A New approach on the solution of a delay differential equation

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Abstract. This paper introduces a new approach for obtaining explicit solutions for a first order linear delay differential equation with constant coefficients. We conjecture that there is a generating function defined over of a specific class of polynomials in the delay that solves the equation, and prove in the main theorem that the conjecture is valid. We also show the advantage of our method as regards the traditional Method of Step Algorithm (MSA).

Keywords: Delay differential equation; Method of step algorithm, MSA; Generating function in the delay

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1. Introduction

Delay differential equations (DDEs) are a special class of functional differential equations. The most important results on existence, uniqueness and the properties of solutions for linear and nonlinear DDEs can be found in [1, 2, 3].

In this paper we present a new approach to obtain the exact solution for a particular linear DDE with constant coefficients, based upon the generating function concept.

The results presented in the Main Theorem concern the solutions of the Basic Initial Problem (BIP),

$$\begin{cases} x'(t) = Bx(t-r), t \geq 0 \\ x(t) = \phi(t), t \in [-r, 0] \end{cases}$$

where B and r are constants, $r > 0$ is the delay, and $\phi(t)$ is a given continuous function on $[-r, 0]$.

Assuming $\phi(t)$ is constant on $[-r, 0]$, and applying the MSA to the BIP, the solutions $x_n(t)$ defined on $A_n = ((n-1)r, nr]$, $n \geq 1$, reveal a kind of a tree structure for the solution $x(t)$ of the problem. This allows to formulate a conjecture concerning the solution for the BIP: $x(t)$ *is the generating function for a sequence of polynomials in the delay,* $P_j^n(rB)$.

In order to prove this, a new specific formulation of the solution of the problem is required. As far as we know, the approach via generating function is new to the relevant literature. When compared with the MSA, the advantage of the Main Theorem is to provide an explicite formula for $x_n(t)$ on the A_n interval without the use of the information on all solutions $x_{n-1}(t)$ defined on the previous intervals A_{n-1}. If we used the MSA, each solution $x_n(t)$ would depend upon the solution $x_{n-1}(t)$. Our main theorem proves that this is possible if we introduce polynomials $P_j^n(rB)$.

The present paper is organized as follows. Section 2 describes the MSA, and presents the conjecture. Section 3 constructs the alternative method to obtain the BIP's solution. Section 4 contains the two fundamental propositions with which we obtain the calculating formulas for any solution $x_n(t)$ defined on A_n with $n \geq 2$. Section 5 is devoted to the Main Theorem. Section 6 consists of the proof of the lemma, which is the basis of the new solution's method. An example is given to illustrate the theorem in section 7. Section 8 presents future directions that this work allowed to persue.

2. Preliminaries

2.1. The Method of Step Algorithm

Consider the Basic Initial Problem

$$\begin{cases} x'(t) = Bx(t-r), t \geq 0 \\ x(t) = \phi(t), t \in [-r, 0], \end{cases} \tag{1}$$

where $B \in \Re$, $r > 0$ is the delay, and $\phi(t)$ is a given continuous function on $[-r, 0]$.

The Method of Step Algorithm (MSA) can be described as follows.

Step 1

Consider $x'(t) = f(x(t-r))$. Given $\phi(t)$ on $[-r, 0]$, we can determine $x(t)$ on the interval $[0, r]$ by solving the ODE

$$\begin{cases} x'(t) = f(\phi(t-r)) \\ x(0) = \phi(0). \end{cases}$$

Denote its solution by $x_1(t)$.

Step n

For each integer $n \geq 2$, given the solution $x_{n-1}(t)$ on $[(n-2)\,r, (n-1)\,r]$, we can determine $x(t)$ on the interval $[(n-1)\,r, nr]$ by solving the ODE

$$\begin{cases} x'(t) = f(x_{n-1}(t-r)) \\ x((n-1)\,r) = x_{n-1}((n-1)\,r). \end{cases}$$

Denote its solution by $x_n(t)$.

Conclusion

We can define the solution of

$$\begin{cases} x'(t) = f(x(t-r)), t \geq 0 \\ x(t) = \phi(t), t \in [-r, 0] \end{cases}$$

on each interval $A_n = [(n-1)\,r, nr]$, $n \geq 1$, by

$$x_n(t) = x_{n-1}\left((n-1)\,r\right) + \int_{(n-1)r}^{t} f\left(x_{n-1}\left(s-r\right)\right)ds,$$

where $x_0\,(.) \equiv \phi\,(.)$.

For $j = 1, ..., 5$, let $x_j(t)$ be the solution of (1) defined on the interval A_j obtained with the MSA. Assuming $\phi(t) = \phi(0)$ is a constant,

