dr}$ and $Z_s = v(s,X_s^{t,x})$, for $s \geq t$. Noting that the stochastic integral in the Itô process Y_s is null, integrating by parts Y_sZ_s we obtain

$$\begin{aligned} e^{\int_t^s c(r,X_r^{t,x})dr} v(s,X_s^{t,x}) &= v(t,x) + \int_t^s v(r,X_r^{t,x}) d(e^{\int_t^r c(q,X_q^{t,x})dq}) \\ &\quad + \int_t^s e^{\int_t^r c(q,X_q^{t,x})dq} dv(r,X_r^{t,x}) \\ &= v(t,x) + \int_t^s e^{\int_t^r c(q,X_q^{t,x})dq} c(r,X_r^{t,x}) v(r,X_r^{t,x}) dr \\ &\quad + \int_t^s e^{\int_t^r c(q,X_q^{t,x})dq} dv(r,X_r^{t,x}). \end{aligned} \tag{3.29}$$

Owing to Theorem 2.1.27 (multidimensional Itô formula), we have

$$\begin{aligned} v(s,X_s^{t,x}) &= v(t,x) + \int_t^s (Av)(r,X_r^{t,x}) + \frac{\partial v}{\partial t}(r,X_r^{t,x}) dr \\ &\quad + \int_t^s v_{x^i}(r,X_r^{t,x}) \sigma^{ij}(r,X_r^{t,x}) dW_r^j, \end{aligned} \tag{3.30}$$

where

$$A(t,x) = \frac{1}{2}(\sigma\sigma')^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t,x)\frac{\partial}{\partial x^i}.$$

Using (3.30), from (3.29) we obtain

$$\begin{aligned} e^{\int_t^s c(r,X_r^{t,x})dr} v(s,X_s^{t,x}) &= v(t,x) + \int_t^s e^{\int_t^r c(q,X_q^{t,x})dq} (Lv + \frac{\partial}{\partial t} v)(r,X_r^{t,x}) dr \\ &\quad + \int_t^s e^{\int_t^r c(q,X_q^{t,x})dq} v_{x^i}(r,X_r^{t,x}) \sigma^{ij}(r,X_r^{t,x}) dW_r^j. \end{aligned} \tag{3.31}$$

Making $s = T$ in equation (3.31) and taking the expectation,

$$v(t,x) = \mathbf{E}(e^{\int_t^T c(s,X_s^{t,x})ds} v(T, X_T^{t,x})) - \mathbf{E}(\int_t^T e^{\int_t^s c(r,X_r^{t,x})dr}(Lv + \frac{\partial}{\partial t} v)(s, X_s^{t,x})ds)$$
$$= \mathbf{E}(e^{\int_t^T c(s,X_s^{t,x})ds} g(X_T^{t,x})) + \mathbf{E}(\int_t^T e^{\int_t^s c(r,X_r^{t,x})dr} f(s, X_s^{t,x})ds). \qquad (3.32)$$

We note that the expectation of the stochastic integral in (3.31) is zero owing to the assumptions over the growth of $v_x(t,x)$ and $a(t,x)$, the moment estimate (3.28) in Remark 3.3.5 and Theorem 2.1.22. The assumptions over the growth of f and g and the moment estimate (3.28) guarantee that the expectations in (3.32) exist. □

Theorem 3.3.7. *Let (1)–(2) in Assumption 3.3.2 be satisfied. Assume that parabolic problem (3.23) has a unique solution u in $C^{1,2}([0,T] \times U)$. Assume also that there exists a unique solution X_t of the stochastic problem (3.24)–(3.25) in $[0,T]$. Then the unique solution u of problem (3.23) is given by*

$$u(t,x) = \mathbf{E}\left(\mathbf{1}_{\{\tau=T\}} e^{\int_t^\tau c(s,X_s^{t,x})ds} g(X_T^{t,x})\right) + \mathbf{E}\left(\int_t^\tau e^{\int_t^s c(r,X_r^{t,x})dr} f(s, X_s^{t,x})ds\right),$$

for all $t \in [0,T]$, $x \in \bar{U}$, where $X_s^{t,x}$, with $s \geq t$, is the solution of equation (3.24) starting from x at time t and $\tau := \inf\{s \geq t : X_s^{t,x}$ is not in $U\} \wedge T$.

Proof This proof follows the one by A. Friedman (Friedman [18], pp. 145-146) for the stochastic representation of the solution of a boundary-value elliptic PDE problem.

Let $\bar{V}_\varepsilon \in \mathbb{R}^d$ be the closed $\varepsilon-$neighborhood of ∂U and denote $U_\varepsilon = U \setminus \bar{V}_\varepsilon$. Let w be a $C^{1,2}([0,T] \times \mathbb{R}^d)$ function such that $w = u$ in $[0,T] \times U_{\varepsilon/2}$. We consider the processes $Y_s = e^{\int_t^s c(r,X_r^{t,x})dr}$ and $Z_s = w(s, X_s^{t,x})$, for $s \geq t$.

Following the same steps as in the proof of Theorem 3.3.6, we obtain

$$e^{\int_t^s c(r,X_r^{t,x})dr} w(s, X_s^{t,x}) = w(t,x) + \int_t^s e^{\int_t^r c(q,X_q^{t,x})dq}(Lw + \frac{\partial}{\partial t} w)(r, X_r^{t,x})dr$$
$$+ \int_t^s e^{\int_t^r c(q,X_q^{t,x})dq} w_{x^i}(r, X_r^{t,x})\sigma^{ij}(r, X_r^{t,x})dW_r^j. \qquad (3.33)$$

Let $\varsigma \geq t$ a stopping time with respect to $X_s^{t,x}$. Making $s = \varsigma$ in equation (3.33) and taking the expectation, we have

$$w(t,x) = \mathbf{E}\,(e^{\int_t^\varsigma c(s,X_s^{t,x})ds} w(\varsigma, X_\varsigma^{t,x}))$$
$$- \mathbf{E}\,(\int_t^\varsigma e^{\int_t^s c(r,X_r^{t,x})dr}(Lw + \frac{\partial}{\partial t} w)(s, X_s^{t,x})ds). \qquad (3.34)$$

Note that, owing to Theorem 2.1.22, the expectation of the stochastic integral in (3.33) is zero.

Let $\varsigma = \varsigma_\varepsilon \wedge T$, where $\varsigma_\varepsilon \geq t$ is the hitting time of U_ε. We then have $w(s, X_s^{t,x}) = u(s, X_s^{t,x})$ for all $t \leq s \leq \varsigma_\varepsilon \wedge T$ and (3.34) still holds when w is replaced by u. Taking the limit when $\varepsilon \to 0$ and using Lebesgue dominated convergence, we obtain

$$\begin{aligned} u(t,x) &= \mathbf{E}\,(e^{\int_t^{\varsigma\wedge T} c(s,X_s^{t,x})ds} u(\varsigma\wedge T, X_{\varsigma\wedge T}^{t,x})) \\ &\quad + \mathbf{E}\,(\int_t^{\varsigma\wedge T} e^{\int_t^s c(r,X_r^{t,x})dr} f(s, X_s^{t,x})ds), \end{aligned} \tag{3.35}$$

where ς is the exit time of U.

Define $\tau := \varsigma \wedge T$. From (3.35) we have

$$\begin{aligned} u(t,x) &= \mathbf{E}\,(e^{\int_t^{\tau} c(s,X_s^{t,x})ds} u(\tau, X_\tau^{t,x})) + \mathbf{E}\,(\int_t^{\tau} e^{\int_t^s c(r,X_r^{t,x})dr} f(s, X_s^{t,x})ds) \\ &= \mathbf{E}(\mathbf{1}_{\{\tau=T\}} e^{\int_t^{\tau} c(s,X_s^{t,x})ds} u(\tau, X_\tau^{t,x})) + \mathbf{E}(\mathbf{1}_{\{\tau<T\}} e^{\int_t^{\tau} c(s,X_s^{t,x})ds} u(\tau, X_\tau^{t,x})) \\ &\quad + \mathbf{E}(\int_t^{\tau} e^{\int_t^s c(r,X_r^{t,x})dr} f(s, X_s^{t,x})ds) \\ &= \mathbf{E}(\mathbf{1}_{\{\tau=T\}} e^{\int_t^{\tau} c(s,X_s^{t,x})ds} u(T, X_T^{t,x})) + \mathbf{E}(\int_t^{\tau} e^{\int_t^s c(r,X_r^{t,x})dr} f(s, X_s^{t,x})ds) \\ &= \mathbf{E}(\mathbf{1}_{\{\tau=T\}} e^{\int_t^{\tau} c(s,X_s^{t,x})ds} g(X_T^{t,x})) + \mathbf{E}(\int_t^{\tau} e^{\int_t^s c(r,X_r^{t,x})dr} f(s, X_s^{t,x})ds). \end{aligned} \tag{3.36}$$

The term $\mathbf{E}(\mathbf{1}_{\{\tau<T\}} e^{\int_t^{\tau} c(s,X_s^{t,x})ds} u(\tau, X_\tau^{t,x}))$ in the above computations vanishes due to the zero boundary condition for the PDE problem. Also, as $u \in C^{1,2}([0,T]\times U)$ and f and g are bounded functions therefore the expectations in (3.36) exist. The result is proved. □

Remark 3.3.8. Note that no smoothness assumption over the space-boundary ∂U was needed for the stochastic representation of problem (3.23).

We consider now a particular localization of problem (3.22), in order to ensure compatibility between the Cauchy problem and the localized problem. Let $\eta \in C_0^\infty\big([0,\infty)\big)$ be a non-increasing function such that

$$\eta(r) = \begin{cases} 1, & r \leq 1 \\ 0, & r > 2. \end{cases}$$

Here the notation $C_0^\infty\big([0,\infty)\big)$ stands for the set of all infinitely differentiable functions on $[0,\infty)$ with compact support.

We localize problem (3.22) in the following way (a particular case of problem (3.23)):

$$Lu + u_t + f_R = 0 \text{ in } Q_R, \quad u(T,x) = g_R(x) \text{ for } x \in \bar{U}_R, \quad u = 0 \text{ on } \partial_x Q_R, \tag{3.37}$$

where $Q_R=[0,T]\times U_R$, $U_R=\{x\in\mathbb{R}^d:|x|<2R\}$, $f_R(t,x):=\eta(|x|/R)f(t,x)$ in $[0,T]\times\mathbb{R}^d$ and $g_R(x):=\eta(|x|/R)g(x)$ in $\mathbb{R}^d$, with $T\in(0,\infty)$, $R>0$ constants. Note that if g_R is continuous in $\bar{U}_R$, the consistency condition (1) in Assumption 3.1.11 is satisfied. We estimate the localization error.

Theorem 3.3.9. *Let the hypothesis of Theorems 3.3.6 and 3.3.7 be satisfied. Let v be the unique solution of problem (3.22) in $C^{1,2}([0,T]\times\mathbb{R}^d)$ and u_R the unique solution of problem (3.37) in $C^{1,2}([0,T]\times U_R)$. Then, for all $q\geq 1$, $t\in[0,T]$, $x\in\bar{U}_R$.*

$$|u_R(t,x)-v(t,x)|\leq N\big(1+|x|^{q+m}+|x|^qR^m\big)R^{-q},$$

where N is a constant depending on T, q, the constant M in Proposition 3.3.4, and the constants K, m in the growth conditions imposed over both functions f and g.

Proof We estimate $|u_R(t,x)-v(t,x)|$ taking the stochastic representation of v and u_R given in Theorems 3.3.6 and 3.3.7, respectively. We have

$$\begin{aligned}
&|u_R(t,x)-v(t,x)|\\
&=|\mathbf{E}(\mathbf{1}_{\{\tau=T\}}e^{\int_t^\tau c(s,X_s^{t,x})ds}g_R(X_T^{t,x})-e^{\int_t^T c(s,X_s^{t,x})\,ds}g(X_T^{t,x}))\\
&\quad+\mathbf{E}(\int_t^\tau e^{\int_t^s c(r,X_r^{t,x})dr}f_R(s,X_s^{t,x})ds-\int_t^T e^{\int_t^s c(r,X_r^{t,x})dr}f(s,X_s^{t,x})ds)|\\
&\leq\mathbf{E}(|\mathbf{1}_{\{\tau=T\}}e^{\int_t^\tau c(s,X_s^{t,x})ds}g_R(X_T^{t,x})-e^{\int_t^T c(s,X_s^{t,x})\,ds}g(X_T^{t,x})|)\\
&\quad+\mathbf{E}(|\int_t^\tau e^{\int_t^s c(r,X_r^{t,x})dr}f_R(s,X_s^{t,x})ds-\int_t^T e^{\int_t^s c(r,X_r^{t,x})dr}f(s,X_s^{t,x})ds|).
\end{aligned}\tag{3.38}$$

For the first term in (3.38), as $c\leq 0$ by Assumption 3.3.2 and noting that, by construction, $g_R(x)=g(x)$ if $|x|\leq R$ and $|g_R(x)|\leq|g(x)|$ for all $x\in\mathbb{R}^d$, we obtain

$$\begin{aligned}
&\mathbf{E}(|\mathbf{1}_{\{\tau=T\}}e^{\int_t^\tau c(s,X_s^{t,x})ds}g_R(X_T^{t,x})-e^{\int_t^T c(s,X_s^{t,x})\,ds}g(X_T^{t,x})|)\\
&\leq\mathbf{E}(|e^{\int_t^T c(s,X_s^{t,x})\,ds}(g_R(X_T^{t,x})-g(X_T^{t,x}))|\mathbf{1}_{\{\tau=T\}})+\mathbf{E}(|e^{\int_t^T c(s,X_s^{t,x})\,ds}g(X_T^{t,x})|\mathbf{1}_{\{\tau<T\}})\\
&\leq\mathbf{E}(|g_R(X_T^{t,x})-g(X_T^{t,x})|\mathbf{1}_{\{\tau=T\}})+\mathbf{E}(|g(X_T^{t,x})|\mathbf{1}_{\{\tau<T\}})\\
&\leq\mathbf{E}(\sup_{R\leq|x|\leq 2R}|g_R(x)-g(x)|\mathbf{1}_{\{\sup_{t\leq s\leq T}|X_s^{t,x}|\geq R\}})+\mathbf{E}(|g(X_T^{t,x})|\mathbf{1}_{\{\tau<T\}})\\
&\leq 2\mathbf{E}(\sup_{R\leq|x|\leq 2R}|g(x)|\mathbf{1}_{\{\sup_{t\leq s\leq T}|X_s^{t,x}|\geq R\}})+\mathbf{E}(|g(X_T^{t,x})|\mathbf{1}_{\{\tau<T\}}).
\end{aligned}\tag{3.39}$$

For the second term in (3.38), as $c \leq 0$ and also as $f_R(t,x) = f(t,x)$ for all (t,x) such that $|x| \leq R$ and $|f_R(x)| \leq |f(x)|$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$, we have

$$
\begin{aligned}
&\mathbf{E}(|\int_t^\tau e^{\int_t^s c(r,X_r^{t,x})dr} f_R(s,X_s^{t,x})ds - \int_t^T e^{\int_t^s c(r,X_r^{t,x})dr} f(s,X_s^{t,x})ds|) \\
&\leq \mathbf{E}(\int_t^T |e^{\int_t^s c(r,X_r^{t,x})dr}(f_R(s,X_s^{t,x}) - f(s,X_s^{t,x}))ds|\mathbf{1}_{\{\tau=T\}}) \\
&\quad +\mathbf{E}(\int_t^\tau |e^{\int_t^s c(r,X_r^{t,x})dr}(f_R(s,X_s^{t,x}) - f(s,X_s^{t,x}))ds|\mathbf{1}_{\{\tau<T\}}) \\
&\quad +\mathbf{E}(\int_\tau^T |e^{\int_t^s c(r,X_r^{t,x})dr}(f(s,X_s^{t,x}))ds|\mathbf{1}_{\{\tau<T\}}) \\
&\leq \mathbf{E}(\int_t^T \sup_{R\leq|x|\leq 2R} |f_R(s,x) - f(s,x)|ds\, \mathbf{1}_{\{\sup_{t\leq s\leq T}|X_s^{t,x}|\geq R\}}) \\
&\quad +\mathbf{E}(\int_t^\tau \sup_{R\leq|x|\leq 2R} |f_R(s,x) - f(s,x)|ds\mathbf{1}_{\{\tau<T\}}) + \mathbf{E}(\int_\tau^T |f(s,X_s^{t,x})|ds\mathbf{1}_{\{\tau<T\}}) \\
&\leq 2\mathbf{E}(\int_t^T \sup_{R\leq|x|\leq 2R} |f(s,x)|ds\, \mathbf{1}_{\{\sup_{t\leq s\leq T}|X_s^{t,x}|\geq R\}}) \\
&\quad +2\mathbf{E}(\int_t^\tau \sup_{R\leq|x|\leq 2R} |f(s,x)|ds\mathbf{1}_{\{\tau<T\}}) + \mathbf{E}(\int_\tau^T |f(s,X_s^{t,x})|ds\mathbf{1}_{\{\tau<T\}}). \qquad (3.40)
\end{aligned}
$$

Putting together estimates (3.38), (3.39) and (3.40),

$$
\begin{aligned}
&|u_R(t,x) - v(t,x)| \\
&\leq 2\mathbf{E}(\sup_{R\leq|x|\leq 2R} |g(x)|\mathbf{1}_{\{\sup_{t\leq s\leq T}|X_s^{t,x}|\geq R\}}) + \mathbf{E}(|g(X_T^{t,x})|\mathbf{1}_{\{\tau<T\}}) \\
&\quad +2\mathbf{E}(\int_t^T \sup_{R\leq|x|\leq 2R} |f(s,x)|ds\, \mathbf{1}_{\{\sup_{t\leq s\leq T}|X_s^{t,x}|\geq R\}}) \\
&\quad +2\mathbf{E}(\int_t^\tau \sup_{R\leq|x|\leq 2R} |f(s,x)|ds\mathbf{1}_{\{\tau<T\}}) + \mathbf{E}(\int_\tau^T |f(s,X_s^{t,x})|ds\, \mathbf{1}_{\{\tau<T\}}) \\
&\leq 2(\sup_{R\leq|x|\leq 2R} |g(x)|)\, \mathbf{P}^{1/2}(\sup_{t\leq s\leq T} |X_s^{t,x}| \geq R) + \mathbf{E}^{1/2}(|g(X_T^{t,x})|^2)\, \mathbf{P}^{1/2}(\tau < T) \\
&\quad +2(\int_t^T \sup_{R\leq|x|\leq 2R} |f(s,x)|ds)\, \mathbf{P}^{1/2}(\sup_{t\leq s\leq T} |X_s^{t,x}| \geq R) \\
&\quad +2(\int_t^\tau \sup_{R\leq|x|\leq 2R} |f(s,x)|ds)\, \mathbf{P}^{1/2}(\tau < T) \\
&\quad +\mathbf{E}^{1/2}((\int_\tau^T |f(s,X_s^{t,x})|\, ds)^2)\, \mathbf{P}^{1/2}(\tau < T). \qquad (3.41)
\end{aligned}
$$

We estimate $\mathbf{P}(\tau < T)$ and $\mathbf{P}(\sup_{t\leq s\leq T} |X_s^{t,x}| \geq R)$ in (3.41)

$$
\begin{aligned}
\mathbf{P}(\tau < T) = \mathbf{P}(\exists s \in [t,T) : |X_s^{t,x}| \geq R) &\leq \mathbf{P}(\exists s \in [t,T] : |X_s^{t,x}| \geq R) \\
&= \mathbf{P}(\sup_{t\leq s\leq T} |X_s^{t,x}| \geq R) \\
&\leq \frac{1}{R^{2q}}\, \mathbf{E}\, (\sup_{t\leq s\leq T} |X_s^{t,x}|^{2q}), \qquad (3.42)
\end{aligned}
$$

owing to Chebyshev's inequality, with $q > 1$ a constant.

Using the assumptions over the growth of g and f we estimate $\mathbf{E}(|g(X_T^{t,x})|^2)$ and $\mathbf{E}((\int_\tau^T |f(s,X_s^{t,x})|\, ds)^2)$ in (3.41),

$$\begin{aligned}\mathbf{E}(|g(X_T^{t,x})|^2) \leq \mathbf{E}(K(1+|X_T^{t,x}|^m)^2) &\leq K(1+\mathbf{E}(|X_T^{t,x}|^{2m})) \\ &\leq K(1+\mathbf{E}(\sup_{t\leq s\leq T}|X_s^{t,x}|^{2m})) \qquad (3.43)\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}((\int_\tau^T |f(s,X_s^{t,x})|\, ds)^2) &\leq \int_\tau^T \mathbf{E}(|f(s,X_s^{t,x})|^2)\, ds \leq K \int_\tau^T \mathbf{E}((1+|X_s^{t,x}|^m)^2) ds \\ &\leq K\int_\tau^T (1+\mathbf{E}(|X_s^{t,x}|^{2m}))ds \leq K(1+\mathbf{E}(\sup_{\tau\leq s\leq T}|X_s^{t,x}|^{2m})) \qquad (3.44)\end{aligned}$$

where K, m are positive constants.

Due to the same assumptions we estimate the remaining expressions in (3.41)

$$\sup_{R\leq|x|\leq 2R} |g(x)| \leq \sup_{R\leq|x|\leq 2R} (K(1+|x|^m)) \leq K(1+R^m) \qquad (3.45)$$

and

$$\begin{aligned}\int_t^\tau \sup_{R\leq|x|\leq 2R} |f(s,x)| ds \leq \int_t^T \sup_{R\leq|x|\leq 2R} |f(s,x)| ds &\leq \int_t^T \sup_{R\leq|x|\leq 2R} (K(1+|x|^m)) ds \\ &\leq K(1+R^m), \qquad (3.46)\end{aligned}$$

where K, m are positive constants.

From (3.41)−(3.46) and using the moment estimate (3.28) in Remark 3.3.5 we obtain the estimate for the localization error

$$\begin{aligned}|u_R(t,x)-v(t,x)| &\leq K\Big(\big(1+\mathbf{E}(\sup_{t\leq s\leq T}|X_s^{t,x}|^{2m})\big)^{1/2}+(1+R^m)\Big) \\ &\quad \times\Big(\frac{1}{R^{2q}}\,\mathbf{E}\,(\sup_{t\leq s\leq T}|X_s^{t,x}|^{2q})\Big)^{1/2} \\ &\leq N\frac{\big((1+|x|^m)+(1+R^m)\big)(1+|x|^q)}{R^q} \\ &\leq N\frac{1+|x|^{q+m}+|x|^q R^m}{R^q}\,,\end{aligned}$$

where N is a constant depending on T, q, M, K and m. □

Until now, we have assumed the existence of the solutions for the parabolic Cauchy problem (3.22), for its localized version (3.23) and for the stochastic problem (3.24)−(3.25). Then, under some hypothesis, we deduced the parabolic problems' stochastic representation and the localization error estimate.

We will now study the stochastic representation considering the conditions under which problems (3.22), (3.23) and (3.24)−(3.25) are solvable.

In Section 3.1, we considered the solvability of the PDE problem when the operator L defined by (3.2) is, for each $t \in [0,T]$, uniformly elliptic in space with bounded Hölder continuous coefficients and the coefficient c is non-positive. Under these hypothesis, Assumption 3.3.2 is satisfied.

The following well known result on the stochastic representation of the solution of problem (3.22) (see e.g. Friedman [18], p. 148) is obtained immediately from Theorem 3.3.6, using Theorem 3.1.14 (with t replaced by $T-t$) and Theorem 2.1.31 for the existence and uniqueness of the solution of problems (3.22) and (3.24)−(3.25), respectively.

