Optimal reinsurance for variance related premium calculation principles[1]

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Abstract: In this paper we deal with the numerical computation of the optimal form of reinsurance from the ceding company point of view, when the cedent seeks to maximize the adjustment coefficient of the retained risk and the reinsurance loading is an increasing function of the variance.

We compare the optimal treaty with the best stop loss policy. The optimal arrangement can provide a significant improvement in the adjustment coefficient when compared to the best stop loss treaty. Further, it is substantially more robust with respect to choice of retention level than stop-loss treaties.

Keywords: adjustment coefficient, expected utility of wealth, optimal reinsurance, stop loss, standard deviation premium principle, variance premium principle.

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1 Introduction

This paper deals with optimal reinsurance when the insurer seeks to maximize the adjustment coefficient of the retained risk and the reinsurer prices reinsurance using a loading which is an increasing function g of the variance of the accepted risk. Important instances of such pricing principles are the variance and the standard deviation principles.

Guerra and Centeno (2008) studied the problem of determining the optimal reinsurance policy using as optimality criterion the adjustment coefficient. Assuming that the reinsurance premium is convex and satisfies some very general regularity assumptions, it was shown that the optimal reinsurance scheme always exists and it is unique "up to an economic equivalent treaty". A necessary optimality condition was found, that in principle allows for the computation of the optimal treaty.

The proofs in Guerra and Centeno (2008) were obtained by relating the adjustment coefficient with the expected utility of wealth for an exponential utility function. The type of reinsurance arrangement that maximizes the expected utility of wealth is the same type that maximizes the adjustment coefficient and vice versa. Further, the optimal policies for both problems coincide when the risk aversion coefficient is equal to the adjustment coefficient of the retained risk. For example, if for a given premium functional P, stop loss maximizes the adjustment coefficient (which will be the case when P is calculated according to the expected value principle), then stop loss is also optimal for the expected utility problem, and vice-versa. The retention limit on the expected utility problem will depend of course on the risk aversion coefficient of the exponential utility function. When the risk aversion coefficient equals the adjustment coefficient of the retained risk, then that particular adjustment coefficient is maximal and the same retention limit maximizes the expected utility and the adjustment coefficient.

In the case that concerns us specifically in the present paper, namely when the loading is an increasing function of the variance, it was shown that the optimal arrangement is a nonlinear function of a type previously unknown in the reinsurance literature.

We have three objectives in the present paper: first to characterize the functions g that provide convex premium calculation principles, second to show that the solution mentioned above can easily be computed by standard numerical methods and third to compare the performance of the optimal treaty with the best stop-loss policy, under fairly realistic reinsurance loadings and claim distributions.

Comparison with stop-loss treaties is meaningful because it is by far the most widely known type of aggregate treaty that guarantees existence of the adjustment coefficient for the retained risk in cases when de distribution of the aggregate claims has an heavy tail, as is usually the case in practical applications. Further, there are well known results in the literature showing that stop-loss is the optimal treaty for various types of optimality criteria and several assumptions on the reinsurance premium. Such results go back to Borch (1960) and Arrow (1963) which considered the variance and the expected utility of wealth, respectively, as optimality criteria. Hesselager (1990) proved an equivalent result using the adjustment coefficient as optimality criterion. Some recent results in favor of stop-loss treaties are found in Kaluszka (2004).

The text is organized as follows: Section 2 contains the main assumptions and characterizes convex variance-related premium principles. Section 3 contains the statement of the problem and a short overview of the main results in Guerra and Centeno (2008) concerning specifically the case when the reinsurance loading is an increasing function of the variance. This overview is kept to the minimum required to make the paper self-contained. Interested readers are referred to Guerra and Centeno (2008) for a full theoretical analysis of the interrelated problems of maximizing the expected utility of the insurer's wealth and maximizing the adjustment coefficient of the retained risk. Some theoretical details which are useful in the computation of optimal treaties are added in the present paper. Section 4 contains an analysis of the main issues arising in the numerical computation of optimal treaties. We show that thought the solution given in Section 3 is in an implicit form, it can be numerically computed using classical methods. In Section 5 we compare the optimal policy with the best stop loss policy with respect to a standard deviation principle for two different claim distributions. The distributions are chosen to have identical first two moments but quite different tails. The results suggest that the optimal policy not only can offer significant improvement in the value of the adjustment coefficient compared to the best stop-loss treaty, but also its performance is much more robust with respect to the retention level. This is an important feature for practical implementation where the data of the problem cannot be known with full accuracy and hence the chosen treaty is in fact suboptimal.

2 Assumptions and preliminaries

Let Y be a non-negative random variable, representing the annual aggregate claims and let us assume that aggregate claims over consecutive periods are i.i.d. random variables. We assume that Y is a continuous random variable, with density function f, and that $E[Y^2] < +\infty$. Let c, $c > E[Y]$, be the corresponding premium income, net of expenses. A map $Z : [0, +\infty) \mapsto [0, +\infty)$ identifies a reinsurance policy. The set of all possible reinsurance programmes is:

$$\mathcal{Z} = \{Z : [0, +\infty) \mapsto \mathbb{R} |\ Z \text{ is measurable and } 0 \leq Z(y) \leq y,\ \forall y \geq 0\}.$$

We do not distinguish between functions which differ only on a set of zero probability. i.e., two measurable functions, ϕ and ϕ' are considered to be the same whenever $\Pr\{\phi(Y) = \phi'(Y)\} = 1$. Similarly, a measurable function, Z, is an element of $\mathcal{Z}$ whenever $\Pr\{0 \leq Z(Y) \leq Y\} = 1$.

For a given reinsurance policy, $Z \in \mathcal{Z}$, the reinsurer charges a premium $P(Z)$ of the type

$$P(Z) = E[Z] + g(Var[Z]), \tag{1}$$

where $g : [0, +\infty) \mapsto [0, +\infty)$ is a continuous function smooth in $(0, +\infty)$ such that $g(0) = 0$ and $g'(x) > 0, \ \forall x \in (0, +\infty)$. Further we assume that P is a convex functional. We call premium calculation principles of this type "variance-related principles". The variance principle and the standard deviation principle are both under these conditions, with $g(x) = \beta x$ and $g(x) = \beta x^{1/2}$, $\beta > 0$, respectively. Convexity of these two principles was proved by Deprez and Gerber (1985), but also follows immediately from the Proposition 1, which characterizes convex variance-related premiums and will be useful in the next section.

Proposition 1 *Let $B = \sup\{Var[Z] : Z \in \mathcal{Z}\}$ and assume that g is twice differentiable in the interval $(0, B)$. $P(Z) = E[Z] + g(Var[Z])$ is a convex functional if and only if*

$$\frac{g''(x)}{g'(x)} \geq -\frac{1}{2x}, \qquad \forall x \in (0, B). \ \square \tag{2}$$

Proof. The proof below is an adaptation of the proof by Deprez and Gerber (1985) for a related result.

First, assume that the map $P : \mathcal{Z} \mapsto \mathbb{R}$ is convex. Fix $Z \in \mathcal{Z} \backslash \{0\}$ and consider the map $t \mapsto P(tZ), \ t \in [0, 1]$. Then

$$\begin{aligned} \frac{d^2}{dt^2} P(tZ) &= \frac{d^2}{dt^2} \left(tE[Z] + g\left(t^2 Var[Z]\right)\right) = \\ &= g''\left(t^2 Var[Z]\right) 4t^2 Var[Z]^2 + g'\left(t^2 Var[Z]\right) 2Var[Z]. \end{aligned}$$

Convexity of P implies convexity of the map $t \mapsto P(tZ), \ t \in [0, 1]$. It follows that $\frac{d^2}{dt^2} P(tZ) \geq 0$, i.e.,

$$\frac{g''\left(t^2 Var[Z]\right)}{g'\left(t^2 Var[Z]\right)} \geq \frac{-1}{2t^2 Var[Z]}$$

must hold for all $t \in (0, 1)$. Since $Z \in \mathcal{Z}$ is arbitrary, inequality (2) follows immediately.