$$\begin{aligned}
x_1(t) &= \phi(0)\,[1 + Bt] \\
x_2(t) &= \phi(0)\left[\frac{(Bt)^2}{2!} + Bt\,(1 - Br) + \frac{(Br)^2}{2!} + 1\right] \\
x_3(t) &= \phi(0)\left[\begin{array}{c} \frac{(Bt)^3}{3!} + \frac{(Bt)^2}{2!}\left(1 - \frac{3.2}{3}Br\right) + Bt\left(1 - Br + \frac{3.2^2}{3!}(Br)^2\right) - \\ -\frac{(Br)^3}{3!}2^3 + \frac{(Br)^2}{2!} + 1 \end{array}\right] \\
x_4(t) &= \phi(0)\left[\begin{array}{c} \frac{(Bt)^4}{4!} + \frac{(Bt)^3}{3!}\left(1 - \frac{4.3}{4}Br\right) + \frac{(Bt)^2}{2!}\left(1 - \frac{3.2}{3}Br + \frac{6.3^2}{4.3}(Br)^2\right) + \\ +Bt\left(1 - Br + \frac{3.2^2}{3!}(Br)^2 - \frac{4.3^3}{4!}(Br)^3\right) + \\ +\frac{(Br)^4}{4!}3^4 - \frac{(Br)^3}{3!}2^3 + \frac{(Br)^2}{2!} + 1 \end{array}\right] \\
x_5(t) &= \phi(0)\left[\begin{array}{c} \frac{(Bt)^5}{5!} + \frac{(Bt)^4}{4!}\left(1 - \frac{5.4}{5}Br\right) + \frac{(Bt)^3}{3!}\left(1 - \frac{4.3}{4}Br + \frac{10.4^2}{5.4}(Br)^2\right) + \\ +\frac{(Bt)^2}{2!}\left(1 - \frac{3.2}{3}Br + \frac{6.3^2}{4.3}(Br)^2 - \frac{10.4^3}{5.4.3}(Br)^3\right) + \\ +Bt\left(1 - Br + \frac{3.2^2}{3!}(Br)^2 - \frac{4.3^3}{4!}(Br)^3 + \frac{5.4^4}{5!}(Br)^4\right) - \\ -\frac{(Br)^5}{5!}4^5 + \frac{(Br)^4}{4!}3^4 - \frac{(Br)^3}{3!}2^3 + \frac{(Br)^2}{2!} + 1 \end{array}\right]
\end{aligned}$$

Analysing the form of these first iterates, we observe a tree structure effect, which allow us to formulate the following conjecture.

2.2. The Conjecture

Definition 1 *(Rainville, [4]): Let c_j, $j \in \mathbb{N}_0$, be a specified sequence independent of r and t. We say that $X(r,t)$ is a generating function of the set $g_j(r)$ if*

$$X(r,t) = \sum_{j\geq 0} c_j g_j(r) t^j.$$

Conjecture 2 *If $x(t)$, $t \geq 0$, is the solution of BIP, then*

$$x(t) \equiv X(r,t) = \sum_{j\geq 0} v_j(r) t^j,$$

i.e., $x(t)$ is a generating function for some sequence $(v_j(r))_{j\geq 0}$ in the delay r.

3. Construction of a New Solution's Method

3.1. A New Solution's Formalization

In order to prove our claim, we will proceed in the following way. Consider the decomposition of $(0,\infty)$ in disjoint subintervals of equal length r. We will consider the restriction of the solution to each of these subintervals, as a generating function of some family of polynomials in r. That is,

$$(0,\infty) = \bigcup_{n\geq 1} A_n, \text{ where for each } n \geq 1,\ A_n = ((n-1)r, nr], \tag{2}$$

$$\varphi(t) = \begin{cases} \phi(t) \text{ if } t \in A_0 = [-r,0] \\ \sum_{j\geq 0} w_j^1(r) t^j \text{ if } t \in A_1 = (0,r] \\ \sum_{j\geq 0} w_j^2(r) t^j \text{ if } t \in A_2 = (r,2r] \\ \vdots \\ \sum_{j\geq 0} w_j^n(r) t^j \text{ if } t \in A_n = ((n-1)r, nr] \\ \vdots \end{cases} \tag{3}$$

Hence, we have $\varphi(t)$ defined on each interval A_n, $n \geq 1$, as

$$\varphi(t)\mid_{t\in A_n} \equiv x_n(t) = \sum_{j\geq 0} w_j^n(r) t^j. \tag{4}$$

If our conjecture is valid we must have $\varphi'(t) = B\varphi(t-r)$ for $t \geq 0$, where the derivative at $t = 0$ represents the right-hand derivative. Two different types of conditions must hold. On one hand, we are concerned with the differenciability at each point $t = nr$, which will guarantee the continuity of the solution. This will be treated in conditions **(2.A).**

On the other hand, we want $x'(t) = Bx(t-r)$ to be satisfied on any interior point of A_n. This will be treated in conditions **(2.B).** In order to do that, we determine which conditions the iterates $w_j^n(r)$ in equation (4) should satisfy in terms of $\varphi(t)$. Meaning

$$\varphi'(0) = B\varphi(-r) \qquad (5)$$

$$x'_n(t) = Bx_{n-1}(t-r) \text{ for } t \in A_n,\ n \geq 1, \text{ where } x_0 \equiv \phi. \qquad (6)$$

3.2. The Constructive Process

(2.A.1) $\varphi'(0) = B\phi(-r)$.

At $t = 0$ we have

$$\varphi'(0) = \lim_{h\to 0^+} \frac{\sum_{j\geq 0} w_j^1(r)\, h^j - \phi(0)}{h} = \lim_{h\to 0^+} \frac{w_0^1(r) + w_1^1(r)\, h + w_2^1(r)\, h^2 + \cdots - \phi(0)}{h}.$$

A sufficient condition for (2.A.1) to hold, is

(2.a.1) $w_0^1(r) = \phi(0)$ and $w_1^1(r) = B\phi(-r)$ and $w_j^1(r)$ takes any value for $j \geq 2$.

(2.B.1) $\varphi'(t) = B\phi(t-r)$, $t \in (0, r)$.