Theorem 3.3.10. *Assume that the coefficients $a(t,x)$, $b(t,x)$ in the operator L are Lipschitz continuous in $[0,T]\times\mathbb{R}^d$. Let functions f, g in (3.22) be such that $[f]_{\delta/2,\delta;[0,T]\times\mathbb{R}^d} < \infty$ and $g \in C(\mathbb{R}^d)$. Assume also that f, g satisfy*

$$|f(t,x)| \leq K(1+|x|^m) \text{ in } [0,T]\times\mathbb{R}^d, \quad |g(t,x)| \leq K(1+|x|^m) \text{ in } \mathbb{R}^d$$

where K, m are positive constants. Then the unique solution v of problem (3.22) is given by

$$v(t,x) = \mathbf{E}\left(e^{\int_t^T c(s,X_s^{t,x})\,ds} g(X_T^{t,x})\right) + \mathbf{E}\left(\int_t^T e^{\int_t^s c(r,X_r^{t,x})dr} f(s,X_s^{t,x})ds\right).$$

for all $t \in [0,T]$, $x \in \mathbb{R}^d$, where $X_s^{t,x}$, with $s \geq t$, is the solution of equation (3.24) starting from x at time t.

Similarly, for the initial-boundary value problem (3.23), the following well known result (see e.g. Friedman [18], p. 147) is obtained as an immediate consequence of Theorem 3.3.7, using Theorems 3.1.16 (with t replaced by $T-t$) and 2.1.31.

Theorem 3.3.11. *Assume that the coefficients $a(t,x)$, $b(t,x)$ in the operator L are Lipschitz continuous in $[0,T]\times\mathbb{R}^d$. Let $U \subset \mathbb{R}^d$ a $C^{2+\delta}$ bounded domain and f, g functions such that $f \in C^{\delta/2,\delta}([0,T]\times U)$ and $g \in C(\bar{U})$. Assume also that the consistency condition (1) in Assumption 3.1.11 is satisfied. Then the unique solution u of problem (3.23) is given by*

$$u(t,x) \quad = \quad \mathbf{E}\left(\mathbf{1}_{\{\tau=T\}} e^{\int_t^\tau c(s,X_s^{t,x})ds} g(X_T^{t,x})\right) + \mathbf{E}\left(\int_t^\tau e^{\int_t^s c(u,X_u^{t,x})du} f(s,X_s^{t,x})ds\right),$$

for all $t \in [0,T]$, $x \in \bar{U}$, where $X_s^{t,x}$, with $s \geq t$, is the solution of equation (3.24) starting from x at time t and $\tau := \inf\{s \geq t : X_s^{t,x} \text{ is not in } U\} \wedge T$.

If we consider the weak solution of the stochastic equation instead of the (strong) solution considered in Theorem 2.1.31, we can formulate two new theorems for the stochastic representation of the two parabolic problems' solutions.

We state a result on the existence of the weak solution (see Krylov [31], p. 87).

Theorem 3.3.12. *Let $b(t,x)$ be a $d-$dimensional vector and $\sigma(t,x)$ a matrix of dimension $d \times d$. Let b, σ be defined for all $(t,x) \in [0,+\infty) \times \mathbb{R}^d$ and bounded. Assume that the matrix σ is positive definite and, moreover, satisfies $(\sigma(t,x)\xi, \xi) \geq \lambda|\xi|^2$, for some constant $\lambda > 0$ for all $t \geq 0$, $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$. Then there exists a probability space, an $\mathcal{F}_t-$Brownian motion W_t on this space, and a continuous process X_t which is progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$, such that almost surely for all $t \geq 0$*

$$X_t = x + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s.$$

We note that it is not needed to impose a Lipschitz condition on b and σ to obtain the existence of the weak solution.

Let σ be such that $1/2(\sigma\sigma) = a$, where a is the coefficient associated with the second-order derivatives in the operator L (σ is the square root of $2a$). It can be shown easily that the ellipticity in space of the operator L implies the coercivity condition imposed over σ in Theorem 3.3.12.

We have then two new theorems on the representation of the solutions of problems (3.22) and (3.23).

Theorem 3.3.13. *Let functions f, g in (3.22) be such that $[f]_{\delta/2,\delta;[0,T]\times\mathbb{R}^d} < \infty$ and $g \in C(\mathbb{R}^d)$. Assume also that f, g satisfy*

$$|f(t,x)| \leq K(1+|x|^m) \ \ in \ \ [0,T]\times\mathbb{R}^d, \ \ |g(t,x)| \leq K(1+|x|^m) \ \ in \ \ \mathbb{R}^d$$

where K, m are positive constants. Then the unique solution v of problem (3.22) is given by

$$v(t,x) = \mathbf{E}\left(e^{\int_t^T c(s,X_s^{t,x})\,ds} g(X_T^{t,x})\right) + \mathbf{E}\left(\int_t^T e^{\int_t^s c(r,X_r^{t,x})dr} f(s,X_s^{t,x})ds\right).$$

for all $t \in [0,T]$, $x \in \mathbb{R}^d$, where $X_s^{t,x}$, with $s \geq t$, is the weak solution of equation (3.24) starting from x at time t.

Theorem 3.3.14. *Let $U \subset \mathbb{R}^d$ a $C^{2+\delta}$ bounded domain and f, g functions such that $f \in C^{\delta/2,\delta}([0,T]\times U)$ and $g \in C(\bar{U})$. Assume also that the consistency condition (1) in Assumption 3.1.11 is satisfied. Then the unique solution u of problem (3.23) is given by*

$$u(t,x) \ = \ \mathbf{E}\left(\mathbf{1}_{\{\tau=T\}} e^{\int_t^\tau c(s,X_s^{t,x})ds} g(X_T^{t,x})\right) + \mathbf{E}\left(\int_t^\tau e^{\int_t^s c(u,X_u^{t,x})du} f(s,X_s^{t,x})ds\right),$$

for all $t \in [0,T]$, $x \in \bar{U}$, where $X_s^{t,x}$, with $s \geq t$, is the weak solution of equation (3.24) starting from x at time t and $\tau := \inf\{s \geq t : \ X_s^{t,x}$ is not in $U\} \wedge T$.

Chapter 4

Parabolic PDE in Sobolev and weighted Sobolev spaces: space discretization

In the previous chapter we studied the parabolic PDE problem in Hölder spaces. Although we could obtain a well defined problem, strong regularity was imposed over the data.

We will now study the Cauchy parabolic PDE problem using the L^2 theory of solvability in Sobolev spaces and in weighted Sobolev spaces. Weaker regularity will be assumed from the data, and the operator coefficients' growth will be allowed.

Under the proper framework, we will proceed to the problem discretization in space. The discretization in time will be considered in Chapter 5.

4.1 Classical results

Let us first establish some facts on the solvability of PDE under a general framework.

Let V be a reflexive separable Banach space embedded continuously and densely into a Hilbert space H with inner product $(\ ,\ )$. Then H^*, the dual space of H, is also continuously and densely embedded into V^*, the dual of V. Let us use the notation $\langle\ ,\ \rangle$ for the duality. Let H^* be identified with H in the usual way, by the help of the inner product. Then we have the so called normal triple

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*,$$

with continuous and dense embeddings.

Let us consider the Cauchy problem

$$L(t)u(t) - \frac{\partial u(t)}{\partial t} + f(t) = 0, \quad u(0) = g, \tag{4.1}$$

where $L(t)$ and $\partial/\partial t$ are linear operators from V to V^* for every $t \geq 0$, $f \in L^2([0,T];V^*)$ with $T \in (0,\infty)$ and $g \in H$.

We make some assumptions.

Assumption 4.1.1. There exist constants $\lambda > 0$, K, M and N such that

1. $\langle L(t)v,\ v\rangle + \lambda |v|_V^2 \leq K|v|_H^2, \ \ \forall v \in V \text{ and } \forall t \in [0,T];$
2. $|L(t)v|_{V^*} \leq M|v|_V, \ \ \forall v \in V \text{ and } \forall t \in [0,T];$
3. $\int_0^T |f(t)|_{V^*}^2 dt \leq N \quad \text{and} \quad |g|_H \leq N.$

We define the generalized solution of problem (4.1).

Definition 4.1.2. We say that $u \in C([0,T];H)$ is a generalized solution of (4.1) on $[0,T]$ if

1. $u \in L^2([0,T];V)$;
2. For all $t \in [0,T]$

$$(u(t),v) = (g,v) + \int_0^t \langle L(s)u(s),v\rangle ds + \int_0^t \langle f(s),v\rangle ds$$

holds for all $v \in V$.

We next state the existence and uniqueness of the solution.

Theorem 4.1.3. *Under (1)–(3) in Assumption 4.1.1, problem (4.1) has a unique generalized solution on* $[0,T]$. *Moreover*

$$\sup_{t\in[0,T]} |u(t)|_H^2 + \int_0^T |u(t)|_V^2 dt \leq N \left(|g|_H^2 + \int_0^T |f(t)|_{V^*}^2 dt \right),$$

where N is a constant.

This theorem is a special case of a more general one proved, for example, in Lions [35] for nonlinear PDE.

The second-order parabolic PDE problem.

We consider now the particular case where L is the second-order operator

$$L(t,x) = a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t,x)\frac{\partial}{\partial x^i} + c\,(t,x), \tag{4.2}$$

with real coefficients. Take a number $T \in (0, \infty)$ and denote $Q = [0, T] \times \mathbb{R}^d$. We consider the Cauchy problem

$$Lu - u_t + f = 0 \text{ in } Q, \quad u(0, x) = g(x) \text{ in } \mathbb{R}^d, \tag{4.3}$$

where f and g are given functions.

To set the proper framework to deal with problem (4.3), we introduce the Sobolev spaces (see e.g. Evans [16], pp. 241-289).

We define the weak derivative of v. Let U be a domain in $\mathbb{R}^d$.

Definition 4.1.4. Suppose $v, w \in L^1_{loc}(U)$ and α is a multi-index. We say that w is the α^{th} weak partial derivative of v, and we write $D^\alpha v = w$, if

$$\int_U v D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U w \phi \, dx,$$

for all functions $\phi \in C_0^\infty(U)$.

Here the notation $L^1_{loc}(U)$ stands for the set of all functions $v : U \to \mathbb{R}$ locally summable, that is, the set of all functions v such that $\int_V |v| dx < \infty$, for every bounded subset V of U; $C_0^\infty(U)$ denotes the set of all infinitely differentiable functions on U with compact support.

The weak derivative is unique.

Proposition 4.1.5. *If a weak α^{th} partial derivative of v exists, it is uniquely defined up to a set of measure zero.*

We define the Sobolev space.

Definition 4.1.6. The Sobolev space $W^{m,2}(U)$, with $m \geq 0$ an integer, consists of all locally summable functions $v : U \to \mathbb{R}$ such that for each multi-index α with $|\alpha| \leq m$, $D^\alpha v$ exists in the weak sense and belongs to $L^2(U)$.

Remark 4.1.7. When $U = \mathbb{R}^d$ we drop the argument in $W^{m,2}(U)$ and denote $W^{m,2}(\mathbb{R}^d) := W^{m,2}$.

Definition 4.1.8. If $v \in W^{m,2}(U)$, we define its norm to be

$$|v|_{W^{m,2}(U)} := (\sum_{|\alpha| \leq m} \int_U |D^\alpha v|^2 dx)^{1/2}.$$

Definition 4.1.9. If $v, w \in W^{m,2}(U)$, we define the inner product

$$(v, w)_{W^{m,2}(U)} := \sum_{|\alpha| \leq m} \int_U D^\alpha v D^\alpha w \, dx.$$

We state the good structure of the Sobolev spaces.

Proposition 4.1.10. *The Sobolev space $W^{m,2}(U)$ is a Hilbert space.*

The following properties hold:

Proposition 4.1.11. *Let $v \in W^{m,2}(U)$, $|\alpha| \leq m$. Then*

1. *If V is an open subset of U, then $v \in W^{m,2}(V)$;*
2. *If $\zeta \in C_0^\infty(U)$, then $\zeta v \in W^{m,2}(U)$ and $D^\alpha(\zeta v) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} v$.*

We have a fundamental result on the embedding in better spaces.

Theorem 4.1.12. (Sobolev's embedding Theorem). *Let U be a bounded domain in $\mathbb{R}^d$ with a C^1 boundary. Let $v \in W^{m,2}(U)$. If $m > \frac{d}{2}$ then $v \in C^{(m-[\frac{d}{2}]-1)+\delta}(U)$, where*

$$\delta = \begin{cases} [\frac{d}{2}] + 1 - \frac{d}{2}, & if\ \frac{d}{2} \text{ is not an integer} \\ \text{any positive number} < 1, & if\ \frac{d}{2} \text{ is an integer.} \end{cases}$$

Moreover

$$|v|_{(m-[\frac{d}{2}]-1)+\delta;U} \leq N|v|_{W^{m,2}(U)},$$

with N a constant depending only on m, d, δ and U.

We recall that we are using the notation $|v|_{k+\delta;U}$ for the norm of $v \in C^{k+\delta}(U)$.

Now, we consider the functions $w : Q \to \mathbb{R}$ as functions of $[0, T]$ with values in $\mathbb{R}^\infty$ such that, for all $t \in [0, T]$, $w(t) := \{w(t, x) : x \in \mathbb{R}^d\}$.

Let us now make some assumptions.

Assumption 4.1.13. Let $m \geq 0$ be an integer. There exist constants $\lambda > 0$, K such that

1. $\sum_{i,j=1}^d a^{ij}(t, x)\xi^i\xi^j \geq \lambda \sum_{i=1}^d |\xi^i|^2$, for all $t \geq 0$, $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$;
2. $|D_x^\alpha a^{ij}| \leq K$ for all $|\alpha| \leq m \vee 1$, $|D_x^\alpha b^i| \leq K$, $|D_x^\alpha c| \leq K$ for all $|\alpha| \leq m$, where D_x^α denotes the α^{th} partial derivative operator with respect to x;
3. $f \in L^2([0, T]; W^{m-1,2})$, $g \in W^{m,2}$.

Remark 4.1.14. (1) in Assumption 4.1.13 states that operator $L - \partial/\partial t$ is (uniformly) parabolic.

Remark 4.1.15. For $m = 0$ we use the notation $W^{m-1,2} = W^{-1,2} = (W^{1,2})^*$, where $(W^{1,2})^*$ is the dual of $W^{1,2}$.

We define the generalized solution of problem (4.3).

Definition 4.1.16. We say that $u \in C([0,T];L^2)$ is a generalized solution of (4.3) on $[0,T]$ if

1. $u \in L^2([0,T];W^{1,2})$;
2. For all $t \in [0,T]$

$$\begin{aligned}(u(t),\varphi) &= (g,\varphi) + \int_0^t \{-(a^{ij}(s)D_i u(s), D_j\varphi) \\ &\quad +(b(s)D_i u(s) - D_j a^{ij}(s)D_i u(s),\varphi) \\ &\quad +(c(s)u(s),\varphi) + \langle f(s),\varphi\rangle\}ds\end{aligned}$$

   holds for all $\varphi \in C_0^\infty(\mathbb{R}^d)$.

The notation (,) in the above definition stands for the inner product in L^2.

Remark 4.1.17. Note that, alternatively to the infinite differentiability of φ in (2) it could be required that $\varphi \in W^{1,2}$.

From the above we have the following well known result, which can be obtained from Theorem 4.1.3 using the appropriate triples of spaces (see e.g. Gyöngy et all [22], p. 67, for the more general case of SPDE with unbounded coefficients).

Theorem 4.1.18. *Under (1)–(3) in Assumption 4.1.13, (4.3) admits a unique generalized solution u on $[0,T]$. Moreover*

$$u \in C([0,T];W^{m,2}) \cap L^2([0,T];W^{m+1,2})$$

and

$$\sup_{0\le t\le T} |u(t)|^2_{W^{m,2}} + \int_0^T |u(t)|^2_{W^{m+1,2}}dt \le N\left(|g|^2_{W^{m,2}} + \int_0^T |f(t)|^2_{W^{m-1,2}}dt\right)$$

with N a constant.

Further results for the unbounded data case.
We still consider problem (4.3) but in the more general case where no boundedness is imposed over the operator's coefficients. We introduce the weighted Sobolev spaces (see e.g. Gyöngy et all [22], pp. 58-65).

Definition 4.1.19. Let $r > 0$, $\rho > 0$ be smooth functions on $\mathbb{R}^d$ and $m \ge 0$ an integer. $W^{m,2}(r,\rho)$, the weighted Sobolev space (on $\mathbb{R}^d$), is the closure of $C_0^\infty(\mathbb{R}^d)$ with respect to the norm

$$|\varphi|_{W^{m,2}(r,\rho)} := (\sum_{|\alpha|\le m} \int_{\mathbb{R}^d} r^2|\rho^{|\alpha|}D^\alpha\varphi|^2dx)^{1/2},$$

for $\varphi \in C_0^\infty(\mathbb{R}^d)$.

Definition 4.1.20. If $v, w \in W^{m,2}(r,\rho)$, we define the inner product

$$(v,w)_{W^{m,2}(r,\rho)} := \sum_{|\alpha|\le m} \int_{\mathbb{R}^d} r^2 \rho^{2|\alpha|} D^\alpha v D^\alpha w \, dx.$$

The weighted Sobolev spaces have a good structure.

Proposition 4.1.21. *The weighted Sobolev space $W^{m,2}(r,\rho)$ is a Hilbert space.*

As before, we consider the functions $w : Q \to \mathbb{R}$ as functions of $[0,T]$ with values in $\mathbb{R}^\infty$ defined by $w(t) := \{w(t,x) : x \in \mathbb{R}^d\}$, for $t \in [0,T]$.

We make some assumptions.

Assumption 4.1.22. Let $r > 0$ and $\rho > 0$ be a smooth functions on $\mathbb{R}^d$ and $m \ge 0$ an integer. There exist constants $\lambda > 0$, K such that

1. $\sum_{i,j=1}^d a^{ij}(t,x)\xi^i\xi^j \ge \lambda\rho^2 \sum_{i=1}^d |\xi^i|^2$, for all $t \ge 0$, $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$;
2. $|D_x^\alpha a^{ij}| \le K\rho^{2-|\alpha|}$ for all $|\alpha| \le m \vee 1$, $|D_x^\alpha b^i| \le K\rho^{1-|\alpha|}$, $|D_x^\alpha c| \le K$ for all $|\alpha| \le m$, where D_x^α denotes the α^{th} partial derivative operator with respect to x;
3. $f \in L^2([0,T]; W^{m-1,2}(r,\rho))$ and $g \in W^{m,2}(r,\rho)$.

Assumption 4.1.23. Let $l \ge 0$ be an integer and $r > 0$ and $\rho > 0$ smooth functions on $\mathbb{R}^d$. There exists a constant K such that

1. $|D^\alpha \rho| \le K\rho^{1-|\alpha|}$, for all multi-indexes α such that $|\alpha| \le l-1$ if $l \ge 2$;
2. $|D^\alpha r| \le K\dfrac{r}{\rho^{|\alpha|}}$, for all multi-indexes α such that $|\alpha| \le l$.

Example 4.1.24. The following functions, taken from Gyöngy et all [22], pp. 63-64, satisfy Assumption 4.1.23:

1. $r(x) = (1+|x|^2)^\beta$, $\beta \in \mathbb{R}$; $\rho(x) = (1+|x|^2)^\gamma$, $\gamma \le \frac{1}{2}$;
2. $r(x) = \exp(\pm(1+|x|^2)^\beta)$, $0 \le \beta \le \frac{1}{2}$; $\rho(x) = (1+|x|^2)^\gamma$, $\gamma \le \frac{1}{2} - \beta$;
3. $r(x) = (1+|x|^2)^\beta$, $\beta \in \mathbb{R}$; $\rho(x) = \ln^\gamma(2+|x|^2)$, $\gamma \in \mathbb{R}$;
4. $r(x) = (1+|x|^2)^\beta \ln^\gamma(2+|x|^2)$, $\beta \ge 0$, $\gamma \ge 0$; $\rho(x) = (1+|x|^2)^\gamma$, $\gamma \le \frac{1}{2}$;
5. $r(x) = (1+|x|^2)^\beta \ln^\gamma(2+x^2)$, $\beta \ge 0$, $\gamma \ge 0$; $\rho(x) = \ln^\gamma(2+|x|^2)$, $\gamma \ge 0$;
6. $\rho(x) = \exp(-(1+|x|^2)^\gamma)$, $\gamma \ge 0$; each weight function $r(x)$ in examples (1)-(5).

We define the generalized solution of problem (4.3).

Definition 4.1.25. We say that $u \in C([0,T]; W^{0,2}(r,\rho))$ is a generalized solution of (4.3) on $[0,T]$ if

1. $u \in L^2([0,T]; W^{1,2}(r,\rho))$;
2. For all $t \in [0,T]$

$$\begin{aligned}(u(t),\varphi) &= (g,\varphi) + \int_0^t \{-(a^{ij}(s)D_i u(s), D_j\varphi) \\ &\quad + (b(s)D_i u(s) - D_j a^{ij}(s) D_i u(s), \varphi) \\ &\quad + (c(s)u(s),\varphi) + \langle f(s),\ \varphi\rangle\} ds\end{aligned}$$

   holds for all $\varphi \in C_0^\infty(\mathbb{R}^d)$.

The notation (,) in the above definition stands for the inner product in $W^{0,2}(r,\rho)$.

We have the following well known result on the existence and uniqueness of the solution.

Theorem 4.1.26. *Under (1)–(3) in Assumption 4.1.22 and (1)–(2) in Assumption 4.1.23, (4.3) admits a unique generalized solution u on $[0,T]$. Moreover*

$$u \in C([0,T]; W^{m,2}(r,\rho)) \cap L^2([0,T]; W^{m+1,2}(r,\rho))$$

and

$$\sup_{0\le t\le T} |u(t)|^2_{W^{m,2}(r,\rho)} + \int_0^T |u(t)|^2_{W^{m+1,2}(r,\rho)} dt \le N\left(|g|^2_{W^{m,2}(r,\rho)} + \int_0^T |f(t)|^2_{W^{m-1,2}(r,\rho)} dt\right),$$

with N a constant.

This result can also be obtained from the general one by using the suitable triples of spaces (see Gyöngy et all [22], p. 67).

4.2 Numerical approximation: bounded data case

We want to discretize in space the problem (4.3). We will set an appropriate framework and show that it is a particular case of the general framework we presented in Section 4.1.

We define the h-grid on $\mathbb{R}^d$, with $h \in (0,1]$

$$Z_h^d = \{x \in \mathbb{R}^d : x = h\sum_{i=1}^d e_i n_i,\ \ n_i = 0, \pm 1, \pm 2, \ldots\}.$$

Denote

$$\partial_i^+ u = \partial_i^+ u(t,x) = h^{-1}(u(t,x+he_i) - u(t,x))$$

and

$$\partial_i^- u = \partial_i^- u(t,x) = h^{-1}(u(t,x) - u(t,x-he_i)),$$

the forward and backward discrete differences in space, respectively. Define the discrete operator

$$L_h(t,x) = a^{ij}(t,x)\partial_j^- \partial_i^+ + b^i(t,x)\partial_i^+ + c\,(t,x).$$

We consider the discrete problem

$$L_h u - u_t + f_h = 0 \text{ in } Q(h), \quad u(0,x) = g_h(x) \text{ in } Z_h^d, \tag{4.4}$$

where $Q(h) = [0,T] \times Z_h^d$, with T a number such that $T \in (0,\infty)$ and f_h and g_h are functions such that $f_h : Q(h) \to \mathbb{R}$ and $g_h : Z_h^d \to \mathbb{R}$.