Now, assume that inequality (2) holds and for each $Z, \ W \in \mathcal{Z}$ consider the map

$$t \mapsto \mathcal{P}_{Z,W}(t) = P(Z + t(W - Z)), \qquad t \in [0, 1].$$

From the definition of convex map, it follows that $Z \mapsto P(Z)$ is convex if and only if for every $Z, \ W \in \mathcal{Z}$ the map $t \mapsto \mathcal{P}_{Z,W}(t)$ is convex. The maps $t \mapsto \mathcal{P}_{Z,W}(t)$ are continuous in $[0, 1]$, twice differentiable in $(0, 1)$, and

$$\begin{aligned} \mathcal{P}''_{Z,W}(t) = \ & 4g''\left(Var[Z + t(W - Z)]\right)\left(Cov[Z, W - Z] + tVar[W - Z]\right)^2 + \\ & + 2g'\left(Var[Z + t(W - Z)]\right) Var[W - Z]. \end{aligned}$$

In particular,

$$\mathcal{P}''_{Z,W}(0) = 4g''\left(Var[Z]\right)Cov[Z,W-Z]^2 + 2g'\left(Var[Z]\right)Var[W-Z] =$$

$$= \ 4g''\left(Var[Z]\right)\left(Cov[Z,W] - Var[Z]\right)^2 +$$

$$+2g'\left(Var[Z]\right)\left(Var[W] - 2Cov[Z,W] + Var[Z]\right) =$$

$$= 2g'\left(Var[Z]\right)\left(\tfrac{2g''(Var[Z])}{g'(Var[Z])}\left(Cov[Z,W] - Var[Z]\right)^2 + Var[W] - 2Cov[Z,W] + Var[Z]\right).$$

By inequality (2), this implies

$$\mathcal{P}''_{Z,W}(0) \geq$$

$$\geq 2g'\left(Var[Z]\right)\left(\tfrac{-1}{Var[Z]}\left(Cov[Z,W] - Var[Z]\right)^2 + Var[W] - 2Cov[Z,W] + Var[Z]\right) =$$

$$= \tfrac{2g'(Var[Z])}{Var[Z]}\left(Var[W]Var[Z] - Cov[Z,W]^2\right).$$

Then, the Cauchy-Schwarz inequality guarantees that

$$\mathcal{P}''_{Z,W}(0) \geq 0, \qquad \forall\, Z, W \in \mathcal{Z}. \tag{3}$$

We conclude the proof by showing that inequality (3) implies the apparently stronger condition

$$\mathcal{P}''_{Z,W}(t) \geq 0, \qquad \forall\, Z, W \in \mathcal{Z}, \forall\, t \in (0,1).$$

In order to do this, notice that

$$\mathcal{P}_{Z+t(W-Z),W}(s) = P\left(Z + t(W-Z) + s\left(W - \left(Z + t(W-Z)\right)\right)\right) = \mathcal{P}_{Z,W}(t + s(1-t))$$

holds for every $Z, W \in \mathcal{Z}$, $t, s \in (0,1)$ and $t, s \in (0,1)$ implies $t + s(1-t) \in (0,1)$. If follows that

$$\frac{d^2}{ds^2}\mathcal{P}_{Z+t(W-Z),W}(s)\bigg|_{s=0} = \frac{d^2}{ds^2}\mathcal{P}_{Z,W}(t + s(1-t))\bigg|_{s=0} = \mathcal{P}''^{2}_{Z,W},$$

which concludes the proof. ■

Remark 1 *Condition (2) holds as an equality for the standard deviation principle and the left hand side of (2) is zero for the variance principle. Hence both principles are convex.*

The net profit, after reinsurance, is

$$L_Z = c - P\left(Z\right) - \left(Y - Z\left(Y\right)\right).$$

We assume that c, P and the claim size distribution are such that

$$\Pr\left\{L_Z < 0\right\} > 0, \qquad \forall Z \in \mathcal{Z}, \tag{4}$$

otherwise there would exist a policy under which the risk of ruin would be zero. This requires the premium loading to be sufficiently high. Namely, for the variance principle it requires that the inequality

$$\beta > (c - E[Y])/Var[Y] \tag{5}$$

holds. In the standard deviation principle case the required condition is

$$\beta > (c - E[Y])/\sqrt{Var[Y]}. \tag{6}$$

Consider the map $G : \mathbb{R} \times \mathcal{Z} \mapsto [0, +\infty]$, defined by

$$G(R, Z) = \int_0^{+\infty} e^{-RL_Z(y)} f(y)\; dy, \qquad R \in \mathbb{R},\; Z \in \mathcal{Z}. \tag{7}$$

Let R_Z denote the adjustment coefficient of the retained risk for a particular reinsurance policy, $Z \in \mathcal{Z}$. R_Z is defined as the strictly positive value of R which solves the equation

$$G(R, Z) = 1, \tag{8}$$

for that particular Z, when such a root exists. Equation (8) can not have more than one positive solution. This means the map $Z \mapsto R_Z$ is a well defined functional in the set

$$\mathcal{Z}^+ = \{Z \in \mathcal{Z} : \; (8) \text{ admits a positive solution}\}.$$

3 Optimal reinsurance policies for variance related premiums

Theorem 1 below, which proof can be seen in Guerra and Centeno (2008), provides the solution, under the assumptions made on Section 2, to the following problem:

Problem 1 *Find* $\left(\hat{R}, \hat{Z}\right) \in (0, +\infty) \times \mathcal{Z}^+$ *such that*

$$\hat{R} = R_{\hat{Z}} = \max\left\{R_Z : Z \in \mathcal{Z}^+\right\}. \square$$

In what follows $\nu \in [0, +\infty)$ denotes the number

$$\nu = \sup\{y : \Pr\{Y \leq y\} = 0\}.$$

Theorem 1 *A solution to Problem 1 always exists.*

a) When g' is a bounded function in a neighborhood of zero, the adjustment coefficient of the retained aggregate claims is maximized when $Z(y)$ satisfies

$$y = Z(y) + \frac{1}{R} \ln \frac{Z(y) + \alpha}{\alpha}, \qquad \forall y \geq 0, \tag{9}$$

where α is a positive solution to

$$h(\alpha) = 0, \tag{10}$$

with

$$h(\alpha) = \alpha + E[Z] - \frac{1}{2g'(Var[Z])}. \tag{11}$$

and R is the unique positive root to equation (8).

