The optimal reinsurance strategy - the individual claim case*

Maria de Lourdes Centeno
CEMAPRE, ISEG -T.U.Lisbon
R. Quelhas 6, 1200-781 Lisboa
lcenteno@iseg.utl.pt

Manuel Guerra
CEOC and ISEG -T.U.Lisbon
R. Quelhas 6, 1200-781 Lisboa
mguerra@iseg.utl.pt

Abstract: This paper is concerned with the optimal form of reinsurance when the cedent seeks to maximize the adjustment coefficient of the retained risk (related to the probability of ultimate ruin) - which we prove to be equivalent to maximizing the expected utility of wealth, with respect to an exponential utility with a certain coefficient of risk aversion - and restricts the reinsurance strategies to functions of the individual claims, which is the case for most nonproportional treaties placed in the market.

*This research has been supported by Fundação para a Ciência e a Tecnologia (FCT) - project PTDC/ECO/66693/2006 - through PIDDAC, partially funded by the Portuguese State Budget.

Assuming that the premium calculation principle is a convex functional we prove existence and uniqueness of solutions and provide a necessary optimality condition (via needle-like perturbations, widely known in optimal control). These results are used to find the optimal reinsurance policy when the reinsurance loading is increasing with the variance. The optimal contract is described by a nonlinear function, of a similar form than in the aggregate case.

KEY WORDS: optimal reinsurance, adjustment coefficient, expected utility, exponential utility function, convex premium principles, risk.

1 Introduction

There are many theoretical results in favor of this or that type of reinsurance, depending on the optimality criterion and the premium principle that has been chosen. Borch (1960) proved that stop loss minimizes the variance of the retained risk if the reinsurer charges a fixed premium dependent only on the expected reinsurance claims. Taking the maximization of the expected utility as the optimality criterion, Arrow (1963) proved a similar result in favor of the stop loss contract. There are some generalizations of Arrow's result, a few of them quite recent (e.g. Kaluszka (2004)). Hesselager (1990) achieved an equivalent result using the adjustment coefficient as optimality criterion.

All these articles in favor of the stop loss contract are based on the assumption that the ceded claims have a fixed expected value, although Borch himself has made, in Borch (1969), a number of negative remarks to his result. In fact the reinsurance premium with the same loading coefficient (=loading divided by the expected claim amount), for all the reinsurance schemes, does not have any practical adherence.

All these articles consider that reinsurance is placed on the aggregate. When we consider that reinsurance is placed on individual terms excess of loss takes the place of the stop loss contract. See, for instance, Bowers et al (1987), Gerber (1979), Gajek (2000) and Kaluszka (2001). The comments made about the reinsurance premium can also apply to individual reinsurance. Froot (2001), using over 4,000 catastrophe reinsurance layers transacted during the period 1970-1998, shows that the loading coefficient increases for higher layers, as it would be expected. This justifies the need of results that deviate from the assumption that the premium is calculated according to the expected value principle.

There are some results on optimal reinsurance that consider that the premium loading is an increasing function with the variance of the ceded risk. Gajek (2000) and Kaluszka (2001) minimize the variance of the retained risk, when the loading of the reinsurance premium used is based on the expected value and/or on the variance of the reinsured risk and the premium is fixed. Kaluszka (2005) generalizes that article to other convex premium calculation principles and other optimality criteria.

Guerra and Centeno (2008) choose as optimality criterion the adjustment coefficient of the retained risk and assume that the reinsurance premium is a convex functional. Note, however that the amount to pay for the reinsurance arrangements is not fixed, as it is the case in all the papers cited above. Part of the difficulty in studying the problem lies on the fact that the adjustment coefficient is defined in an implicit form and its domain has not a structure appropriate to use arguments based on classical implicit function theorems. We overcome that difficulty by showing that to maximize the adjustment coefficient is equivalent to solve a two-step problem. The first step in this new problem consists in maximizing the expected utility of wealth of the retained risk for an exponential utility function, for all positive values of the coefficient of risk aversion. The second step consists in solving a single-variable equation. The optimal adjustment coefficient equals the coefficient of risk aversion for which the maximal expected value of the utility function is -1. The reinsurance policy that maximizes the adjustment coefficient is the treaty that maximizes the expected utility of wealth for that particular value of the risk-aversion coefficient. It turns out that the maximization step in the two-step problem is easier to deal from the mathematical point of view than the original one. Thus, both problems are solved. It is proved that one optimal reinsurance policy always exists and it is given a necessary condition for a policy to be optimal. Stop loss is indeed the optimal form of reinsurance if the reinsurer rates the contracts by the expected value principle, but when the reinsurance loading is an increasing function of the variance (for example, in the variance or standard deviation premium principles), then the optimal form is of a nonlinear type (not an already known typical form), but very easily constructed (see Guerra and Centeno (2007)).

In this article we generalize the results obtained in Guerra and Centeno (2008), by considering that the reinsurance strategies are confined to be per claim reinsurance. We omit the proves of our results whenever they follow exactly the same reasoning that in Guerra and Centeno (2008).

The paper follows the same structure as Guerra and Centeno (2008) and is organized as follows. Section 2 contains the formulation of the problem, the basic notation and the blanket assumptions that will be used. Section 3 contains some essentially technical elements that will be used to obtain the main results. In Section 4 we analyze the relationship between the maximization of the adjustment coefficient of the retained risk and the maximization of the expected value of the utility of the insurer's wealth. In Section 5 we prove existence and uniqueness of optimal policies for the expected utility criterion. This result is used in Section 6 to prove existence and uniqueness of a policy which maximizes the adjustment coefficient. A necessary condition for optimality is obtained in Section 7. In Section 8 we assume that the loading on the reinsurance premium is an increasing function of the variance and provide the optimal necessary conditions. We show that the optimal treaty is broadly of the same type as in the aggregate case but some additional issues concerning the structure of optimal treaties arise in the individual claim case. The relationship between this structure and the distribution of claim numbers is discussed in Section 9.

2 Assumptions

In our model we consider some simplifying assumptions. To start with we consider that the reinsurance market consists of one insurer and one reinsurer, that both the insurer and the reinsurer have the same information on the claim number and claim amount distributions, that the reinsurer fixes the pricing rule and that this rule is of the knowledge of the insurer.

Let N denote the number of claims of a given risk (a policy or portfolio of policies) over a given period of time (say, one year). N is an integer random variable with distribution

$$\Pr\{N = n\} = p(n), \qquad n = 0, 1, 2, 3, \ldots.$$

We assume that

Assumption 1 *The moment-generating function of the random variable N exists (is finite) on a neighborhood of zero, i.e. the radius of convergence of the probability generating function $\pi(t) = E[t^N] = \sum_{n=0}^{+\infty} t^n p(n)$ is strictly greater than 1.* □

Let Y_i represent the value of the i^{th} claim in the period of time being considered. Y_i, $i \in \mathbb{N}$ are assumed to be random variables satisfying the following assumptions:

Assumption 2 $\{Y_i\}_{i=1,2,\ldots}$ *are i.i.d. nonnegative continuous random variables with common density function f, and $E\left[Y_i^2\right] < +\infty$.* □

Assumption 3 $\{Y_i\}_{i=1,2,\ldots}$ *are independent of the random variable N.* □

When we refer to a generic claim (whichever its order of occurrence), we denote it by Y (i.e., Y denotes an arbitrary continuous random variable with the same density f).

Let $\widehat{Y}$ be the gross (of reinsurance) aggregate claim amount for the same period of time, i.e.

$$\hat{Y} = \sum_{i=0}^{N} Y_i,$$

with $Y_0 \equiv 0$. Aggregate claims over consecutive periods are assumed to be i.i.d..

As we are dealing with per claim reinsurance, a reinsurance policy is a function $Z : [0, +\infty[\mapsto [0, +\infty[$, mapping each possible value of a claim Y into the corresponding value refunded under the reinsurance contract. The set of all possible reinsurance policies is:

$$\mathcal{Z} = \{Z : [0, +\infty[\mapsto \mathbb{R} |\ Z \text{ is measurable and } 0 \leq Z(y) \leq y,\ \forall y \geq 0\}.$$

We do not distinguish between functions which differ only on a set of zero probability with respect to the density f. i.e., two measurable functions, ϕ and ϕ' are considered to be the same whenever $\Pr\{\phi(Y) = \phi'(Y)\} = 1$. Similarly, a measurable function Z is an element of $\mathcal{Z}$ whenever

$$\Pr\{0 \leq Z(Y) \leq Y\} = 1.$$

The aggregate claims refunded under the reinsurance contract $Z \in \mathcal{Z}$ is the random variable

$$\hat{Z} = \sum_{i=0}^{N} Z(Y_i).$$

Assume that for each period of time, the premium charged for a reinsurance policy is computed by a real functional $P : \mathcal{Z} \mapsto [0, +\infty]$. Assuming that the insurer's gross premium amount, free of expenses and tax, per unit of time, is c, with $c > E[\hat{Y}]$, and acquires a given reinsurance policy $Z \in \mathcal{Z}$ for the same period, the net result obtained, which for simplicity we call profit is the random variable

$$L_Z = c - P(Z) - \left(\hat{Y} - \hat{Z}\right) = c - P(Z) - \sum_{i=0}^{N} (Y_i - Z(Y_i)),$$

(meaning a loss when its value is negative). Concerning the random variable $\widehat{Y}$ and the premiums c and $P(Z)$, we take the following additional assumptions:

Assumption 4 *No reinsurance policy exists that guarantees a nonnegative profit, i.e.,* $\Pr\{L_Z < 0\} > 0$ *holds for every* $Z \in \mathcal{Z}$. □

Assumption 5 *The reinsurance premium is a convex, non negative functional, such that* $P(0) = 0$. *It is continuous in the mean-squared sense, i.e.,* $\lim_{k\to\infty} P(Z_k) = P(Z')$ *holds for every sequence* $\{Z_k \in \mathcal{Z}\}_{k=1,2,\ldots}$ *such that*

$$\lim_{k\to\infty} \int_0^{+\infty} (Z_k(y) - Z'(y))^2 f(y)\ dy = 0. \ \square$$

The concept of convex premium calculation principles was introduced in the actuarial literature by Deprez and Gerber (1985).

Thought some regularity of the probability measure is necessary, the requirement that Y is a continuous random variable can be much weakened. We provide this assumption in order to simplify the technical content of our proofs, so we focus in the general features of the approach we propose. In contrast, Assumptions 1, 3 and 5 and the requirement that $E[Y^2] < +\infty$ can not be lifted. This is because our approach depends in a fundamental way on Hilbert space arguments and convex optimization.