We introduce the discrete Sobolev spaces.

Consider functions $v : Z_h^d \to \mathbb{R}$. For all functions $v, w : Z_h^d \to \mathbb{R}$, we define the inner product

$$(v,w)_{l^{0,2}} = \sum_{x \in Z_h^d} v(x)w(x)h^d.$$

The function space $l^{0,2}$ is defined by

$$l^{0,2} = \{v : Z_h^d \to \mathbb{R} : \ |v|_{l^{0,2}} < \infty\},$$

where the $|v|_{l^{0,2}}$ is the norm induced by the inner product

$$|v|_{l^{0,2}} = (v,v)_{l^{0,2}}^{1/2} = (\sum_{x \in Z_h^d} |v(x)|^2 h^d)^{1/2}.$$

Remark 4.2.1. It is trivial to check that $(\ ,\ )$ and $|\ \ |_{l^{0,2}}$ as defined above are, respectively, a inner product and a norm. The triangle inequality for the norm is proved using Cauchy-Schwarz inequality.

Next we will show that $l^{0,2}$ is a Hilbert space.

Proposition 4.2.2. *The function space $l^{0,2}$ is a Hilbert space.*

Proof To prove that $l^{0,2}$ is a Hilbert space we have to prove that the inner product space $l^{0,2}$ is complete, i.e., that any Cauchy sequence in $l^{0,2}$ is convergent in the space norm.

Let (v_n) be a Cauchy sequence in $l^{0,2}$. That is, $\forall \varepsilon > 0 \; \exists N$ such that for $m, n > N$

$$|v_m - v_n|_{l^{0,2}} = (\sum_{x \in Z_h^d} |v_m(x) - v_n(x)|^2 h^d)^{1/2} < \varepsilon. \tag{4.5}$$

Then, for every $x \in Z_h^d$ we have

$$|v_m(x) - v_n(x)|^2 h^d < \varepsilon^2, \quad \text{for } m, n > N. \tag{4.6}$$

Let us fix $x = x_0$. From (4.6) we see that $(v_1(x_0), v_2(x_0), \ldots)$ is a Cauchy sequence of real numbers, therefore convergent, say $v_m(x_0) \to v(x_0)$. Using these limits we define $v = v(x)$, for each $x \in Z_h^d$.

Let B be a ball in Z_h^d. From (4.5) we have, for $m, n > N$

$$\sum_{x \in B} |v_m(x) - v_n(x)|^2 h^d < \varepsilon^2.$$

Letting $n \to \infty$, we have for $m > N$

$$\sum_{x \in B} |v_m(x) - v(x)|^2 h^d \leq \varepsilon^2.$$

Letting now the diameter of B go to ∞, we have for $m > N$

$$\sum_{x \in Z_h^d} |v_m(x) - v(x)|^2 h^d \leq \varepsilon^2, \tag{4.7}$$

which shows that $v_m - v \in l^{0,2}$.

As $v_m \in l^{0,2}$, it follows, owing to Minkowski inequality for sums, that

$$v = v_m + (v - v_m) \in l^{0,2}.$$

Finally, (4.7) also implies that $v_m \to v$ and the result is proved. □

For functions $v : Z_h^d \to \mathbb{R}$ we introduce also the function space

$$l^{1,2} = \{v : Z_h^d \to \mathbb{R} : \; |v|_{l^{1,2}} < \infty\}, \quad \text{with } |v|^2_{l^{1,2}} = |v|^2_{l^{0,2}} + \sum_{i=1}^{d} |\partial_i^+ v|^2_{l^{0,2}}.$$

Let us endow this space with the inner product

$$(v, w)_{l^{1,2}} = (v, w)_{l^{0,2}} + \sum_{i=1}^{d} (\partial_i^+ v, \partial_i^+ w)_{l^{0,2}},$$

where v, w are any functions in $l^{1,2}$.

The space $l^{1,2}$ has a good structure.

Proposition 4.2.3. *The function space $l^{1,2}$ is a Hilbert space.*

Proof The proof follows the same steps as for Proposition 4.2.2. □

We note that as $l^{1,2}$ is a Hilbert space therefore it is reflexive. Next we will prove that $l^{1,2}$ is separable.

Proposition 4.2.4. *The function space $l^{1,2}$ is separable.*

Proof We have to prove that $l^{1,2}$ has a countable subset dense in $l^{1,2}$.

Let us consider the set $S = B \cup \{x + he_i : x \in B,\ i = 1, 2, \ldots, d\}$, with B a ball in Z_h^d. Consider the set l of all functions $w(x) \in l^{1,2}$ taking rational values if $x \in S$ and vanishing outside S. This set l of functions is countable.

Let v be an arbitrary function in $l^{1,2}$. Let $x \in B$. For any given $\varepsilon > 0$, we can choose w such that

$$\begin{aligned}
&\sum_x |v(x) - w(x)|^2 h^d + \sum_{i=1}^d \sum_x |\partial_i^+ (v(x) - w(x))|^2 h^d \\
=&\sum_x |v(x) - w(x)|^2 h^d + \sum_{i=1}^d \sum_x |h^{-1}(v(x + he_i) - w(x + he_i) - (v(x) - w(x)))|^2 h^d \\
\le&\sum_x |v(x) - w(x)|^2 h^d + 2\sum_{i=1}^d \sum_x |v(x + he_i) - w(x + he_i)|^2 h^{d-2} \\
&\quad + 2\sum_{i=1}^d \sum_x |v(x) - w(x)|^2 h^{d-2} \\
<&\frac{\varepsilon^2}{2}.
\end{aligned} \tag{4.8}$$

Also, as $|v|^2_{l^{1,2}}$ is a convergent series, for any given $\varepsilon > 0$ we can choose the diameter of B such that, for x outside B we have

$$\sum_x |v(x)|^2 h^d + \sum_{i=1}^d \sum_x |\partial_i^+ v(x)|^2 h^d \quad < \quad \frac{\varepsilon^2}{2}. \tag{4.9}$$

From (4.8) and (4.9) we have

$$|v - w|_{l^{1,2}} < \varepsilon,$$

and the result is proved. □

We now show that $l^{1,2}$ is continuously and densely embeddable in $l^{0,2}$. The continuity follows immediately from

$$|v|_{l^{0,2}} \le |v|_{l^{1,2}}, \quad \text{for all } \ v \in l^{1,2}.$$

For the denseness, we have the following result:

Proposition 4.2.5. *The function space $l^{1,2}$ is densely embeddable in $l^{0,2}$.*

Proof We want to prove that $\overline{l^{1,2}} = l^{0,2}$. Let us take an arbitrary function $v \in l^{0,2}$. Let B be a ball in Z_h^d. We consider the function w such that

$$w(x) = \begin{cases} v(x), & x \in B \\ 0, & \text{otherwise.} \end{cases}$$

This function belongs obviously to $l^{1,2}$. Furthermore, for any given $\varepsilon > 0$, we have

$$|v - w|_{l^{0,2}} < \varepsilon,$$

if the diameter of B is chosen sufficiently large. The result is proved. □

Finally, we consider the functions $w : Q(h) \to \mathbb{R}$ as functions of $[0, T]$ with values in $\mathbb{R}^\infty$ such that, for all $t \in [0, T]$, $w(t) := \{w(t, x) : x \in Z_h^d\}$. For these functions, we consider the subspaces $C([0, T]; l^{0,2})$ and

$$L^2([0, T]; l^{1,2}) = \{w : [0, T] \to l^{1,2} : |w|_{L^2} < \infty\},$$

with $|w|_{L^2}^2 = \int_0^T |w(t)|_{l^{1,2}}^2 \, dt$.

We make some assumptions over the data f_h and g_h in (4.4).

Assumption 4.2.6. We assume

1. $f_h \in L^2([0, T]; l^{0,2})$;
2. $g_h \in l^{0,2}$.

Remark 4.2.7. $f_h \in L^2([0, T]; l^{0,2})$ in Assumption 4.2.6 could be replaced for the weaker assumption $f_h \in L^2([0, T]; (l^{1,2})^*)$, where $(l^{1,2})^*$ denotes the dual space of $l^{1,2}$.

Remark 4.2.8. The boundedness of the discrete difference

$$\partial_i^+ a^{ij} = \partial_i^+ a^{ij}(t, x) = h^{-1}(a^{ij}(t, x + he_i) - a^{ij}(t, x))$$

can be obtained from (2) in Assumption 4.1.13. In fact, as

$$|\partial_i^+ a^{ij}(t, x)| = |h^{-1}(a^{ij}(t, x + he_i) - a^{ij}(t, x))| \leq |\frac{\partial}{\partial x^i} a^{ij}(t, x + \tau e_i)|,$$

for some τ such that $0 < \tau < h$, from the boundedness of $(\partial/\partial x^i)\, a^{ij}$ we have the boundedness of $\partial_i^+ a^{ij}$.

We define the generalized solution of problem (4.4).

Definition 4.2.9. We say that $u \in C([0,T];l^{0,2}) \cap L^2([0,T];l^{1,2})$ is a generalized solution of (4.4) if for all $t \in [0,T]$

$$\begin{aligned}(u(t),\varphi) &= (g_h,\varphi) + \int_0^t \{-(a^{ij}(s)\partial_i^+ u(s), \partial_j^+ \varphi) \\ &\quad +(b^i(s)\partial_i^+ u(s) - \partial_j^+ a^{ij}(s)\partial_i^+ u(s), \varphi) \\ &\quad +(c(s)u(s),\varphi) + \langle f_h(s), \varphi\rangle\} ds,\end{aligned}$$

holds for all $\varphi \in l^{1,2}$.

In the above definition, as in the rest of the present section, $(\, , \,)$ denotes the inner product in $l^{0,2}$.

We state next the existence and uniqueness of the solution for the discrete problem, as a consequence of Theorem 4.1.3. It remains only to show that within the discrete framework we constructed $(1)-(2)$ in Assumption 4.1.1 hold.

Theorem 4.2.10. *Under (1)–(2) in Assumption 4.1.13 and (1)–(2) in Assumption 4.2.6, problem (4.4) admits a unique generalized solution on* $[0,T]$. *Moreover*

$$\sup_{0\le t\le T} |u(t)|^2_{l^{0,2}} + \int_0^T |u(t)|^2_{l^{1,2}}\, dt \;\le\; N\,(|g_h|^2_{l^{0,2}} + \int_0^T |f_h(t)|^2_{l^{0,2}}\, dt),$$

with N *a constant not depending on* h.

Proof Let $L_h(s) : l^{1,2} \to (l^{1,2})^*$ and define for all $\varphi, \psi \in l^{1,2}$

$$\langle L_h(s)\psi,\varphi\rangle := -\,(a^{ij}(s)\partial_i^+\psi, \partial_j^+\varphi) + (b^i(s)\partial_i^+\psi - \partial_j^+ a^{ij}(s)\partial_i^+\psi, \varphi) + (c(s)\psi,\varphi).$$

We will prove that L_h satisfies the following properties:

1. $|\langle L_h(s)\psi,\varphi\rangle| \le K|\psi|_{l^{1,2}} \cdot |\varphi|_{l^{1,2}}, \quad \forall \varphi,\psi \in l^{1,2}, \;\; K$ constant;
2. $\langle L_h(s)\psi,\psi\rangle \le K|\psi|^2_{l^{0,2}} - \lambda|\psi|^2_{l^{1,2}}, \quad \forall \psi \in l^{1,2}, \;\; \lambda > 0, \; K$ constants.

The first property follows immediately from (2) in Assumption 4.1.13 and Cauchy-Schwarz inequality

$$\begin{aligned}
|\langle L_h(s)\psi,\varphi\rangle| &= |-\sum_x\sum_{i,j} a^{ij}(s)\partial_i^+\psi\,\partial_j^+\varphi\,h^d+\sum_x\sum_i b^i(s)\partial_i^+\psi\,\varphi\,h^d \\
&\quad -\sum_x\sum_{i,j}\partial_j^+ a^{ij}(s)\partial_i^+\psi\,\varphi\,h^d+\sum_x c(s)\psi\,\varphi\,h^d| \\
&\le K\sum_x\sum_{i,j}|\partial_i^+\psi\,\partial_j^+\varphi|\,h^d+K\sum_x\sum_i|\partial_i^+\psi\,\varphi|\,h^d \\
&\quad +K\sum_x|\psi\,\varphi|h^d \\
&\le K\sum_i|\partial_i^+\psi|_{l^{0,2}}\sum_j|\partial_j^+\varphi|_{l^{0,2}}+K\sum_i|\partial_i^+\psi|_{l^{0,2}}|\varphi|_{l^{0,2}} \\
&\quad +K|\psi|_{l^{0,2}}|\varphi|_{l^{0,2}} \\
&\le K|\psi|_{l^{1,2}}|\varphi|_{l^{1,2}}.
\end{aligned}$$

In the above, the variable $x\in Z_h^d$ is omitted and $\sum_x$ denotes the summation over Z_h^d.

For the second property, with the same conventions, we have

$$\begin{aligned}
\langle L_h(s)\psi,\psi\rangle &= -\sum_{i,j}\sum_x a^{ij}(s)\partial_i^+\psi\,\partial_j^+\psi\,h^d \\
&\quad +\sum_{i,j}\sum_x(b^i(s)-\partial_j^+a^{ij}(s))\partial_i^+\psi\,\psi\,h^d+\sum_x c(s)\psi\,\psi\,h^d \\
&\le -\lambda\sum_i\sum_x|\partial_i^+\psi|^2h^d+2K\sum_x\sum_i|\partial_i^+\psi\,\psi|h^d+K\sum_x|\psi|^2h^d \\
&= -\lambda\sum_i|\partial_i^+\psi|^2_{l^{0,2}}+2K\sum_i\sum_x|\partial_i^+\psi\,\psi|h^d+K|\psi|^2_{l^{0,2}},
\end{aligned}$$

owing to (1) and (2) in Assumption 4.1.13. Applying the Cauchy's inequality to the second term of last expression, we obtain

$$\begin{aligned}
&\langle L_h(s)\psi,\psi\rangle \\
&\le -\lambda\sum_i|\partial_i^+\psi|^2_{l^{0,2}}+\varepsilon K\sum_i\sum_x|\partial_i^+\psi|^2h^d+\frac{K}{\varepsilon}\sum_i\sum_x|\psi|^2h^d+K|\psi|^2_{l^{0,2}} \\
&= -\lambda\sum_i|\partial_i^+\psi|^2_{l^{0,2}}-\lambda|\psi|^2_{l^{0,2}}+\varepsilon K\sum_i|\partial_i^+\psi|^2_{l^{0,2}}+\frac{K}{\varepsilon}|\psi|^2_{l^{0,2}}+(K+\lambda)|\psi|^2_{l^{0,2}} \\
&\le -\lambda|\psi|^2_{l^{1,2}}+K|\psi|^2_{l^{0,2}},
\end{aligned}$$

with $\lambda>0$, K constants, by taking ε sufficiently small, and the second propriety is proved. Owing to Theorem 4.1.3 the result follows. □

We prove that the partial derivatives are approximated by the discrete differences.

Proposition 4.2.11. *Let m be an integer strictly greater than $d/2$. Let $u(t) \in W^{m+2,2}$, $v(t) \in W^{m+3,2}$, for all $t \in [0,T]$. Then there exists a constant N not depending on h such that*

1. $\sum_x |u_{x^i}(t,x) - \partial_i^+ u(t,x)|^2 h^d \le h^2 N |u(t)|^2_{W^{m+2,2}}$,

2. $\sum_x |v_{x^i x^j}(t,x) - \partial_j^- \partial_i^+ v(t,x)|^2 h^d \le h^2 N |v(t)|^2_{W^{m+3,2}}$,

for all $t \in [0,T]$, with $x \in Z_h^d$ and $\sum_x$ denoting the summation over Z_h^d.

Proof Let us prove (1). By the mean-value theorem we have

$$\partial_i^+ u(t,x) = h^{-1}(u(t,x+he_i) - u(t,x)) = u_{x^i}(t, x+\theta h e_i)$$

and

$$u_{x^i}(t,x) - \partial_i^+ u(t,x) = u_{x^i}(t,x) - u_{x^i}(t,x+\theta h e_i) = h\, u_{x^i x^i}(t, x+\theta' h e_i),$$

for some $0 < \theta' < \theta < 1$. We consider d-cells

$$R_h = \{(x^1, x^2, \ldots, x^d) \in \mathbb{R}^d : \ \ x_h^i < x^i < x_h^i + h, \ \ i = 1,2,\ldots,d\},$$

with $x_h = (x_h^1, x_h^2, \ldots, x_h^d) \in Z_h^d$ fixed.

For every $x_h \in Z_h^d$ we have

$$|u_{x^i x^i}(t, x_h + \theta' h e_i)| \le \sup_{x \in R_h} |u_{x^i x^i}(t,x)|,$$

and then

$$|u_{x^i}(t,x_h) - \partial_i^+ u(t,x_h)|^2 \le h^2 \sup_{x \in R_h} |u_{x^i x^i}(t,x)|^2. \tag{4.10}$$

Let us consider the particular d-cell where $h = 1$ and $x_1 = (0,0,\ldots,0)$ and denote it R_1^0. We have

$$\sup_{x \in R_h} |u_{x^i x^i}(t,x)| = \sup_{x \in R_1^0} |u_{x^i x^i}(t, x_h + hx)|. \tag{4.11}$$

Take open balls B_h such that $B_h \supset R_h$, with the vertices $\{x_h^i, x_h^i + h, \ \ i = 1,2,\ldots,d\}$ on the boundary sphere. Denote B_1^0 the ball containing R_1^0. We have

$$\sup_{x \in R_1^0} |u_{x^i x^i}(t, x_h + hx)|^2 \ \ \le \ \ \sup_{x \in B_1^0} |u_{x^i x^i}(t, x_h + hx)|^2 \tag{4.12}$$

Taking in mind (1) in Proposition 4.1.11, by Theorem 4.1.12 for $m > d/2$ we have

$$\begin{aligned}
\sup_{x\in B_1^0} |u_{x^ix^i}(t,x_h+hx)|^2 &\le N\sum_{|\alpha|\le m}\int_{B_1^0}|D_x^\alpha u_{x^ix^i}(t,x_h+hx)|^2dx \\
&\le N\sum_{|\alpha|\le m+2}\int_{B_1^0}|D_x^\alpha u(t,x_h+hx)|^2dx \\
&= N\sum_{|\alpha|\le m+2}\int_{B_h}|D_x^\alpha u(t,x)|^2h^{-d}h^{2|\alpha|}dx \\
&\le N\sum_{|\alpha|\le m+2}\int_{B_h}|D_x^\alpha u(t,x)|^2h^{-d}dx. \qquad (4.13)
\end{aligned}$$

Then, by (4.10), (4.11), (4.12) and (4.13), we finally obtain

$$\begin{aligned}
\sum_{x_h\in Z_h^d}|u_{x^i}(t,x_h)-\partial_i^+u(t,x_h)|^2h^d &\le Nh^2\sum_{|\alpha|\le m+2}\sum_{x_h\in Z_h^d}\int_{B_h(x_h)}|D_x^\alpha u(t,x)|^2dx \\
&\le Nh^2\sum_{|\alpha|\le m+2}\sum_{x_h\in Z_h^d}\int_{R_h(x_h)}|D_x^\alpha u(t,x)|^2dx \\
&\le h^2N|u(t)|^2_{W^{m+2,2}},
\end{aligned}$$

where $B_h(x_h)=B_h$, $R_h(x_h)=R_h$, and the proof for (1) is complete. The proof for (2) is similar. □

Next we determine a rate of convergence.

Theorem 4.2.12. *Denote u the solution of (4.3) in Theorem 4.1.18 and u_h the solution of (4.4) in Theorem 4.2.10. Let m be an integer strictly greater than $d/2$ and assume that $u\in L^2([0,T];W^{m+3,2})$. Then*

$$\begin{aligned}
&\sup_{0\le t\le T}|u(t)-u_h(t)|^2_{l^{0,2}}+\int_0^T|u(t)-u_h(t)|^2_{l^{1,2}}\,dt \\
&\le h^2N\int_0^T|u(t)|^2_{W^{m+3,2}}dt+N(|g-g_h|^2_{l^{0,2}}+\int_0^T|f(t)-f_h(t)|^2_{l^{0,2}}\,dt),
\end{aligned}$$

for some constant N independent of h.

Proof From (4.3) and (4.4),

$$\begin{cases}
L_h(u-u_h)-\dfrac{d}{dt}(u-u_h)+(L-L_h)u+(f-f_h)=0 \quad \text{in} \quad Q(h)\\
(u-u_h)(0,x)=(g-g_h)(x) \quad \text{in} \quad Z_h^d.
\end{cases}$$

We have that $(f-f_h)\in L^2([0,T];l^{0,2})$ and $(g-g_h)\in l^{0,2}$, obviously. Also if $u\in W^{m+3,2}$ we have that $(L-L_h)u\in L^2([0,T];l^{0,2})$. Then, by Theorem 4.2.10,

$$\begin{aligned}
&\sup_{0\le t\le T}|u(t)-u_h(t)|^2_{l^{0,2}}+\int_0^T|u(t)-u_h(t)|^2_{l^{1,2}}\,dt \\
&\le N(|g-g_h|^2_{l^{0,2}}+\int_0^T|f(t)-f_h(t)|^2_{l^{0,2}}\,dt+\int_0^T|(L-L_h)u(t)|^2_{l^{0,2}}\,dt).
\end{aligned}$$

As

$$\int_0^T |(L-L_h)u(t)|^2_{l^{0,2}}\,dt$$
$$= \int_0^T |a^{ij}(t,x)(\frac{\partial^2}{\partial x^i \partial x^j} - \partial_j^- \partial_i^+)u(t,x) + b^i(t,x)(\frac{\partial}{\partial x^i} - \partial_i^+)u(t,x)|^2_{l^{0,2}}\,dt,$$

owing to Proposition 4.2.11 and to the hypothesis over the boundedness of the coefficients, the result follows. □

Corollary 4.2.13. *Denote u the solution of (4.3) in Theorem 4.1.18 and u_h the solution of (4.4) in Theorem 4.2.10. Let m be an integer strictly greater than $d/2$ and assume that $u \in L^2([0,T]; W^{m+3,2})$. If there is a constant N independent of h such that*

$$|g-g_h|^2_{l^{0,2}} + \int_0^T |f(t)-f_h(t)|^2_{l^{0,2}}\,dt \le h^2 N(|g|^2_{W^{m,2}} + \int_0^T |f(t)|^2_{W^{m-1,2}}dt),$$

then

$$\sup_{0\le t\le T} |u(t)-u_h(t)|^2_{l^{0,2}} + \int_0^T |u(t)-u_h(t)|^2_{l^{1,2}}\,dt$$
$$\le h^2 N(\int_0^T |u(t)|^2_{W^{m+3,2}}dt + |g|^2_{W^{m,2}} + \int_0^T |f(t)|^2_{W^{m-1,2}}dt).$$

Proof The result follows immediately from Theorem 4.2.12. □

Let us compute now a rate of convergence in the special case of one space dimension, where weaker smoothness is demanded from the solution function u.

Theorem 4.2.14. *Denote u the solution of (4.3) in Theorem 4.1.18 and u_h the solution of (4.4) in Theorem4.2.10.Let $d = 1$ and assume that $u \in L^2([0,T]; W^{3,2})$. Then*

$$\sup_{0\le t\le T} |u(t)-u_h(t)|^2_{l^{0,2}} + \int_0^T |u(t)-u_h(t)|^2_{l^{1,2}}\,dt$$
$$\le h^2 N \int_0^T |u(t)|^2_{W^{3,2}}dt + N(|g-g_h|^2_{l^{0,2}} + \int_0^T |f(t)-f_h(t)|^2_{l^{0,2}}\,dt),$$

for some constant N independent of h.