When g' is unbounded in any neighborhood of zero, then either a contract satisfying (8), (9) and (10) is optimal or the optimal treaty is $Z(y) = 0$, $\forall y$ (no reinsurance at all) and no solution to (8), (9), (10) exists.

b) If $\nu = 0$, the solution is unique. If $\nu > 0$ then all solutions are of the form $Z(y) + x$, where $Z(y)$ is the treaty described in a) and x is any constant such that $-Z(\nu) \leq x \leq \nu - Z(\nu)$. $\square$

Theorem 1 evokes some simple remarks:

Remark 2 *Under the optimal treaty the direct insurer always retains some part of the tail of the risk distribution. Off course, the retained tail must always be "light" since the corresponding adjustment coefficient is guaranteed to exist.*

Remark 3 *If the optimal treaty is not unique (i.e., if $\nu > 0$) then any two optimal treaties differ only by a constant. This implies that all optimal treaties provide the same profit ($L_{Z+x} = L_Z$, since $P(Z + x) = P(Z) + x$), and hence are indifferent from the economic point of view.*

Remark 4 *Let Z satisfy (9). Although Z is not an explicitly function of Y, its distribution function can easily be calculated. Since the left-hand side of (9) is strictly increasing with respect to Z, the distribution function of Z is*

$$F_Z(\zeta) = \Pr\left\{Y \leq \zeta + \frac{1}{R}\ln\frac{\zeta+\alpha}{\alpha}\right\} = F\left(\zeta + \frac{1}{R}\ln\frac{\zeta+\alpha}{\alpha}\right). \tag{12}$$

Therefore its density function is

$$f_Z(\zeta) = f\left(\zeta + \frac{1}{R}\ln\frac{\zeta+\alpha}{\alpha}\right)\frac{1+R(\zeta+\alpha)}{R(\zeta+\alpha)}. \tag{13}$$

Theorem 1 leaves some ambiguity about the number of roots of equation (10). We will show below that this equation has at most one solution.

First, let us introduce the functions

$$\Phi_k\left(R,\alpha\right) = \int_0^{+\infty}\left(1 + R\left(Z\left(y\right)+\alpha\right)\right)^k f\left(y\right)\ dy, \qquad k \in \mathbb{Z}, \tag{14}$$

where $Z(y)$ is such that (9) holds for the particular (R, α) indicated. These functions are useful to prove the properties below. They are also convenient to deal with issues related to numerical computation of optimal treaties.

Remark 5 *Since we assume that $E[Y^2] < +\infty$, Φ_k is finite for all $k \leq 2$, $\alpha > 0$, $R > 0$.*

Remark 6 *For $k \geq 0$, it is clear that Φ_k is a linear combination of the moments of order $\leq k$ of Z. Indeed, a simple computation shows that for the first two moments we have:*

$$E[Z] = \frac{1}{R}\left(\Phi_1 - (1 + R\alpha)\right), \tag{15}$$

$$Var[Z] = \frac{1}{R^2}\left(\Phi_2 - \Phi_1^2\right). \tag{16}$$

The reason why we use the functions Φ_k, instead of formulating the following arguments in terms of moments of Z, is that due to Proposition 2 below, functions Φ_k with $k < 0$ turn out naturally in our proof of Proposition 4, making some expressions far more complicated when expressed in terms of moments. Further, the same functions appear again when Newton-type algorithms are considered to compute the numerical solutions of the problem.

Derivatives of Φ_k with respect to the parameter α can be easily computed:

Proposition 2 *For $k \leq 2$, $R > 0$ the map $\alpha \mapsto \Phi_k(R, \alpha)$ is smooth and*

$$\frac{\partial \Phi_k}{\partial \alpha} = k\left(\frac{1}{\alpha} + R\right)\left(\Phi_{k-1} - \Phi_{k-2}\right). \square \tag{17}$$

Proof. From (9) it follows that

$$\frac{\partial Z(y)}{\partial \alpha} = \frac{Z(y)}{\alpha(1 + R(Z(y) + \alpha)} = \frac{1}{\alpha R} - \frac{1 + \alpha R}{\alpha R}\frac{1}{1 + R(Z(y) + \alpha)}.$$

Then,

$$\begin{aligned} \frac{\partial \Phi_k}{\partial \alpha} &= \int_0^{+\infty} k(1 + R(Z(y) + \alpha)^{k-1} R\left(\frac{\partial Z}{\partial \alpha} + 1\right) f(y)dy = \\ &= \frac{k(1 + \alpha R)}{\alpha}\int_0^{+\infty}\left((1 + R(Z(y) + \alpha))^{k-1} - (1 + R(Z(y) + \alpha))^{k-2}\right) f(y)dy, \end{aligned}$$

from where follows (17). ■

This Proposition allows us to state the following:

Proposition 3

$$\frac{\partial E[Z]}{\partial \alpha} = \frac{1}{R\alpha} - \frac{1 + R\alpha}{R\alpha}\Phi_{-1}, \tag{18}$$

$$\frac{\partial Var[Z]}{\partial \alpha} = \frac{2(1 + R\alpha)}{R^2\alpha}\left(\Phi_1\Phi_{-1} - 1\right). \square \tag{19}$$

Using the material above we are able to prove the following uniqueness result:

Proposition 4 *Suppose that g is twice differentiable in the interval $(0,+\infty)$. For each $R\in(0,+\infty)$ (fixed) equation (10) has at most one solution, $\alpha_R>0$.*
If such a solution exists, then $h'(\alpha_R)>0$ holds. Therefore $h(\alpha)$ is strictly negative for $\alpha\in(0,\alpha_R)$, and it is strictly positive for $\alpha\in(\alpha_R,+\infty)$. $\square$

Proof. Throughout this proof we consider $Z(y)$ defined by (9).
Differentiating $h(\alpha)$, for $\alpha>0$, we get

$$\frac{\partial h}{\partial\alpha}(\alpha)=1+\frac{\partial E[Z]}{\partial\alpha}+\frac{1}{2}\frac{g''(Var[Z])}{(g'(Var[Z]))^2}\frac{\partial Var[Z]}{\partial\alpha}. \tag{20}$$

At the points where $h(\alpha)=0$, we must have

$$\frac{1}{2}=(E[Z]+\alpha)g'(Var[Z]) \tag{21}$$

and hence

$$\left.\frac{\partial h}{\partial\alpha}(\alpha)\right|_{h(\alpha)=0}=1+\frac{\partial E[Z]}{\partial\alpha}+(E[Z]+\alpha)\frac{g''(Var[Z]}{g'(Var[Z])}\frac{\partial Var[Z]}{\partial\alpha}. \tag{22}$$

Noticing that $E[Z]+\alpha$ and $\partial Var[Z]/\partial\alpha$ (given by (19)) are positive and using Proposition 1 we have that

$$\left.\frac{\partial h}{\partial\alpha}(\alpha)\right|_{h(\alpha)=0}\geq 1+\frac{\partial E[Z]}{\partial\alpha}-\frac{(E[Z]+\alpha)}{2Var[Z]}\frac{\partial Var[Z]}{\partial\alpha}. \tag{23}$$

Now, using (15), (16), (18) and (19) we get

$$\begin{aligned}
\left.\frac{\partial h}{\partial\alpha}(\alpha)\right|_{h(\alpha)=0} &\geq \left(1+\frac{1}{R\alpha}\right)(1-\Phi_{-1})-\frac{E[Z]+\alpha}{\Phi_2-\Phi_1^2}\left(\frac{1}{\alpha}+R\right)(\Phi_1\Phi_{-1}-1)= \\
&= \left(1+\frac{1}{R\alpha}\right)(1-\Phi_{-1})-\frac{\frac{1}{R}(\Phi_1-1)}{\Phi_2-\Phi_1^2}\left(\frac{1}{\alpha}+R\right)(\Phi_1\Phi_{-1}-1)= \\
&= \left(1+\frac{1}{R\alpha}\right)\left(1-\Phi_{-1}-\frac{(\Phi_1-1)(\Phi_1\Phi_{-1}-1)}{\Phi_2-\Phi_1^2}\right)= \\
&= \frac{1+R\alpha}{R\alpha(\Phi_2-\Phi_1^2)}\left((1-\Phi_{-1})(\Phi_2-\Phi_1^2)-(\Phi_1-1)(\Phi_1\Phi_{-1}-1)\right)= \\
&= \frac{1+R\alpha}{R\alpha(\Phi_2-\Phi_1^2)}\left((\Phi_2-\Phi_1)(1-\Phi_{-1})-(\Phi_1-1)^2\right).
\end{aligned}$$