Consider the map $G : \mathbb{R} \times \mathcal{Z} \mapsto [0, +\infty]$, defined by

$$G(R, Z) = E\left[e^{-RL_Z}\right], \qquad R \in \mathbb{R},\ Z \in \mathcal{Z}.$$

Let R_Z denote the adjustment coefficient of the retained risk for a particular reinsurance policy, $Z \in \mathcal{Z}$. R_Z is defined as the strictly positive value of R which solves the equation

$$G(R, Z) = 1, \tag{1}$$

for that particular Z, when such a root exists. It comes as a Corollary of Lemma 1 below that (1) cannot have more than one positive solution. This means that the map $Z \mapsto R_Z$ is a well defined functional in the set

$$\mathcal{Z}^+ = \{Z \in \mathcal{Z} : (1) \quad \text{admits a positive solution}\}.$$

Now, suppose that the insurance company detains a certain amount of reserves, $u > 0$, to cover eventual losses. If a reinsurance policy $Z \in \mathcal{Z}$ is in force year after year, then the probability of ultimate ruin is

$$\psi_Z(u) = \Pr\left\{u + \sum_{k=1}^{n} L_{Z,k} < 0,\ \text{for some } n = 1, 2, ...\right\},$$

where $L_{Z,k}$ denotes the profit obtained by the insurer in the year k (all $L_{Z,k}$ are i.i.d. with the same distribution as L_Z). As it is well known (see for example Gerber (1979)) the probability of ruin satisfies the Lundberg inequality:

$$\psi_Z(u) \leq \exp(-uR_Z).$$

Further, the behavior of $\psi_Z(u)$ as a function of u is quite similar to the behavior of $\exp(-uR_Z)$ for most common reinsurance forms (see Centeno (1997)). Therefore, the established practice of seeking to maximize R_Z instead of minimizing $\psi_Z(u)$ itself is acceptable.

Thus, we deal with the following optimization problem:

Problem 1 *Find* $\left(\hat{R}, \hat{Z}\right) \in \left]0, +\infty\right[\times \mathcal{Z}^+$ *such that*

$$\hat{R} = R_{\hat{Z}} = \max\left\{R_Z : Z \in \mathcal{Z}^+\right\}.\square$$

A policy $\hat{Z} \in \mathcal{Z}$ is said to be **optimal for the adjustment coefficient criterion** if $\left(R_{\hat{Z}}, \hat{Z}\right)$ solves this problem. We remind that the problem was solved in Guerra and Centeno (2008) for the case $N \equiv 1$, i.e. for aggregate reinsurance.

Given the formulation of Problem 1, it becomes clear that Assumption 4 is required in order to make the problem nontrivial: If there exists some policy satisfying $\Pr\{L_Z < 0\} = 0$, then the risk of ruin under this policy is obviously zero.

It is also clear that Assumption 1 is necessary to make the problem well posed. Indeed it can be checked that

$$G(R, Z) = \pi\left(E\left[e^{R(Y-Z)}\right]\right) e^{R(P(Z)-c)}, \tag{2}$$

where $\pi(\cdot)$ is the probability generating function of the number of claims N. Therefore, if the radius of convergence of $\pi(t)$ equals 1, then $Z \equiv Y$ is the unique reinsurance treaty satisfying $G(R, Z) < +\infty$ and the problem does not admit any solution.

3 Preliminaries

In this section we study some properties of the map $(R, Z) \mapsto G(R, Z)$ and the existence of the adjustment coefficient.

Lemma 1 *Fix $Z \in \mathcal{Z}$, and suppose there exists some $R > 0$ such that $G(R, Z) < +\infty$.*

Then, there exists a constant $\eta_Z \in]0, +\infty]$ such that map $R \mapsto G(R, Z)$ is smooth in $[0, \eta_Z[$, and $G(R, Z) = +\infty$ for all $R > \eta_Z$. Further,

$$\lim_{R \to \eta_Z^-} G(R, Z) = G(\eta_Z, Z). \tag{3}$$

For $R \in [0, \eta_Z[$, we have

$$\frac{\partial^k G(R, Z)}{\partial R^k} = E\left[(-L_Z)^k e^{-RL_Z}\right], \qquad k \geq 0. \ \square \tag{4}$$

Proof. Existence of η_Z and equality (3) follow directly from (2) and Lemma 1 in Guerra and Centeno (2008). The rest of the proof is in all similar to the proof of Lemma 1 in Guerra and Centeno (2008). ■

Lemma 1 has the following immediate Corollary:

Corollary 1 *For any* $Z \in \mathcal{Z}$ *the map* $R \mapsto G(R,Z)$ *is strictly convex in* $[0, \eta_Z]$*. Hence, equation* (1) *admits at most one positive solution. It admits no positive solution if* $E[L_Z] \leq 0$*.* $\square$

The existence of a reinsurance policy such that $E[L_Z] > 0$ and $G(\eta_Z, Z) < 1$ can not be ruled out. Let ρ be the radius of convergence of the probability generating function of N, $\pi(\cdot)$. Then it can be shown that the cases in which such phenomenon occurs are exactly the following:

1. $E\left[e^{\eta_Z(Y-Z)}\right] < \rho$ and $G(\eta_Z, Z) < 1$ hold, but $E\left[e^{R(Y-Z)}\right] = +\infty$ holds for all $R > \eta_Z$;

2. $E\left[e^{\eta_Z(Y-Z)}\right] = \rho < +\infty$ and $G(\eta_Z, Z) < 1$.

This implies that the map $Z \mapsto R_Z$ is a functional defined in implicit form whose domain $\mathcal{Z}^+$ lacks a convenient structure to allow for optimization methods based on the implicit function theorem. The remaining results presented in this section will be used in later sections to overcome this difficulty.

The following Proposition shows that for each $Z \in \mathcal{Z}$ such that $E[L_Z] > 0$ and $G(\eta_Z, Z) < 1$, there exists $\tilde{Z} \in \mathcal{Z}^+$ such that $R_{\tilde{Z}} > \eta_Z$.

Proposition 1 *Fix* $(R, Z) \in \left]0, +\infty\right[\times \mathcal{Z}$ *such that* $G(R,Z) < e^{R(P(Y)-c)}$*. For every sufficiently small* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that, for any* $\tilde{R} \in [R, R+\delta]$ *and any* $\gamma \in \left[G(R,Z) + \varepsilon, e^{R(P(Y)-c)}\right]$*, there exists* $\tilde{Z} \in \mathcal{Z}$ *such that*

$$G\left(\tilde{R}, \tilde{Z}\right) = \gamma. \ \square$$

Proof. Fix $(R, Z) \in]0, +\infty[\times \mathcal{Z}$ such that $G(R, Z) < e^{R(P(Y)-c)}$ and fix $\varepsilon > 0$ such that $G(R, Z) + \varepsilon \leq e^{R(P(Y)-c)}$. For each $M \in]0, +\infty[$, let

$$Z_M(y) = \max\{Z(y),\ y - M\}.$$

The dominated convergence theorem guarantees that for every $M_0 \in [0, +\infty[$, the equality

$$\lim_{M \to M_0} \int_0^{+\infty} (Z_M(y) - Z_{M_0}(y))^2 f(y)\ dy = 0$$

holds. The dominated convergence theorem also guarantees that

$$\lim_{M \to +\infty} \int_0^{+\infty} (Z_M(y) - Z(y))^2 f(y)\ dy = 0.$$

Therefore, Assumption 5 guarantees that the map $M \mapsto P(Z_M)$ is continuous, and $\lim_{M \to +\infty} P(Z_M) = P(Z)$. Also, the dominated convergence theorem guarantees that the map $M \mapsto E\left[e^{R(Y-Z_M)}\right]$ is continuous nondecreasing in $[0, +\infty[$ with $E\left[e^{R(Y-Z_0)}\right] = 1$ and $\lim_{M \to +\infty} E\left[e^{R(Y-Z_M)}\right] = E\left[e^{R(Y-Z)}\right] \leq \rho$. For any $M > 0$ (fixed), the map $\tilde{R} \mapsto E\left[e^{\tilde{R}(Y-Z_M)}\right]$ is continuous and finite for all $\tilde{R} > 0$. Hence, there exists $M_0 \in]0, +\infty[$ such that $G(R, Z_{M_0}) < G(R, Z) + \frac{\varepsilon}{2}$, It follows that there exists $\delta > 0$ such that $G\left(\tilde{R}, Z_{M_0}\right) < G(R, Z) + \varepsilon$ holds for all $\tilde{R} \in [R, R + \delta]$. Assumption 4 implies that $P(Y) - c > 0$. Therefore, Bolzano's Theorem guarantees that the set

$$\left\{\left(\tilde{R}, G\left(\tilde{R}, Z_M\right)\right) : \tilde{R} \in [R, R + \delta],\ M \in [0, M_0]\right\}$$

covers the rectangle $[R, R + \delta] \times \left[G(R, Z) + \varepsilon, e^{R(P(Y)-c)}\right]$. ■

Corollary 2 *Fix* $Z \in \mathcal{Z}$. *If* $G(\eta_Z, Z) < 1$, *then there exists* $R > \eta_Z$, $\tilde{Z} \in \mathcal{Z}$ *such that* $G\left(R, \tilde{Z}\right) = 1$.$\square$

Proof. Assumption 4 implies that $P(Y) - c > 0$. Hence $1 \in \left]G(\eta_Z, Z), e^{\eta_Z(P(Z)-c)}\right[$. Therefore, the Corollary follows immediately from Proposition 1. $\blacksquare$

The following Proposition studies the convexity of the map $Z \mapsto G(R, Z)$.