Since

$$\varphi'(t) = \sum_{j\geq 0} (j+1)\, w_{j+1}^1(r)\, t^j,$$

we can establish the following statement. A sufficient condition for having **(2.B.1)** is

(2.b.1) $B\phi(-r) + \sum_{j\geq 2} j w_j^1(r)\, t^{j-1} = B\phi(t-r).$

We want to emphasize an important statement that later will lead us to the Main Theorem. If the initial function, $\phi(t)$, is constant, combining **(2.a.1)** and **(2.b.1),** we can choose $w_j^1(r) = (\phi(0),\ B\phi(-r),\ 0,\ 0, \cdots)$. In fact, condition (2.b.1) implies $\phi(-r) = \phi(t-r)$ since $t - r \in (-r, 0)$, and on this interval the function is constant.

Therefore, the solution on the interval A_1can be defined as

$$x_1(t) = \sum_{j\geq 0} w_j^1(r)\, t^j = \phi(0)\,[1 + Bt],$$

where $\phi(t) = \phi(0)$ for $t \in [-r, 0]$, which is exactly the solution obtained by the MSA.

Returning to a continuous initial function $\phi(t)$, we can state the following proposition.

Proposition 3 *If there exists $w_j^1(r)$ satisfying **(2.a.1)** and **(2.b.1),** then $x_1(t) = \sum_{j\geq 0} w_j^1(r)\, t^j$ satisfies (6) on the interval $(0, r)$.*

Proof. For $t \in (0, r)$, let

$x_1(t) = \sum_{j \geq 0} w_j^1(r) t^j = w_0^1(r) + w_1^1(r) t + \sum_{j \geq 2} w_j^1(r) t^j$.

If $w_j^1(r)$ satisfies **(2.a.1),** then

$x_1(t) = \phi(0) + B\phi(-r) t + \sum_{j \geq 2} w_j^1(r) t^j$.

By differentiation we obtain

$x_1'(t) = B\phi(-r) + \sum_{j \geq 2} j w_j^1(r) t^{j-1}$,

and if **(2.b.1)** holds, then

$x_1'(t) = B\phi(t - r)$. ■

From now on we will use the following lemma whose proof can be seen in Section 6.

Lemma 4 *For $t \neq 0$ and $t \neq r$*

$$\sum_{j \geq 0} f_j(r)(t - r)^j = \sum_{j \geq 0} \frac{t^j}{j!} \left(\sum_{i \geq 0} f_{j+i}(r) \frac{(-r)^i}{i!} (j + i)! \right).$$

(2.A.2) $\varphi'(r) = B\phi(0)$.

This equality requires: the existence of the derivative at $t = r$, the derivative has the value $B\phi(0)$.

- **(ai)** To prove the existence of $\varphi'(r)$, notice

$$\begin{aligned}
\varphi'(r^-) &= \lim_{h \to 0^-} \frac{\varphi(r+h) - \varphi(r)}{h} = \\
&= \lim_{h \to 0^-} \frac{\sum_{j \geq 0} w_j^1(r)(r+h)^j - \sum_{j \geq 0} w_j^1(r) r^j}{h} = \\
&= \lim_{h \to 0^-} \frac{\sum_{j \geq 0} \frac{h^j}{j!} \left(\sum_{i \geq 0} w_{j+i}^1(r) \frac{r^i}{i!} (j+i)! \right) - \sum_{j \geq 0} w_j^1(r) r^j}{h} = \\
&= \lim_{h \to 0^-} \frac{\sum_{i \geq 0} w_i^1(r) r^i + h \sum_{i \geq 0} w_{1+i}^1(r) r^i (1+i) + \sum_{j \geq 2} \frac{h^j}{j!} \left(\sum_{i \geq 0} w_{j+i}^1(r) \frac{r^i}{i!} (j+i)! \right) - \sum_{j \geq 0} w_j^1(r) r^j}{h} = \\
&= \sum_{i \geq 0} w_{1+i}^1(r) r^i (1 + i) + \lim_{h \to 0^-} \sum_{j \geq 2} \frac{h^{j-1}}{j!} \left(\sum_{i \geq 0} w_{j+i}^1(r) \frac{r^i}{i!} (j + i)! \right).
\end{aligned}$$

If convergence of the series is ensured, then the left-hand derivative at $t = r$, is equal to

$$\varphi'(r^-) = \sum_{i \geq 0} w_{1+i}^1(r) r^i (1 + i).$$

Proceeding in a similar way, we have

$$\begin{aligned}\varphi'\left(r^{+}\right) &= \lim_{h\to 0^{+}} \frac{\varphi(r+h)-\varphi(r)}{h} = \\ &= \lim_{h\to 0^{+}} \frac{\sum_{j\geq 0} w_j^2(r)(r+h)^j - \sum_{j\geq 0} w_j^1(r) r^j}{h} = \\ &= \lim_{h\to 0^{+}} \frac{\sum_{j\geq 0}\frac{h^j}{j!}\left(\sum_{i\geq 0} w_{j+i}^2(r)\frac{r^i}{i!}(j+i)!\right) - \sum_{j\geq 0} w_j^1(r) r^j}{h} = \\ &= \lim_{h\to 0^{+}} \frac{\sum_{i\geq 0} w_i^2(r) r^i + h\sum_{i\geq 0} w_{1+i}^2(r) r^i(1+i) + \sum_{j\geq 2}\frac{h^j}{j!}\left(\sum_{i\geq 0} w_{j+i}^2(r)\frac{r^i}{i!}(j+i)!\right) - \sum_{j\geq 0} w_j^1(r) r^j}{h}.\end{aligned}$$

The right-hand derivative of φ at $t=r$ exists, if:
$\sum_{i\geq 0} w_i^2(r)\, r^i = \sum_{j\geq 0} w_j^1(r)\, r^j$,
$\sum_{i\geq 0} w_{1+i}^2(r)\, r^i(1+i) = \sum_{i\geq 0} w_{1+i}^1(r)\, r^i(1+i)$,
and the series $\sum_{i\geq 0} w_{j+i}^k(r)\frac{r^i}{i!}(j+i)!$, $k=1,2$, converge.