Noticing that

$$\begin{aligned}
\Phi_2-\Phi_1 &= \int_0^{+\infty} R(Z(y)+\alpha)\left(1+R\left(Z(y)+\alpha\right)\right)f(y)\ dy, \\
1-\Phi_{-1} &= \int_0^{+\infty}\frac{R\left(Z(y)+\alpha\right)}{1+R(Z(y)+\alpha)}f(y)dy, \\
\Phi_1-1 &= \int_0^{+\infty} R(Z(y)+\alpha)f(y)\ dy= \\
&= \int_0^{+\infty}\sqrt{R(Z(y)+\alpha)(1+R(Z(y)+\alpha))}\sqrt{\frac{R(Z(y)+\alpha)}{(1+R(Z(y)+\alpha))}}\ f(y)\,dy,
\end{aligned}$$

and recalling that the Cauchy–Schwarz inequality states that

$$E^2[X_1X_2] \leq E[X_1^2]E[X_2^2],$$

holds for any random variables such that $E[X_1^2] < +\infty$ and $E[X_2^2] < +\infty$, with strict inequality when X_1, X_2 are linearly independent, we conclude that $\frac{\partial h}{\partial \alpha}(\alpha)\big|_{h(\alpha)=0} > 0$. Hence there exists at most a positive solution to equation (10), in which case $h(\alpha) < 0$ holds for every α between zero and the root of (10). ■

Remark 7 *For the variance premium calculation principle we have $g(x) = \beta x$, $\beta > 0$. Therefore $g' \equiv \beta$ is bounded in a neighborhood of zero. Therefore, Theorem 1 guarantees that the optimal reinsurance policy is always a nonzero policy. Since the solution for (8), (9), (10) is unique, it gives indeed the optimal solution (and not any other critical point of the adjustment coefficient).*

This contrasts with the case of the standard deviation principle where $g(x) = \beta x^{1/2}$, $\beta > 0$, and hence $g'(x) = \frac{\beta}{2}x^{-1/2}$ is unbounded in any neighborhood of zero. In this case the optimal policy may be not to reinsure any risk, but this can only happen when the tail of the distribution of Y is light such that the moments generating function of Y is finite in some neighborhood of zero. In any case, if the optimal policy is different from no reinsurance it will be given again by the unique solution for (8), (9), (10).

4 Numerical calculation of optimal treaties

Let $G(R, \alpha)$ be defined as $G(R, Z)$ with Z satisfying (9) for that particular (R, α). The following proposition gives a convenient expression for $G(R, \alpha)$:

Proposition 5 *$G(R, \alpha)$ can be computed by*

$$G(R, \alpha) = \frac{1}{\alpha}\left(E[Z] + \alpha\right) e^{R(P(Z)-c)}. \ \square$$

Proof.

$$\begin{aligned} G(R, \alpha) &= e^{R(P(Z)-c)} \int_0^{+\infty} e^{R(y-Z(y))} f(y)dy = \\ &= e^{R(P(Z)-c)} \int_0^{+\infty} \frac{Z(y) + \alpha}{\alpha} f(y)dy = \\ &= \frac{1}{\alpha}\left(E[Z] + \alpha\right) e^{R(P(Z)-c)}. \end{aligned}$$

■

Now we are ready to proceed into the discussion of numerical solution of the system (8), (10). We will show that this can be achieved by a simple combination of standard algorithms for quadrature and for solution of nonlinear equations.

Notice that, for any $R \in (0, +\infty)$ (fixed) the functions $Z(y)$ satisfying (9) converge pointwise to $Z(y) = y$ when $\alpha \to +\infty$ and converge pointwise to $Z \equiv 0$ when $\alpha \to 0^+$. It follows that $\lim_{\alpha \to +\infty} h(\alpha) = +\infty$ and hence Proposition 4 implies that equation (10) has a solution if and only if $h(\alpha) < 0$ for some $\alpha > 0$.

This observation shows that it is quite easy to devise numerical schemes that for any $R > 0$ (fixed) always converge to the corresponding solution of (10) if such a solution exists and converge to $\alpha = 0$ if such a solution does not exist. For an example of such a procedure (thought not a very efficient one), consider the bisection algorithm starting with a sufficiently large interval $[0, \alpha_0]$ and use formulae (15), (16) to compute $h(\alpha)$.

For each $R > 0$, let α_R denote the unique solution of (10) if it exists, otherwise $\alpha_R = 0$. The results in Guerra and Centeno (2008) guarantee that the one-variable equation

$$G(R, \alpha_R) = 1 \tag{24}$$

has one unique solution $R^* \in (0, +\infty)$ and that (R^*, α_{R^*}) is the solution of (8), (10). Further, $G(R, \alpha_R) < 1$ holds for every $R \in (0, R^*)$, while $G(R, \alpha_R) > 1$ holds for every $R \in (R^*, +\infty)$. Thus, it is also easy to devise algorithms to solve (24) that always converge to the solution.

The discussion above shows that algorithms that always converge to the solution of our problem require only evaluations $h(\alpha)$ and $G(R, \alpha)$, which can be obtained using (15), (16) and Proposition 5. Typically, such methods have the drawback of having quite slow convergence rates. Therefore, one may wish to use Newton-type algorithms which have good properties of rapid (local) convergence and high accuracy, provided that the left-hand side of the equations and its first order partial derivatives can be quickly and accurately computed.

In order to see that this can also be achieved, we introduce the functions

$$\Psi_k(R, \alpha) = \int_0^{+\infty} (1 + R\,(Z(y) + \alpha))^k \ln\left(\frac{Z(y) + \alpha}{\alpha}\right) f(y)dy, \qquad R > 0,\ \alpha > 0,\ k \in \mathbb{Z}.$$

Using Proposition 2 it is straightforward to obtain convenient expressions for the partial derivative with respect to α of the left-hand side of (8), (10). The following proposition does the same for the partial derivative with respect to R.

Proposition 6 *For $k \leq 2$, $\alpha > 0$ the map $R \mapsto \Phi_k(R, \alpha)$ is smooth and*

$$\frac{\partial \Phi_k}{\partial R} = \frac{k}{R}\left(\Phi_k - \Phi_{k-1} + \Psi_{k-1} - \Psi_{k-2}\right). \ \square$$

Proof. By differentiating (9) with respect to R, we obtain

$$0 = \frac{\partial Z}{\partial R} - \frac{1}{R^2} \ln\left(\frac{Z+\alpha}{\alpha}\right) + \frac{1}{R}\frac{1}{Z+\alpha}\frac{\partial Z}{\partial R}.$$

Solving this with respect to $\frac{\partial Z}{\partial R}$ yields

$$\frac{\partial Z}{\partial R} = \frac{1}{R^2}\left(1 - \frac{1}{1+R(Z+\alpha)}\right)\ln\left(\frac{Z+\alpha}{\alpha}\right).$$