Proposition 2 *The map* $Z \mapsto G(R, Z)$ *is convex in* $\mathcal{Z}$. *It is strictly convex unless the random variable* N *is concentrated at a unique integer* k. *In that case the map* $\left(\frac{P(Z)}{k} - Z\right) \mapsto G(R, Z)$ *is strictly convex.* $\square$

Proof. Given that $\widehat{Y}$ and $\widehat{Z}$ are both compound random variables, with the same claim number distribution, for every $(R, Z) \in \left]0, +\infty\right[\times \mathcal{Z}$, we have that

$$G(R, Z) = e^{R(P(Z)-c)} \sum_{n=0}^{+\infty} E\left[e^{R(Y-Z)}\right]^n p(n). \tag{5}$$

Consider two reinsurance treaties $Z_1, Z_2 \in \mathcal{Z}$, with $Z_1 \neq Z_2$. Fix $\lambda \in \left]0, 1\right[$. Then,

$$\begin{aligned} G(R, \lambda Z_1 + (1-\lambda) Z_2) &= e^{R(P(\lambda Z_1 + (1-\lambda)Z_2) - c)} p(0) + \\ &+ \sum_{n=1}^{\infty} E\left[e^{R\left(\frac{P(\lambda Z_1 + (1-\lambda)Z_2) - c}{n} + Y - \lambda Z_1 - (1-\lambda) Z_2(Y)\right)}\right]^n p(n). \end{aligned}$$

Convexity of the premium implies that

$$\begin{aligned} G(R, \lambda Z_1 + (1-\lambda) Z_2) &\leq e^{R(\lambda(P(Z_1)-c) + (1-\lambda)(P(Z_2)-c))} p(0) + \\ &+ \sum_{n=1}^{+\infty} E\left[e^{R\left(\lambda\left(\frac{P(Z_1)-c}{n} + Y - Z_1\right) + (1-\lambda)\left(\frac{P(Z_2)-c}{n} + Y - Z_2\right)\right)}\right]^n p(n). \end{aligned}$$

Therefore, strict convexity of the exponential and of powers x^n $(n \geq 1,\ x > 0)$ imply

$$\begin{aligned} & G\left(R, \lambda Z_1 + (1-\lambda) Z_2\right) \leq \\ \leq \quad & (\lambda e^{R(P(Z_1)-c)} + (1-\lambda) e^{R(P(Z_2)-c)}) p(0) + \\ & + \sum_{n=1}^{+\infty} \left(\lambda E\left[e^{R\left(\frac{P(Z_1)-c}{n} + Y - Z_1\right)} \right]^n + (1-\lambda) E\left[e^{R\left(\frac{P(Z_2)-c}{n} + Y - Z_2\right)} \right]^n \right) p(n) = \\ = \quad & \lambda G(R, Z_1) + (1-\lambda) G(R, Z_2), \end{aligned}$$

with strict inequality holding unless

$$P(Z_1) - nZ_1 = P(Z_2) - nZ_2$$

holds for every $n \geq 0$ such that $p(n) \neq 0$.

It is easy to check that this last condition can be satisfied only when N is a degenerate random variable taking one single value k with probability 1. ■

4 Maximization of the expected utility of wealth

It was pointed in the previous Section that the map $Z \mapsto R_Z$ is a functional defined in implicit form whose domain, $\mathcal{Z}^+$, lacks a convenient structure to allow for optimization methods based on the implicit function theorem. It turns out that this theoretical obstacle can be avoided by exploiting the close relationship between maximizing of the adjustment coefficient of the retained risk (Problem 1) and maximizing of the expected utility of wealth with an arbitrary coefficient of risk aversion (Problem 2, below). In this Section we discuss the relationship between the two problems.

Consider the exponential utility function with coefficient of risk aversion $R > 0$:

$$U_R(w) = -e^{-Rw}.$$

For any given coefficient of risk aversion, $R > 0$, the expected utility of the profit obtained by the insurance company in a given unit of time is

$$E[U_R(L_Z)] = -G(R, Z). \tag{6}$$

We consider the maximization problem:

Problem 2 *Find $Z^* \in \mathcal{Z}$, such that*

$$E[U_R(L_{Z^*})] = \max\{E[U_R(L_Z)] : Z \in \mathcal{Z}\}.$$

Here $R > 0$ is a given constant (fixed).□

A policy $Z \in \mathcal{Z}$ is said to be **optimal for the expected utility criterion** with coefficient of risk aversion R if it solves Problem 2 for that particular R. When it is clear from the context which is the coefficient of risk aversion being considered, we will just say that the policy is optimal for the expected utility criterion.

It follows immediately from (6) that a policy is optimal for the expected utility criterion if and only if it is a minimizer of the functional $Z \mapsto G(R, Z)$, with the same (fixed) value of R being considered. The following relationship between Problem 1 and Problem 2 is the key to our approach and its proof is similar to the corresponding result for the aggregate case, see Guerra and Centeno (2008).

Proposition 3 *A pair* $(R^*, Z^*) \in]0, +\infty[\times \mathcal{Z}$ *solves Problem 1 (i.e.,* Z^* *is optimal for the adjustment coefficient criterion) if and only if it satisfies the following conditions:*

1. Z^* *is optimal for the expected utility criterion (i.e., it solves Problem 2) with coefficient of risk aversion* $R = R^*$*;*

2. $G(R^*, Z^*) = 1$.

Proof. The only substantial difference to the aggregate case is that the Lemma 2 of Section 3 in Guerra and Centeno (2008) does not hold in the individual case claim. However, the Lemma can be replaced by Proposition 1 from the previous section and then all other arguments hold. ■

Proposition 3 shows that, Problem 1 can be solved in two steps:

1. For each $R \in]0, +\infty[$ find Z_R, the respective optimal policy for the expected utility criterion. Equivalently, find

$$Z_R = \arg\min \{G(R, Z) : Z \in \mathcal{Z}\};$$

2. Solve the equation with one single real variable

$$G(R, Z_R) = 1.$$

We will always adhere to the notation used above:

Notation 1 *For each* $Z \in \mathcal{Z}^+$*,* R_Z *denotes the positive solution of the equation* $G(R, Z) = 1$*, for the particular (fixed)* Z *being considered.*
For each $R > 0$*,* Z_R *denotes the optimal policy for the expected utility criterion with the particular coefficient of risk aversion* R*.*

Below we show that the map $R \mapsto Z_R$ is well defined for $R \in]0, +\infty[$.

5 Existence and uniqueness of optimal policies for the expected utility criterion

Theorem 1 *For each* $R \in]0, +\infty[$ *there exists an optimal policy for the expected utility criterion.*
If $\Pr\{N = k\} = 1$*, for some* $k \geq 1$*, then given an optimal policy* $Z^* \in \mathcal{Z}$*, any other policy* $\tilde{Z} \in \mathcal{Z}$ *is optimal if and only if*

$$\Pr\left\{\tilde{Z}(Y) - Z^*(Y) = \frac{P\left(\tilde{Z}\right) - P(Z^*)}{k}\right\} = 1,$$

otherwise the optimal policy is unique.□

Proof. In our proof we consider the equivalent problem of minimizing the functional $Z \mapsto G(R, Z)$, for the particular value of R being considered. Proposition 2 states that the functional $Z \mapsto G(R, Z)$ is convex. If $\Pr\{N = k\} = 1$, for some integer $k \geq 1$ then the map $\left(\frac{P(Z)}{k} - Z\right) \mapsto G(R, Z)$ is well defined and strictly convex. Otherwise the map $Z \mapsto G(R, Z)$ is strictly convex. Since $\mathcal{Z}$ is convex, this proves the part of the Theorem concerning uniqueness of solutions.

Existence of a minimizer is a consequence of the classical Banach-Alaoglu Theorem from functional analysis (see e.g. Rudin (1991)). The key points of the argument are presented in Guerra and Centeno (2008). ■

6 Existence and uniqueness of optimal policy for the adjustment coefficient criterion

In this Section we use the results above to prove existence and uniqueness of solutions of Problem 1.

Theorem 2 *There exists an optimal policy for the adjustment coefficient criterion.*

If $\Pr\{N = k\} = 1$, *for some* $k \geq 1$, *then given an optimal policy* $Z^* \in \mathcal{Z}$, *any other policy* $\tilde{Z} \in \mathcal{Z}$ *is optimal if and only if*

$$\Pr\left\{\tilde{Z}\left(Y\right) - Z^*\left(Y\right) = \frac{P\left(\tilde{Z}\right) - P\left(Z^*\right)}{k}\right\} = 1, \tag{7}$$

otherwise the optimal policy is unique.□

Proof. Uniqueness is a straightforward consequence of Proposition 3 and Theorem 1. To see this, suppose the adjustment coefficient admits two different global maximizers, $Z^*, \tilde{Z} \in \mathcal{Z}$. Proposition 3 states that Z^* and $\tilde{Z}$ are both optimal policies for the expected utility criterion with the particular coefficient of risk aversion $R = R_{Z^*} = R_{\tilde{Z}}$. Then, Theorem 1 states that either the random variable N is such that $\Pr\{N=k\}=1$, for some $k\geq 1$, in which case Z^*, $\tilde{Z}$ satisfy (7), or $\tilde{Z}=Z^*$.

In order to prove existence, we will proceed like in Guerra and Centeno (2008): first we prove that the set $\{R\in\,]0,+\infty[\,:G(R,Z_R)\geq 1\}$ is non-empty. Then, we prove that the infimum of this set solves the equation $G(R,Z_R)=1$.

Suppose that $G(R,Z_R)<1$ holds for all $R\in\,]0,+\infty[$. Consider a sequence $\{R_k\}\to+\infty$, and the corresponding sequence $\{Z_{R_k}\}$. The argument used in Guerra and Centeno (2008) can be applied in the present case to show that $\{R_k\}$ can be chosen in such a way that $\{Z_{R_k}\}$ converges in the weak sense towards some $\tilde{Z}\in\mathcal{Z}$, and

$$G\left(R,\tilde{Z}\right)\leq\lim_{k\to\infty}G\left(R,Z_{R_k}\right) \tag{8}$$

holds for every $R\in\,]0,+\infty[$. Since the map $R\mapsto G(R,Z)$ can not cross the line $G=1$ more than once in the positive semiaxis, the hypothesis $G(R_k,Z_{R_k})<1$ implies that $G\left(R,Z_{R_{k+m}}\right)<1$ holds for every $k,m\in\mathbb{N}$, $R\leq R_k$. Therefore, (8) implies $G\left(R,\tilde{Z}\right)\leq 1$, $\forall R\in\,]0,+\infty[$, which implies $\Pr\{L_{\tilde{Z}}<0\}=0$. Since this contradicts Assumption 4, we conclude that $G(R,Z_R)\geq 1$ must hold for some finite R.

Let $R^*=\inf\{R>0:G(R,Z_R)\geq 1\}$. Suppose that $G(R^*,Z_{R^*})<1$. Then, Proposition 1 shows that there exists $\varepsilon>0$ such that $G(R^*+\delta,Z_{R^*+\delta})<1$ holds for all $\delta\in[0,\varepsilon[$. This is in contradiction with the definition of R^*, therefore $G(R^*,Z_{R^*})\geq 1$ must hold.