Proof We have

$$\begin{cases} L_h(u-u_h) - \dfrac{d}{dt}(u-u_h) + (L-L_h)u + (f-f_h) = 0 \quad \text{in} \quad [0,T]\times Z_h \\ (u-u_h)(0,x) = (g-g_h)(x) \quad \text{in} \ \ Z_h. \end{cases}$$

We have that $(f - f_h) \in L^2([0,T]; l^{0,2})$ and $(g - g_h) \in l^{0,2}$, obviously. Also as $u \in W^{3,2}$ we have that $(L - L_h)u \in L^2([0,T]; l^{0,2})$. Then, by Theorem 4.2.10,

$$\sup_{0 \le t \le T} |u(t) - u_h(t)|^2_{l^{0,2}} + \int_0^T |u(t) - u_h(t)|^2_{l^{1,2}} \, dt$$
$$\le N(|g - g_h|^2_{l^{0,2}} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}} \, dt + \int_0^T |(L - L_h)u(t)|^2_{l^{0,2}} \, dt).$$

As we are considering $d = 1$

$$\int_0^T |(L - L_h)u(t)|^2_{l^{0,2}} \, dt$$
$$= \int_0^T |a(t,x)(\frac{\partial^2}{\partial x^2} - \partial^- \partial^+)u(t,x) + b(t,x)(\frac{\partial}{\partial x} - \partial^+)u(t,x)|^2_{l^{0,2}} \, dt.$$

Now, as

$$\partial^+ u(t,x) = h^{-1}(u(t, x+h) - u(t,x)) = \int_0^1 \frac{\partial}{\partial x} \, u(t, x + hq) dq$$

and

$$\partial^- u(t,x) = h^{-1}(u(t,x) - u(t, x-h)) = \int_0^1 \frac{\partial}{\partial x} \, u(t, x - hs) ds$$

and then

$$\begin{aligned} \partial^- \partial^+ u(t,x) &= \partial^- \int_0^1 \frac{\partial}{\partial x} \, u(t, x + hq) dq \\ &= \int_0^1 \left(\frac{\partial}{\partial x} \int_0^1 \frac{\partial}{\partial x} \, u(t, x + hq - hs) dq \right) ds \\ &= \int_0^1 \int_0^1 \frac{\partial^2}{\partial x^2} \, u(t, x + h(q - s)) ds dq, \end{aligned}$$

we have

$$\begin{aligned} (\frac{\partial}{\partial x} - \partial^+)u(t,x) &= \int_0^1 (\frac{\partial}{\partial x} u(t,x) - \frac{\partial}{\partial x} \, u(t, x + hq)) dq \\ &= h \int_0^1 \int_0^1 q \frac{\partial^2}{\partial x^2} \, u(t, x + hqs) ds dq \end{aligned} \tag{4.14}$$

and

$$(\frac{\partial^2}{\partial x^2} - \partial^- \partial^+)u(t,x)$$
$$= \int_0^1 \int_0^1 (\frac{\partial^2}{\partial x^2}(t,x) - \frac{\partial^2}{\partial x^2} \, u(t, x + h(q - s))) ds dq$$
$$= h \int_0^1 \int_0^1 \int_0^1 (q - s) \frac{\partial^3}{\partial x^3} \, u(t, x + hv(q - s)) \, dv ds dq. \tag{4.15}$$

For the first-order term (4.14) we have, using Jensen's inequality,

$$\begin{aligned}
\left|(\frac{\partial}{\partial x} - \partial^+)u(t,x)\right|^2 &\leq h^2 \int_0^1 \int_0^1 q^2 \left|\frac{\partial^2}{\partial x^2}\, u(t, x+hqs)\right|^2 dsdq \\
&= h^2 \int_0^1 \int_0^{hq} \frac{q}{h} \left|\frac{\partial^2}{\partial x^2}\, u(t,x+v)\right|^2 dvdq \\
&\leq h \int_0^1 qdq \int_0^h \left|\frac{\partial^2}{\partial x^2}\, u(t,x+v)\right|^2 dv \\
&\leq \frac{h}{2} \int_0^h \left|\frac{\partial^2}{\partial x^2}\, u(t,x+v)\right|^2 dv \\
&= \frac{h}{2} \int_x^{x+h} \left|\frac{\partial^2}{\partial z^2}\, u(t,z)\right|^2 dz,
\end{aligned}$$

with $v = hqs$.

Finally we obtain, with N a constant independent of h,

$$\sum_{x \in Z_h^d} \left|(\frac{\partial}{\partial x} - \partial^+)u(t,x)\right|^2 h \leq h^2 N |u(t)|^2_{W^{2,2}}.$$

For the second-order term (4.15) we have, also using Jensen's inequality,

$$\begin{aligned}
\left|(\frac{\partial^2}{\partial x^2} - \partial^-\partial^+)u(t,x)\right|^2 &\leq h^2 \int_0^1 \int_0^1 \int_0^1 |q-s|^2 \left|\frac{\partial^3}{\partial x^3} u(t, x+hv(q-s))\right|^2 dvdsdq \\
&= h^2 \int_0^1 \int_0^1 \int_0^{h(q-s)} \frac{q-s}{h} \left|\frac{\partial^3}{\partial x^3}\, u(t,x+w)\right|^2 dwdsdq \\
&\leq h \int_0^1 \int_0^1 |q-s| dsdq \int_0^h \left|\frac{\partial^3}{\partial x^3}\, u(t,x+w)\right|^2 dw \\
&\leq h \int_0^h \left|\frac{\partial^3}{\partial x^3}\, u(t,x+w)\right|^2 dw \\
&\leq h \int_x^{x+h} \left|\frac{\partial^3}{\partial z^3}\, u(t,z)\right|^2 dz,
\end{aligned}$$

with $w = hv(q-s)$.

Finally,

$$\sum_{x \in Z_h^d} \left|(\frac{\partial^2}{\partial x^2} - \partial^-\partial^+)u(t,x)\right|^2 h \leq h^2 N |u(t)|^2_{W^{3,2}}\,,$$

with N a constant independent of h. The result follows. □

Next result is an immediate consequence of Theorem 4.2.14.

Corollary 4.2.15. *Denote u the solution of (4.3) in Theorem 4.1.18 and u_h the solution of (4.4) in Theorem 4.2.10. Let $d = 1$, m a positive integer and assume that $u \in L^2([0,T];W^{3,2})$. If there is a constant N independent of h such that*

$$|g - g_h|^2_{l^{0,2}} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}}\, dt \le h^2 N(|g|^2_{W^{m,2}} + \int_0^T |f(t)|^2_{W^{m-1,2}} dt),$$

then

$$\sup_{0\le t\le T} |u(t) - u_h(t)|^2_{l^{0,2}} + \int_0^T |u(t) - u_h(t)|^2_{l^{1,2}}\, dt$$
$$\le h^2 N(\int_0^T |u(t)|^2_{W^{3,2}} dt + |g|^2_{W^{m,2}} + \int_0^T |f(t)|^2_{W^{m-1,2}} dt).$$

4.3 Numerical approximation: unbounded data case

To be able to consider unbounded data, we set a new discrete framework, which is still a particular case of the general framework presented in Section 4.1.

As before, we consider the discrete problem (4.4), discretized version of the problem (4.3).

Consider functions $v : Z_h^d \to \mathbb{R}$. We introduce the function space

$$l^{0,2}(r) = \{v : Z_h^d \to \mathbb{R} :\ |v|_{l^{0,2}(r)} < \infty\},$$

where the norm $|v|_{l^{0,2}(r)}$ is defined by

$$|v|_{l^{0,2}(r)} = (\sum_{x\in Z_h^d} r^2 |v(x)|^2 h^d)^{1/2}.$$

Define the inner product

$$(v, w)_{l^{0,2}(r)} = \sum_{x\in Z_h^d} r^2 v(x)w(x)h^d, \ \ \forall v, w \in l^{0,2}(r).$$

We introduce also the function space

$$l^{1,2}(r, \rho) = \{v : Z_h^d \to \mathbb{R} :\ |v|_{l^{1,2}(r,\rho)} < \infty\},$$

with

$$|v|^2_{l^{1,2}(r,\rho)} = |v|^2_{l^{0,2}(r)} + \sum_{i=1}^d |\rho\, \partial_i^+ v|^2_{l^{0,2}(r)}.$$

We endow $l^{1,2}(r,\rho)$ with the inner product

$$(v,w)_{l^{1,2}(r,\rho)} = (v,w)_{l^{0,2}(r)} + \sum_{i=1}^{d} (\partial_i^+ v, \partial_i^+ w)_{l^{0,2}(r)},$$

where v, w are any functions in $l^{1,2}(r,\rho)$.

Finally, we consider the functions $w : Q(h) \to \mathbb{R}$ as functions of $[0,T]$ with values in $\mathbb{R}^\infty$, defined by $w(t) = \{w(t,x) : x \in Z_h^d\}$, for all $t \in [0,T]$. For these functions, we consider the subspaces $C([0,T]; l^{0,2}(r))$ and

$$L^2([0,T]; l^{1,2}(r,\rho)) = \{w : [0,T] \to l^{1,2}(r,\rho) : |w|_{L^2} < \infty\},$$

with $|w|_{L^2}^2 = \int_0^T |w(t)|_{l^{1,2}(r,\rho)}^2 \, dt$.

In the same way we have done for $l^{0,2}$ and $l^{1,2}$, it could be shown that $l^{0,2}(r)$ is a Hilbert space, $l^{1,2}(r,\rho)$ is a reflexive and separable Banach space and that $l^{1,2}(r,\rho)$ is continuously and densely embedded into $l^{0,2}(r)$.

We make some assumptions over the data f_h and g_h in (4.4).

Assumption 4.3.1. Let $r > 0$ and $\rho > 0$ be smooth functions on $\mathbb{R}^d$. We assume

1. $f_h \in L^2([0,T]; l^{0,2}(r))$;
2. $g_h \in l^{0,2}(r)$.

Remark 4.3.2. The assumption $f_h \in L^2([0,T]; l^{0,2}(r))$ could be replaced for the weaker assumption $f_h \in L^2([0,T]; (l^{1,2}(r,\rho))^*)$, where $(l^{1,2}(r,\rho))^*$ denotes the dual space of $l^{1,2}(r,\rho)$.

Remark 4.3.3. We can obtain $|\partial_i^+ a^{ij}| \le K\rho$ from (2) in Assumption 4.1.22. This follows from

$$|\partial_i^+ a^{ij}(t,x)| = |h^{-1}(a^{ij}(t,x+he_i) - a^{ij}(t,x))| \le |\frac{\partial}{\partial x^i} \, a^{ij}(t,x+\tau e_i)|,$$

for some τ such that $0 < \tau < h$.

We define the generalized solution of problem (4.4).

Definition 4.3.4. We say that $u \in C([0,T]; l^{0,2}(r)) \cap L^2([0,T]; l^{1,2}(r,\rho))$ is a generalized solution of (4.4) if for all $t \in [0,T]$

$$\begin{aligned}(u(t),\ \varphi) \;=\; & (g_h),\varphi) + \int_0^t \{-(a^{ij}(s)\partial_i^+ u(s), \partial_j^+ \varphi) \\ & +(b^i(s)\partial_i^+ u(s) - \partial_j^+ a^{ij}(s)\partial_i^+ u(s), \varphi) \\ & +(c(s)u(s),\varphi) + \langle f_h(s),\varphi\rangle\} ds,\end{aligned}$$

holds for $\varphi \in l^{1,2}(r,\rho)$.

In the above, (,) denotes the inner product in $l^{0,2}(r)$. We keep this simplification for the rest of the present section,

As in Section 4.2, we will show that $(1)-(2)$ in Assumption 4.1.1 hold within this discrete framework and obtain a result on the existence and uniqueness of the solution for the discrete problem.

Theorem 4.3.5. *Under (1)–(2) in Assumption 4.1.22 and (1)–(2) in Assumption 4.3.1, problem (4.4) has a unique generalized solution in* $[0,T]$. *Moreover*

$$\sup_{0\le t\le T} |u(t)|^2_{l^{0,2}(r)} + \int_0^T |u(t)|^2_{l^{1,2}(r,\rho)}\,dt \ \le\ N\,(|g_h)|^2_{l^{0,2}(r)} + \int_0^T |f_h(t)|^2_{l^{0,2}(r)}\,dt),$$

with N *a constant not depending on* h.

Proof Let $L_h(s) : l^{1,2}(r,\rho) \to (l^{1,2}(r,\rho))^*$. We define for all $\varphi, \psi \in l^{1,2}(r,\rho)$

$$\langle L_h(s)\psi, \varphi\rangle := -\,(a^{ij}(s)\partial_i^+\psi, \partial_j^+\varphi) + (b^i(s)\partial_i^+\psi - \partial_j^+ a^{ij}(s)\partial_i^+\psi, \varphi) + (c(s)\psi, \varphi).$$

We will prove that L_h satisfies the properties:

1. $|\langle L_h(s)\psi,\ \varphi\rangle| \le K|\psi|_{l^{1,2}(r,\rho)} \cdot |\varphi|_{l^{1,2}(r,\rho)},\ \ \forall\varphi,\psi \in l^{1,2}(r,\rho),\ K$ constant;
2. $\langle L_h(s)\psi,\ \psi\rangle \le K|\psi|_{l^{0,2}(r)} - \lambda|\psi|_{l^{1,2}(r,\rho)},\ \ \forall\psi \in l^{1,2}(r,\rho),\ \lambda > 0,\ K$ constants.

The first property follows from (2) in Assumption 4.1.22 and Cauchy-Schwarz inequality

$$\begin{aligned}
|\langle L_h(s)\psi,\ \varphi\rangle| &= |-\sum_x\sum_{i,j} r^2 a^{ij}(s)\partial_i^+\psi\ \partial_j^+\varphi\ h^d + \sum_x\sum_i r^2 b^i(s)\partial_i^+\psi\ \varphi\ h^d \\
&\quad -\sum_x\sum_{i,j} r^2\partial_j^+ a^{ij}(s)\partial_i^+\psi\ \varphi\ h^d + \sum_x r^2 c(s)\psi\ \varphi\ h^d| \\
&\le K\sum_x\sum_{i,j} r^2|\rho^2\partial_i^+\psi\ \partial_j^+\varphi|\ h^d + K\sum_x\sum_i r^2|\rho\partial_i^+\psi\ \varphi|\ h^d \\
&\quad +K\sum_x r^2|\psi\ \varphi|h^d \\
&\le K\sum_i |\rho\partial_i^+\psi|_{l^{0,2}(r)} \sum_j |\rho\partial_j^+\varphi|_{l^{0,2}(r)} \\
&\quad +K\sum_i |\rho\partial_i^+\psi|_{l^{0,2}(r)}\ |\varphi|_{l^{0,2}(r)} + K|\psi|_{l^{0,2}(r)}\ |\varphi|_{l^{0,2}(r)} \\
&\le K|\psi|_{l^{1,2}(r,\rho)} \cdot |\varphi|_{l^{1,2}(r,\rho)},
\end{aligned}$$

where the variable $x \in Z_h^d$ is omitted and $\sum_x$ denotes the summation over Z_h^d.

For the second property, with the same conventions, we have

$$\begin{aligned}\langle L_h(s)\psi,\psi\rangle =& -\sum_{i,j}\sum_x r^2 a^{ij}(s)\partial_i^+\psi\,\partial_j^+\psi\,h^d \\ &+\sum_i\sum_x r^2(b^i(s)-\partial_j^+a^{ij}(s))\partial_i^+\psi\,\psi\,h^d+\sum_x r^2c(s)\psi\,\psi\,h^d \\ \le& -\lambda\sum_i\sum_x r^2|\rho\partial_i^+\psi|^2h^d+2K\sum_i\sum_x r^2\rho|\partial_i^+\psi\,\psi|h^d+K\sum_x r^2|\psi|^2h^d \\ =& -\lambda\sum_i|\rho\partial_i^+\psi|^2_{l^{0,2}(r)}+2K\sum_i\sum_x r^2\rho|\partial_i^+\psi\,\psi|h^d+K|\psi|^2_{l^{0,2}(r)},\end{aligned}$$

owing to (1) and (2) in Assumption 4.1.22.

Applying the Cauchy's inequality to the second term of last expression, we obtain

$$\begin{aligned}&\langle L_h(s)\psi,\psi\rangle \\ &\le -\lambda\sum_i|\rho\partial_i^+\psi|^2_{l^{0,2}(r)}+\varepsilon K\sum_i\sum_x r^2|\rho\partial_i^+\psi|^2h^d+\frac{K}{\varepsilon}\sum_i\sum_x r^2|\psi|^2h^d+K|\psi|^2_{l^{0,2}(r)} \\ &= -\lambda\sum_i|\rho\partial_i^+\psi|^2_{l^{0,2}(r)}-\lambda|\psi|^2_{l^{0,2}(r)}+\varepsilon K\sum_i|\rho\partial_i^+\psi|^2_{l^{0,2}(r)}+\frac{K}{\varepsilon}|\psi|^2_{l^{0,2}(r)}+(K+\lambda)|\psi|^2_{l^{0,2}(r)} \\ &\le -\lambda|\psi|^2_{l^{1,2}(r,\rho)}+K|\psi|^2_{l^{0,2}(r)},\end{aligned}$$

with $\lambda > 0$, K constants, by taking ε sufficiently small.

Owing to Theorem 4.1.3 the result follows. □

We prove that the discrete differences approximate the partial derivatives.

Proposition 4.3.6. *Let $r > 0$ and $\rho > 0$ be functions on $\mathbb{R}^d$ and m be an integer strictly greater than $d/2$. Assume that (1)–(2) in Assumption 4.1.23 are satisfied, and that, additionally, $\rho(x) \ge C$ on $\mathbb{R}^d$, with $C > 0$ a constant. Let $u(t) \in W^{m+2,2}(r,\rho)$, $v(t) \in W^{m+3,2}(r,\rho)$, for all $t \in [0,T]$. Then there exists a constant N not depending on h such that*

1. $\sum_x r^2(x)|u_{x^i}(t,x)-\partial_i^+u(t,x)|^2\rho^2(x)h^d \le h^2N|u(t)|^2_{W^{m+2,2}(r,\rho)}$,
2. $\sum_x r^2(x)|v_{x^ix^j}(t,x)-\partial_j^-\partial_i^+v(t,x)|^2\rho^4(x)h^d \le h^2N|v(t)|^2_{W^{m+3,2}(r,\rho)}$,

for all $t \in [0,T]$, with $x \in Z_h^d$ and $\sum_x$ denoting the summation over Z_h^d.

Proof Let us prove (1). We consider the d-cells R_h, R_1^0 and the balls B_h and B_1^0 as defined in the proof of Proposition 4.2.11. In the same way as in this proof we obtain the inequality

$$
\begin{aligned}
&r^2(x_h)|u_{x^i}(t,x_h)-\partial_i^+u(t,x_h)|^2\rho^2(x_h)\\
&\le h^2\sup_{x\in R_1^0} r^2(x_h+hx)|u_{x^ix^i}(t,x_h+hx)|^2\rho^2(x_h+hx)\\
&\le h^2\sup_{x\in B_1^0} r^2(x_h+hx)|u_{x^ix^i}(t,x_h+hx)|^2\rho^2(x_h+hx).
\end{aligned}
$$

Taking in mind $(1)-(2)$ in Proposition 4.1.11, by Theorem 4.1.12 for $m>d/2$ we have

$$
\begin{aligned}
&\sup_{x\in B_1^0}|r(x_h+hx)u_{x^ix^i}(t,x_h+hx)\rho(x_h+hx)|^2\\
&\le N\sum_{|\alpha|\le m}\int_{B_1^0}|D_x^\alpha(r(x_h+hx)u_{x^ix^i}(t,x_h+hx)\rho(x_h+hx))|^2dx, \quad (4.16)
\end{aligned}
$$

with N a constant.

We note that

$$
\begin{aligned}
|D_x^\alpha(ru_{x^ix^i}\rho)| &= \Big|\sum_{\beta\le\alpha}\binom{\alpha}{\beta}D^\beta(r\rho)D_x^{\alpha-\beta}u_{x^ix^i}\Big|\\
&= \Big|\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\Big(\sum_{\gamma\le\beta}\binom{\beta}{\gamma}D^\gamma rD^{\beta-\gamma}\rho\Big)D_x^{\alpha-\beta}u_{x^ix^i}\Big|, \quad (4.17)
\end{aligned}
$$

where the arguments of r, ρ and $u_{x^ix^i}$ are omitted.

As, owing to Assumption 4.1.23,

$$
|D^\gamma r|\le Kr\rho^{-|\gamma|} \quad \text{and} \quad |D^{\beta-\gamma}\rho|\le K\rho^{1-(|\beta|-|\gamma|)},
$$

with K a constant, and then

$$
\Big|\sum_{\gamma\le\beta}\binom{\beta}{\gamma}D^\gamma rD^{\beta-\gamma}\rho\Big|\le N\sum_{\gamma\le\beta}\binom{\beta}{\gamma}r\rho^{-|\gamma|}\rho^{1-(|\beta|-|\gamma|)}\le Nr\rho^{1-|\beta|}, \quad (4.18)
$$

with N a constant.

Therefore, by (4.16), (4.17) and (4.18), we get

$$
\begin{aligned}
&\sup_{x\in B_1^0}|r(x_h+hx)u_{x^ix^i}(t,x_h+hx)\rho(x_h+hx)|^2\\
&\le N\sum_{|\alpha|\le m}\sum_{\beta\le\alpha}\int_{B_1^0}r^2(x_h+hx)|\rho^{1-|\beta|}(x_h+hx)|^2|D_x^{\alpha-\beta}u_{x^ix^i}(t,x_h+hx)|^2dx (4.19)
\end{aligned}
$$

Finally, owing to Hölder inequality and to the hypotheses over function ρ, we estimate the integral in (4.19)

$$\int_{B_1^0} r^2(x_h+hx)|\rho^{1-|\beta|}(x_h+hx)|^2 D_x^{\alpha-\beta}u_{x^ix^i}(t,x_h+hx)|^2dx$$
$$\leq N\int_{B_1^0} r^2(x_h+hx)|\rho^{2+|\alpha|-|\beta|}(x_h+hx)|^2 D_x^{\alpha-\beta}u_{x^ix^i}(t,x_h+hx)|^2dx$$
$$\cdot \sup_{x\in B_1^0}|\rho^{-1-|\alpha|}(x_h+hx)|^2$$
$$\leq N\int_{B_1^0} r^2(x_h+hx)|\rho^{2+|\alpha|-|\beta|}(x_h+hx)|^2 D_x^{\alpha-\beta}u_{x^ix^i}(t,x_h+hx)|^2dx. \quad (4.20)$$

Now, by (4.19) and (4.20),

$$\sup_{x\in B_1^0}|r(x_h+hx)u_{x^ix^i}(t,x_h+hx)\rho(x_h+hx)|^2$$
$$\leq N\sum_{|\alpha|\leq m}\sum_{\beta\leq\alpha}\int_{B_1^0} r^2(x_h+hx)|\rho^{|\alpha|-|\beta|}(x_h+hx)|D_x^{\alpha-\beta}u_{x^ix^i}(t,x_h+hx)|^2dx$$
$$\leq N\sum_{|\alpha|\leq m}\int_{B_1^0} r^2(x_h+hx)|\rho^{|\alpha|}(x_h+hx)|D_x^{\alpha}u_{x^ix^i}(t,x_h+hx)|^2dx$$
$$\leq N\sum_{|\alpha|\leq m+2}\int_{B_1^0} r^2(x_h+hx)|\rho^{|\alpha|}(x_h+hx)D_x^{\alpha}u(t,x_h+hx)|^2dx$$
$$= N\sum_{|\alpha|\leq m+2}\int_{B_h} r^2(x_h+hx)|\rho^{|\alpha|}(x_h+hx)D_x^{\alpha}u(t,x_h+hx)|^2h^{-d}h^{2|\alpha|}dx$$
$$\leq N\sum_{|\alpha|\leq m+2}\int_{B_h} r^2(x)|\rho^{|\alpha|}(x)D_x^{\alpha}u(t,x)|^2h^{-d}dx.$$

Finally,

$$\sum_{x\in Z_h^d} r^2(x)|u_{x^i}(t,x)-\partial_i^+u(t,x)|^2\rho^2(x)h^d$$
$$\leq Nh^2\sum_{|\alpha|\leq m+2}\sum_{x_h\in Z_h^d}\int_{B_h(x_h)} r^2(x)|\rho^{|\alpha|}(x)D_x^{\alpha}u(t,x)|^2dx$$
$$\leq Nh^2\sum_{|\alpha|\leq m+2}\sum_{x_h\in Z_h^d}\int_{R_h(x_h)} r^2(x)|\rho^{|\alpha|}(x)D_x^{\alpha}u(t,x)|^2dx \leq h^2N|u(t)|^2_{W^{m+2,2}(r,\rho)},$$

where $B_h(x_h) := B_h$, $R_h(x_h) := R_h$, and the proof for (1) is complete. The proof for (2) is similar. □

Next we determine a rate of convergence.