- **(aii)** $\varphi'(r) = B\phi(0)$

We notice that the second condition

$$\sum_{i\geq 0} w_{1+i}^2(r)\, r^i(1+i) = \sum_{i\geq 0} w_{1+i}^1(r)\, r^i(1+i)$$

represents the equality between, respectively, $\varphi'(r^{+})$ and $\varphi'(r^{-})$. In order to have $\varphi'(r) = B\phi(0)$, it suffices to have

$$\sum_{i\geq 0} w_{1+i}^2(r)\, r^i(1+i) = B\phi(0).$$

The next proposition tells us the behaviour $w_j^2(r)$ must have, so that the delay differential equation is satisfied at $t=r$. We remark that in Proposition 3, we have established an equivalent result for the interior points of A_1.

Proposition 5 *If there exists $w_j^2(r)$ satisfying*

$$\boxed{(2.2i))\quad \begin{cases} \sum_{j\geq 0} w_j^2(r)\, r^j = \sum_{j\geq 0} w_j^1(r)\, r^j \\ \sum_{j\geq 0} w_{1+j}^2(r)\, r^j(1+j) = \sum_{j\geq 0} w_{1+j}^1(r)\, r^j(1+j) \end{cases}}$$

and

$$\boxed{(2.2ii))\quad \sum_{j\geq 0} w_{1+j}^2(r)\, r^j(1+j) = B\phi(0),}$$

then $\varphi'(r) = B\phi(0)$ and equality $(2.b.1)$ holds at $t=r$.

Proof. We have already seen that (2.2i)) and (2.2ii)) imply $\varphi'(r) = B\phi(0)$. We want to prove

$$B\phi(-r) + \sum_{j\geq 2} j w_j^1(r) r^{j-1} - B\phi(0) = 0.$$

If (2.2ii)) holds, then we can write the first member as

$$\begin{aligned} & B\phi(-r) + \sum_{i\geq 0}(2+i) w_{2+i}^1(r) r^{i+1} - \sum_{j\geq 0} w_{1+j}^2(r) r^j (1+j) = \\ = \quad & B\phi(-r) + \sum_{i\geq 0}(2+i) w_{2+i}^1(r) r^{i+1} - \sum_{j\geq 0} w_{1+j}^1(r) r^j (1+j) = \\ = \quad & \sum_{j\geq 0} w_{1+j}^1(r) r^j (1+j) - \sum_{j\geq 0} w_{1+j}^1(r) r^j (1+j), \end{aligned}$$

taking into account that $B\phi(-r) = w_1^1(r)$ and associating the terms in a appropriate way. ■

The procedure we have just described, can be repeated in an inductive way. Hence we can proceed in the following way;

(2.B.2) $\varphi'(t) = B\varphi(t-r)$, $t \in (r, 2r)$.
Since

$$\varphi'(t) = \sum_{j\geq 0}(j+1) w_{j+1}^2(r) t^j$$

and

$$\varphi(t-r) = \sum_{j\geq 0} w_j^1(r)(t-r)^j = \sum_{j\geq 0}\frac{t^j}{j!}\left(\sum_{i\geq 0} w_{j+i}^1(r)\frac{(-r)^i}{i!}(j+i)!\right),$$

we can state that a sufficient condition for having **(2.B.2)** is

(2.b.2) $w_{j+1}^2(r) = \frac{B}{(j+1)!}\sum_{i\geq 0} w_{j+i}^1(r)\frac{(-r)^i}{i!}(j+i)!$, for each $j \geq 0$.

We have finished the analysis of the solution defined on interior points of A_2.

4. The Fundamental Propositions

In a structural point of view, conditions (2.2i)), (2.2ii)) and (**2.b.2**) are identical in each interval A_n, for $n \geq 2$. Then we can state two fundamental propositions which establish sufficient conditions on $w_j^n(r)$, $n \geq 2$, in order for (6) to hold. The first one concerns with interior points, and the second one concerns with end points.

Proposition 6 *If for each* $n \geq 2$ *and* $j \geq 0$, *there exist a sequence* $w_j^n(r)$ *verifying*

$$w_{j+1}^n(r) = \frac{B}{(j+1)!} \sum_{i \geq 0} w_{j+i}^{n-1}(r) \frac{(-r)^i}{i!} (j+i)! \tag{7}$$

then $x_n'(t) = Bx_{n-1}(t-r)$, *for* $t \in int\ A_n$.