Now, chose a sequence $\{R_k<R^*\}$, converging to R^*. Without loss of gener-

ality, we can chose the sequence $\{R_k\}$ such that the corresponding sequence $\{Z_{R_k}\}$ converges weakly to some $\tilde{Z} \in \mathcal{Z}$, and

$$G\left(R, \tilde{Z}\right) \leq \lim_{k\to\infty} G\left(R, Z_{R_k}\right) \leq 1 \tag{9}$$

holds for every $R < R^*$. The inequality (9) implies that $R^* \leq \eta_{\tilde{Z}}$ and $\lim_{R\to R^{*-}} G\left(R, \tilde{Z}\right) \leq 1$. Therefore $G\left(R^*, \tilde{Z}\right) > 1$ would be a contradiction with Lemma 1. This proves that $G\left(R^*, \tilde{Z}\right) = 1$ and therefore, $\tilde{Z}$ is optimal for the adjustment coefficient criterion. ■

7 Necessary condition for optimality

Fix a reinsurance policy $Z \in \mathcal{Z}$. We consider needle-like perturbations (see e.g. Gamkrelidze (1978)), i.e., we consider reinsurance policies of type

$$Z_{\upsilon,\alpha,\varepsilon}(y) = \begin{cases} Z(y), & \text{if } y \notin [\upsilon, \upsilon+\varepsilon]; \\ \alpha y, & \text{if } y \in [\upsilon, \upsilon+\varepsilon]. \end{cases}$$

In what follows, we assume that the expression

$$\Delta P_Z(\upsilon) = \lim_{\alpha\to\frac{Z(\upsilon)}{\upsilon}} \lim_{\varepsilon\to 0^+} \frac{P\left(Z_{\upsilon,\alpha,\varepsilon}\right) - P(Z)}{\varepsilon\left(\alpha\upsilon - Z(\upsilon)\right)}$$

is a well defined function in a domain with probability equal to 1 with respect to the density f.

Indeed, in order to obtain some of the following results we also need to consider compositions of needle-like perturbations. I.e., we will consider treaties of the type

$$Z_{\upsilon_1,\alpha_1,\varepsilon_1|\upsilon_2,\alpha_2,\varepsilon_2}(y)=\begin{cases} Z(y), & \text{if } y\notin[\upsilon_1,\upsilon_1+\varepsilon_1]\cup[\upsilon_2,\upsilon_2+\varepsilon_2]; \\ \alpha_1 y, & \text{if } y\in[\upsilon_1,\upsilon_1+\varepsilon_1]; \\ \alpha_2 y & \text{if } y\in[\upsilon_2,\upsilon_2+\varepsilon_2], \end{cases}$$

for arbitrary $\upsilon_1\neq\upsilon_2$ and sufficiently small ε_1,ε_2. We assume that the function ΔP_Z suitably approximates the effect of double needle-like perturbations on the reinsurance premium, in the sense that the estimate

$$\begin{aligned} & P\left(Z_{\upsilon_1,\alpha_1,\varepsilon_1|\upsilon_2,\alpha_2,\varepsilon_2}\right)-P(Z)= \\ = \ & \varepsilon_1(\alpha_1\upsilon_1-Z(\upsilon_1))\Delta P_Z(\upsilon_1)+\varepsilon_1 o(\alpha_1\upsilon_1-Z(\upsilon_1))+o(\varepsilon_1)+ \\ & +\varepsilon_2(\alpha_2\upsilon_2-Z(\upsilon_2))\Delta P_Z(\upsilon_2)+\varepsilon_2 o(\alpha_2\upsilon_2-Z(\upsilon_2))+o(\varepsilon_2) \end{aligned} \tag{10}$$

holds for every $\upsilon_1\neq\upsilon_2$ chosen in a set of probability equal to one with respect to the density f. It can be shown that important premium calculation principles like the expected value principle and the variance-related principles studied in the next section satisfy these conditions.

Under the assumptions above, the following holds:

Theorem 3 *Let $\rho\in\,]0,+\infty]$ denote the radius of convergence of the series $\pi(t)=\sum_{n=0}^{+\infty}t^n p(n)$. Fix $R>0$, and let $Z_R\in\mathcal{Z}$ be optimal for the expected utility criterion.*

There exists a constant $C\in\,]0,+\infty[$ such that Z_R satisfies

$$\begin{cases} \Delta P_Z(y)\leq Cf(y), & \text{if } Z(y)=y; \\ \Delta P_Z(y)=Ce^{R(y-Z_R(y))}f(y), & \text{if } 0<Z(y)<y; \\ \Delta P_Z(y)\geq Ce^{Ry}f(y), & \text{if } Z(y)=0, \end{cases} \tag{11}$$

with probability equal to one with respect to the density f.

Further,

$$C = \frac{\pi'\left(E\left[e^{R(Y-Z_R)}\right]\right)}{\pi\left(E\left[e^{R(Y-Z_R)}\right]\right)}$$

holds whenever $E\left[e^{R(Y-Z_R)}\right] < \rho$, while

$$C \geq \frac{\pi'(\rho)}{\pi(\rho)} \tag{12}$$

holds if $E\left[e^{R(Y-Z_R)}\right] = \rho$. $\square$

Proof. Fix $\upsilon \in [0, +\infty[$, a Lebesgue point of the functions f, $e^{-RZ_R}f$, with $f(\upsilon) > 0$.

Suppose that $Z_R(\upsilon) < \upsilon$ and consider a perturbation $Z_{\upsilon,\alpha,\varepsilon}$ with

$$\alpha\upsilon > Z_R(\upsilon). \tag{13}$$

Optimality of Z_R implies that

$$0 \leq G(R, Z_{\upsilon,\alpha,\varepsilon}) - G(R, Z_R), \tag{14}$$

and (13) implies that $E\left[e^{R(Y-Z_{\upsilon,\alpha,\varepsilon})}\right] < E\left[e^{R(Y-Z_R)}\right] \leq \rho$. Therefore the function $t \mapsto \pi(t) = \sum_{n=0}^{\infty} t^n p(n)$ is continuous in the interval $\left[0, E\left[e^{R(Y-Z_R)}\right]\right]$, differentiable in $\left[0, E\left[e^{R(Y-Z_R)}\right]\right[$. It follows by the mean-value theorem that for each α satisfying (13) and each sufficiently small $\varepsilon > 0$ there exists $\theta \in]0,1[$ such that

$G(R, Z_{\upsilon,\alpha,\varepsilon}) - G(R, Z_R) =$

$$= \quad e^{R((1-\theta)P(Z_R)+\theta P(Z_{\upsilon,\alpha,\varepsilon})-c)}\pi\left((1-\theta)\,E\left[e^{R(Y-Z_R)}\right]+\theta E\left[e^{R(Y-Z_{\upsilon,\alpha,\varepsilon})}\right]\right)\times$$

$$\times R\left(P\left(Z_{\upsilon,\alpha,\varepsilon}\right)-P\left(Z_R\right)\right)+$$

$$+e^{R((1-\theta)P(Z_R)+\theta P(Z_{\upsilon,\alpha,\varepsilon})-c)}\pi'\left((1-\theta)\,E\left[e^{R(Y-Z_R)}\right]+\theta E\left[e^{R(Y-Z_{\upsilon,\alpha,\varepsilon})}\right]\right)\times$$

$$\times\left(E\left[e^{R(Y-Z_{\upsilon,\alpha,\varepsilon})}\right]-E\left[e^{R(Y-Z_R)}\right]\right).$$

Since υ is a Lebesgue point of the functions f, $e^{-RZ_R}f$, we have

$$E\left[e^{R(Y-Z_{\upsilon,\alpha,\varepsilon})}\right]-E\left[e^{R(Y-Z_R)}\right]=$$

$$=\int_{\upsilon}^{\upsilon+\varepsilon}\left(e^{R(y-\alpha y)}-e^{R(y-Z_R(y))}\right)f\left(y\right)\;dy=$$

$$=\varepsilon\left(e^{R(\upsilon-\alpha\upsilon)}-e^{R(\upsilon-Z_R(\upsilon))}\right)f\left(\upsilon\right)+o\left(\varepsilon\right).$$

Therefore, inequality (14) reduces to

$$-e^{R((1-\theta)P(Z_R)+\theta P(Z_{\upsilon,\alpha,\varepsilon})-c)}\pi\left((1-\theta)\,E\left[e^{R(Y-Z_R)}\right]+\theta E\left[e^{R(Y-Z_{\upsilon,\alpha,\varepsilon})}\right]\right)\times$$

$$\times R\tfrac{P(Z_{\upsilon,\alpha,\varepsilon})-P(Z_R)}{\varepsilon}\leq$$

$$\leq e^{R((1-\theta)P(Z_R)+\theta P(Z_{\upsilon,\alpha,\varepsilon})-c)}\pi'\left((1-\theta)\,E\left[e^{R(Y-Z_R)}\right]+\theta E\left[e^{R(Y-Z_{\upsilon,\alpha,\varepsilon})}\right]\right)\times$$

$$\times\left(e^{R(\upsilon-\alpha\upsilon)}-e^{R(\upsilon-Z_R(\upsilon))}\right)f\left(\upsilon\right)+\tfrac{o(\varepsilon)}{\varepsilon}.$$

Taking limits when $\varepsilon\to 0^+$, this yields

$$\begin{aligned}&-e^{R(P(Z_R)-c)}\pi\left(E\left[e^{R(Y-Z_R)}\right]\right)R\lim_{\varepsilon\to 0^+}\tfrac{P(Z_{\upsilon,\alpha,\varepsilon})-P(Z_R)}{\varepsilon}\leq\\&\leq e^{R(P(Z_R)-c)}\pi'\left(E\left[e^{R(Y-Z_R)}\right]\right)\left(e^{R(\upsilon-\alpha\upsilon)}-e^{R(\upsilon-Z_R(\upsilon))}\right)f\left(\upsilon\right).\end{aligned}\tag{15}$$

Due to (13), this is

$$\begin{aligned}&-e^{R(P(Z_R)-c)}\pi\left(E\left[e^{R(Y-Z_R)}\right]\right)R\lim_{\varepsilon\to 0^+}\tfrac{P(Z_{\upsilon,\alpha\varepsilon})-P(Z_R)}{\varepsilon(\alpha\upsilon-Z_R(\upsilon))}\leq\\&\leq e^{R(P(Z_R)-c)}\pi'\left(E\left[e^{R(Y-Z_R)}\right]\right)\tfrac{e^{R(\upsilon-\alpha\upsilon)}-e^{R(\upsilon-Z_R(\upsilon))}}{\alpha\upsilon-Z_R(\upsilon)}f\left(\upsilon\right).\end{aligned}$$

Taking limits when $\alpha \to \left(\frac{Z_R(\upsilon)}{\upsilon}\right)^+$, this yields

$$-e^{R(P(Z_R)-c)}\pi\left(E\left[e^{R(Y-Z_R)}\right]\right)R\Delta P_{Z_R}(\upsilon) \leq$$
$$\leq e^{R(P(Z_R)-c)}\pi'\left(E\left[e^{R(Y-Z_R)}\right]\right)(-R)\,e^{R(\upsilon-Z_R(\upsilon))}f(\upsilon),$$

i.e.,

$$\frac{\pi'\left(E\left[e^{R(Y-Z_R)}\right]\right)}{\pi\left(E\left[e^{R(Y-Z_R)}\right]\right)}e^{R(\upsilon-Z_R(\upsilon))}f(\upsilon) \leq \Delta P_{Z_R}(\upsilon) \tag{16}$$

holds for almost every υ such that $Z_R(\upsilon) < \upsilon$.