The assumption $E[Y^2] < +\infty$ implies that Ψ_k is finite for all $k \le 1$, $R > 0$, $\alpha > 0$. It follows that

$$\begin{aligned}
\frac{\partial \Phi_k}{\partial R} &= \int_0^{+\infty} k\left(1+R\left(Z+\alpha\right)\right)^{k-1}\left(Z+\alpha+R\frac{\partial Z}{\partial R}\right) f dy = \\
&= \int_0^{+\infty} k\left(1+R\left(Z+\alpha\right)\right)^{k-1} \times \\
&\qquad \times \left(Z+\alpha+\tfrac{1}{R}\left(1-\tfrac{1}{1+R(Z+\alpha)}\right)\ln\left(\tfrac{Z+\alpha}{\alpha}\right)\right) f dy = \\
&= \frac{k}{R}\int_0^{+\infty} \left(1+R\left(Z+\alpha\right)\right)^{k-1}\left(1+R(Z+\alpha)-1\right) f dy + \\
&\quad + \frac{k}{R}\int_0^{+\infty} \left(1+R\left(Z+\alpha\right)\right)^{k-1}\left(1-\frac{1}{1+R(Z+\alpha)}\right)\ln\left(\frac{Z+\alpha}{\alpha}\right) f dy = \\
&= \frac{k}{R}\left(\Phi_k - \Phi_{k-1} + \Psi_{k-1} - \Psi_{k-2}\right).
\end{aligned}$$

■

We see that the system (8), 10) can be solved by algorithms using derivatives provided we find a way to compute Φ_2, Φ_1, Ψ_1, Ψ_0, Ψ_{-1}. Hence, the important point that remains open is to show that the functions Φ_k, Ψ_k can be computed by standard integration methods. The following proposition shows that Φ_k, Ψ_k can be given an explicit form thought Z is given only in the implicit form (9).

Proposition 7 *The functions Φ_k, Ψ_k can be represented as the integrals:*

$$\Phi_k(R,\alpha) = \frac{1}{R}\int_0^{+\infty} \frac{\left(1+R\left(\zeta+\alpha\right)\right)^{k+1}}{\zeta+\alpha} f\left(\zeta + \frac{1}{R}\ln\frac{\zeta+\alpha}{\alpha}\right) d\zeta, \qquad (25)$$

$$\Psi_k(R,\alpha) = \frac{1}{R}\int_0^{+\infty} \frac{\left(1+R\left(\zeta+\alpha\right)\right)^{k+1}}{\zeta+\alpha} \ln\left(\frac{\zeta+\alpha}{\alpha}\right) f\left(\zeta + \frac{1}{R}\ln\frac{\zeta+\alpha}{\alpha}\right) d\zeta. \; \square \qquad (26)$$

Proof. Using the change of variable $y = \zeta + \frac{1}{R}\ln\frac{\zeta+\alpha}{\alpha}$, $\zeta \in [0, +\infty[$, we obtain

$$\begin{aligned}
\Phi_k &= \int_0^{+\infty} \left(1+R\left(Z\left(y\right)+\alpha\right)\right)^k f\left(y\right)\, dy = \\
&= \int_0^{+\infty} \left(1+R\left(\zeta+\alpha\right)\right)^k f\left(\zeta+\frac{1}{R}\ln\frac{\zeta+\alpha}{\alpha}\right)\left(1+\frac{1}{R\left(\zeta+\alpha\right)}\right) d\zeta = \\
&= \frac{1}{R}\int_0^{+\infty} \frac{\left(1+R\left(\zeta+\alpha\right)\right)^{k+1}}{\zeta+\alpha} f\left(\zeta+\frac{1}{R}\ln\frac{\zeta+\alpha}{\alpha}\right) d\zeta.
\end{aligned}$$

The proof of equality (26) is analogous. ■

It is well known that, generically speaking, integrals of smooth functions in compact intervals are much easier to evaluate numerically than other types of integrals. We conclude this section by giving a procedure that allows for the reduction of the integrals (25), (26) into sums of integrals of smooth functions in compact intervals, provided the density f satisfies suitable regularity assumptions, which are usually met in practical applications.

Notice that our blanket assumption that $E[Y^2] < +\infty$ guarantees that $\lim_{y\to+\infty} y^3 f(y) = 0$ holds. In the following we need a stronger condition, namely, existence of some $\varepsilon > 0$ such that

$$\lim_{y\to+\infty} y^{3+\varepsilon} f(y) = 0 \tag{27}$$

holds. Notice that condition (27) is sufficient but not necessary for $E[Y^2] < +\infty$. Also, we will assume that the density f is a continuous function in $[0, +\infty)$, with the possible exception of a finite set of points, where it has right and left limits (possibly infinite). Thus, the density f can be unbounded but only in neighborhoods of a finite number of points of discontinuity. In this case, there is a partition $0 = a_0 < a_1 < ... < a_m < +\infty$ such that the map $\zeta \mapsto f\left(\zeta + \frac{1}{R}\ln\frac{\zeta+\alpha}{\alpha}\right)$ is continuous and bounded in $[a_m, +\infty)$ and for each $i \in \{1, 2, ..., m\}$ it is continuous in one semiclosed interval, $(a_{i-1}, a_i]$ or $[a_{i-1}, a_i)$. Thus, we only need to reduce integrals over the intervals $[a_m, +\infty)$ and $(a_{i-1}, a_i]$ or $[a_{i-1}, a_i)$, $i = 1, 2, ..., m$.

If condition (27) holds then the condition

$$\lim_{y\to+\infty} y^{2+\varepsilon} \ln(y)\, f(y) = 0 \tag{28}$$

also holds. Using the change of variable $\zeta = t^{-\frac{1}{\varepsilon}} - 1$, we obtain

$$\begin{aligned}\int_{a_m}^{+\infty} &\frac{(1+R(\zeta+\alpha))^{k+1}}{\zeta+\alpha} f\left(\zeta + \frac{1}{R}\ln\left(\frac{\zeta+\alpha}{\alpha}\right)\right) d\zeta = \\ &= \frac{1}{\varepsilon}\int_0^{(1+a_m)^{-\varepsilon}} \frac{(1+R(t^{-\frac{1}{\varepsilon}}-1+\alpha))^{k+1}}{t^{-\frac{1}{\varepsilon}}-1+\alpha} \times \\ &\qquad \times f\left(t^{-\frac{1}{\varepsilon}} - 1 + \tfrac{1}{R}\ln\left(\tfrac{t^{-\frac{1}{\varepsilon}}-1+\alpha}{\alpha}\right)\right) t^{-\frac{1}{\varepsilon}(1+\varepsilon)} dt,\end{aligned} \tag{29}$$

$$\begin{aligned}\int_{a_m}^{+\infty} &\frac{(1+R(\zeta+\alpha))^{k+1}}{\zeta+\alpha} \ln\left(\frac{\zeta+\alpha}{\alpha}\right) f\left(\zeta + \frac{1}{R}\ln\left(\frac{\zeta+\alpha}{\alpha}\right)\right) d\zeta = \\ &= \frac{1}{\varepsilon}\int_0^{(1+a_m)^{-\varepsilon}} \frac{(1+R(t^{-\frac{1}{\varepsilon}}-1+\alpha))^{k+1}}{t^{-\frac{1}{\varepsilon}}-1+\alpha} \ln\left(\tfrac{t^{-\frac{1}{\varepsilon}}-1+\alpha}{\alpha}\right) \times \\ &\qquad \times f\left(t^{-\frac{1}{\varepsilon}} - 1 + \tfrac{1}{R}\ln\left(\tfrac{t^{-\frac{1}{\varepsilon}}-1+\alpha}{\alpha}\right)\right) t^{-\frac{1}{\varepsilon}(1+\varepsilon)} dt.\end{aligned} \tag{30}$$

We can check that the integrand on the right-hand side of (29) is bounded for $k \le 2$ and the integrand on the right-hand side of (30) is bounded for $k \le 1$.