Proof. Let $x_n(t) = \sum_{j \geq 0} w_j^n(r) t^j$. If $t \in int A_n$ and $n \geq 2$ then

$$\begin{aligned} x_n'(t) &= \sum_{j \geq 0} (1+j) w_{1+j}^n(r) t^j = \sum_{j \geq 0} (1+j) t^j \left(\frac{B}{(j+1)!} \sum_{i \geq 0} w_{j+i}^{n-1}(r) \frac{(-r)^i}{i!} (j+i)! \right) \\ &= B \sum_{j \geq 0} \frac{t^j}{j!} \left(\sum_{i \geq 0} w_{j+i}^{n-1}(r) \frac{(-r)^i}{i!} (j+i)! \right) = \\ &= B \sum_{j \geq 0} w_j^{n-1}(r) (t-r)^j = Bx_{n-1}(t-r), \end{aligned}$$

where we have considered (7) and lemma 4. ■

Proposition 7 *If for each* $n \geq 2$ *there exist a sequence* $w_j^n(r)$ *that satisfies the conditions*

$$\begin{cases} \sum_{j \geq 0} (nr)^j w_j^{n+1}(r) = \sum_{j \geq 0} (nr)^j w_j^n(r) \\ \sum_{j \geq 0} (1+j)(nr)^j w_{1+j}^{n+1}(r) = \sum_{j \geq 0} (1+j)(nr)^j w_{1+j}^n(r) \end{cases} \tag{8}$$

and

$$\sum_{j \geq 0} (1+j)(nr)^j w_{1+j}^{n+1}(r) = B \sum_{j \geq 0} w_j^{n-1}(r) [(n-1)r]^j \tag{9}$$

then $x_n'(nr) = Bx_{n-1}((n-1)r)$.

Proof. Let $n \geq 2$ and $t = nr$

We start by showing the existence of derivative at points $t = nr$ for $n \geq 2$.

- left-hand derivative:

$$
\begin{aligned}
x_n'\left(nr^-\right) &= \lim_{h\to 0^-}\frac{x_n\left(nr+h\right)-x_n\left(nr\right)}{h} = \lim_{h\to 0^-}\frac{\sum\limits_{j\geq 0} w_j^n\left(r\right)\left(nr+h\right)^j - \sum\limits_{j\geq 0} w_j^n\left(r\right)\left(nr\right)^j}{h} \\
&= \lim_{h\to 0^-}\frac{\sum\limits_{j\geq 0}\frac{h^j}{j!}\left(\sum\limits_{i\geq 0} w_{j+i}^n\left(r\right)\frac{(nr)^i}{i!}\left(j+i\right)!\right) - \sum\limits_{j\geq 0} w_j^n\left(r\right)\left(nr\right)^j}{h} \\
&= \lim_{h\to 0^-}\left\{\begin{array}{c} \frac{\sum\limits_{i\geq 0} w_i^n(r)(nr)^i + h\sum\limits_{i\geq 0} w_{1+i}^n(r)(nr)^i(1+i)}{h} + \\ + \frac{\sum\limits_{j\geq 2}\frac{h^j}{j!}\left(\sum\limits_{i\geq 0} w_{j+i}^n(r)\frac{(nr)^i}{i!}(j+i)!\right) - \sum\limits_{j\geq 0} w_j^n(r)(nr)^j}{h}\end{array}\right\} = \\
&= \sum_{i\geq 0} w_{1+i}^n\left(r\right)\left(nr\right)^i\left(1+i\right),
\end{aligned}
$$

assuming that $\sum\limits_{i\geq 0} w_{j+i}^n\left(r\right)\frac{(nr)^i}{i!}\left(j+i\right)!$ converge for $j\geq 1$.

- right-hand derivative:

$$
\begin{aligned}
x_n'\left(nr^+\right) &= \lim_{h\to 0^+}\frac{x_n\left(nr+h\right)-x_n\left(nr\right)}{h} = \lim_{h\to 0^+}\frac{\sum\limits_{j\geq 0} w_j^{n+1}\left(r\right)\left(nr+h\right)^j - \sum\limits_{j\geq 0} w_j^n\left(r\right)\left(nr\right)^j}{h} = \\
&= \lim_{h\to 0^+}\frac{\sum\limits_{j\geq 0}\frac{h^j}{j!}\left(\sum\limits_{i\geq 0} w_{j+i}^{n+1}\left(r\right)\frac{(nr)^i}{i!}\left(j+i\right)!\right) - \sum\limits_{j\geq 0} w_j^n\left(r\right)\left(nr\right)^j}{h} = \\
&= \lim_{h\to 0^+}\left\{\frac{\sum\limits_{i\geq 0} w_i^{n+1}\left(r\right)\left(nr\right)^i + h\sum\limits_{i\geq 0} w_{1+i}^{n+1}\left(r\right)\left(nr\right)^i\left(1+i\right)}{h} + \right. \\
&\qquad \left. + \frac{\sum\limits_{j\geq 2}\frac{h^j}{j!}\left(\sum\limits_{i\geq 0} w_{j+i}^{n+1}\left(r\right)\frac{(nr)^i}{i!}\left(j+i\right)!\right) - \sum\limits_{j\geq 0} w_j^n\left(r\right)\left(nr\right)^j}{h}\right\} \\
&= \sum_{i\geq 0} w_{1+i}^{n+1}\left(r\right)\left(nr\right)^i\left(1+i\right),
\end{aligned}
$$

assuming that (8) holds, and $\sum\limits_{i\geq 0} w_{j+i}^{n+1}\left(r\right)\frac{(nr)^i}{i!}\left(j+i\right)!$ converges for $n\geq 1$ and $j\geq 1$.