Theorem 4.3.7. *Assume that (1)–(2) in Assumption 4.1.23 are satisfied, and that, additionally, $\rho(x) \geq C$ on $\mathbb{R}^d$, with $C > 0$ a constant. Let m be an integer strictly greater than $d/2$ and denote u the solution of (4.3) in Theorem 4.1.26 and u_h the solution of (4.4) in Theorem 4.3.5. Assume that $u \in L^2([0,T]; W^{m+3,2}(r,\rho))$. Then*

$$\sup_{0\leq t\leq T} |u(t) - u_h(t)|^2_{l^{0,2}(r)} + \int_0^T |u(t) - u_h(t)|^2_{l^{1,2}(r,\rho)}\, dt$$
$$\leq h^2 N \int_0^T |u(t)|^2_{W^{m+3,2}(r,\rho)} dt + N(|g - g_h|^2_{l^{0,2}(r)} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}(r)}\, dt),$$

with N a constant independent of h.

Proof From (4.3) and (4.4), we obtain

$$\begin{cases} L_h(u-u_h) - \dfrac{d}{dt}(u-u_h) + (L - L_h)u + (f - f_h) = 0 \quad \text{in} \quad Q(h) \\ (u-u_h)(0,x) = (g - g_h)(x) \quad \text{in} \quad Z^d_h. \end{cases}$$

We have that $(f - f_h) \in L^2([0,T]; l^{0,2}(r))$ and $(g - g_h) \in l^{0,2}(r)$. Also if $u \in W^{m+3,2}(r,\rho)$ we have $(L - L_h)u \in L^2([0,T]; l^{0,2}(r))$. Then, by Theorem 4.3.5,

$$\sup_{0\leq t\leq T} |u(t) - u_h(t)|^2_{l^{0,2}(r)} + \int_0^T |u(t) - u_h(t)|^2_{l^{1,2}(r,\rho)}\, dt$$
$$\leq \; N(|g - g_h|^2_{l^{0,2}(r)} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}(r)}\, dt + \int_0^T |(L - L_h)u(t)|^2_{l^{0,2}(r)}\, dt).$$

As

$$\int_0^T |(L - L_h)u(t)|^2_{l^{0,2}(r)}\, dt$$
$$= \int_0^T |a^{ij}(t,x)(\frac{\partial^2}{\partial x^i \partial x^j} - \partial_j^- \partial_i^+)u(t,x) + b^i(t,x)(\frac{\partial}{\partial x^i} - \partial_i^+)u(t,x)|^2_{l^{0,2}(r)}\, dt,$$

owing to Proposition 4.3.6 and to the hypothesis over the coefficients, the result follows. □

Corollary 4.3.8. *Denote u the solution of (4.3) in Theorem 4.1.26 and u_h the solution of (4.4) in Theorem 4.3.5. Let the hypothesis of Theorem 4.3.7 be satisfied. If there is a constant N independent of h such that*

$$|g - g_h|^2_{l^{0,2}(r)} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}(r)}\, dt \;\leq\; h^2 N(|g|^2_{W^{m,2}(r,\rho)} + \int_0^T |f(t)|^2_{W^{m-1,2}(r,\rho)} dt),$$

then

$$\sup_{0\leq t\leq T} |u(t) - u_h(t)|^2_{l^{0,2}(r)} + \int_0^T |u(t) - u_h(t)|^2_{l^{1,2}(r,\rho)}\, dt$$
$$\leq h^2 N(\int_0^T |u(t)|^2_{W^{m+3,2}(r,\rho)} dt + |g|^2_{W^{m,2}(r,\rho)} + \int_0^T |f(t)|^2_{W^{m-1,2}(r,\rho)} dt).$$

Proof The result is an immediate consequence of Theorem 4.3.7. □

Chapter 5

Evolution equations in abstract spaces: time discretization

In Chapter 4, we studied the discretization in space of the second-order parabolic PDE problem in Sobolev and weighted Sobolev half spaces. In the present chapter, we will proceed to the discretization in time under a more general framework, using both the implicit and the explicit schemes. The approximation of the solution of the parabolic PDE problem will be given as an example.

5.1 Numerical approximation under a general framework

We consider the general framework we presented in Chapter 4 - Section 4.1.

Briefly, we consider the normal triple

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*,$$

with V a reflexive separable Banach space embedded continuously and densely into a Hilbert space H with inner product $(\ ,\ )$ and H^* and V^* the dual spaces of H and V, respectively. The notation $\langle\ ,\ \rangle$ is used for the duality. H^* is identified with H, by the help of the inner product.

We still consider the problem

$$L(t)u(t) - \frac{\partial u(t)}{\partial t} + f(t) = 0, \quad u(0) = g, \tag{5.1}$$

where $L(t)$ and $\partial/\partial t$ are linear operators from V to V^* for every $t \geq 0$, $f \in L^2([0,T];V^*)$, with $T \in (0,\infty)$, and $g \in H$, and make the same set of assumptions:

Assumption 5.1.1. There exist constants $\lambda > 0$, K, M, N, such that

1. $\langle L(t)v, v\rangle + \lambda|v|_V^2 \le K|v|_H^2$, $\forall v \in V$ and $\forall t \in [0,T]$;
2. $|L(t)v|_{V^*} \le M|v|_V$, $\forall v \in V$ and $\forall t \in [0,T]$:
3. $\int_0^T |f(t)|_{V^*}^2 dt \le N$ and $|g|_H \le N$.

Under Assumption 5.1.1, problem (5.1) has an unique weak solution

$$u \in C([0,T]; H) \cap L^2([0,T]; V)$$

on $[0,T]$, as stated in Theorem 4.1.3.

Implicit scheme.
Take a number $T \in (0,\infty)$, a non-negative integer n such that $T/n \in (0,1]$ and define the n-grid on $[0,T]$

$$T_n = \{t \in [0,T]: \ t = k\varepsilon, \ \ k = 0,1,\dots,n\}, \tag{5.2}$$

where $\varepsilon := T/n$. Denote $t_k = k\varepsilon$ for $k = 0,1,\dots,n$.

For all $z \in V$, we introduce the backward discrete difference in time

$$\Delta^- z(t_{j+1}) = \varepsilon^{-1}(z(t_{j+1}) - z(t_j)), \quad j = 0,1,\dots,n-1.$$

Let L_ε, f_ε be some time-discrete versions of L and f, respectively. $\forall z \in V$, denote $L_{\varepsilon,j+1}z = L_\varepsilon(t_{j+1})z$, $f_{\varepsilon,j+1} = f_\varepsilon(t_{j+1})$, $j = 0,1,\dots,n-1$.

For each $n \ge 1$ fixed, we define $v_j = v(t_j)$, $j = 0,1,\dots,n$, vectors in V satisfying

$$\Delta^- v_{i+1} = L_{\varepsilon,i+1}v_{i+1} + f_{\varepsilon,i+1} \quad \text{for } i = 0,1,\dots,n-1, \quad v_0 = g. \tag{5.3}$$

Problem (5.3) is a time-discrete version of problem (5.1).

Assumption 5.1.2.

1. $\langle L_{\varepsilon,j+1}v, v\rangle + \lambda|v|_V^2 \le K|v|_H^2$, $\forall v \in V$, $j = 0,1,\dots,n-1$
2. $|L_{\varepsilon,j+1}v|_{V^*} \le M|v|_V$, $\forall v \in V$, $j = 0,1,\dots,n-1$
3. $\sum_{j=0}^{n-1} |f_{\varepsilon,j+1}|_{V^*}^2 \varepsilon \le N$ and $|g|_H \le N$,

where λ, K, M and N are the constants in Assumption 5.1.1.

We have an existence and uniqueness result for the solution of problem (5.3).

Theorem 5.1.3. *Under Assumption 5.1.2,* $\forall n \in \mathbb{N}$ $\exists!$ $v_0, v_1, \ldots, v_n$ *in* V *satisfying (5.3).*

To prove this result, we consider a well known lemma even for a class of non-linear operators (see Zeidler [48]).

Lemma 5.1.4. *Let* $B : V \to V^*$ *be a bounded linear operator. Assume* $\exists \lambda > 0$ *such that* $\langle Bv, v \rangle \geq \lambda |v|_V^2$, $\forall v \in V$. *Then* $Bv = v^*$ *has a unique solution* $v \in V$ *for every given* $v^* \in V^*$.

We will prove now Theorem 5.1.3.

Proof From (5.3), we have $(I - \varepsilon\, L_{\varepsilon,1})v_1 = g + f_{\varepsilon,1}\varepsilon$ and $(I - \varepsilon\, L_{\varepsilon,i+1})v_{i+1} = v_i + f_{\varepsilon,i+1}\varepsilon$, for $i = 0, 1, \ldots, n-1$.

We first check that the operators $I - \varepsilon\, L_{\varepsilon,j+1}$, $j = 0, 1, \ldots, n-1$ satisfy the hypothesis of Lemma 5.1.4. These operators are obviously bounded. We have to show they satisfy: $\exists \lambda > 0$ such that $\langle (I - \varepsilon\, L_{\varepsilon,j+1})v, v \rangle \geq \lambda |v|_V^2$, $\forall v \in V$, $j = 0, 1, \ldots, n-1$. We have

$$
\begin{aligned}
\langle (I - \varepsilon\, L_{\varepsilon,j+1})v, v \rangle &= \langle Iv - \varepsilon\, L_{\varepsilon,j+1}v, v \rangle \\
&= |v|_H^2 - \varepsilon \langle L_{\varepsilon,j+1}v, v \rangle \\
&\geq |v|_H^2 - \varepsilon K |v|_H^2 + \varepsilon \lambda |v|_V^2,
\end{aligned}
$$

using Assumption 5.1.2. Then, with ε sufficiently small,

$$
\langle (I - \varepsilon\, L_{\varepsilon,j+1})v, v \rangle \geq \varepsilon \lambda |v|_V^2,
$$

and the hypothesis of Lemma 5.1.4 are satisfied.

Now, for v_1 we have $(I - \varepsilon\, L_{\varepsilon 1})v_1 = g + f_{\varepsilon 1}\varepsilon$. This equation has a unique solution by Lemma 5.1.4. Suppose now that equation $(I - \varepsilon\, L_{\varepsilon,i})v_i = v_{i-1} + f_{\varepsilon,i}\varepsilon$ has a unique solution. Then equation $(I - \varepsilon\, L_{\varepsilon,i+1})v_{i+1} = v_i + f_{\varepsilon,i+1}\varepsilon$ has also a unique solution, again by Lemma 5.1.4. The result is proved by induction. $\square$

Next result will be used to obtain the discrete version of Gronwall Lemma.

Lemma 5.1.5. *Let* $a_1^n, a_2^n, \ldots, a_n^n$ *be a finite sequence of numbers for every integer* $n \geq 1$ *such that*

$$
0 \leq a_j^n \leq c_0 + C \sum_{1 \leq i \leq j-1} a_i^n,
$$

for all $j = 1, 2, \ldots, n$, *where* C *is a positive constant and* $c_0 \geq 0$ *is some real number. Then*

$$
a_j^n \leq (C+1)^{j-1} c_0,
$$

for all $j = 1, 2, \ldots, n$.

Proof Let $b_j^n := c_0 + C\sum_{1\le i\le j-1} b_i^n,\ \ j = 1,2,\ldots,n$. Then $a_j^n \le b_j^n$ for all $j \ge 1$. Indeed for $j = 1$ we have $a_1^n \le b_1^n = c_0$. Assume that $a_i^n \le b_i^n$ for all $i \le j$. Then

$$b_{j+1}^n = c_0 + C\sum_{1\le i\le j} b_i^n \ge c_0 + C\sum_{1\le i\le j} a_i^n \ge a_{j+1}^n,$$

which proves by induction that $a_j^n \le b_j^n$ for all $j \ge 1$. It is easy to see that $b_{j+1}^n - b_j^n = Cb_j^n,\ \ j \ge 1$, which gives

$$a_{j+1}^n \le b_{j+1}^n = (C+1)b_j^n = (C+1)^2\, b_{j-1}^n = \ldots = (C+1)^j\, b_1^n = (C+1)^j\, c_0,$$

and the result is proved. □

Corollary 5.1.6. (Discrete Gronwall Lemma). *Let $a_0^n, a_1^n, \ldots, a_n^n$ be a finite sequence of numbers for every integer $n \ge 1$ such that*

$$0 \le a_j^n \le a_0^n + K\sum_{1\le i\le j} a_i^n \varepsilon, \tag{5.4}$$

holds for every $j = 1,2,\ldots,n$, with $\varepsilon := T/n$ and K a positive number such that $K\varepsilon =: q < 1$, with q a fixed constant. Then

$$a_j^n \le a_0^n e^{K_q T},$$

for all integers $n \ge 1$ and $j \in \{1,2,\ldots,n\}$, where $K_q := -K\ln(1-q)/q$.

Proof From (5.4), as $K\varepsilon < 1$ for $j = 1,2,\ldots,n$ we have

$$(1-K\varepsilon)a_j^n \le a_0^n + K\sum_{1\le i\le j-1} a_i^n\varepsilon \Leftrightarrow a_j^n \le \frac{a_0^n}{1-K\varepsilon} + \frac{K\varepsilon}{1-K\varepsilon}\sum_{1\le i\le j-1} a_i^n.$$

Applying Lemma 5.1.5 to the previous inequality with

$$c_0 = \frac{a_0^n}{1-K\varepsilon} \quad \text{and} \quad C = \frac{K\varepsilon}{1-K\varepsilon}$$

we obtain

$$a_j^n \le \left(\frac{K\varepsilon}{1-K\varepsilon}+1\right)^{j-1}\frac{a_0^n}{1-K\varepsilon} = \frac{a_0^n}{(1-K\varepsilon)^j} \le \frac{a_0^n}{(1-K\varepsilon)^n}$$

and, noting that

$$(1-K\varepsilon)^n = \exp(n\ln(1-K\varepsilon)) = \exp\left(nK\varepsilon\frac{\ln(1-q)}{q}\right) = \exp\left(KT\frac{\ln(1-q)}{q}\right),$$

the result is proved. □

We have an estimate for the solution of the discrete problem (5.3).

Theorem 5.1.7. *Let* $v_{\varepsilon,j}$ *, with* $j = 0, 1, \ldots, n$ *be the unique solution of problem (5.3). Let Assumption 5.1.2 be verified and assume the constant* K *in Assumption 5.1.1 satisfies:* $2K\varepsilon < 1$*. Then there exists a constant* N *independent of* ε *such that*

1. $\sup_{n\geq 1} \max_{0\leq j\leq n} |v_{\varepsilon,j}|_H^2 \leq N$;
2. $\sup_{n\geq 1} \sum_{0\leq j\leq n} |v_{\varepsilon,j}|_V^2 \varepsilon \leq N$.

Proof For $i = 0, 1, \ldots, n-1$, we have

$$|v_{\varepsilon,i+1}|_H^2 - |v_{\varepsilon,i}|_H^2 = 2\left\langle v_{\varepsilon,i+1}, v_{\varepsilon,i+1} - v_{\varepsilon,i}\right\rangle - |v_{\varepsilon,i+1} - v_{\varepsilon,i}|_H^2. \tag{5.5}$$

Summing up both members of equation (5.5) we obtain, for $j = 1, 2, \ldots, n$,

$$|v_{\varepsilon,j}|_H^2 = |v_{\varepsilon,0}|_H^2 + \sum_{i=0}^{j-1} 2\left\langle v_{\varepsilon,i+1}, v_{\varepsilon,i+1} - v_{\varepsilon,i}\right\rangle - \sum_{i=0}^{j-1} |v_{\varepsilon,i+1} - v_{\varepsilon,i}|_H^2.$$

Hence

$$\begin{aligned} |v_{\varepsilon,j}|_H^2 &\leq |v_{\varepsilon,0}|_H^2 + \sum_{i=0}^{j-1} 2\left\langle v_{\varepsilon,i+1}, v_{\varepsilon,i+1} - v_{\varepsilon,i}\right\rangle \\ &= |v_{\varepsilon,0}|_H^2 + \sum_{i=0}^{j-1} 2\left\langle v_{\varepsilon,i+1}, L_{\varepsilon,i+1} v_{\varepsilon,i+1}\varepsilon + f_{\varepsilon,i+1}\varepsilon\right\rangle. \end{aligned}$$

As, by Cauchy's inequality,

$$2\langle v_{\varepsilon,i+1}, f_{\varepsilon,i+1}\rangle\varepsilon \leq \lambda |v_{\varepsilon,i+1}|_V^2 \varepsilon + \frac{1}{\lambda}|f_{\varepsilon,i+1}|_{V^*}^2 \varepsilon,$$

with $\lambda > 0$, owing to Assumption 5.1.2 we have

$$|v_{\varepsilon,j}|_H^2 \leq |v_{\varepsilon,0}|_H^2 + 2K\sum_{i=0}^{j-1} |v_{\varepsilon,i+1}|_H^2 \varepsilon - \lambda \sum_{i=0}^{j-1} |v_{\varepsilon,i+1}|_V^2 \varepsilon + \frac{1}{\lambda}\sum_{i=0}^{j-1} |f_{\varepsilon,i+1}|_{V^*}^2 \varepsilon.$$

Hence

$$|v_{\varepsilon,j}|_H^2 + \lambda \sum_{1\leq i\leq j} |v_{\varepsilon,i}|_V^2 \varepsilon \leq |v_{\varepsilon,0}|_H^2 + 2K \sum_{1\leq i\leq j} |v_{\varepsilon,i}|_H^2 \varepsilon + \frac{1}{\lambda}\sum_{1\leq i\leq n} |f_{\varepsilon,i}|_{V^*}^2 \varepsilon. \tag{5.6}$$

In particular

$$|v_{\varepsilon,j}|_H^2 \leq |v_{\varepsilon,0}|_H^2 + 2K \sum_{1\leq i\leq j} |v_{\varepsilon,i}|_H^2 \varepsilon + \frac{1}{\lambda}\sum_{1\leq i\leq n} |f_{\varepsilon,i}|_{V^*}^2 \varepsilon, \tag{5.7}$$

and, using Corollary 5.1.6,

$$|v_{\varepsilon,j}|_H^2 \leq \left(|v_{\varepsilon,0}|_H^2 + \frac{1}{\lambda}\sum_{1\leq i\leq n} |f_{\varepsilon,i}|_{V^*}^2 \varepsilon\right) e^{2K_q T}, \tag{5.8}$$

where K_q is the constant defined in Corollary 5.1.6. We have proved (1).

From (5.6), (5.7) and (5.8) we obtain

$$|v_{\varepsilon,j}|_H^2 + \lambda \sum_{1\le i\le j} |v_{\varepsilon,i}|_V^2 \varepsilon \le \left(|v_{\varepsilon,0}|_H^2 + \frac{1}{\lambda} \sum_{1\le i\le n} |f_{\varepsilon,i}|_{V^*}^2 \varepsilon \right) e^{2K_qT}$$

and

$$\sum_{1<i\le j} |v_{\varepsilon,i}|_V^2 \varepsilon \le \left(|v_{\varepsilon,0}|_H^2 + \frac{1}{\lambda} \sum_{1\le i\le n} |f_{\varepsilon,i}|_{V^*}^2 \varepsilon \right) \frac{1}{\lambda}\, e^{2K_qT},$$

and (2) is proved. □

We now determine the rate of convergence for the scheme we constructed. Let $u = u(t),\ t \in [0,T]$ with $T \in (0,\infty)$, be the solution of problem (5.1), where u is a weakly continuous function of t with values in V.

Assumption 5.1.8. Let u be the unique solution of problem (5.1). There exist a fixed number $\delta \in (0,1]$ and a constant C such that

$$\frac{1}{\varepsilon} \int_{t_i}^{t_{i+1}} |u(t_{i+1}) - u(s)|_V ds \le C|\varepsilon|^{\delta/2},$$

for all $i = 0, 1, \ldots, n-1$.

Remark 5.1.9. Assume that u satisfies the following condition: "There exists a fixed number $\delta \in (0,1]$ and a constant C such that

$$|u(t) - u(s)|_V \le C|t-s|^{\delta/2}, \ \ \forall s,t \in [0,T]."$$

Then Assumption 5.1.8 obviously holds.

Theorem 5.1.10. *Let $u(t)$ and $v_{\varepsilon,j}$, $j = 0,1,\ldots,n$, be the unique solutions of (5.1) and (5.3), respectively. Let Assumptions 5.1.1, 5.1.2 and 5.1.8 be verified. Assume the constant K in Assumption 5.1.1 satisfies: $2K\varepsilon < 1$. Then there exists a constant N independent of ε such that*

1. $\max_{0\le j\le n} |v_{\varepsilon,j} - u(t_j)|_H^2 \le N|\varepsilon|^\delta + N\sum_{1\le j\le n} \frac{1}{\varepsilon} |L_{\varepsilon,j}u(t_j)\varepsilon - \int_{t_{j-1}}^{t_j} L(s)u(t_j)ds|_{V^*}^2$

$+N\sum_{1\le j\le n} \frac{1}{\varepsilon} |f_{\varepsilon,j}\varepsilon - \int_{t_{j-1}}^{t_j} f(s)ds|_{V^*}^2;$

2. $\sum_{0\le j\le n} |v_{\varepsilon,j} - u(t_j)|_V^2\, \varepsilon \le N|\varepsilon|^\delta + N\sum_{1\le j\le n} \frac{1}{\varepsilon} |L_{\varepsilon,j}u(t_j)\varepsilon - \int_{t_{j-1}}^{t_j} L(s)u(t_j)ds|_{V^*}^2$

$+N\sum_{1\le j\le n} \frac{1}{\varepsilon} |f_{\varepsilon,j}\varepsilon - \int_{t_{j-1}}^{t_j} f(s)ds|_{V^*}^2.$

Proof Define $w(t_i) := v_{\varepsilon,i} - u(t_i)$, $i = 0, 1, \dots, n$. For $i = 0, 1, \dots, n-1$ we have

$$\begin{aligned} w(t_{i+1}) - w(t_i) &= L_{\varepsilon,i+1} w(t_{i+1})\varepsilon + f_{\varepsilon,i+1}\varepsilon - u(t_{i+1}) + u(t_i) + L_{\varepsilon,i+1} u(t_{i+1})\varepsilon \\ &= L_{\varepsilon,i+1} w(t_{i+1})\varepsilon + \varphi(t_{i+1}), \end{aligned}$$

denoting $\varphi(t_{i+1}) := f_{\varepsilon,i+1}\varepsilon - u(t_{i+1}) + u(t_i) + L_{\varepsilon,i+1} u(t_{i+1})\varepsilon$.