Now, consider the case when $\zeta \mapsto f\left(\zeta + \frac{1}{R}\ln\frac{\zeta+\alpha}{\alpha}\right)$ is continuous in $(a_{i-1}, a_i]$ (resp., $[a_{i-1}, a_i)$) but

$$\lim_{y \to (a_{i-1} + \frac{1}{R}\ln\frac{a_{i-1}+\alpha}{\alpha})^+} f(y) = +\infty \qquad \left(\text{resp.,} \lim_{y \to (a_i + \frac{1}{R}\ln\frac{a_i+\alpha}{\alpha})^-} f(y) = +\infty\right).$$

Since f is integrable, it follows that

$$\lim_{y \to (a_{i-1} + \frac{1}{R}\ln\frac{a_{i-1}+\alpha}{\alpha})^+} f(y)\sqrt{y - (a_{i-1} + \frac{1}{R}\ln\frac{a_{i-1}+\alpha}{\alpha})} = 0$$

$$\left(\text{resp.,} \lim_{y \to (a_i + \frac{1}{R}\ln\frac{a_i+\alpha}{\alpha})^-} f(y)\sqrt{a_i + \frac{1}{R}\ln\frac{a_i+\alpha}{\alpha} - y} = 0\right)$$

must hold.

Using the change of variable $\zeta = a_{i-1} + (a_i - a_{i-1})t^2$ (resp., $\zeta = a_i - (a_i - a_{i-1})t^2$), we transform the integrals

$$\int_{a_{i-1}}^{a_i} \frac{(1 + R(\zeta+\alpha))^{k+1}}{\zeta+\alpha} f\left(\zeta + \frac{1}{R}\ln\left(\frac{\zeta+\alpha}{\alpha}\right)\right) d\zeta,$$
$$\int_{a_{i-1}}^{a_i} \frac{(1 + R(\zeta+\alpha))^{k+1}}{\zeta+\alpha} \ln\left(\frac{\zeta+\alpha}{\alpha}\right) f\left(\zeta + \frac{1}{R}\ln\left(\frac{\zeta+\alpha}{\alpha}\right)\right) d\zeta$$

into integrals of continuous functions over the interval $[0, 1]$.

Further, if f has continuous derivatives up to order n in the intervals $(0, a_1)$, (a_1, a_2),..., $a_{m-1}, a_m)$, $(a_m, +\infty)$, then the same holds for the integrands after the changes of variables introduced above. In that case all the integrals can be computed by Gaussian quadrature or any other standard method based on smooth interpolation. Note that adaptative quadrature based on these methods allows for easy estimates of the truncation error.

5 Examples

In this section we give two examples for the standard deviation principle. In the first example we consider that Y follows a Pareto distribution. In the second example we consider a generalized gamma distribution. The parameters of these distributions where chosen such that $E[Y] = 1$ and both distributions have the same variance (which was set to $Var[Y] = \frac{16}{5}$, for convenience of the choice of parameters). Notice that thought they have the same mean and variance, the tails of the two distributions are rather different. However, none of them has moment generating function defined in any neighborhood of the origin. Hence the optimal solution must always be different than no reinsurance.

In both examples we consider the same premium income $c = 1.2$ and the same loading coefficient $\beta = 0.25$.

Table 1: Y - Pareto random variable

	Optimal Treaty	Best Stop Loss
	$\alpha = 1.74411$	$M = 67.4436$
R	0.055406	0.047703
$E[Z]$	0.098018	0.001050
$Var[Z]$	0.212089	0.160269
$P(Z)$	0.213151	0.101134
$E[L_Z]$	0.084867	0.099916

Example 1 *We consider that Y follows the Pareto distribution*

$$f(y) = \frac{32 \times 21^{32/11}}{(21 + 11y)^{43/11}}, \qquad y > 0.$$

The first column of Table 1 shows the optimal value of α and the corresponding values of R, $E[Z]$, $Var[Z]$, $P(Z)$, and $E[L_Z]$, while the second column shows the corresponding values for the best (in terms of the adjustment coefficient) stop loss treaty. The optimal policy improves the adjustment coefficient by 16.1% *with respect to the best stop loss treaty, at the cost of an increase of* 111% *in the reinsurance premium. However, notice that the relative contribution of the loading to the total reinsurance premium is much smaller in the optimal policy, compared with the best stop loss. Hence, thought a larger premium is ceded under the optimal treaty than under the best stop loss, this is made mainly through the pure premium, rather than the premium loading, so the expected profits are not dramatically different.*

Figure 1 shows the optimal reinsurance arrangement versus the best stop loss treaty $Z_M(y) = \max\{0, y - M\}$. It shows that the improved performance of the optimal policy is achieved partly by compensating a lower level of reinsurance against very high losses (which occur rarely) by reinsuring a substantial part of moderate losses, which occur more frequently but are inadequately covered or not covered at all by the stop-loss treaty.

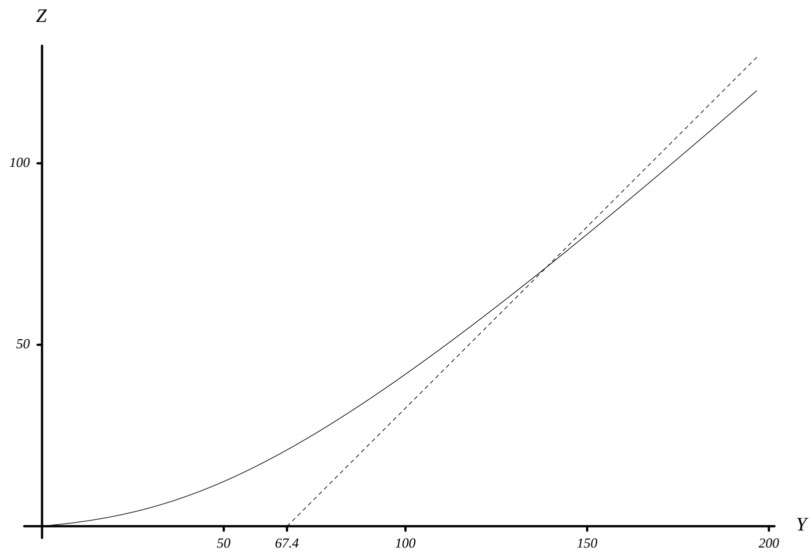

Figure 1: *Optimal policy (full line) versus best stop loss (dashed line): the Pareto case.*

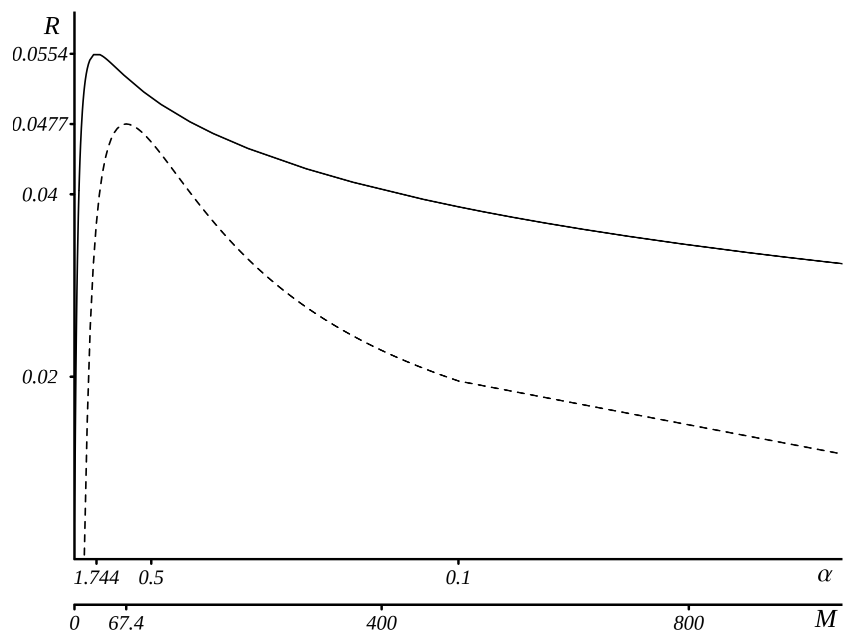

Figure 2: *Adjustment coefficient as a function of treaty parameter for policies of type (9) (full line) compared with stop loss policies (dashed line) in the Pareto case. In both cases the horizontal axis represents the policy parameter (α and M, resp., scales not comparable).*