We have proved the existence of derivative of $x_n\left(t\right)$ at $t=nr$, $n\geq 2$, and

$$
x_n'\left(nr\right) = \sum_{j\geq 0} w_{1+j}^{n+1}\left(r\right)\left(nr\right)^j\left(1+j\right) = \sum_{j\geq 0} w_{1+j}^n\left(r\right)\left(nr\right)^j\left(1+j\right).
$$

Next, we will show that $x_n'(t) = Bx_{n-1}(t-r)$ at $t = nr$, $n \geq 2$.

$$x_n'(nr) = \sum_{j\geq 0} w_{1+j}^{n+1}(r)(nr)^j(1+j) = B\sum_{j\geq 0} w_j^{n-1}(r)\left[(n-1)r\right]^j = Bx_{n-1}((n-1)r),$$

where we have considered (9). ■

We point out, that equalities, (7), (8) and (9) provide calculating formulas for all the terms of the sequences $w_j^n(r)$ for $n \geq 2$.

Corollary 8 *Equality (9) is a direct consequence of (7) and (8).*

Proof.

$$\begin{aligned}
&\sum_{j\geq 0}(1+j)(nr)^j w_{1+j}^{n+1}(r) \underset{(8)}{=} \sum_{j\geq 0}(1+j)(nr)^j w_{1+j}^{n}(r) = \\
= \ & \sum_{j\geq 0}(1+j)(nr)^j\left(\frac{B}{(j+1)!}\sum_{i\geq 0} w_{i+j}^{n-1}(r)\frac{(-r)^i}{i!}(i+j)!\right) = \\
= \ & B\sum_{j\geq 0}\frac{(nr)^j}{j!}\left(\sum_{i\geq 0} w_{i+j}^{n-1}(r)\frac{(-r)^i}{i!}(i+j)!\right) = \\
&= B\sum_{j\geq 0} w_j^{n-1}(r)(nr-r)^j,
\end{aligned}$$

where we have considered (7) and the Lemma 4. ■

We also have a correspondent result to (2.2ii), which refers to $n = 1$.

Corollary 9 *Equality (2.2ii) is a direct consequence of (2.2i) and (2.b.1), being the last one applied to $t = r$.*

Proof. As a consequence of proposition 5, (2.b.1) is verified at $t = r$, so

$$B\phi(-r) + \sum_{j\geq 2} j w_j^1(r) r^{j-1} = B\phi(0).$$

Then,

$$\begin{aligned}
&\sum_{j\geq 0} w_{1+j}^2(r) r^j(1+j) = \sum_{j\geq 0} w_{1+j}^1(r) r^j(1+j) \\
= \ & w_1^1(r) + \sum_{j\geq 1} w_{1+j}^1(r) r^j(1+j) = w_1^1(r) + B\phi(0) - B\phi(-r) = B\phi(0),
\end{aligned}$$

since $w_1^1(r) = B\phi(-r)$. ■

These two corolaries suggest that during the constructive process of the solution, some conditions with distinct functions emerge.

5. The Main Theorem

From now on we consider $\phi(t) = \phi(0) = C$ for $t \in [-r, 0]$, where C is a real constant.

Proposition 10 *If*

$$w_0^1(r) = C, \; w_1^1(r) = BC \quad \text{and} \quad w_j^1(r) = 0 \quad \text{for } j \geq 2 \tag{10}$$

then

$$x_1(t) = \sum_{j \geq 0} w_j^1(r) t^j = C(1 + Bt)$$

is the solution of problem (1) defined on $A_1 = (0, r]$.

Proof. Equalities (10) verify (2.a.1) and (2.b.1), implying $x_1'(t) = BC$ for $t \in [0, r)$. According to (4), we can then write

$$x_1(t) = \sum_{j \geq 0} w_j^1(r) t^j = C(1 + Bt) \quad \text{for} \quad t \in (0, r).$$

By proposition 5, the result is also true at $t = r$. ■

The main result of this paper is the following Theorem.

Theorem 11 *The solution of problem (1) with* $\phi(t) = C$ *if* $t \in [-r, 0]$, *can be written as*

$$X(r, t) = \sum_{j \geq 0} v_j(r) t^j,$$

for $t \geq 0$. *The sequence* $v_j(r)$ *is defined by*

$$v_j(r) = C \frac{B^j}{j!} P_j^n(rB),$$

where the polynomials $P_j^n(rB)$ *are defined by*

$$P_j^n(rB) = \begin{cases} 1 + \sum\limits_{i=0}^{n-(j+1)} \frac{(-rB)^{i+1}}{(i+1)!} (i+j)^{i+1} & \text{if } j \leq n-1 \\ 1 & \text{if } j = n \\ 0 & \text{if } j \geq n+1. \end{cases}$$

The proof of this Theorem is divided into four stages: Propositions (12) and (13), and Corollaries (14) and (15). In Proposition (12) we will obtain the calculating formulas to obtain all terms of the sequences, $w_j^n(r)$, $n \geq 2$. Moreover, we will show that these formulas do not depend on the fact that the initial function is constant. This fact makes this procedure an alternative method to solve problem (1). In Proposition (13) and their Corollaries, we will show the consequences of taking constant the initial function constant.