If $E\left[e^{R(Y-Z_R)}\right] < \rho$, then $E\left[e^{R(Y-Z_{\upsilon,\alpha\varepsilon})}\right] < \rho$ also holds for any $\alpha \in [0,1]$, provided $\varepsilon > 0$ is sufficiently small. Therefore in this case, inequality (15) holds for almost every υ such that $Z_R(\upsilon) > 0$. Therefore, dividing both sides of (15) by $\alpha\upsilon - Z_R(\upsilon)$ and taking limits when $\alpha \to \left(\frac{Z_R(\upsilon)}{\upsilon}\right)^-$, we obtain

$$\frac{\pi'\left(E\left[e^{R(Y-Z_R)}\right]\right)}{\pi\left(E\left[e^{R(Y-Z_R)}\right]\right)}e^{R(\upsilon-Z_R(\upsilon))}f(\upsilon) \geq \Delta P_{Z_R}(\upsilon). \tag{17}$$

This proves the Proposition in the case when $E\left[e^{R(Y-Z_R)}\right] < \rho$.

Now, consider the case when $E\left[e^{R(Y-Z_R)}\right] = \rho$. In this case we have

$$E\left[e^{R(Y-Z_{\upsilon,\alpha,\varepsilon})}\right] > \rho$$

and therefore $G(R, Z_{\upsilon,\alpha,\varepsilon}) = +\infty$ for any needle-like perturbation with $\alpha\upsilon < Z_R(\upsilon)$ and sufficiently small support. Thus we can not use the argument above to prove an inequality analogous to (17). Instead we use double needle-like perturbations such that $E\left[e^{R\left(Y-Z_{\upsilon_1,\alpha_1,\varepsilon_1|\upsilon_2,\alpha_2,\varepsilon_2}\right)}\right] = \rho$. Like before, $\upsilon_1, \upsilon_2 \in \left]0,+\infty\right[$ are Lebesgue points of the functions f, $e^{-RZ_R}f$, such that $\upsilon_1 \neq \upsilon_2$ and $f(\upsilon_i) > 0$, $i = 1,2$. Due to Assumption 4, we can choose υ_1 such that $Z_R(\upsilon_1) < \upsilon_1$ and fix α_1 such that $\alpha_1\upsilon_1 > Z_R(\upsilon_1)$. For any

$\upsilon_2 \neq \upsilon_1$ such that $Z_R(\upsilon_2) > 0$, fix α_2 such that $\alpha_2 \upsilon_2 < Z_R(\upsilon_2)$. Then, we have

$$E\left[e^{R\left(Y - Z_{\upsilon_1,\alpha_1,\varepsilon_1|\upsilon_2,\alpha_2,\varepsilon_2}\right)}\right] = E\left[e^{R(Y-Z_R)}\right] + \\ + \int_{\upsilon_1}^{\upsilon_1+\varepsilon_1} \left(e^{R(y-\alpha_1 y)} - e^{R(y - Z_R(y))}\right) f(y)\ dy + \\ + \int_{\upsilon_2}^{\upsilon_2+\varepsilon_2} \left(e^{R(y-\alpha_2 y)} - e^{R(y - Z_R(y))}\right) f(y)\ dy.$$

Since υ_1, υ_2 are Lebesgue points of the functions f, $e^{-RZ_R} f$, it follows that

$$\begin{aligned} & E\left[e^{R\left(Y - Z_{\upsilon_1,\alpha_1,\varepsilon_1|\upsilon_2,\alpha_2,\varepsilon_2}\right)}\right] = \\ = \ & \rho + \varepsilon_1 \left(e^{R(\upsilon_1 - \alpha_1 \upsilon_1)} - e^{R(\upsilon_1 - Z_R(\upsilon_1))}\right) f(\upsilon_1) + o(\varepsilon_1) + \\ & + \varepsilon_2 \left(e^{R(\upsilon_2 - \alpha_2 \upsilon_2)} - e^{R(\upsilon_2 - Z_R(\upsilon_2))}\right) f(\upsilon_2) + o(\varepsilon_2) = \\ = \ & \rho - \varepsilon_1 (\alpha_1 \upsilon_1 - Z_R(\upsilon_1)) R e^{R(\upsilon_1 - Z_R(\upsilon_1))} f(\upsilon_1) + \varepsilon_1 o(\alpha_1 \upsilon_1 - Z_R(\upsilon_1)) + o(\varepsilon_1) - \\ & - \varepsilon_2 (\alpha_2 \upsilon_2 - Z_R(\upsilon_2)) R e^{R(\upsilon_2 - Z_R(\upsilon_2))} f(\upsilon_2) + \varepsilon_2 o(\alpha_2 \upsilon_2 - Z_R(\upsilon_2)) + o(\varepsilon_2). \end{aligned}$$

Therefore, an implicit function-type argument shows that for each sufficiently small $\varepsilon_2 > 0$, $(\alpha_1 \upsilon_1 - Z_R(\upsilon_1))$ and $(\alpha_2 \upsilon_2 - Z_R(\upsilon_2))$ there exists a unique $\varepsilon_1 > 0$ such that $E\left[e^{R\left(Y - Z_{\upsilon_1,\alpha_1,\varepsilon|\upsilon_2,\alpha_2,\varepsilon}\right)}\right] = \rho$. Further, such ε_1 satisfies

$$\begin{aligned} \varepsilon_1 = \ & -\frac{e^{R(\upsilon_2 - Z_R(\upsilon_2))}(\alpha_2 \upsilon_2 - Z_R(\upsilon_2)) f(\upsilon_2)}{e^{R(\upsilon_1 - Z_R(\upsilon_1))}(\alpha_1 \upsilon_1 - Z_R(\upsilon_1)) f(\upsilon_1)} \varepsilon_2 + \\ & + (o(\alpha_1 \upsilon_1 - Z_R(\upsilon_1)) + o(\alpha_2 \upsilon_2 - Z_R(\upsilon_2))) \varepsilon_2 + o(\varepsilon_2). \end{aligned} \tag{18}$$

Since $E\left[e^{R(Y-Z_R)}\right] = E\left[e^{R\left(Y - Z_{\upsilon_1,\alpha_1,\varepsilon_1|\upsilon_2,\alpha_2,\varepsilon_2}\right)}\right] = \rho$, we have

$$\begin{aligned} G(R, Z_R) &= e^{R(P(Z_R) - c)} \pi(\rho); \\ G\left(R, Z_{\upsilon_1,\alpha_1,\varepsilon_1|\upsilon_2,\alpha_2,\varepsilon_2}\right) &= e^{R\left(P\left(Z_{\upsilon_1,\alpha_1,\varepsilon_1|\upsilon_2,\alpha_2,\varepsilon_2}\right) - c\right)} \pi(\rho). \end{aligned}$$

Therefore, optimality of Z_R implies that

$$P\left(Z_{\upsilon_1,\alpha_1,\varepsilon_1|\upsilon_2,\alpha_2,\varepsilon_2}\right) - P\left(Z_R\right) \geq 0.$$

Substituting (18) in (10), this inequality becomes

$$\left(\alpha_2\upsilon_2 - Z_R\left(\upsilon_2\right)\right)\left(\Delta P_{Z_R}\left(\upsilon_2\right) - \tfrac{e^{R\left(\upsilon_2 - Z_R\left(\upsilon_2\right)\right)}f(\upsilon_2)}{e^{R\left(\upsilon_1 - Z_R\left(\upsilon_1\right)\right)}f(\upsilon_1)}\Delta P_{Z_R}\left(\upsilon_1\right)\right) +$$
$$+ o\left(\alpha_1\upsilon_1 - Z_R\left(\upsilon_1\right)\right) + o\left(\alpha_2\upsilon_2 - Z_R\left(\upsilon_2\right)\right) + \tfrac{o(\varepsilon_2)}{\varepsilon_2} \geq 0.$$

Setting $\left(\alpha_1\upsilon_1 - Z_R\left(\upsilon_1\right)\right) = -\left(\alpha_2\upsilon_2 - Z_R\left(\upsilon_2\right)\right)$ and making $\varepsilon_2 \to 0^+$, we obtain

$$\left(\alpha_2\upsilon_2 - Z_R\left(\upsilon_2\right)\right)\left(\Delta P_{Z_R}\left(\upsilon_2\right) - \tfrac{e^{R\left(\upsilon_2 - Z\left(\upsilon_2\right)\right)}f(\upsilon_2)}{e^{R\left(\upsilon_1 - Z_R\left(\upsilon_1\right)\right)}f(\upsilon_1)}\Delta P_{Z_R}\left(\upsilon_1\right)\right) +$$
$$+ o\left(\alpha_2\upsilon_2 - Z_R\left(\upsilon_2\right)\right) \geq 0.$$

Making $\left(\alpha_2\upsilon_2 - Z_R\left(\upsilon_2\right)\right) \to 0^-$, we see that this implies

$$\frac{\Delta P_{Z_R}\left(\upsilon_2\right)}{e^{R\left(\upsilon_2 - Z_R\left(\upsilon_2\right)\right)}f\left(\upsilon_2\right)} \leq \frac{\Delta P_{Z_R}\left(\upsilon_1\right)}{e^{R\left(\upsilon_1 - Z_R\left(\upsilon_1\right)\right)}f\left(\upsilon_1\right)}. \tag{19}$$

This shows that (19) holds for any pair of Lebesgue points of the functions $y \mapsto f\left(y\right)$, $y \mapsto e^{-RZ_R(y)}f\left(y\right)$ such that $f\left(\upsilon_i\right) > 0$, $0 \leq Z_R\left(\upsilon_1\right) < \upsilon_1$, $0 < Z_R\left(\upsilon_2\right) \leq \upsilon_2$. Let C denote the infimum value of $\frac{\Delta P_{Z_R}(\upsilon)}{e^{R(\upsilon - Z_R(\upsilon))}f(\upsilon)}$ over all the Lebesgue points of f, $e^{-RZ_R}f$ such that $Z_R\left(\upsilon\right) < \upsilon$. Inequality (16) shows that $C \geq \frac{\pi'(\rho)}{\pi(\rho)}$ and $\Delta P_{Z_R}\left(\upsilon\right) \geq Ce^{R(\upsilon - Z_R(\upsilon))}f\left(\upsilon\right)$ holds whenever $Z_R\left(\upsilon\right) < \upsilon$. Since (19) guarantees that $\Delta P_{Z_R}\left(\upsilon\right) \leq Ce^{R(\upsilon - Z_R(\upsilon))}f\left(\upsilon\right)$ holds whenever $Z_R\left(\upsilon\right) > 0$, the proof is complete. ■