In general, it can be expected that the treaty selected in a practical context is suboptimal. Supposing that the direct insurer is allowed to chose a treaty of the type (9), numerical errors and incomplete knowledge *about the distribution of claims ensure that the choice of the value for the parameter α can not be made with complete accuracy. Therefore, it is interesting to see how the adjustment coefficients of treaties of type (9) and stop loss treaties behave as functions of the treaty parameters (resp., α and M). For this purpose we present some additional figures.*

Figure 2 plots values of the adjustment coefficient against the treaty parameters. In order to make the retained risk to increase in the same direction (from left to right) in both curves, we plot the α parameter of the treaties (9) in inverse scale (i.e., we plot $\frac{1}{\alpha}$). The curves corresponding to both types of treaties have the same overall shape, decreasing smoothly to the right of a well defined maximum. However, notice that the horizontal scales of these curves is not comparable because the parameters M and $\frac{1}{\alpha}$ do not have any common interpretation.

In order to make the comparison more meaningful we present two other plots in which the horizontal axis has the same meaning for both treaties. In Figure 3 we show the adjustment coefficient plotted as a function of the ceded risk ($E[Z]$). We see that while stop loss policies exhibit a very sharp maximum corresponding to a small value of $E[Z]$, the policies of type (9) exhibit a broad maximum. The adjustment coefficient of stop loss policies decreases very steeply when $E[Z]$ departs in either way from the optimum (this can be seen in some detail in Figure 4). Such behavior contrasts with policies of type (9) which keep a good performance even when the amount of risk ceded differs substantially from the optimum.

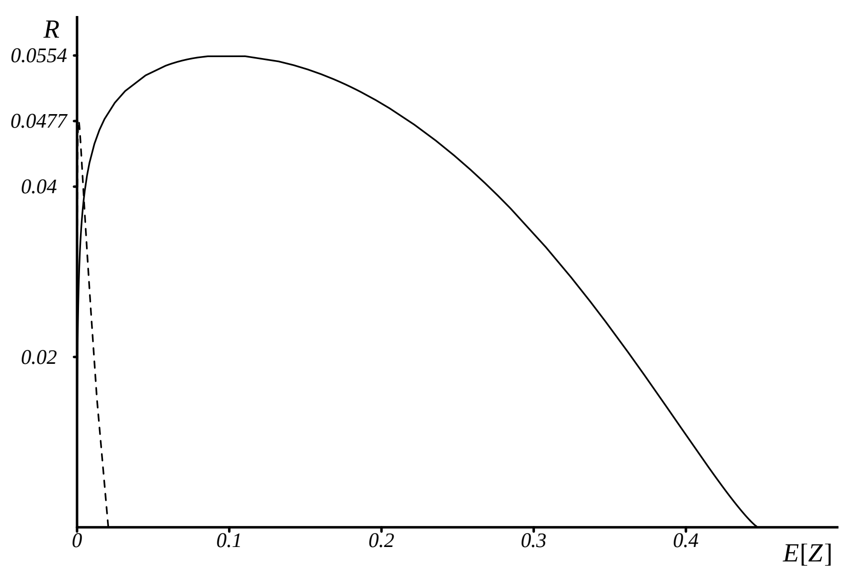

Figure 3: *Adjustment coefficient as a function of ceded risk ($E[Z]$) for policies of type (9) (full line) compared with stop loss policies (dashed line) in the Pareto case.*

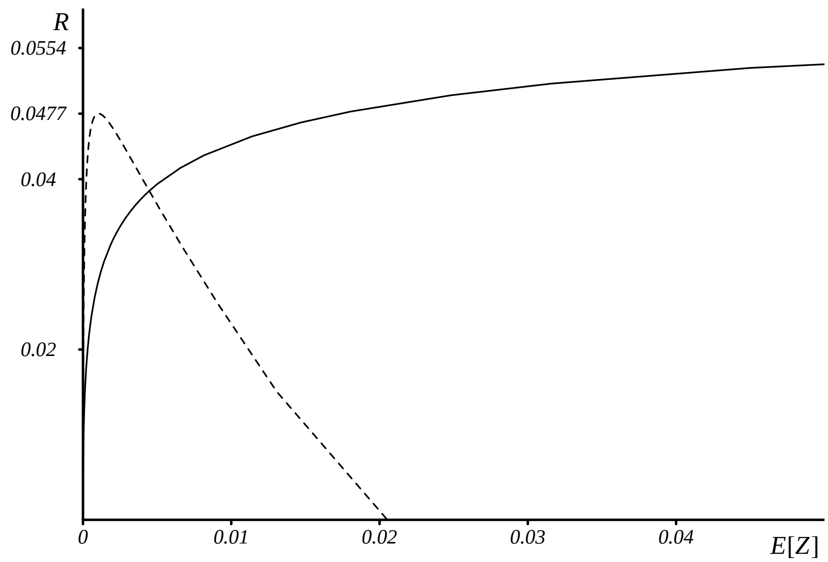

Figure 4: *Detail of figure 3.*

The presence of a sharp maximum is due to the fact that when stop loss policies are considered, the expected profit decreases very sharply when the ceded risk increases. By contrast, using policies of type (9) it is possible to cede a larger amount of risk with a moderate decrease in the expected profit.

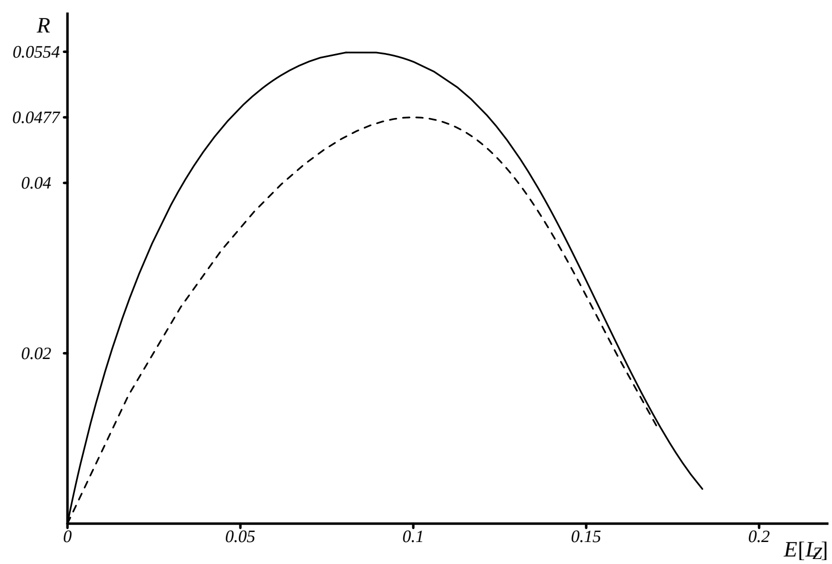

Figure 5: *Adjustment coefficient as a function of expected profit ($E[L_Z]$) for policies of type (9) (full line) compared with stop loss policies (dashed line) in the Pareto case.*

Figure 5 shows the adjustment coefficient plotted as a function of the expected profit ($E\left[L_Z\right]$). Recall that the adjustment coefficient is defined only for policies satisfying $E\left[L_Z\right] > 0$ and due to the choice of our examples

$E[L_Z] \leq 0.2$ holds for all $Z \in \mathcal{Z}$. Therefore we see that the policies of type (9) significantly outperform the comparable stop loss policies except at very high or very low values of expected profit (i.e., except in situations of very strong over-reinsurance or sub-reinsurance).