We then have

$$\begin{aligned} |w(t_{i+1})|_H^2 - |w(t_i)|_H^2 &= 2\langle w(t_{i+1}), w(t_{i+1}) - w(t_i)\rangle - |w(t_{i+1}) - w(t_i)|_H^2 \\ &\le 2\langle w(t_{i+1}), L_{\varepsilon,i+1} w(t_{i+1})\rangle\varepsilon + 2\langle w(t_{i+1}), \varphi(t_{i+1})\rangle \\ &\le -2\lambda |w(t_{i+1})|_V^2 \varepsilon + 2K |w(t_{i+1})|_H^2 \varepsilon \\ &\quad + 2|\langle w(t_{i+1}), \varphi(t_{i+1})\rangle|, \end{aligned} \tag{5.9}$$

owing to Assumption 5.1.2.

Noting that $\varphi(t_{i+1})$ can be written

$$\varphi(t_{i+1}) = \int_{t_i}^{t_{i+1}} L(s)(u(t_{i+1}) - u(s))ds + \varphi_1(t_{i+1}) + \varphi_2(t_{i+1}),$$

where $\varphi_1(t_{i+1}) := L_{\varepsilon,i+1} u(t_{i+1})\varepsilon - \int_{t_i}^{t_{i+1}} L(s) u(t_{i+1}) ds$ and $\varphi_2(t_{i+1}) := f_{\varepsilon,i+1}\varepsilon - \int_{t_i}^{t_{i+1}} f(s) ds$, for the last term in (5.9) we have

$$\begin{aligned} 2|\langle w(t_{i+1}), \varphi(t_{i+1})\rangle| \le\, & 2\left|\left\langle w(t_{i+1}), \int_{t_i}^{t_{i+1}} L(s)(u(t_{i+1}) - u(s))ds\right\rangle\right| \\ & + 2|\langle w(t_{i+1}), \varphi_1(t_{i+1})\rangle| + 2|\langle w(t_{i+1}), \varphi_2(t_{i+1})\rangle|. \end{aligned} \tag{5.10}$$

For the term $2|\langle w(t_{i+1}), \int_{t_i}^{t_{i+1}} L(s)(u(t_{i+1}) - u(s))ds\rangle|$ in (5.10), we have

$$\begin{aligned} & 2\left|\left\langle w(t_{i+1}), \int_{t_i}^{t_{i+1}} L(s)(u(t_{i+1}) - u(s))ds\right\rangle\right| \\ & \le 2\int_{t_i}^{t_{i+1}} |\langle w(t_{i+1}), L(s)(u(t_{i+1}) - u(s))\rangle| ds \\ & \le 2M |w(t_{i+1})|_V \int_{t_i}^{t_{i+1}} |u(t_{i+1}) - u(s)|_V ds \\ & \le \frac{\lambda}{3} |w(t_{i+1})|_V^2 \varepsilon + \frac{3M^2}{\lambda\varepsilon} \left(\int_{t_i}^{t_{i+1}} |u(t_{i+1}) - u(s)|_V ds \right)^2, \end{aligned} \tag{5.11}$$

with $\lambda > 0$, using Assumption 5.1.1 and Cauchy's inequality.

For the terms $2|\langle w(t_{i+1}), \varphi_1(t_{i+1})\rangle|$ and $2|\langle w(t_{i+1}), \varphi_2(t_{i+1})\rangle|$ in (5.10), we have

$$2|\langle w(t_{i+1}), \varphi_1(t_{i+1})\rangle| \le \frac{\lambda}{3} |w(t_{i+1})|_V^2 \varepsilon + \frac{3}{\lambda\varepsilon} |\varphi_1(t_{i+1})|_{V^*}^2, \tag{5.12}$$

and

$$2|\langle w(t_{i+1}), \varphi_2(t_{i+1})\rangle| \;\leq\; \frac{\lambda}{3}|w(t_{i+1})|_V^2\varepsilon + \frac{3}{\lambda\varepsilon}|\varphi_2(t_{i+1})|_{V^*}^2, \tag{5.13}$$

with $\lambda > 0$, using Cauchy's inequality.

From (5.10), (5.11), (5.12) and (5.13) we have

$$\begin{aligned} 2|\langle w(t_{i+1}), \varphi(t_{i+1})\rangle| \;\leq\; & \lambda|w(t_{i+1})|_V^2\varepsilon + \frac{3M^2}{\lambda\varepsilon}\left(\int_{t_i}^{t_{i+1}}|u(t_{i+1}) - u(s)|_V ds\right)^2 \\ & + \frac{3}{\lambda\varepsilon}|\varphi_1(t_{i+1})|_{V^*}^2 + \frac{3}{\lambda\varepsilon}|\varphi_2(t_{i+1})|_{V^*}^2. \end{aligned} \tag{5.14}$$

Putting together estimates (5.9) and (5.14) and using Assumption 5.1.8, we have

$$\begin{aligned} |w(t_{i+1})|_H^2 - |w(t_i)|_H^2 \;\leq\; & -\lambda|w(t_{i+1})|_V^2\varepsilon + 2K|w(t_{i+1})|_H^2\varepsilon + \frac{3M^2}{\lambda}|\varepsilon|^{\delta+1} \\ & + \frac{3}{\lambda\varepsilon}|\varphi_1(t_{i+1})|_{V^*}^2 + \frac{3}{\lambda\varepsilon}|\varphi_2(t_{i+1})|_{V^*}^2. \end{aligned}$$

Summing up, we have, for $j = 1, 2, \ldots, n$,

$$\begin{aligned} |w(t_j)|_H^2 + \lambda\sum_{i=0}^{j-1}|w(t_{i+1})|_V^2\varepsilon \;\leq\; & 2K\sum_{i=0}^{j-1}|w(t_{i+1})|_H^2\varepsilon + \frac{3M^2}{\lambda}\sum_{i=0}^{j-1}|\varepsilon|^{\delta+1} \\ & + \frac{3}{\lambda\varepsilon}\sum_{i=0}^{j-1}|\varphi_1(t_{i+1})|_{V^*}^2 + \frac{3}{\lambda\varepsilon}\sum_{i=0}^{j-1}|\varphi_2(t_{i+1})|_{V^*}^2. \end{aligned}$$

Hence

$$\begin{aligned} |w(t_j)|_H^2 + \lambda\sum_{1\leq i\leq j}|w(t_i)|_V^2\varepsilon \;\leq\; & 2K\sum_{1\leq i\leq j}|w(t_i)|_H^2\varepsilon + N\,|\varepsilon|^\delta \\ & + N\sum_{1\leq i\leq n}\frac{1}{\varepsilon}\,|L_{\varepsilon,i}u(t_i)\varepsilon - \int_{t_{i-1}}^{t_i}L(s)u(t_i)ds|_{V^*}^2 \\ & + N\sum_{1\leq i\leq n}\frac{1}{\varepsilon}\,|f_{\varepsilon,i}\varepsilon - \int_{t_{i-1}}^{t_i}f(s)ds|_{V^*}^2, \end{aligned} \tag{5.15}$$

with N a constant.

In particular

$$\begin{aligned} |w(t_j)|_H^2 \;\leq\; & 2K\sum_{1\leq i\leq j}|w(t_i)|_H^2\varepsilon + N|\varepsilon|^\delta \\ & + N\sum_{1\leq i\leq n}\frac{1}{\varepsilon}\,|L_{\varepsilon,i}u(t_i)\varepsilon - \int_{t_{i-1}}^{t_i}L(s)u(t_i)ds|_{V^*}^2 \\ & + N\sum_{1\leq i\leq n}\frac{1}{\varepsilon}\,|f_{\varepsilon,i}\varepsilon - \int_{t_{i-1}}^{t_i}f(s)ds|_{V^*}^2, \end{aligned} \tag{5.16}$$

and, using Corollary 5.1.6,

$$\begin{aligned}|w(t_j)|_H^2 &\leq Ne^{2K_qT}|\varepsilon|^\delta \\ &+Ne^{2K_qT}\sum_{1\leq i\leq n}\frac{1}{\varepsilon}|L_{\varepsilon,i}u(t_i)\varepsilon-\int_{t_{i-1}}^{t_i}L(s)u(t_i)ds|_{V^*}^2 \\ &+Ne^{2K_qT}\sum_{1\leq i\leq n}\frac{1}{\varepsilon}|f_{\varepsilon,i}\varepsilon-\int_{t_{i-1}}^{t_i}f(s)ds|_{V^*}^2, \qquad (5.17)\end{aligned}$$

with K_q the constant defined in Corollary 5.1.6. We have proved (1).

From (5.15), (5.16) and (5.17) we obtain

$$\begin{aligned}|w(t_j)|_H^2+\lambda\sum_{1\leq i\leq j}|w(t_i)|_V^2\varepsilon &\leq Ne^{2K_qT}|\varepsilon|^\delta \\ &+Ne^{2K_qT}\sum_{1\leq i\leq n}\frac{1}{\varepsilon}|L_{\varepsilon,i}u(t_i)\varepsilon-\int_{t_{i-1}}^{t_i}L(s)u(t_i)ds|_{V^*}^2 \\ &+Ne^{2K_qT}\sum_{1\leq i\leq n}\frac{1}{\varepsilon}|f_{\varepsilon,i}\varepsilon-\int_{t_{i-1}}^{t_i}f(s)ds|_{V^*}^2,\end{aligned}$$

and

$$\begin{aligned}\sum_{1\leq i\leq j}|w(t_i)|_V^2\varepsilon &\leq \frac{N}{\lambda}e^{2K_qT}|\varepsilon|^\delta \\ &+\frac{N}{\lambda}e^{2K_qT}\sum_{1\leq i\leq n}\frac{1}{\varepsilon}|L_{\varepsilon,i}u(t_i)\varepsilon-\int_{t_{i-1}}^{t_i}L(s)u(t_i)ds|_{V^*}^2 \\ &+\frac{N}{\lambda}e^{2K_qT}\sum_{1\leq i\leq n}\frac{1}{\varepsilon}|f_{\varepsilon,i}\varepsilon-\int_{t_{i-1}}^{t_i}f(s)ds|_{V^*}^2,\end{aligned}$$

and (2) is proved. □

Corollary 5.1.11. *Let $u(t)$ and $v_{\varepsilon,j}$, $j=0,1,\ldots,n$, be the unique solutions of (5.1) and (5.3), respectively. Assume the hypothesis of Theorem 5.1.10 are verified. If there exists a constant N independent of ε such that*

$$|L_{\varepsilon,j}u(t_j)-\frac{1}{\varepsilon}\int_{t_{j-1}}^{t_j}L(s)u(t_j)ds|_{V^*}^2+|f_{\varepsilon,j}-\frac{1}{\varepsilon}\int_{t_{j-1}}^{t_j}f(s)ds|_{V^*}^2\leq N|\varepsilon|^\delta,\ j=1,2,\ldots,n,$$

then

1. $\max_{0\leq j\leq n}|v_{\varepsilon,j}-u(t_j)|_H^2\leq N|\varepsilon|^\delta$;
2. $\sum_{0\leq j\leq n}|v_{\varepsilon,j}-u(t_j)|_V^2\varepsilon\leq N|\varepsilon|^\delta$.

Proof The result follows immediately from Theorem 5.1.10. □

We consider briefly the particular case where the operators L and f in problem (5.1) are approximated in time respectively by

$$\forall z \in V,\ \bar{L}_\varepsilon(t_{j+1})z := \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} L(s)zds \quad \text{and} \quad \bar{f}_\varepsilon(t_{j+1}) := \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} f(s)ds,$$

for $j = 0, 1, \ldots, n-1$.

We denote

$$\bar{L}_{\varepsilon,j+1}z = \bar{L}_\varepsilon(t_{j+1})z, \quad \bar{f}_{\varepsilon,j+1} = \bar{f}_\varepsilon(t_{j+1}), \quad j = 0, 1, \ldots, n-1.$$

We have now the particular time-discrete version of problem (5.1)

$$\Delta^- v_{i+1} = \bar{L}_{\varepsilon,i+1}v_{i+1} + \bar{f}_{\varepsilon,i+1} \quad \text{for } i = 0, 1, \ldots, n-1, \quad v_0 = g \tag{5.18}$$

with $n \geq 1$.

The following result holds:

Lemma 5.1.12. *Under Assumption 5.1.1 the operators $\bar{L}_\varepsilon$ and $\bar{f}_\varepsilon$ satisfy*

1. $\langle \bar{L}_{\varepsilon,j+1}v,\ v\rangle + \lambda|v|_V^2 \leq K|v|_H^2, \quad \forall v \in V, \ j = 0, 1, \ldots, n-1,$
2. $|\bar{L}_{\varepsilon,j+1}v|_{V^*} \leq M|v|_V, \quad \forall v \in V, \ j = 0, 1, \ldots, n-1,$
3. $\sum_{j=0}^{n-1} |\bar{f}_{\varepsilon,j+1}|_{V^*}^2 \varepsilon \leq N,$

where λ, K, M and N are the constants in Assumption 5.1.1.

Proof $\forall v \in V, \ j = 0, 1, \ldots, n-1$, we have

$$\begin{aligned}
\langle \bar{L}_{\varepsilon,j+1}v,\ v\rangle = \langle \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} L(s)vds,\ v\rangle &= \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} \langle L(s)v,\ v\rangle ds \\
&\leq \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} (K|v|_H^2 - \lambda|v|_V^2)ds \\
&= K|v|_H^2 - \lambda|v|_V^2,
\end{aligned}$$

using Assumption 5.1.1, and (1) is proved. For (2), $\forall v \in V, \ j = 0, 1, \ldots, n-1$, we have

$$\begin{aligned}
|\bar{L}_{\varepsilon,j+1}v|_{V^*} = |\frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} L(s)vds|_{V^*} &\leq \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} |L(s)v|_{V^*}ds \\
&\leq \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} M|v|_V ds = M|v|_V,
\end{aligned}$$

using again Assumption 5.1.1. For (3),

$$\sum_{j=0}^{n-1} |\bar{f}_{\varepsilon,j+1}|^2_{V^*}\varepsilon = \sum_{j=0}^{n-1} |\frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} f(s)ds|^2_{V^*}\varepsilon \leq \sum_{j=0}^{n-1} \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} |f(s)|^2_{V^*}ds\ \varepsilon$$
$$= \int_0^T |f(s)|^2_{V^*}ds \leq N,$$

using Assumption 5.1.1 and Jensen's inequality. The result is proved. □

By the previous result we have that the operators $\bar{L}_\varepsilon$ and $\bar{f}_\varepsilon$, satisfy Assumption 5.1.2.

Next two results are corollaries of Theorems 5.1.7 and 5.1.10, respectively.

Corollary 5.1.13. *Let $v_{\varepsilon,j}$, with $j = 0, 1, \dots, n$ be the unique solution of problem (5.18). Let the hypothesis of Lemma 5.1.12, be verified and assume the constant K in Assumption 5.1.1 satisfies: $2K\varepsilon < 1$. Then there exists a constant N independent of ε such that*

1. $\sup_{n\geq 1} \max_{0\leq j\leq n} |v_{\varepsilon,j}|^2_H \leq N|g|^2_H$;
2. $\sup_{n\geq 1} \sum_{0\leq j\leq n} |v_{\varepsilon,j}|^2_V\varepsilon \leq N|g|^2_H$.

Proof The result is an immediate consequence of Theorem 5.1.7. □

Corollary 5.1.14. *Let $u(t)$ and $v_{\varepsilon,j}$, $j = 0, 1, \dots, n$, be the unique solutions of (5.1) and (5.18) respectively. Let Assumption 5.1.8 and the hypothesis of Lemma 5.1.12 be verified. Assume the constant K in Assumption 5.1.1 satisfies: $2K\varepsilon < 1$. Then there exists a constant N independent of ε such that*

1. $\max_{0\leq j\leq n} |v_{\varepsilon,j} - u(t_j)|^2_H \leq N|\varepsilon|^\delta$;
2. $\sum_{0\leq j\leq n} |v_{\varepsilon,j} - u(t_j)|^2_V\ \varepsilon \leq N|\varepsilon|^\delta$.

Proof The result follows immediately from Theorem 5.1.10. □

Explicit scheme.

We consider a particular case of problem (5.1)

$$L_h(t)u(t) - \frac{\partial u(t)}{\partial t} + f_h(t) = 0, \quad u(0) = g_h, \tag{5.19}$$

in the spaces V_h and H_h, space-discrete versions of V and H, and with $L_h(t)$, $f_h(t)$ and g_h space-discrete versions of $L(t)$, $f(t)$ and g, respectively.

For the discretization, we consider the time-grid T_n as defined in (5.2). For all $z \in V_h$, we introduce the forward discrete difference in time

$$\Delta^+ z(t_j) = \varepsilon^{-1}(z(t_{j+1}) - z(t_j)), \quad j = 0, 1, \ldots, n-1.$$

Let $L_{h\varepsilon}$, $f_{h\varepsilon}$ be some time-discrete versions of L_h and f_h, respectively and denote

$$\forall z \in V_h, \;\; L_{h\varepsilon,j+1} z = L_{h\varepsilon}(t_{j+1})z, \;\; f_{h\varepsilon,j+1} = f_{h\varepsilon}(t_{j+1}), \quad j = 0, 1, \ldots, n-1.$$

For each $n \geq 1$ fixed, we consider the time-discrete version of (5.19),

$$\Delta^+ v_i = L_{h\varepsilon,i}\, v_i + f_{h\varepsilon,i} \;\text{ for }\; i = 0, 1, \ldots, n-1, \quad v_0 = g_h, \tag{5.20}$$

with $v_j = v(t_j)$, $j = 0, 1, \ldots, n$, vectors in V_h.

Problem (5.20) can be solved uniquely by recursion

$$v_j = g_h + \sum_{i=0}^{j-1} L_{h\varepsilon,i}\, v_i \varepsilon + \sum_{i=0}^{j-1} f_{h\varepsilon,i}\varepsilon \;\text{ for }\; j = 1, \ldots, n, \quad v_0 = g_h.$$

We make some assumptions.

Assumption 5.1.15.

1. $\langle L_{h\varepsilon,j} v,\ v\rangle_h + \lambda |v|^2_{V_h} \leq K|v|^2_{H_h}, \quad \forall v \in V_h, \;\; j = 0, 1, \ldots, n-1$
2. $|L_{h\varepsilon,j} v|_{V_h^*} \leq M|v|_{V_h}, \quad \forall v \in V_h, \;\; j = 0, 1, \ldots, n-1$
3. $\sum_{j=0}^{n-1} |f_{h\varepsilon,j}|^2_{V_h^*}\varepsilon \leq N$ and $|g_h|_{H_h} \leq N$,

where λ, K, M and N are the constants in Assumption 5.1.1.

We have a version of the discrete Gronwall Lemma:

Lemma 5.1.16. *Let $a_0^n, a_1^n, \ldots, a_n^n$ be a finite sequence of numbers for every integer $n \geq 1$ such that*

$$0 \leq a_j^n \leq a_0^n + K \sum_{0 \leq i \leq j-1} a_i^n \varepsilon, \tag{5.21}$$

holds for every $j = 0, 1, \ldots, n$, with $\varepsilon := T/n$ and K a positive number such that $K\varepsilon =: q < 1$, with q a fixed constant. Then

$$a_j^n \leq a_0^n e^{K_q T},$$

for all integers $n \geq 1$ and $j \in \{0, 1, \ldots, n\}$, where $K_q := -K\ln(1-q)/q$.

Proof The result is a consequence of Corollary 5.1.6.

From (5.21), for $j = 1, 2, \ldots, n,$ we have

$$(1 + K\varepsilon)a_j^n \leq (1 + K\varepsilon)a_0^n + K \sum_{1\leq i\leq j} a_i^n \varepsilon \leq (1 + K\varepsilon)a_0^n e^{K_q T},$$

owing to Corollary 5.1.6. The result follows. □

Assumption 5.1.17. There exists a constant C_h, dependent of the space-step h, such that $|w|_{H_h} \leq C_h |w|_{V_h^*} \quad \forall w \in V_h$.

We give an estimate for the solution of problem (5.20).

Theorem 5.1.18. *Let $v_{h\varepsilon,j}$, with $j = 0, 1, \ldots, n$ be the unique solution of problem (5.20). Let Assumptions 5.1.15 and 5.1.17 be verified and λ, K, M, C_h the constants defined in Assumptions 5.1.1 and 5.1.17. Assume the constant K satisfies: $2K\varepsilon < 1$. If there exists a number p such that $M^2 C_h^2 \varepsilon \leq p < \lambda$ then there exists a constant N, independent of ε and h, such that*

1. $\sup_{n\geq 1} \max_{0\leq j\leq n} |v_{h\varepsilon,j}|^2_{H_h} \leq N$;
2. $\sup_{n\geq 1} \sum_{0\leq j\leq n} |v_{h\varepsilon,j}|^2_{V_h} \varepsilon \leq N$.

Proof For $i = 0, 1, \ldots, n - 1$ we have

$$|v_{h\varepsilon,i+1}|^2_{H_h} - |v_{h\varepsilon,i}|^2_{H_h} = 2 \langle v_{h\varepsilon,i}, v_{h\varepsilon,i+1} - v_{h\varepsilon,i}\rangle_h + |v_{h\varepsilon,i+1} - v_{h\varepsilon,i}|^2_{H_h}. \tag{5.22}$$

Summing up both members of equation (5.22) we obtain, for $j = 1, 2, \ldots, n$,

$$\begin{aligned}
|v_{h\varepsilon,j}|^2_{H_h} &= |v_{h\varepsilon,0}|^2_{H_h} + \sum_{i=0}^{j-1} 2\langle v_{h\varepsilon,i}, v_{h\varepsilon,i+1} - v_{h\varepsilon,i}\rangle_h + \sum_{i=0}^{j-1} |v_{h\varepsilon,i+1} - v_{h\varepsilon,i}|^2_{H_h} \\
&= |v_{h\varepsilon,0}|^2_{H_h} + \sum_{i=0}^{j-1} 2\langle v_{h\varepsilon,i}, L_{h\varepsilon,i} v_{h\varepsilon,i} + f_{h\varepsilon,i}\rangle_h \varepsilon + \sum_{i=0}^{j-1} |L_{h\varepsilon,i} v_{h\varepsilon,i} + f_{h\varepsilon,i}|^2_{H_h} \varepsilon^2 \\
&= |v_{h\varepsilon,0}|^2_{H_h} + \sum_{i=0}^{j-1} 2\langle v_{h\varepsilon,i}, L_{h\varepsilon,i} v_{h\varepsilon,i}\rangle_h \varepsilon \\
&\quad + \sum_{i=0}^{j-1} 2\langle v_{h\varepsilon,i}, f_{h\varepsilon,i}\rangle_h \varepsilon + \sum_{i=0}^{j-1} |L_{h\varepsilon,i} v_{h\varepsilon,i} + f_{h\varepsilon,i}|^2_{H_h} \varepsilon^2 \\
&\leq |v_{h\varepsilon,0}|^2_{H_h} + 2K \sum_{i=0}^{j-1} |v_{h\varepsilon,i}|^2_{H_h} \varepsilon - 2\lambda \sum_{i=0}^{j-1} |v_{h\varepsilon,i}|^2_{V_h} \varepsilon \\
&\quad + \lambda \sum_{i=0}^{j-1} |v_{h\varepsilon,i}|^2_{V_h} \varepsilon + \frac{1}{\lambda} \sum_{i=0}^{j-1} |f_{h\varepsilon,i}|^2_{V_h^*} \varepsilon + \sum_{i=0}^{j-1} |L_{h\varepsilon,i} v_{h\varepsilon,i} + f_{h\varepsilon,i}|^2_{H_h} \varepsilon^2,
\end{aligned} \tag{5.23}$$

with $\lambda > 0$, using Assumption 5.1.15 and Cauchy's inequality.