8 The optimal solution for variance related premium calculation principles

In this section we apply the results obtained in the previous sections to the case where the premium principle $P : \mathcal{Z} \mapsto [0, +\infty)$ is a convex variance-related functional, i.e.

$$P(Z) = E\left[\widehat{Z}\right] + g\left(Var\left[\widehat{Z}\right]\right), \tag{20}$$

with $g : [0, +\infty) \mapsto [0, +\infty)$ continuous, smooth in $(0, +\infty)$, such that

$$g(0) = 0, \qquad g'(x) > 0 \tag{21}$$

and

$$\frac{g''(x)}{g'(x)} \geq -\frac{1}{2x}, \qquad \forall x \in]0, B[, \tag{22}$$

with $B = \sup\{Var[\widehat{Z}] : Z \in \mathcal{Z}\}$. Following Guerra and Centeno (2007) we can say that if g is twice differentiable and satisfies (21), then the principle (20) is convex if and only if (22) is fulfilled. It can be checked that such premium principles are continuous in mean-squared sense, as required by Assumption 5. The most important examples of convex variance related premium principles are the standard deviation and the variance principles.

The following proposition provides an expression for ΔP_Z.

Proposition 4 *If the reinsurance premium is computed by a functional of* *the form (20), then the following equality holds on a set of probability one*

$$\Delta P_Z(\upsilon) = \begin{cases} E[N] f(\upsilon) + \\ +2g'\left(Var\left[\widehat{Z}\right]\right)\left(E[N](Z(\upsilon) - E[Z]) + Var[N] E[Z]\right) f(\upsilon), \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{if } \Pr\{Z \neq 0\} > 0; \\ E[N] f(\upsilon), \qquad\qquad \textit{if } Z \equiv 0 \textit{ and } g'(0^+) < +\infty; \\ +\infty, \qquad\qquad \textit{if } Z \equiv 0 \textit{ and } g'(0^+) = +\infty. \square \end{cases}$$

Proof. Fix $\upsilon > 0$, a Lebesgue point of the maps $\upsilon \to f(\upsilon)$, $\upsilon \to Z(\upsilon)f(\upsilon)$ and $\upsilon \to Z(\upsilon)^2 f(\upsilon)$, such that $f(\upsilon) > 0$.

By the definition of ΔP_Z, we have

$$\begin{aligned} \Delta P_Z(\upsilon) &= \lim_{\alpha \to \frac{Z(\upsilon)}{\upsilon}} \lim_{\varepsilon \to 0^+} \frac{P(Z_{\upsilon,\alpha,\varepsilon}) - P(Z)}{(\alpha\upsilon - Z(\upsilon))\varepsilon} = \\ &= \lim_{\alpha \to \frac{Z(\upsilon)}{\upsilon}} \lim_{\varepsilon \to 0^+} \frac{E\left[\hat{Z}_{\upsilon,\alpha,\varepsilon}\right] - E\left[\hat{Z}\right] + g\left(Var\left[\hat{Z}_{\upsilon,\alpha,\varepsilon}\right]\right) - g\left(Var\left[\hat{Z}\right]\right)}{(\alpha\upsilon - Z(\upsilon))\varepsilon}. \quad (23) \end{aligned}$$

The mean-value theorem states that, for each $\upsilon > 0$, $\alpha \in [0,1]$, $\varepsilon > 0$, there exists $\theta \in]0,1[$ such that

$$\begin{aligned} & g\left(Var\left[\hat{Z}_{\upsilon,\alpha,\varepsilon}\right]\right) - g\left(Var\left[\hat{Z}\right]\right) = \\ = \; & g'\left((1-\theta) Var\left[\hat{Z}\right] + \theta Var\left[\hat{Z}_{\upsilon,\alpha,\varepsilon}\right]\right) \times \left(Var\left[\hat{Z}_{\upsilon,\alpha,\varepsilon}\right] - Var\left[\hat{Z}\right]\right). \end{aligned}$$

Recall that the first two moments of $\hat{Z}$ can be calculated as

$$\begin{aligned} E\left[\hat{Z}\right] &= E[N] E[Z]; \\ Var\left[\hat{Z}\right] &= E[N] Var[Z] + Var[N] E[Z]^2. \end{aligned}$$

Since υ is a Lebesgue point, we have

$$E\left[Z_{\upsilon,\alpha,\varepsilon}\right]-E\left[Z\right]=\int_{\upsilon}^{\upsilon+\varepsilon}\left(\alpha y-Z\left(y\right)\right)f\left(y\right)\,dy=\varepsilon\left(\alpha\upsilon-Z\left(\upsilon\right)\right)f\left(\upsilon\right)+o\left(\varepsilon\right),$$

and

$$\begin{aligned}
& Var\left[Z_{\upsilon,\alpha,\varepsilon}\right]-Var\left[Z\right]= \\
= \; & \left(E\left[Z_{\upsilon,\alpha,\varepsilon}^2\right]-E\left[Z^2\right]\right)-\left(E\left[Z_{\upsilon,\alpha,\varepsilon}\right]^2-E\left[Z\right]^2\right)= \\
= \; & \int_{\upsilon}^{\upsilon+\varepsilon}\left(\left(\alpha y\right)^2-Z\left(y\right)^2\right)f\left(y\right)\,dy-\left(E[Z_{\upsilon,\alpha,\varepsilon}]-E\left[Z\right]\right)\left(E[Z_{\upsilon,\alpha,\varepsilon}]+E\left[Z\right]\right)= \\
= \; & \varepsilon\left(\alpha^2\upsilon^2-Z\left(\upsilon\right)^2\right)f\left(\upsilon\right)-\varepsilon\left(\alpha\upsilon-Z\left(\upsilon\right)\right)f\left(\upsilon\right)2E\left[Z\right]+o\left(\varepsilon\right)= \\
= \; & \varepsilon\left(\alpha\upsilon-Z\left(\upsilon\right)\right)\left(\alpha\upsilon+Z\left(\upsilon\right)\right)f\left(\upsilon\right)-\varepsilon\left(\alpha\upsilon-Z\left(\upsilon\right)\right)f\left(\upsilon\right)2E\left[Z\right]+o\left(\varepsilon\right)= \\
= \; & \varepsilon\left(\alpha\upsilon-Z\left(\upsilon\right)\right)\left(\alpha\upsilon+Z\left(\upsilon\right)-2E\left[Z\right]\right)f\left(\upsilon\right)+o\left(\varepsilon\right).
\end{aligned}$$

It follows that

$$E\left[\hat{Z}_{\upsilon,\alpha,\varepsilon}\right]-E\left[\hat{Z}\right]=\varepsilon\left(\alpha\upsilon-Z\left(\upsilon\right)\right)E\left[N\right]f\left(\upsilon\right)+o\left(\varepsilon\right); \tag{24}$$

$$\begin{aligned}
& Var\left[\hat{Z}_{\upsilon,\alpha,\varepsilon}\right]-Var\left[\hat{Z}\right]= \\
& =E\left[N\right]\left(Var\left[Z_{\upsilon,\alpha,\varepsilon}\right]-Var\left[Z\right]\right)+Var\left[N\right]\left(E\left[Z_{\upsilon,\alpha,\varepsilon}\right]^2-E\left[Z\right]^2\right)= \\
& = \; \varepsilon\left(\alpha\upsilon-Z\left(\upsilon\right)\right)\times \qquad (25) \\
& \qquad \times\left(E\left[N\right]\left(\alpha\upsilon+Z\left(\upsilon\right)-2E\left[Z\right]\right)+2Var\left[N\right]E\left[Z\right]\right)f\left(\upsilon\right)+o\left(\varepsilon\right).
\end{aligned}$$

In the case when $\Pr\left\{Z>0\right\}>0$, substitution of (24)-(25) in (23) yields immediately the desired equality.

In the case when $\Pr\left\{Z>0\right\}=0$, the equalities (24)-(25) show that

$$\begin{aligned}
& \lim_{\varepsilon\to0^+}\frac{P\left(Z_{\upsilon,\alpha,\varepsilon}\right)-P\left(Z\right)}{\left(\alpha\upsilon-Z\left(\upsilon\right)\right)\varepsilon}=\frac{\alpha\upsilon E\left[N\right]+g'\left(0^+\right)E\left[N\right]\left(\alpha\upsilon\right)^2}{\alpha\upsilon}f\left(\upsilon\right)= \\
& =\begin{cases}\left(E\left[N\right]+g'\left(0^+\right)E\left[N\right]\alpha\upsilon\right)f\left(\upsilon\right), & \text{if } g'\left(0^+\right)<+\infty; \\ +\infty, & \text{if } g'\left(0^+\right)=+\infty.\end{cases}
\end{aligned}$$

This completes the proof. ■

Using Proposition 4 we can state the following Corollary to Theorem 3:

Corollary 3 *Fix $R > 0$, and let $Z = Z_R \in \mathcal{Z}$ be optimal for the expected utility criterion. If g' is bounded in a neighborhood of zero, then the following set of conditions holds with probability equal to one with respect to the density f:*

$$y \leq \alpha_1 + \alpha_2, \qquad if\ Z(y) = y; \tag{26}$$

$$y = Z(y) + \frac{1}{R} \ln \frac{Z(y) - \alpha_2}{\alpha_1}, \qquad if\ 0 < Z(y) < y; \tag{27}$$

$$y \leq \frac{1}{R} \ln \frac{-\alpha_2}{\alpha_1}, \qquad if\ Z(y) = 0. \tag{28}$$

α_1, α_2 are constants satisfying

$$\begin{aligned}
\alpha_1 &= \frac{\pi'\left(E\left[e^{R(Y-Z)}\right]\right)}{E[N]\,\pi\left(E\left[e^{R(Y-Z)}\right]\right)\ 2g'\left(Var\left(\widehat{Z}\right)\right)}, && \text{if } E\left[e^{R(Y-Z)}\right] < \rho; \\
\alpha_1 &\geq \frac{\pi'\left(E\left[e^{R(Y-Z)}\right]\right)}{E[N]\,\pi\left(E\left[e^{R(Y-Z)}\right]\right)\ 2g'\left(Var\left(\widehat{Z}\right)\right)}, && \text{if } E\left[e^{R(Y-Z)}\right] = \rho; \\
\alpha_2 &= \frac{E[N] - Var[N]}{E(N)} E[Z] - \frac{1}{2g'\left(Var\left(\widehat{Z}\right)\right)}.
\end{aligned}$$

where ρ is the radius of convergence of $\pi(\cdot)$, the probability generating function of N.