Example 2 *In this example, Y follows the generalized gamma distribution with density*

$$f(y) = \frac{b}{\Gamma(k)\theta}\left(\frac{y}{\theta}\right)^{kb-1} e^{-\left(\frac{y}{\theta}\right)^b}, \qquad y > 0,$$

with $b = 1/3$, $k = 4$ and $\theta = 3!/6!$. Table 2 shows the results for this example. The general features are similar to Example 1 but the improvement with respect to the best stop loss is smaller (the optimal policy increases the adjustment coefficient by about 7.8% with respect to the best stop loss). The optimal policy presents a larger increase in the sharing of risk and profits and a sharp increase in the reinsurance premium (more than seven-fold) with respect to the best stop loss. However, in both cases the amount of the risk and of the profits which is ceded under the reinsurance treaty is substantially smaller than in the Pareto case.

Table 2: Y - Generalized gamma random variable

	Optimal Treaty	Best Stop Loss
	α =0.813383	$M = 47.8468$
R	0.084709	0.078571
$E[Z]$	0.076969	0.000204
$Var[Z]$	0.049546	0.004951
$P(Z)$	0.132616	0.017794
$E[L_Z]$	0.144353	0.182410

Our comments on Example 1 comparing the performance of treaties of type (9) with stop loss treaties remain valid for the present example.

Notice that the plots of the adjustment coefficients functions of the expected profit in the present example (figure 8) are skewed to the right compared with the corresponding plot in Example 1 (figure 5). In figure 8 the stop loss treaty presents a sharper maximum than in figure 5, while the opposite is true for the treaties of type (9).

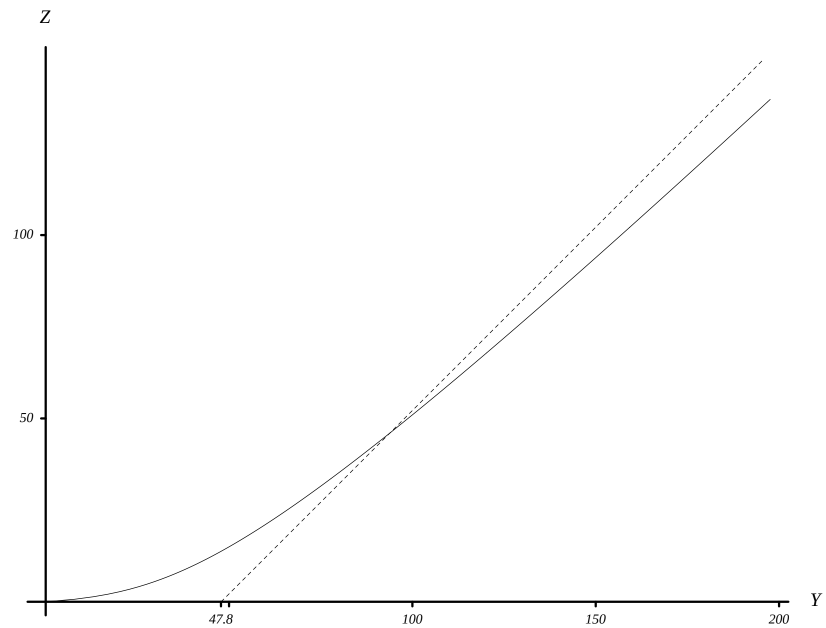

Figure 6: *Optimal policy (full line) versus best stop loss (dashed line): the generalized gamma case.*

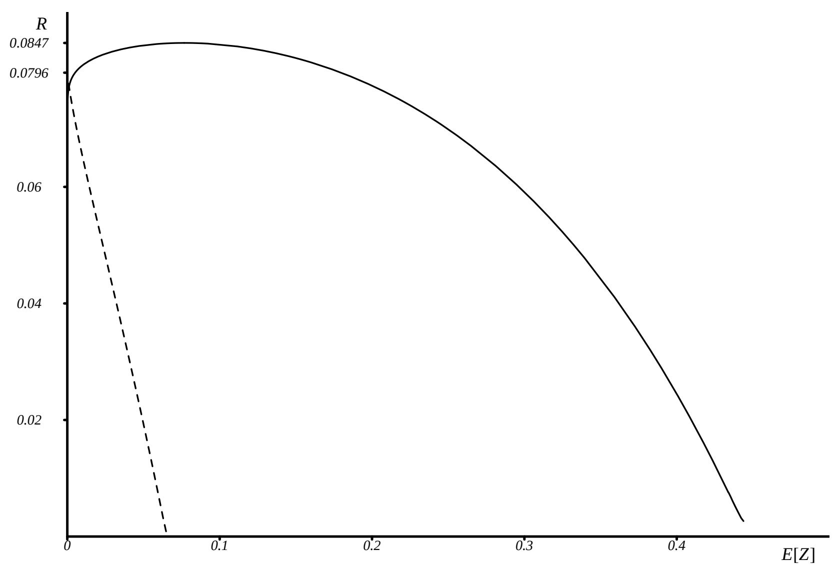

Figure 7: *Adjustment coefficient as a function of ceded risk ($E[Z]$) for policies of type (9) (full line) compared with stop loss policies (dashed line) in the generalized gamma case.*

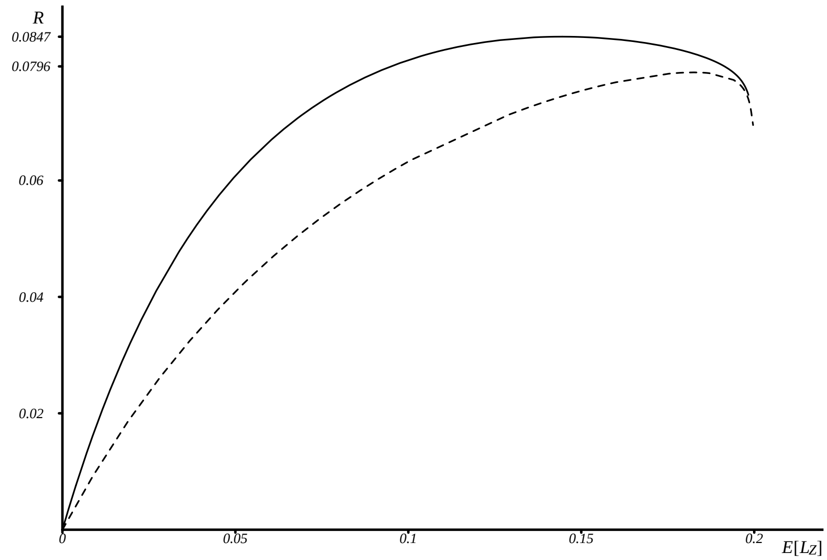

Figure 8: *Adjustment coefficient as a function of expected profit ($E[L_Z]$) for policies of type (9) (full line) compared with stop loss policies (dashed line) in the generalized gamma case.*

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