For the term $\sum_{i=0}^{j-1} |L_{h\varepsilon,i} v_{h\varepsilon,i} + f_{h\varepsilon,i}|^2_{H_h} \varepsilon^2$ in inequality (5.23) we have

$$\begin{aligned}
\sum_{i=0}^{j-1} |L_{h\varepsilon,i} v_{h\varepsilon,i} + f_{h\varepsilon,i}|^2_{H_h} \varepsilon^2 = & \sum_{i=0}^{j-1} |L_{h\varepsilon,i} v_{h\varepsilon,i}|^2_{H_h} \varepsilon^2 + \sum_{i=0}^{j-1} |f_{h\varepsilon,i}|^2_{H_h} \varepsilon^2 \\
& +2 \sum_{i=0}^{j-1} \langle f_{h\varepsilon,i}, L_{h\varepsilon,i} v_{h\varepsilon,i} \rangle_h \, \varepsilon^2 \\
\le & \sum_{i=0}^{j-1} |L_{h\varepsilon,i} v_{h\varepsilon,i}|^2_{H_h} \varepsilon^2 + \sum_{i=0}^{j-1} |f_{h\varepsilon,i}|^2_{H_h} \varepsilon^2 \\
& + \frac{1}{\mu} \sum_{i=0}^{j-1} |f_{h\varepsilon,i}|^2_{H_h} \varepsilon^2 + \mu \sum_{i=0}^{j-1} |L_{h\varepsilon,i} v_{h\varepsilon,i}|^2_{H_h} \varepsilon^2.
\end{aligned}$$

with $\mu > 0$, using Cauchy's inequality.

As, owing to Assumptions 5.1.15 and 5.1.17,

$$\sum_{i=0}^{j-1} |L_{h\varepsilon,i} v_{h\varepsilon,i}|^2_{H_h} \varepsilon^2 \le C_h^2 \varepsilon \sum_{i=0}^{j-1} |L_{h\varepsilon,i} v_{h\varepsilon,i}|^2_{V_h^*} \varepsilon \le M^2 C_h^2 \varepsilon \sum_{i=0}^{j-1} |v_{h\varepsilon,i}|^2_{V_h} \varepsilon,$$

and

$$\sum_{i=0}^{j-1} |f_{h\varepsilon,i}|^2_{H_h} \varepsilon^2 \le C_h^2 \varepsilon \sum_{i=0}^{j-1} |f_{h\varepsilon,i}|^2_{V_h^*} \varepsilon,$$

then we have

$$\begin{aligned}
\sum_{i=0}^{j-1} |L_{h\varepsilon,i} v_{h\varepsilon,i} + f_{h\varepsilon,i}|_{H_h} \varepsilon^2 \quad \le \quad & (1+\mu) M^2 C_h^2 \varepsilon \sum_{i=0}^{j-1} |v_{h\varepsilon,i}|^2_{V_h} \varepsilon \\
& + \left(1 + \frac{1}{\mu}\right) C_h^2 \varepsilon \sum_{i=0}^{j-1} |f_{h\varepsilon,i}|^2_{V_h^*} \varepsilon. \qquad (5.24)
\end{aligned}$$

Putting estimates (5.23) and (5.24) together,

$$\begin{aligned}
|v_{h\varepsilon,j}|^2_{H_h} \quad \le \quad & |v_{h\varepsilon,0}|^2_{H_h} + 2K \sum_{i=0}^{j-1} |v_{h\varepsilon,i}|^2_{H_h} \varepsilon + \left((1+\mu) M^2 C_h^2 \varepsilon - \lambda\right) \sum_{i=0}^{j-1} |v_{h\varepsilon,i}|^2_{V_h} \varepsilon \\
& + \left(\frac{1}{\lambda} + \left(1 + \frac{1}{\mu}\right) C_h^2 \varepsilon\right) \sum_{i=0}^{j-1} |f_{h\varepsilon,i}|^2_{V_h^*} \varepsilon. \qquad (5.25)
\end{aligned}$$

If there is a constant p such that

$$M^2 C_h^2 \varepsilon \le p < \lambda,$$

implying that, for μ sufficiently small,

$$(1+\mu) M^2 C_h^2 \varepsilon - \lambda \le (1+\mu) p - \lambda < 0,$$

then from (5.25) we can estimate

$$|v_{h\varepsilon,j}|^2_{H_h} + (\lambda - (1+\mu)p)\sum_{i=0}^{j-1}|v_{h\varepsilon,i}|^2_{V_h}\varepsilon \;\leq\; |v_{h\varepsilon,0}|^2_{H_h} + 2K\sum_{i=0}^{j-1}|v_{h\varepsilon,i}|^2_{H_h}\varepsilon + L\sum_{i=0}^{n-1}|f_{h\varepsilon,i}|^2_{V_h^*}\varepsilon, \qquad (5.26)$$

where $L := (\mu M^2 + \lambda(1+\mu)p)/\lambda\mu M^2$.

In particular,

$$|v_{h\varepsilon,j}|^2_{H_h} \;\leq\; |v_{h\varepsilon,0}|^2_{H_h} + 2K\sum_{i=0}^{j-1}|v_{h\varepsilon,i}|^2_{H_h}\varepsilon + L\sum_{i=0}^{n-1}|f_{h\varepsilon,i}|^2_{V_h^*}\varepsilon, \qquad (5.27)$$

and, using Lemma 5.1.16, we obtain

$$|v_{h\varepsilon,j}|^2_{H_h} \leq \left(|v_{h\varepsilon,0}|^2_{H_h} + L\sum_{i=0}^{n-1}|f_{h\varepsilon,i}|^2_{V_h^*}\varepsilon\right)e^{2K_qT}, \qquad (5.28)$$

where K_q is the constant defined in Lemma 5.1.16. We have proved (1).

From (5.26), (5.27) and (5.28) we obtain

$$|v_{h\varepsilon,j}|^2_{H_h} + (\lambda - (1+\mu)p)\sum_{i=0}^{j-1}|v_{h\varepsilon,i}|^2_{V_h}\varepsilon \leq \left(|v_{h\varepsilon,0}|^2_{H_h} + L\sum_{i=0}^{n-1}|f_{h\varepsilon,i}|^2_{V_h^*}\varepsilon\right)e^{2K_qT},$$

and (2) follows. □

We next determine a rate of convergence.

Theorem 5.1.19. *Let $u_h(t)$ and $v_{h\varepsilon,j}$, with $j = 0,1,\ldots,n$, be the unique solutions of problems (5.19) and (5.20), respectively. Let Assumptions 5.1.8, 5.1.15 and 5.1.17 be verified and λ, K, M, C_h the constants defined in Assumptions 5.1.1 and 5.1.17. Assume the constant K satisfies: $2K\varepsilon < 1$. If there exists a number p such that $M^2C_h^2\varepsilon \leq p < \lambda$ then there exists a constant N, independent of ε and h, such that*

1. $\max_{0\leq j\leq n}|v_{h\varepsilon,j}-u_h(t_j)|^2_{H_h}\leq N(|\varepsilon|^\delta+\sum_{j=0}^{n-1}\frac{1}{\varepsilon}|L_{h\varepsilon,j}u(t_j)\varepsilon-\int_{t_j}^{t_{j+1}}L_h(s)u_h(t_j)ds|^2_{V_h^*}$
$+\sum_{j=0}^{n-1}\frac{1}{\varepsilon}|f_{h\varepsilon,j}\varepsilon-\int_{t_j}^{t_{j+1}}f_h(s)ds|^2_{V_h^*})$;

2. $\sum_{j=0}^{n}|v_{h\varepsilon,j}-u_h(t_j)|^2_{V_h}\varepsilon \quad \leq N(|\varepsilon|^\delta+\sum_{j=0}^{n-1}\frac{1}{\varepsilon}|L_{h\varepsilon,j}u(t_j)\varepsilon-\int_{t_j}^{t_{j+1}}L_h(s)u_h(t_j)ds|^2_{V_h^*}$
$+\sum_{j=0}^{n-1}\frac{1}{\varepsilon}|f_{h\varepsilon,j}\varepsilon-\int_{t_j}^{t_{j+1}}f_h(s)ds|^2_{V_h^*})$.

Proof Define $w(t_i) := v_{h\varepsilon,i} - u_h(t_i),\ \ i = 0, 1, \ldots, n$. For $i = 0, 1, \ldots, n-1$, we have

$$\begin{aligned} w(t_{i+1}) - w(t_i) &= L_{h\varepsilon,i}w(t_i)\varepsilon + f_{h\varepsilon,i}\varepsilon - u_h(t_{i+1}) + u_h(t_i) + L_{h\varepsilon,i}u_h(t_i)\varepsilon \\ &= L_{h\varepsilon,i}w(t_i)\varepsilon + \varphi(t_i), \end{aligned}$$

denoting $\varphi(t_i) := f_{h\varepsilon,i}\varepsilon - u_h(t_{i+1}) + u_h(t_i) + L_{h\varepsilon,i}\, u_h(t_i)\varepsilon$.

We have

$$\begin{aligned} |w(t_{i+1})|^2_{H_h} - |w(t_i)|^2_{H_h} &= 2\ \langle w(t_i), w(t_{i+1}) - w(t_i)\rangle_h + |w(t_{i+1}) - w(t_i)|^2_{H_h} \\ &= 2\ \langle w(t_i), L_{h\varepsilon,i}w(t_i)\varepsilon + \varphi(t_i)\rangle_h \\ &\quad + |L_{h\varepsilon,i}w(t_i)\varepsilon + \varphi(t_i)|^2_{H_h} \\ &= 2\ \langle w(t_i), L_{h\varepsilon,i}w(t_i)\rangle_h\varepsilon + 2\ |\langle w(t_i), \varphi(t_i)\rangle_h| \\ &\quad + |L_{h\varepsilon,i}w(t_i)\varepsilon + \varphi(t_i)|^2_{H_h}. \end{aligned} \tag{5.29}$$

For the first term in (5.29) we have

$$2\ \langle w(t_i), L_{h\varepsilon,i}w(t_i)\rangle_h\varepsilon \le -2\lambda|w(t_i)|^2_{V_h}\varepsilon + 2K|w(t_i)|^2_{H_h}\varepsilon, \tag{5.30}$$

using Assumption 5.1.15.

Noting that $\varphi(t_i)$ can be written

$$\varphi(t_i) = \int_{t_i}^{t_{i+1}} L_h(s)(u_h(t_i) - u_h(s))ds + \varphi_1(t_i) + \varphi_2(t_i),$$

where $\varphi_1(t_i) := L_{h\varepsilon,i}u_h(t_i)\varepsilon - \int_{t_i}^{t_{i+1}} L_h(s)u_h(t_i)ds$ and $\varphi_2(t_i) := f_{h\varepsilon,i}\varepsilon - \int_{t_i}^{t_{i+1}} f_h(s)ds$, for the second term in (5.29) we have

$$\begin{aligned} 2\ |\langle w(t_i), \varphi(t_i)\rangle_h| &\le 2|\langle w(t_i), \int_{t_i}^{t_{i+1}} L_h(s)(u_h(t_i) - u_h(s))ds\rangle_h| \\ &\quad + 2|\langle w(t_i), \varphi_1(t_i)\rangle_h| + 2|\langle w(t_i), \varphi_2(t_i)\rangle_h|. \end{aligned} \tag{5.31}$$

For the term $2|\langle w(t_i), \int_{t_i}^{t_{i+1}} L_h(s)(u_h(t_i) - u_h(s))ds\rangle_h|$ in (5.31) we have

$$\begin{aligned} & 2|\langle w(t_i), \int_{t_i}^{t_{i+1}} L_h(s)(u_h(t_i) - u_h(s))ds\rangle_h| \\ & \le 2\int_{t_i}^{t_{i+1}} |\langle w(t_i), L_h(s)(u_h(t_i) - u_h(s))\rangle_h|ds \\ & \le 2M|w(t_i)|_{V_h} \int_{t_i}^{t_{i+1}} |u_h(t_i) - u_h(s)|_{V_h}ds \\ & \le \frac{\lambda}{3}|w(t_i)|^2_{V_h}\varepsilon + \frac{3M^2}{\lambda\varepsilon}\left(\int_{t_i}^{t_{i+1}} |u_h(t_i) - u_h(s)|_{V_h}ds\right)^2, \end{aligned} \tag{5.32}$$

with $\lambda > 0$, using Assumption 5.1.1 and Cauchy's inequality.

For the terms $2|\langle w(t_i),\varphi_1(t_i)\rangle_h|$ and $2|\langle w(t_i),\varphi_2(t_i)\rangle_h|$ in (5.31) we have

$$2|\langle w(t_i),\varphi_1(t_i)\rangle_h| \;\le\; \frac{\lambda}{3}|w(t_i)|^2_{V_h}\varepsilon+\frac{3}{\lambda\varepsilon}|\varphi_1(t_i)|^2_{V_h^*}, \tag{5.33}$$

and

$$2|\langle w(t_i),\varphi_2(t_i)\rangle_h| \;\le\; \frac{\lambda}{3}|w(t_i)|^2_{V_h}\varepsilon+\frac{3}{\lambda\varepsilon}|\varphi_2(t_i)|^2_{V_h^*}, \tag{5.34}$$

with $\lambda>0$, using Cauchy's inequality.

From (5.31), (5.32), (5.33) and (5.34) we have

$$\begin{aligned}2|\langle w(t_i),\varphi(t_i)\rangle_h| \;\le\; &\lambda|w(t_i)|^2_{V_h}\varepsilon+\frac{3M^2}{\lambda\varepsilon}\left(\int_{t_i}^{t_{i+1}}|u_h(t_i)-u_h(s)|_{V_h}ds\right)^2\\ &+\frac{3}{\lambda\varepsilon}|\varphi_1(t_i)|^2_{V_h^*}+\frac{3}{\lambda\varepsilon}|\varphi_2(t_i)|^2_{V_h^*}.\end{aligned} \tag{5.35}$$

For the last term in (5.29),

$$|L_{h\varepsilon,i}w(t_i)\varepsilon+\varphi(t_i)|^2_{H_h}=|L_{h\varepsilon,i}w(t_i)|^2_{H_h}\varepsilon^2+|\varphi(t_i)|^2_{H_h}+2\langle L_{h\varepsilon,i}w(t_i),\varphi(t_i)\rangle_h\varepsilon. \tag{5.36}$$

For the term $|L_{h\varepsilon,i}w(t_i)|^2_{H_h}\varepsilon^2$ in (5.36),

$$|L_{h\varepsilon,i}w(t_i)|^2_{H_h}\varepsilon^2\le C_h^2|L_{h\varepsilon,i}w(t_i)|^2_{V_h^*}\varepsilon^2\le M^2C_h^2\varepsilon|w(t_i)|^2_{V_h}\varepsilon, \tag{5.37}$$

owing to Assumptions 5.1.15 and 5.1.17.

For the term $|\varphi(t_i)|^2_{H_h}$ in (5.36),

$$\begin{aligned}|\varphi(t_i)|^2_{H_h} \;=\; &|\int_{t_i}^{t_{i+1}}L_h(s)(u_h(t_i)-u_h(s))ds|^2_{H_h}+|\varphi_1(t_i)|^2_{H_h}+|\varphi_2(t_i)|^2_{H_h}\\ &+2\langle\int_{t_i}^{t_{i+1}}L_h(s)(u_h(t_i)-u_h(s))ds,\varphi_1(t_i)\rangle_h\\ &+2\langle\int_{t_i}^{t_{i+1}}L_h(s)(u_h(t_i)-u_h(s))ds,\varphi_2(t_i)\rangle_h\\ &+2\langle\varphi_1(t_i)\,,\,\varphi_2(t_i)\rangle_h\\ \le\; &M^2C_h^2\left(\int_{t_i}^{t_{i+1}}|u_h(t_i)-u_h(s)|_{V_h}ds\right)^2+C_h^2|\varphi_1(t_i)|^2_{V_h^*}+C_h^2|\varphi_2(t_i)|^2_{V_h^*}\\ &+\mu M^2C_h^2\left(\int_{t_i}^{t_{i+1}}|u_h(t_i)-u_h(s)|_{V_h}ds\right)^2+\frac{1}{\mu}C_h^2|\varphi_1(t_i)|^2_{V_h^*}\\ &+\frac{1}{\mu}M^2C_h^2\left(\int_{t_i}^{t_{i+1}}|u_h(t_i)-u_h(s)|_{V_h}ds\right)^2+\mu C_h^2|\varphi_2(t_i)|^2_{V_h^*}\\ &+\mu C_h^2|\varphi_1(t_i)|^2_{V_h^*}+\frac{1}{\mu}C_h^2|\varphi_2(t_i)|^2_{V_h^*}\\ =\; &\left(1+\mu+\frac{1}{\mu}\right)M^2C_h^2\left(\int_{t_i}^{t_{i+1}}|u_h(t_i)-u_h(s)|_{V_h}ds\right)^2\\ &+\left(1+\mu+\frac{1}{\mu}\right)C_h^2|\varphi_1(t_i)|^2_{V_h^*}+\left(1+\mu+\frac{1}{\mu}\right)C_h^2|\varphi_2(t_i)|^2_{V_h^*},\end{aligned} \tag{5.38}$$

with $\mu > 0$, using Cauchy's inequality and Assumptions 5.1.1 and 5.1.17.

For the term $2\langle L_{h\varepsilon,i}w(t_i), \varphi(t_i)\rangle_h\varepsilon$ in (5.36),

$$
\begin{aligned}
2\langle L_{h\varepsilon,i}w(t_i), \varphi(t_i)\rangle_h\varepsilon \le\; & 2|\langle L_{h\varepsilon,i}w(t_i)\varepsilon, \int_{t_i}^{t_{i+1}} L_h(s)(u_h(t_i) - u_h(s))ds\rangle_h| \\
& +2|\langle L_{h\varepsilon,i}w(t_i)\varepsilon, \varphi_1(t_i)\rangle_h| + 2|\langle L_{h\varepsilon,i}w(t_i)\varepsilon, \varphi_2(t_i)\rangle_h| \\
\le\; & \frac{\nu}{3}M^2C_h^2\varepsilon|w(t_i)|_{V_h}^2\varepsilon + \frac{3}{\nu}M^2C_h^2\left(\int_{t_i}^{t_{i+1}}|u_h(t_i) - u_h(s)|_{V_h}ds\right)^2 \\
& +\frac{\nu}{3}M^2C_h^2\varepsilon|w(t_i)|_{V_h}^2\varepsilon + \frac{3}{\nu}C_h^2|\varphi_1(t_i)|_{V_h^*}^2 \\
& +\frac{\nu}{3}M^2C_h^2\varepsilon|w(t_i)|_{V_h}^2\varepsilon + \frac{3}{\nu}C_h^2|\varphi_2(t_i)|_{V_h^*}^2 \\
=\; & \nu M^2C_h^2\varepsilon|w(t_i)|_{V_h}^2\varepsilon + \frac{3}{\nu}M^2C_h^2\left(\int_{t_i}^{t_{i+1}}|u_h(t_i) - u_h(s)|_{V_h}ds\right)^2 \\
& +\frac{3}{\nu}C_h^2|\varphi_1(t_i)|_{V_h^*}^2 + \frac{3}{\nu}C_h^2|\varphi_2(t_i)|_{V_h^*}^2, \qquad (5.39)
\end{aligned}
$$

with $\nu > 0$, using Cauchy's inequality and Assumptions 5.1.1, 5.1.15 and 5.1.17.

From (5.36), (5.37), (5.38) and (5.39) we have

$$
\begin{aligned}
|L_{h\varepsilon,i}w(t_i)\varepsilon+\varphi(t_i)|_{H_h}^2 \;\le\; & (1+\nu)M^2C_h^2\varepsilon|w(t_i)|_{V_h}^2\varepsilon \\
& + \left(1+\mu+\frac{1}{\mu}+\frac{3}{\nu}\right)M^2C_h^2\left(\int_{t_i}^{t_{i+1}}|u_h(t_i) - u_h(s)|_{V_h}ds\right)^2 \\
& + \left(1+\mu+\frac{1}{\mu}+\frac{3}{\nu}\right)C_h^2|\varphi_1(t_i)|_{V_h^*}^2 \\
& + \left(1+\mu+\frac{1}{\mu}+\frac{3}{\nu}\right)C_h^2|\varphi_2(t_i)|_{V_h^*}^2. \qquad (5.40)
\end{aligned}
$$

Putting estimates (5.30), (5.35) and (5.40) together and owing to Assumption 5.1.8,

$$
\begin{aligned}
|w(t_{i+1})|_{H_h}^2 - |w(t_i)|_{H_h}^2 \le\; & 2K|w(t_i)|_{H_h}^2\varepsilon + ((1+\nu)M^2C_h^2\varepsilon - \lambda)|w(t_i)|_{V_h}^2\varepsilon \\
& +M^2C^2\left(\left(1+\mu+\frac{1}{\mu}+\frac{3}{\nu}\right)C_h^2\varepsilon + \frac{3}{\lambda}\right)|\varepsilon|^{\delta+1} \\
& +\left(\left(1+\mu+\frac{1}{\mu}+\frac{3}{\nu}\right)C_h^2 + \frac{3}{\lambda\varepsilon}\right)|\varphi_1(t_i)|_{V_h^*}^2 \\
& +\left(\left(1+\mu+\frac{1}{\mu}+\frac{3}{\nu}\right)C_h^2 + \frac{3}{\lambda\varepsilon}\right)|\varphi_2(t_i)|_{V_h^*}^2.
\end{aligned}
$$

Summing up, for $j = 0, 1, \ldots, n$, we have

$$
\begin{aligned}
|w(t_j)|_{H_h}^2 \leq & 2K \sum_{i=0}^{j-1} |w(t_i)|_{H_h}^2 \varepsilon + ((1+\nu)M^2 C_h^2 \varepsilon - \lambda) \sum_{i=0}^{j-1} |w(t_i)|_{V_h}^2 \varepsilon \\
& + M^2 C^2 \left(\left(1 + \mu + \frac{1}{\mu} + \frac{3}{\nu} \right) C_h^2 \varepsilon + \frac{3}{\lambda} \right) |\varepsilon|^\delta \\
& + \left(\left(1 + \mu + \frac{1}{\mu} + \frac{3}{\nu} \right) C_h^2 \varepsilon + \frac{3}{\lambda} \right) \sum_{i=0}^{j-1} \frac{1}{\varepsilon} |\varphi_1(t_i)|_{V_h^*}^2 \\
& + \left(\left(1 + \mu + \frac{1}{\mu} + \frac{3}{\nu} \right) C_h^2 \varepsilon + \frac{3}{\lambda} \right) \sum_{i=0}^{j-1} \frac{1}{\varepsilon} |\varphi_2(t_i)|_{V_h^*}^2. \qquad (5.41)
\end{aligned}
$$

As we assume that there is a constant p such that $M^2 C_h^2 \varepsilon \leq p < \lambda$, we have that, for ν sufficiently small,

$$
(1+\nu)M^2 C_h^2 \varepsilon - \lambda \leq (1+\nu)p - \lambda < 0.
$$

Then from (5.41) we can estimate

$$
\begin{aligned}
|w(t_j)|_{H_h}^2 + (\lambda - (1+\nu)p) \sum_{i=0}^{j-1} |w(t_i)|_{V_h}^2 \varepsilon \leq & 2K \sum_{i=0}^{j-1} |w(t_i)|_{H_h}^2 \varepsilon + L\, |\varepsilon|^\delta \\
& + L \sum_{i=0}^{n-1} \frac{1}{\varepsilon} |L_{h\varepsilon,i} u_h(t_i)\varepsilon - \int_{t_i}^{t_{i+1}} L_h(s) u_h(t_i) ds|_{V_h^*}^2 \\
& + L \sum_{i=0}^{n-1} \frac{1}{\varepsilon} |f_{h\varepsilon,i}\varepsilon - \int_{t_i}^{t_{i+1}} f_h(s) ds|_{V_h^*}^2, \qquad (5.42)
\end{aligned}
$$

where $L := (3\mu\nu M^2 + \lambda((1+\mu)\nu + \mu(\mu\nu + 3))p)/\lambda\mu\nu M^2$.