If g' is unbounded in any neighborhood of zero, then the optimal treaty must be either a function of the type described above or $Z \equiv 0$ (no reinsurance at all). □

Remark 1 *Notice that in the Corollary above α_1 is always strictly positive while α_2 may be either positive or negative.*

Also, remark that if (26) holds for some $y > 0$, then $\alpha_1 + \alpha_2 > 0$ must hold. Similarly, if (28) holds for some $y > 0$, then $\alpha_1 + \alpha_2 < 0$ must hold. It follows that, provided the optimal treaty is not identically zero, then it must be a function satisfying either (26)-(27) or (27)-(28).

Proof. First, suppose that $\Pr\{Z > 0\}$ holds, and let $C \in \left]0, +\infty\right[$ be as stated in Theorem 3. Due to Proposition 4 the optimality conditions become

$$\begin{cases} 2g'\left(Var\left[\widehat{Z}\right]\right)\left(E\left[N\right]\left(y - E\left[Z\right]\right) + Var\left[N\right]E\left[Z\right]\right) + \\ \quad + E\left[N\right] \leq C, & \text{if } Z\left(y\right) = y; \\ 2g'\left(Var\left[\widehat{Z}\right]\right)\left(E\left[N\right]\left(Z\left(y\right) - E\left[Z\right]\right) + Var\left[N\right]E\left[Z\right]\right) + \\ \quad + E\left[N\right] = Ce^{R(y - Z(y))} & \text{if } 0 < Z\left(y\right) < y; \\ 2g'\left(Var\left[\widehat{Z}\right]\right)\left(-E\left[N\right]E\left[Z\right] + Var\left[N\right]E\left[Z\right]\right) + \\ \quad + E\left[N\right] \geq Ce^{Ry} & \text{if } Z\left(y\right) = 0. \end{cases}$$

This is

$$\begin{cases} y \leq \frac{C}{2g'\left(Var\left(\widehat{Z}\right)\right)E[N]} + \frac{E[N] - Var(N)}{E[N]}E\left[Z\right] - \frac{1}{2g'\left(Var\left(\widehat{Z}\right)\right)}, & \text{if } Z\left(y\right) = y; \\ e^{R(y - Z(y))} = \frac{Z(y) - \left(\frac{E[N] - Var(N)}{E[N]}E[Z] - \frac{1}{2g'\left(Var\left(\widehat{Z}\right)\right)}\right)}{C/\left(2g'\left(Var\left(\widehat{Z}\right)\right)E[N]\right)} & \text{if } 0 < Z\left(y\right) < y; \\ e^{R(y)} \leq -\frac{\frac{E[N] - Var(N)}{E[N]}E[Z] - \frac{1}{2g'\left(Var\left(\widehat{Z}\right)\right)}}{C/\left(2g'\left(Var\left(\widehat{Z}\right)\right)E[N]\right)} & \text{if } Z\left(y\right) = 0. \end{cases}$$

Hence the result follows immediately from Theorem 3 by making $\alpha_1 = \frac{C}{2g'\left(Var\left(\widehat{Z}\right)\right)E[N]}$, $\alpha_2 = \frac{E[N] - Var(N)}{E[N]}E\left[Z\right] - \frac{1}{2g'\left(Var\left(\widehat{Z}\right)\right)}$.

Now, consider the case when $g'(x)$ is unbounded in any neighborhood of $x = 0$, and let $Z \equiv 0$. In that case, for each v, a Lebesgue point of the function $y \mapsto f(y)$, and each $\alpha \in]0,1]$, we can choose a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that $\varepsilon_k \to 0^+$ and $\lim_{k\to\infty} \frac{P\left(Z_{v,\alpha,\varepsilon_k}\right)-P(0)}{\varepsilon_k(\alpha v - Z(v))} = +\infty$. Therefore, Theorem 3 does not exclude the possibility that $Z \equiv 0$ be optimal. ■

9 Structure of the optimal treaty: dependence on the distribution of claim numbers

According to Corollary 3, if the optimal treaty for a variance-related principle is not identically zero, then it is a member of the family of functions with two parameters $(\alpha_1, \alpha_2) \in (0, +\infty) \times \mathbb{R}$, satisfying (26)-(28). It can be shown that in the aggregate claim case an optimal treaty satisfying $\alpha_1 + \alpha_2 = 0$ always exists (the limit case $\alpha_1 = \alpha_2 = 0$ corresponding to the case when zero reinsurance is optimal) – see Guerra and Centeno (2008).

Below we show that a similar result holds for the individual claim case when the number of claims N follows a distribution belonging to the so-called Katz family, also known in actuarial literature as the Panjer or $(a, b, 0)$ family of distributions. This family is important for practical applications, including some of the most widely used claim number models. However, we also show that both the cases $\alpha_1 + \alpha_2 > 0$ and $\alpha_1 + \alpha_2 < 0$ do occur in individual claim reinsurance, provided the number of claims follows appropriate distributions.

We start with an interesting relationship between the structure of $\pi(\cdot)$ -

the probability generating function of N - and the sign of $\alpha_1 + \alpha_2$.

Theorem 4 *Suppose that the optimal treaty is not identically zero and satisfies $E\left[e^{R(Y-Z)}\right] < \rho$, the radius of convergence of the series $\pi(t)$.*

(a) If the function $q(t) = \frac{\pi(t)}{\pi'(t)}$ is convex, then $\alpha_1 + \alpha_2 \leq 0$ holds;

(b) If q is strictly convex, then $\alpha_1 + \alpha_2 < 0$ holds;

(c) If q is concave, then $\alpha_1 + \alpha_2 \geq 0$ holds;

(d) If q is strictly concave, then $\alpha_1 + \alpha_2 > 0$ holds. $\square$

Proof. It easy to check that

$$E[N] = \pi'(1), \qquad Var[N] = \pi''(1) + \pi'(1) - \pi'(1)^2.$$

Therefore, Corollary 3 states that

$$\alpha_1 = \frac{q(1)}{q(x)} \frac{1}{2g'(Var[\widehat{Z}])}; \tag{29}$$

$$\alpha_2 = \frac{q'(1)}{q(1)} E[Z] - \frac{1}{2g'(Var[\widehat{Z}])}, \tag{30}$$

where $x = E\left[e^{R(Y-Z)}\right]$. Equation (30) is equivalent to

$$\frac{1}{2g'(Var[\widehat{Z}])} = \frac{q'(1)}{q(1)} E[Z] - \alpha_2.$$

Substituting in (29), we obtain

$$q(x)\alpha_1 = q'(1)E[Z] - q(1)\alpha_2. \tag{31}$$

Now, suppose that q is convex and $\alpha_1+\alpha_2 \geq 0$ holds. Corollary 3 implies that

$$\begin{aligned} x &= E\left[e^{R(Y-Z)}\right] = \int_0^{\alpha_1+\alpha_2} dF(y) + \int_{\alpha_1+\alpha_2}^{+\infty} \frac{Z(y)-\alpha_2}{\alpha_1} dF(y) = \\ &= \frac{1}{\alpha_1}\left(E[Z]-\alpha_2+\int_0^{\alpha_1+\alpha_2}(\alpha_1+\alpha_2-y)dF(y)\right). \end{aligned}$$

This is

$$E[Z] = x\alpha_1+\alpha_2-\int_0^{\alpha_1+\alpha_2}(\alpha_1+\alpha_2-y)dF(y).$$

Substituting in (31) and rearranging we obtain

$$\begin{aligned} \left(q(x)-q(1)-q'(1)(x-1)\right)\alpha_1 + \left(q(1)-q'(1)\right)(\alpha_1+\alpha_2) = \\ = -q'(1)\int_0^{\alpha_1+\alpha_2}(\alpha_1+\alpha_2-y)dF(y). \end{aligned} \tag{32}$$

Notice that $\int_0^{\alpha_1+\alpha_2}(\alpha_1+\alpha_2-y)dF(y) \leq \alpha_1+\alpha_2$, with strict inequality holding unless $\alpha_1+\alpha_2=0$. If $q'(1)\leq 0$ then the second term on the left-hand side of (32) is non-negative and no smaller that the right-hand side term. By convexity, the first term on the left-hand side is non-negative (strictly positive if q is strictly convex). Hence (32) can hold with $q'(1) \leq 0$ only if $\alpha_1+\alpha_2=0$ and q is linear in $[0,x]$.

Now, note that

$$q(1)-q'(1) = \frac{Var[N]}{E[N]^2} > 0. \tag{33}$$

Therefore, if $q'(1)>0$ then the second term in the left-hand side of (32) is non-negative while the term on the right-hand side is nonpositive and (32) can hold only if $\alpha_1+\alpha_2=0$ and q is linear in $[0,x]$. It cannot hold if q is strictly convex.