Proposition 12 *For $n \geq 2$ the solution $x_n(t) = \sum_{j \geq 0} w_j^n(r)\, t^j$, defined on each interval A_n, is obtained through the application of the following formulas, in the following order*

$$w_{j+1}^n(r) = \frac{B}{(j+1)!} \sum_{i \geqslant 0} w_{i+j}^{n-1}(r) \frac{(-r)^i}{i!} (i+j)! \tag{11}$$

and

$$w_0^{n+1}(r) = w_0^n(r) - \sum_{j \geqslant 1} \left[w_j^{n+1}(r) - w_j^n(r) \right] (nr)^j . \tag{12}$$

Proof. Equality (11) is the sufficient condition (7) in proposition 6, and equality (12) is obtained from the first equality of (8). ■

From now on, we consider $w_j^1(r) = 0$ for $j \geq 2$.

Combining equalities (11) and (12) we will obtain the sequence $w_j^2(r)$.

According to (11), for $n \geq 2$

$$w_1^2(r) = B \sum_{i \geq 0} w_i^1(r)(-r)^i = B \sum_{i=0}^{1} w_i^1(r)(-r)^i ,$$

and since $w_j^1(r) = 0$ for $j \geq 2$,

$$w_1^2(r) = B(C - rBC).$$

On the other hand

$$w_2^2(r) = \frac{B}{2!} \sum_{i \geq 0} w_{i+1}^1(r)(-r)^i (i+1) = \frac{B}{2!} BC.$$

It is easy to check $w_j^2(r) = 0$ for $j \geq 3$. This will lead us to assume that for $j \geq n+1, w_j^n(r) = 0$. We will prove this fact in the next Proposition.

It remains to calculate the first term. According to (12)

$$w_0^2(r) = w_0^1(r) - \sum_{j \geq 1} \left(w_j^2(r) - w_j^1(r) \right) r^j = C - \sum_{j=1}^{2} \left(w_j^2(r) - w_j^1(r) \right) r^j ,$$

and since $w_j^2(r) = 0$ for $j \geq 3$,

$$w_0^2(r) = C - r\left(w_1^2(r) - w_1^1(r) \right) - r^2 w_2^2(r) = C\left(1 + \frac{r^2 B^2}{2} \right).$$

Finally, we obtain the solution $x_2(t)$ defined on $A_2 = (r, 2r]$

$$x_2(t) = \sum_{j=0}^{2} w_j^2(r)\, t^j = C\left(1 + \frac{(rB)^2}{2} + t\left(B - rB^2\right) + \frac{t^2 B^2}{2!} \right).$$

We can verify that

$$x_1(r) = x_2(r) = C(1 + rB).$$

Also, notice that the form of $x_2(t)$, obtained with our calculating formulas, has exactly the same form like the one obtained with MSA.

Proposition 13 *Consider problem (1), where $\phi(t) = C$ for $t \in [-r, 0]$.*
If $w_j^1(r) = 0$ for $j \geq 2$, then

$$w_j^n(r) = 0 \text{ for } j \geq n + 1. \tag{13}$$

Proof. We will use induction on n. The case $n = 1$ is obviously true. Assuming $w_j^n(r) = 0$ for $j \geq n + 1$, consider $j \geq n + 2$. As a consequence of (11)

$$w_j^{n+1}(r) = \frac{B}{j!} \sum_{i \geq 0} w_{i+j-1}^n(r) \frac{(-r)^i}{i!} (i + j - 1)!$$

By the induction step, $w_j^{n+1}(r) = 0$ if $i + j - 1 \geq n + 1$. Since $i \geq 0$ we can conclude $w_j^{n+1}(r) = 0$ for $j \geq n + 2$ as wanted. ■

Corollary 14 *In the above conditions, if $n \geq 1$ then*

$$w_n^n(r) = C\frac{B^n}{n!}. \tag{14}$$

Proof. We will use induction on n. Since $w_1^1(r) = BC$, the result holds for $n = 1$. Assuming
$w_n^n(r) = C\frac{B^n}{n!}$.

we have

$$\begin{aligned} w_{n+1}^{n+1}(r) &= \frac{B}{(n+1)!} \sum_{i \geq 0} w_{i+n}^n(r) \frac{(-r)^i}{i!} (i + n)! = \\ &= \frac{B}{(n+1)!} w_n^n(r)\, n! = \frac{B}{(n+1)!} C \frac{B^n}{n!} n! = C\frac{B^{n+1}}{(n+1)!}, \end{aligned}$$

where we used (11), (13) and the induction step. ■

Corollary 15 *In the conditions of proposition (13), if $j \leq n - 1$ then*

$$w_j^n(r) = C\frac{B^j}{j!} \left(1 + \sum_{i=0}^{n-(j+1)} \frac{(-rB)^{i+1}}{(i+1)!} (i + j)^{i+1}\right). \tag{15}$$

Proof. To prove (15), we will use induction reasoning applied to $j = n - k$ for the sucessive values $k = 1, 2, \cdots, n$. So, we will do it, first considering $k = 1$, then $k = 2$, and finally by an induction reasoning.