In particular

$$
\begin{aligned}
|w(t_j)|_{H_h}^2 \leq & 2K \sum_{i=0}^{j-1} |w(t_i)|_{H_h}^2 \varepsilon + L\, |\varepsilon|^\delta + L \sum_{i=0}^{n-1} \frac{1}{\varepsilon} |L_{h\varepsilon,i} u_h(t_i)\varepsilon - \int_{t_i}^{t_{i+1}} L_h(s) u_h(t_i) ds|_{V_h^*}^2 \\
& + L \sum_{i=0}^{n-1} \frac{1}{\varepsilon} |f_{h\varepsilon,i}\varepsilon - \int_{t_i}^{t_{i+1}} f_h(s) ds|_{V_h^*}^2,
\end{aligned}
$$

and, using Lemma 5.1.16,

$$
\begin{aligned}
|w(t_j)|_{H_h}^2 \leq & L e^{2K_q T} |\varepsilon|^\delta + L e^{2K_q T} \sum_{i=0}^{n-1} \frac{1}{\varepsilon} |L_{h\varepsilon,i} u_h(t_i)\varepsilon - \int_{t_i}^{t_{i+1}} L_h(s) u_h(t_i) ds|_{V_h^*}^2 \\
& + L e^{2K_q T} \sum_{i=0}^{n-1} \frac{1}{\varepsilon} |f_{h\varepsilon,i}\varepsilon - \int_{t_i}^{t_{i+1}} f_h(s) ds|_{V_h^*}^2, \qquad (5.43)
\end{aligned}
$$

with K_q the constant defined in Corollary 5.1.6. Claim (1) is proved.

From (5.42) and (5.43) we obtain

$$|w(t_j)|^2_{H_h} + (\lambda - (1+\nu)p)\sum_{i=0}^{j-1}|w(t_i)|^2_{V_h}\varepsilon$$
$$\leq Le^{2K_qT}\,|\varepsilon|^\delta + Le^{2K_qT}\sum_{i=0}^{n-1}\frac{1}{\varepsilon}|L_{h\varepsilon,i}u_h(t_i)\varepsilon - \int_{t_i}^{t_{i+1}}L_h(s)u_h(t_i)ds|^2_{V_h^*}$$
$$+Le^{2K_qT}\sum_{i=0}^{n-1}\frac{1}{\varepsilon}|f_{h\varepsilon,i}\varepsilon - \int_{t_i}^{t_{i+1}}f_h(s)ds|^2_{V_h^*},$$

and (2) follows. □

Corollary 5.1.20. *Let $u_h(t)$ and $v_{h\varepsilon,j}$, with $j = 0, 1, \ldots, n$, be the unique solutions of problems (5.19) and (5.20), respectively. Assume the hypothesis of Theorem 5.1.19 are verified. If there exists a constant N independent of ε such that*

$$|L_{h\varepsilon,j}u_h(t_j) - \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}}L_h(s)u_h(t_j)ds|^2_{V_h^*} + |f_{h\varepsilon,j} - \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}}f_h(s)ds|^2_{V_h^*} \leq N|\varepsilon|^\delta,$$

for $j = 0, 1, \ldots, n-1$, then

1. $\max_{0\leq j\leq n}|v_{h\varepsilon,j} - u_h(t_j)|^2_{H_h} \leq N|\varepsilon|^\delta$;
2. $\sum_{j=0}^{n}|v_{h\varepsilon,j} - u_h(t_j)|^2_{V_h}\varepsilon \leq N|\varepsilon|^\delta$.

Proof The result follows immediately from Theorem 5.1.19. □

We consider now the case where the operators L_h and f_h in problem (5.19) have the particular time-discretization, respectively

$$\forall z \in V_h,\ \bar{L}_{h\varepsilon}(t_j)z := \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}}L_h(s)zds \quad \text{and} \quad \bar{f}_{h\varepsilon}(t_j) := \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}}f_h(s)ds,$$

for $j = 0, 1, \ldots, n-1$.

Denote

$$\bar{L}_{h\varepsilon,j}\,z = \bar{L}_{h\varepsilon}(t_j)z,\ \ \bar{f}_{h\varepsilon,j} = \bar{f}_{h\varepsilon}(t_j),\ \ j = 0, 1, \ldots, n-1.$$

We consider the particular time-discrete version of (5.19)

$$\Delta^+ v_i = \bar{L}_{h\varepsilon,i}\,v_i + \bar{f}_{h\varepsilon,i} \quad \text{for } i = 0, 1, \ldots, n-1, \quad v_0 = g_h, \tag{5.44}$$

with $n \geq 1$.

We have the following result:

Lemma 5.1.21. *Under Assumption 5.1.1, the operators $\bar{L}_{h\varepsilon}$ and $\bar{f}_{h\varepsilon}$ satisfy*

1. $\langle \bar{L}_{h\varepsilon,j}\, v,\ v\rangle_h + \lambda |v|^2_{V_h} \leq K|v|^2_{H_h}, \quad \forall v \in V_h,\ \ j = 0, 1, \dots, n-1,$
2. $|\bar{L}_{h\varepsilon,j}\, v|_{V_h^*} \leq M|v|_{V_h}, \quad \forall v \in V_h,\ \ j = 0, 1, \dots, n-1,$
3. $\sum_{j=0}^{n-1} |\bar{f}_{h\varepsilon,j}|^2_{V_h^*}\varepsilon \leq N,$

where λ, K, M and N are the constants in Assumption 5.1.1.

Proof The operators $\bar{L}_{h\varepsilon}$ and $\bar{f}_{h\varepsilon}$ coincide with the operators $\bar{L}_\varepsilon$ and $\bar{f}_\varepsilon$, replacing L and f for L_h and f_h, respectively, in the integral arguments. The result follows then from Lemma 5.1.12. □

We have then that the operators $\bar{L}_h$ and $\bar{f}_h$, satisfy Assumption 5.1.15.

Next we present two results which are corollaries of Theorems 5.1.18 and 5.1.19, respectively.

Corollary 5.1.22. *Let $v_{h\varepsilon,j}$, with $j = 0, 1, \dots, n$, be the unique solution of problem (5.44). Let the hypothesis of Lemma 5.1.21 and Assumption 5.1.17 be verified and λ, K, M, C_h the constants defined in Assumptions 5.1.1 and 5.1.17. Assume the constant K satisfies: $2K\varepsilon < 1$. If there exists a number p such that $M^2C_h^2\varepsilon \leq p < \lambda$ then there exists a constant N, independent of ε and h, such that*

1. $\sup_{n\geq 1} \max_{0\leq j\leq n} |v_{h\varepsilon,j}|^2_{H_h} \leq N|g_h|^2_{H_h};$
2. $\sup_{n\geq 1} \sum_{0\leq j\leq n} |v_{h\varepsilon,j}|^2_{V_h}\varepsilon \leq N|g_h|^2_{H_h}.$

Proof The result is an immediate consequence of Theorem 5.1.18. □

Corollary 5.1.23. *Let $u_h(t)$ and $v_{h\varepsilon,j}$, with $j = 0, 1, \dots, n$, be the unique solutions of problems (5.19) and (5.44), respectively. Let the hypothesis of Lemma 5.1.21 and Assumptions 5.1.8 and 5.1.17 be verified and λ, K, M, C_h the constants defined in Assumptions 5.1.1 and 5.1.17. Assume the constant K satisfies: $2K\varepsilon < 1$. If there exists a number p such that $M^2C_h^2\varepsilon \leq p < \lambda$ then there exists a constant N, independent of ε and h, such that*

1. $\max_{0\leq j\leq n} |v_{h\varepsilon,j} - u_h(t_j)|^2_{H_h} \leq N|\varepsilon|^\delta;$
2. $\sum_{0\leq j\leq n} |v_{h\varepsilon,j} - u_h(t_j)|^2_{V_h}\,\varepsilon \leq N|\varepsilon|^\delta.$

Proof The result follows from Theorem 5.1.19. □

5.2 An example: the second-order parabolic PDE problem in weighted Sobolev spaces

In Section 4.3, we considered the following problem, discrete in space:

$$L_h u - u_t + f_h = 0 \text{ in } Q(h), \quad u(0,x) = g_h(x) \text{ in } Z_h^d, \tag{5.45}$$

where $Q(h) = [0,T] \times Z_h^d$ ($T > 0$ a number and Z_h^d a h-grid on $\mathbb{R}^d$) and L_h is the discrete operator

$$L_h(t,x) = a^{ij}(t,x)\partial_j^- \partial_i^+ + b^i(t,x)\partial_i^+ + c\,(t,x),$$

with ∂_i^+ and ∂_j^-t he forward and backward discrete differences in space, respectively.

To handle unbounded data, we considered the spaces $l^{0,2}(r)$ and $l^{1,2}(r,\rho)$ and set a framework, discrete in space, which is a particular case of the general framework we presented in Section 4.1 and recalled in Section 5.1.

Let

$$\Delta^- v_{i+1} = L_{h\varepsilon,i+1} v_{i+1} + f_{h\varepsilon,i+1} \quad \text{for } i = 0,1,\ldots,n-1, \quad v_0 = g_h$$

and

$$\Delta^+ v_i = L_{h\varepsilon,i}\, v_i + f_{h\varepsilon,i} \quad \text{for } i = 0,1,\ldots,n-1, \quad v_0 = g_h$$

be, respectively, the implicit and explicit schemes, as set in Section 5.1, for the time discretization of problem (5.45).

From the above, under the assumptions we made in Section 5.1, the results we then obtained still hold.

Chapter 6

Conclusion and further research

We studied the numerical approximation of the parabolic PDE multidimensional problem for the general case where the coefficients b and σ of the underlying stochastic equation are time and space-dependent.

With the approach of the problem in weighted Sobolev spaces, we could consider PDE with unbounded coefficients (with the corresponding coefficients b and σ in the stochastic equation growing linearly). This implies assuming that, in the European option model, the underlying asset drift and volatility are bounded functions, what does not seem to be a strong restriction for the financial application. When the logarithmic transformation of the diffusion X_t, considered in Chapter 2, is available, even the linear growth of drift and volatility can be allowed.

We make some remarks concerning the numerical schemes' implementation. The parabolic problem arising from the stochastic modelling is a Cauchy problem in half spaces. In Chapters 4 and 5 we produced numerical schemes for its approximation in Sobolev and weighted Sobolev spaces. Nevertheless, when the discretization in time is obtained with the implicit scheme, the problem localization is needed for implementation purpose. The approximation of the localized problem in Sobolev spaces as well as the estimate of the corresponding localization error were not considered in the present research.

We outline further research directions from the present study:

- Approximation of the initial-boundary value problem in Sobolev spaces with the localization error estimate.

- Implementation of the discrete schemes we have constructed and testing with real financial data.

- Acceleration of the numerical schemes:

  - Using the Crank and Nicholson finite differences scheme;
  - Applying the splitting-up method, following Richardson's idea to accelerate numerical schemes (see Gyöngy et all [19] and [20]);
  - Using other numerical methods, namely the finite elements method, and more complex grids.

- Another direction is the direct approximation of the SDE by Monte Carlo methods.

- Including the discrete dividend payment and transaction costs in the European option modelling.

- Finally, extending the study to other types of financial options with no early exercise.

Appendix A

Notation

Notation for matrices.

$a = (a^{ij})$ denotes the $d \times p$ matrix with $(i,j)^{th}$ element a^{ij}.

$a' =$ transpose of the matrix a.

$|a|^2 = \sum_{1\le i\le d,\ 1\le j\le p}(a^{ij})^2$.

Sometimes we use the notation a^{ij} for $\sum_{1\le i\le d,\ 1\le j\le p} a^{ij}$.

Geometric notation.

$\mathbb{R}^d = d-$dimensional Euclidean space of points $x = (x^1, \ldots, x^d)$.

$e_i = (0, \ldots, 0, 1, 0, \ldots, 0) = i^{th}$ standard coordinate vector.

$(x, y) = \sum_{i=1}^d x^i y^i,\ |x|^2 = \sum_{i=1}^d (x^i)^2,$ for all $x, y \in \mathbb{R}^d$.

U, V usually denote domains in $\mathbb{R}^d$, meaning open subsets of $\mathbb{R}^d$.

$\partial U =$ boundary of U.

$\bar{U} = U \cup \partial U =$ closure of U.

$B_R(x_0) =$ the open ball in $\mathbb{R}^d$ with center x_0 and radius R.

$\mathbb{R}^d_+ = \{(x', x^d) : x' = (x^1, \ldots, x^{d-1}) \in \mathbb{R}^{d-1},\ x^d > 0\}$, p. 31.

$\mathbb{R}^{d+1} = \{(t, x) : t \in \mathbb{R},\ x \in \mathbb{R}^d\}$, p. 29.

$\mathbb{R}^{d+1}_+ = \{(t, x) : t \ge 0,\ x \in \mathbb{R}^d\}$, p. 32.

$\mathbb{R}^\infty =$ infinite dimensional Euclidean space of points $x = (x^1, x^2, \ldots)$.

Q usually denotes $[0, T] \times \mathbb{R}^d$ or $[0, T] \times U$.

$\partial Q =$ the parabolic boundary of Q, p. 32.

$\partial_x Q =$ the space-boundary of $Q = [0,T] \times U$, p. 32.

$\partial_t Q =$ the time-boundary of $Q = [0,T] \times U$, p. 32.

$T_n =$ the grid on $[0,T]$, p. 82.

$Z_h^d =$ the grid on $\mathbb{R}^d$, p. 62.

$Z_h^{d+1} =$ the grid on $\mathbb{R}_+^{d+1}$, p. 37.

$Q(h) = Q \cap Z_h^{d+1}$, p. 37.

$Q^0(h) =$ the discrete "interior" of $Q(h)$, p. 37.

$\partial' Q(h) =$ the discrete boundary of $Q(h)$, p. 37.

$\partial'_x Q(h) =$ the discrete space-boundary of $Q(h)$, p. 37.

$\partial'_t Q(h) =$ the discrete time-boundary of $Q(h)$, p. 37.

Notation for functions.

Multi-index notation: A vector $\alpha = (\alpha_1, \ldots, \alpha_d)$ of non-negative integers $\alpha_k = 0, 1, 2, \ldots$ is called a multi-index of order $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$.

$D^\alpha = D_1^{\alpha_1} \cdots D_d^{\alpha_d} = D_{x^1}^{\alpha_1} \cdots D_{x^d}^{\alpha_d} = \partial^{|\alpha|} / \partial (x^1)^{\alpha_1} \cdots \partial (x^d)^{\alpha_d}$.

$D_t^\alpha D_x^\beta = D_t^\alpha D_{x^1}^{\beta_1} \cdots D_{x^d}^{\beta_d}$.

$D^k = \{D^\alpha : |\alpha| = k\}$.

$\nabla = (\partial/\partial x^1, \ldots, \partial/\partial x^d)$.

The indicator function of E: $\mathbf{1}_E(x) = \begin{cases} 1, & x \in E \\ 0, & otherwise \end{cases}$

The sign function: $\mathrm{sign}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$

If $u : U \to \mathbb{R}$, $u^+ = \max(u, 0)$, $u^- = -\min(u, 0)$, $u = u^+ - u^-$, $|u| = u^+ + u^-$.

$u : U \to \mathbb{R}$ is called Lipschitz continuous if $|u(x) - u(y)| \le K|x - y|$, with K a constant, for all $x, y \in U$.

L_h = the space-discrete operator, p. 63.

$\mathcal{L}_h$ = the discrete parabolic operator, p. 37.

∂_i^+ = the forward discrete difference operator in space, p. 63.

∂_i^- = the backward discrete difference operator in space, p. 63.

Δ^+ = the forward discrete difference operator in time, p. 92.

Δ^- = the backward discrete difference operator in time, p. 82.

Notation for function spaces.

The notation $|\ \ |$ is used for the norm. Unless there is no risk of confusion, the corresponding space is identified. For instance, the norm in L^2 is denoted $|\ \ |_{L^2}$. The same applies to the inner product notation $(\ ,\ )$. The notation $\langle\ ,\ \rangle$ is used for the duality.

$C^k_{loc}(U)$, p. 29.

$C^k(U)$, p. 29.

$C^\infty(U) = \{u : U \to \mathbb{R} :\ u \text{ is infinitely differentiable}\}$.

$C_0^\infty\big([0,\infty)\big)$, p. 50.

$C_0^\infty(U)$, p. 58.

$C^{k+\delta}(U)$, the Hölder space, p. 29.

$C^{\delta/2,\delta}(Q)$, p. 30.

$C^{1,2}(Q) = \{u : Q \to \mathbb{R} :\ u,\ D_x u,\ D_x^2 u,\ u_t \in C(Q)\}$.

$C^{1+\delta/2,2+\delta}(Q)$, the parabolic Hölder space, p. 30.

$U \in C^r$ (or $\partial U \in C^r$), p. 31.

$|u|_{0;U} = [u]_{0;U}$, p. 29.

$[u]_{k;U}$, p. 29.

$|u|_{k;U}$, p. 29.

$[u]_{\delta;U}$, p. 29.

$[u]_{k+\delta;U}$, p. 29.

$|u|_{k+\delta;U}$, the Hölder norm, p. 29.

$[u]_{\delta/2,\delta;Q}$, p. 29.

$|u|_{\delta/2,\delta;Q}$, p. 29.

$[u]_{1+\delta/2,2+\delta;Q}$, p. 30.

$|u|_{1+\delta/2,2+\delta;Q}$, the parabolic Hölder norm, p. 30.

H denotes the Hilbert space.

$H^* =$ the dual space of H.

$V \hookrightarrow H$ denotes the embedding of space V in space H.

$L^1_{loc}(U)$, p. 58.

$L^2(U)$ denotes the set of all Lebesgue measurable functions $u : U \to \mathbb{R}$ such that $|u|_{L^2(U)} = (\int_U |u|^2 dx)^{1/2} < \infty$.

$W^{m,2}(U)$ denotes a Sobolev space, p. 58.

$W^{m,2}(r, \rho)$ denotes a weighted Sobolev space, p. 60.

$l^{0,2}$, $l^{1,2}$ denote discrete Sobolev spaces, p. 63 and 64.

$l^{0,2}(r)$, $l^{1,2}(r, \rho)$ denote discrete weighted Sobolev spaces, p. 74.

$(u, v)_{l^{0,2}}$, p. 63.

$|u|^2_{l^{0,2}}$, p. 63.

$(u, v)_{l^{1,2}}$, p. 64.

$|u|^2_{l^{1,2}}$, p. 64.

$(u, v)_{l^{0,2}(r)}$, p. 74.

$|u|^2_{l^{0,2}(r)}$, p. 74.

$(u, v)_{l^{1,2}(r,\rho)}$, p. 75.

$|u|^2_{l^{1,2}(r,\rho)}$, p. 74.

p $a.e.$ = the propriety p holds except for sets of measure zero.

Notation for stochastic processes.

$(\Omega, \mathcal{A}, \mathbf{P})$ = the probability space, where Ω is an abstract space, $\mathcal{A}$ is a $\sigma-$algebra of Ω and $\mathbf{P}$ is a probability measure on $\mathcal{A}$.

$(X_t)_{t\geq 0}$ = a stochastic process, p. 5.

$(\mathcal{F}_t)_{t\geq 0}$ = a filtration in $\mathcal{A}$, p. 5.

$(W_t)_{t\geq 0}$, $(B_t)_{t\geq 0}$ denote a standard Brownian motion, p. 6.

$(M_t)_{t\geq 0}$ usually denotes a martingale, p. 7.

$\mathbf{E}(X) = \int X d\mathbf{P}$, the expectation of X.

$\mathbf{E}(X|\mathcal{B})$, with $\mathcal{B}$ a $\sigma-$algebra, denotes the conditional expectation of X.

τ usually denotes a stopping time, p. 6.

$X_s^{t,x}$, $s \geq t$ = the solution of a stochastic differential equation starting from x at time t, p. 15.

$p \quad \mathbf{P}$ *a.s.* = the propriety p holds except for $\mathbf{P}-$null sets.

Notation for estimates.

We usually use the letters K, L, M, and N to denote a constant depending explicitly on known quantities. In many cases during the computations, we use the same letter even if the constant's value changes from one step to the next.

Appendix B

Useful results

Basic inequalities.

Jensen's inequality: Assume $f : \mathbb{R} \to \mathbb{R}$ is convex and U is open bounded subset of $\mathbb{R}^d$. Let $u : U \to \mathbb{R}$ be summable. Then

$$f(\frac{1}{|U|}\int_U u dx) \leq \frac{1}{|U|}\int_U f(u)dx.$$

Cauchy's inequality: $\quad ab \leq \frac{a^2}{2} + \frac{b^2}{2}\,, \quad a, b \in \mathbb{R}.$

Cauchy's inequality with ε: $\quad ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}\,, \quad a, b > 0,\ \varepsilon > 0.$

Minkowski's inequality for sums:

$$(\sum_{i=1}^{\infty} |v_i + w_i|^2)^{1/2} \leq (\sum_{i=1}^{\infty} |v_i|^2)^{1/2} + (\sum_{i=1}^{\infty} |w_i|^2)^{1/2},$$

with $v_i, w_i \in \mathbb{R}$, $\sum_{i=1}^{\infty} |v_i|^2 < \infty$, $\sum_{i=1}^{\infty} |w_i|^2 < \infty$.

Cauchy-Schwarz inequality: $\quad |(x, y)| \leq |x||y|\,, \quad x, y \in \mathbb{R}^d.$

Chebyshev's inequality: $\quad \mathbf{P}(|X| \geq k) \leq \frac{\mathbf{E}(X^2)}{k^2}\,, \quad X$ a random variable, $k > 0$.

Convergence theorems for integrals.

Monotone Convergence Theorem: Assume the functions $\{f_k\}_{k=1}^{\infty}$ are measurable with $f_1 \leq f_2 \leq \cdots \leq f_k \leq f_{k+1} \leq \cdots$ Then

$$\int_{\mathbb{R}^d} \lim_{k\to\infty} f_k dx = \lim_{k\to\infty} \int_{\mathbb{R}^d} f_k dx.$$

Dominated Convergence Theorem: Assume the functions $\{f_k\}_{k=1}^{\infty}$ are integrable, $f_k \to f$ *a.e.* and $|f_k| \leq g$ *a.e.*, for some summable function g. Then

$$\int_{\mathbb{R}^d} f_k dx \to \int_{\mathbb{R}^d} f dx.$$

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