To prove *(c)* and *(d)*, suppose that q is concave and $\alpha_1 + \alpha_2 \leq 0$ holds. In this case, Corollary 3 implies that

$$\begin{aligned} x &= \int_0^{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}} e^{Ry} dF(y) + \int_{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}}^{+\infty} \frac{Z(y) - \alpha_2}{\alpha_1} dF(y) = \\ &= \frac{1}{\alpha_1}\left(E[Z] - \alpha_2 - \alpha_1 \int_0^{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}} \left(\frac{-\alpha_2}{\alpha_1} - e^{Ry} \right) dF(y) \right). \end{aligned}$$

This is

$$E[Z] = \left(x + \int_0^{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}} \left(\frac{-\alpha_2}{\alpha_1} - e^{Ry} \right) dF(y) \right) \alpha_1 + \alpha_2.$$

Substituting in (31) and rearranging we obtain

$$\begin{aligned} q(x) - q(1) - q'(1)(x-1) = q'(1) \int_0^{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}} \left(\frac{-\alpha_2}{\alpha_1} - e^{Ry} \right) dF(y) + \\ +(q(1) - q'(1)) \left(\frac{-\alpha_2}{\alpha_1} - 1 \right). \end{aligned} \tag{34}$$

By convexity the left-hand side is nonpositive (strictly negative if q is strictly concave). If $q'(1) \geq 0$, then the right-hand side of (34) is non-negative, being zero only if $\alpha_1 + \alpha_2 = 0$. If $q'(1) < 0$ then rearrange the right-hand side of (34) to obtain

$$\begin{aligned} q(x) - q(1) - q'(1)(x-1) = q(1) \left(\frac{-\alpha_2}{\alpha_1} - 1 \right) - \\ -q'(1) \left(\frac{-\alpha_2}{\alpha_1} - 1 - \int_0^{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}} \left(\frac{-\alpha_2}{\alpha_1} - e^{Ry} \right) dF(y) \right), \end{aligned}$$

where the right-hand side is obviously non-negative, being zero only if $\alpha_1 + \alpha_2 = 0$. Hence we see that (34) holds only if $\alpha_1 + \alpha_2 = 0$ and q is linear in $[0, x]$. ■

Remark 2 *When the distribution of claim values has a heavy tail, some part of the risk must be ceded under the optimal arrangement, irrespective of the distribution of claim numbers (otherwise the adjustment coefficient would not exist). Therefore the possibility of zero reinsurance being optimal indicated at the beginning of the Theorem can be immediately excluded in such cases.*

Theorem 4 shows that the case in which the function q is linear has the special property $\alpha_1 + \alpha_2 = 0$. One such case is aggregate claim reinsurance, analyzed in Guerra and Centeno (2008). Indeed, it is clear that the random variable $N \equiv 1$ has probability generating function $\pi(t) = t$ and hence $q(t) = t$ is simultaneously concave and convex. It turns out that there exists also an important family of nondegenerate distributions with this property. This is the so-called Katz family (see Johnson et al (1993, pg. 38)). It consists of the binomial, Poisson and negative binomial distributions.

Corollary 4 *Assume that the distribution of the number of claims N belongs to the Katz family.*

If g' is bounded in a neighborhood of zero, then the optimal policy satisfies

$$y = Z(y) + \frac{1}{R} \ln \frac{Z(y)+\alpha}{\alpha};$$

$$\alpha = \frac{\pi'\left(E\left[e^{R(Y-Z)}\right]\right)}{E[N]\pi\left(E\left[e^{R(Y-Z)}\right]\right) \; 2g'\left(Var\left(\widehat{Z}\right)\right)}$$

If g' is unbounded near zero, then either $Z \equiv 0$ is optimal or the optimal policy satisfies the conditions above. □

Proof. Recall that Katz-type distributions are characterized by having a probability generating function of the type

$$\pi(t) = \left(\frac{1-at}{1-a} \right)^{-(a+b)/a}, \tag{35}$$

with $a < 1$, $a + b > 0$ and $b = -(k+1)a$ for some $k \in \mathbb{N}$ when $a < 0$ (in the case $a = 0$, it is $\pi(t) = \lim_{a \to 0} \left(\frac{1-at}{1-a} \right)^{-(a+b)/a} = e^{b(t-1)}$). It follows that q is the linear function

$$q(t) = \frac{1-at}{a+b},$$

hence it is simultaneously concave and convex. The radius of convergence of π is $\rho = \frac{1}{a}$ if $a \in (0,1)$ or $\rho = +\infty$ if $a \leq 0$. In the case when $a \in (0,1)$, we have $\lim\limits_{x \to \rho^-} \pi(x) = +\infty$. It follows that the optimal reinsurance policy must satisfy $E\left[e^{R(Y-Z)}\right] < \rho$ and the result follows immediately from Corollary 3 and Theorem 4. ■

In order to see that there are cases where $\alpha_1 + \alpha_2 < 0$ and cases where $\alpha_1 + \alpha_2 > 0$ holds, recall that the Katz family of distributions can be embedded in the larger Sundt & Jewell family (Sundt & Jewell 1981). These are combinations of a Katz random variable with a random variable concentrated at $N = 0$.

The probability generating function of a Sundt & Jewell distribution is

$$\pi(t) = c + (1-c) \left(\frac{1-at}{1-a} \right)^{-(a+b)/a},$$

with a, b like in (35) and $\frac{(1-a)^{(a+b)/a}}{(1-a)^{(a+b)/a} - 1} \leq c < 1$. Since this is a combination of a Katz random variable with the null random variable, the proof of Corollary 4 shows that the optimal treaty must satisfy $E\left[e^{R(Y-Z)}\right] < \rho$. Therefore, due to Theorem 4, we only need to check the convexity of $q = \frac{\pi}{\pi'}$.

A simple computation shows that a Sundt & Jewell distribution satisfies

$$q''(t) = \frac{c(2a+b)}{(1-at)(\pi(t)-c)}.$$

Since $a+b>0$ and $b=-(k+1)a$ must hold when $a<0$, we see that $2a+b>0$ holds except in the case when $a<0$, $b=-2a$. In this case N is a Bernoulli random variable and hence any mixture with the null random variable is still Bernoulli .

Therefore we see that, except in the Bernoulli case, q is strictly convex (i.e., $\alpha_1+\alpha_2<0$) if $c>0$ and it is strictly concave (i.e., $\alpha_1+\alpha_2>0$) if $\frac{(1-a)^{(a+b)/a}}{(1-a)^{(a+b)/a}-1} \leq c < 0$. It is linear (i.e., $\alpha_1+\alpha_2=0$) if $c=0$.

The discussion above shows that the case $\alpha_1+\alpha_2=0$ is a very particular one, being destroyed by small perturbations of the distribution of claim numbers. To see this consider that N follows a Katz distribution and it is not Bernoulli. This is a Sundt & Jewell distribution with $c=0$. By changing the value of c by any small amount (i.e., by changing $\Pr\{N=0\}$ by any small amount, adjusting proportionally the remaining probabilities in such a way to obtain a probability function) we can either obtain the case $\alpha_1+\alpha_2<0$ or $\alpha_1+\alpha_2>0$, according to the sign of c.

Notice that, according to the proof of Theorem 4, the sign of $\alpha_1+\alpha_2$ depends on the sign of the remainder of the Taylor expansion of the function $q(\cdot)$ (convexity being just one condition that ensures that this remainder has the appropriate sign). Therefore, one can expect to find distributions of the number of claims such that the sign of $\alpha_1+\alpha_2$ changes when the reinsurance loading increases.

To see this, consider a family of variance related principles

$$P_\beta(Z) = E[Z] + \beta g(Var[Z]), \qquad \beta \in]0, +\infty[,$$

where $g : [0, +\infty) \mapsto [0, +\infty)$ is continuous, smooth in $(0, +\infty)$ and satisfies (21)-(22) (say, $g(t) = \sqrt{t}$, in which case $\{P_\beta\}_{\beta>0}$ is the family of standard deviation principles). Assuming that all the remaining data of the problem remains unchanged, one can expect that for small values of the parameter β a large proportion of the risk is ceded under the optimal treaty and hence the value of $x = E\left[e^{R(Y-Z)}\right]$ is close to one. Conversely, when β is large, a small part of the total risk is ceded under the optimal treaty, making $x = E\left[e^{R(Y-Z)}\right]$ large.

Now suppose that the distribution of the number of claims is such that the function q is (say) strictly concave in the interval $[1, x_0]$ and strictly convex in $[x_0, \rho)$. Clearly, the Taylor remainder $q(x) - q(1) - q'(1)(x - 1)$ is negative whenever $x \in (1, x_1)$, for some $x_1 > x_0$. However, we cannot exclude that $x_1 < +\infty$ and the remainder becomes positive for $x \in (x_1, +\infty)$. In such a case optimal treaties would satisfy $\alpha_1 + \alpha_2 > 0$ for small loadings (i.e., small β and hence a large amount of risk is ceded under the optimal treaty) and $\alpha_1 + \alpha_2 < 0$ for higher loadings (large β, and a small amount of risk is ceded).

References

Arrow K.J. (1963). Uncertainty and the Welfare of Medical Care. *The American Economic Review*, LIII, 941-973.

Borch, K. (1960). An Attempt to Determine the Optimum Amount of Stop Loss Reinsurance. *Transactions of the 16th International Congress of Actuaries*, 597-610.

Borch, K. (1969). The optimum reinsurance treaty. *ASTIN Bulletin,* 5, 293-297.

Bowers, N. L., Gerber, H. U., Hickman, J. C., Jones, D. A. and Nesbitt, C. J. (1987). *Actuarial Mathematics*, Society of Actuaries, Chicago.

Centeno, M.L. (1997). Excess of Loss Reinsurance and the Probability of Ruin in Finite Horizon. *ASTIN Bulletin*, Vol. 27, No1, 59-70.

Deprez, O. and Gerber, H.U. (1985). On convex principles of premium calculation. *Insurance: Mathematics and Economics*, 4, 179-189.

Froot, K.A., (2001). The Market for catastrophe risk: a clinical examination. *Journal of Financial Economics,* 60, 529-571.

Gajek, L. and Zagrodny, D. (2000). Insurer's optimal reinsurance strategies. *Insurance: Mathematics and Economics,* 27, 105-112.

Gamkrelidze, R. V. (1978). *Principles of Optimal Control Theory.* Plenum Press.

Gerber, H.U. (1979). *An Introduction to Mathematical Risk Theory.* Huebner Foundation Monographs 8. Homewood Illinois: Richard D. Irwin Inc. (dist.).

Guerra, M. and Centeno, M.L. (2007). Optimal reinsurance for variance related premium calculation principles. Paper presented at the 37th ASTIN Colloquium. Orlando.
http://actuaries.org/ASTIN/Colloquia/Orlando/Papers/Guerra.pdf

Guerra, M. and Centeno, M.L. (2008). Optimal reinsurance policy: the adjustment coefficient and the expected utility criteria. *Insurance: Mathematics and Economics*, 42/2, 529-539.
http://dx.doi.org/10.1016/j.insmatheco.2007.02.008.

Hesselager, O. (1990). Some results on optimal reinsurance in terms of the adjustment coefficient. *Scandinavian Actuarial Journal,* 1, 80-95.

Johnson, N., Kotz, S. and Kemp, A. (1993). *Univariate discrete distributions* (2nd Ed), John Wiley & Sons.

Kaluszka, M. (2001). Optimal reinsurance under mean-variance premium principles. *Insurance: Mathematics and Economics*, 28, pp. 61-67.

Kaluszka, M. (2004). An extension of Arrow's result on optimality of a stop loss contract. *Insurance: Mathematics and Economics*, 35, 524-536.

Kaluszka, M. (2005). Optimal reinsurance under convex principles of premium calculation. *Insurance: Mathematics and Economics*, 36, 375-398.

Rudin, W. (1991). *Functional Analysis*. Singapore: McGraw-Hill International.

Sundt, B. and Jewell, W. (1981). Further results on recursive evaluation of compound distributions. *ASTIN Bulletin*, 12, pp. 27-39.