1. If $j = n-1$, $n \geq 1$, we want to prove that

$$w_{n-1}^{n}(r) = C\frac{B^{n-1}}{(n-1)!}\left[1 + (-rB)(n-1)\right].$$

Using induction on n, the case $n = 1$ is valid as a consequence of (2.a.1). Assuming

$w_{n-1}^{n}(r) = C\frac{B^{n-1}}{(n-1)!}\left[1 + (-rB)(n-1)\right]$.

we have

$$w_{n}^{n+1}(r) = \frac{B}{n!}\sum_{i\geq 0} w_{i+n-1}^{n}(r)\frac{(-r)^{i}}{i!}(i+n-1)! =$$

$$= \frac{B}{n!}\sum_{i=0}^{1} w_{i+n-1}^{n}(r)\frac{(-r)^{i}}{i!}(i+n-1)! =$$

$$= \frac{B}{n!}\left\{w_{n-1}^{n}(r)(n-1)! + w_{n}^{n}(r)(-r)n!\right\} =$$

$$= B\left\{\frac{1}{n}\frac{CB^{n-1}}{(n-1)!}\left[1 + (-rB)(n-1)\right] + (-r)C\frac{B^{n}}{n!}\right\} =$$

$$= C\frac{B^{n}}{n!}\left[1 - rBn + rB - rB\right] = C\frac{B^{n}}{n!}\left[1 + (-rB)n\right].$$

where we used (11) for $j = n-1$, (13), (14) and the induction step.

2. If $j = n-2$ for $n \geq 2$, we want to prove that

$$w_{n-2}^{n}(r) = C\frac{B^{n-2}}{(n-2)!}\left[1 + (-rB)(n-2) + \frac{(-rB)^{2}}{2!}(n-1)^{2}\right].$$

For $n = 2$, using (12), we have

$$w_{0}^{2}(r) = w_{0}^{1}(r) - \sum_{j\geq 1}\left[w_{j}^{2}(r) - w_{j}^{1}(r)\right]r^{j} =$$

$$= C - \left[w_{1}^{2}(r) - w_{1}^{1}(r)\right]r - \left[w_{2}^{2}(r) - 0\right]r^{2} - 0 =$$

$$= C - \left[B(C - rBC) - CB\right]r - C\tfrac{B^{2}}{2!}r^{2} = C\left(1 + \tfrac{(-rB)^{2}}{2!}\right).$$

Assuming

$w_{n-2}^{n}(r) = C\frac{B^{n-2}}{(n-2)!}\left[1 + (-rB)(n-2) + \frac{(-rB)^{2}}{2!}(n-1)^{2}\right]$,

We have

$$w_{n-1}^{n+1}(r) = \frac{B}{(n-1)!}\sum_{i\geq 0} w_{i+n-2}^{n}(r)\frac{(-r)^{i}}{i!}(i+n-2)! =$$

$$= \frac{B}{(n-1)!}\left\{w_{n-2}^{n}(r)(n-2)! + w_{n-1}^{n}(r)(-r)(n-1)! + w_{n}^{n}(r)\frac{(-r)^{2}}{2!}n!\right\} =$$

$$= C\frac{B^{n-1}}{(n-1)!}\left[1 + (-rB)(n-1) + \frac{(-rB)^{2}}{2!}n^{2}\right],$$

where we have used (11) for $j = n-2$, (13), (14) and the induction step.

It can be assumed by induction, that

$$\begin{aligned} \text{for} \quad k &= 1,2,\cdots,n \quad \text{and} \quad n \geq k, \\ w_{n-k}^{n}(r) &= C\frac{B^{n-k}}{(n-k)!}\left(1+\sum_{i=0}^{k-1}\frac{(-rB)^{i+1}}{(i+1)!}(i+n-k)^{i+1}\right). \end{aligned}$$

■

The proof of the Theorem is now complete.

Hence, if we fix $n \in \mathbb{N}$, we can calculate $x_n(t)$ with $t \in A_n = ((n-1)r, nr]$ using

$$x_n(t) = \sum_{j\geq 0} w_j^n(r)\, t^j = C\sum_{j\geq 0}\frac{B^j}{j!}P_j^n(rB)\, t^j, \tag{16}$$

where $P_j^n(rB)$ are defined in theorem 11.

The solution found by this new method coincides with the one obtained by the method of steps, the recurrences formulas (11) and (12) can be replaced with (16), whenever $\phi(t)$ is a constant, and finally the solution of problem (1) is the generating fuction for $\{w_j^n(r)\}_{j=0,1,\dots}$.

6. Proof of Lemma 4

For $r > 0$, $t \neq 0$ and $t \neq r$

$$\sum_{j\geq 0} f_j(r)(t-r)^j = \sum_{j\geq 0}\frac{t^j}{j!}\left(\sum_{i\geq 0} f_{j+i}(r)\frac{(-r)^i}{i!}(j+i)!\right).$$

Proof.

$$\begin{aligned}
\sum_{j\geq 0} f_j(r)\,(t-r)^j &= f_0(r)+f_1(r)(t-r)+f_2(r)(t-r)^2+f_3(r)(t-r)^3+...\\
&\quad +f_p(r)(t-r)^p+f_{p+1}(r)(t-r)^{p+1}+...\\
&= \left[\begin{array}{c} f_0(r)+f_1(r)(-r)+f_2(r)(-r)^2+f_3(r)(-r)^3+...\\ +f_p(r)(-r)^p+f_{p+1}(r)(-r)^{p+1}+...\end{array}\right]\\
&\quad +t\left[\begin{array}{c} f_1(r)-2rf_2(r)+3r^2f_3(r)-4r^3f_4(r)+...\\ +\binom{p}{p-1}(-r)^{p-1}f_p(r)+f_{p+1}(r)(-r)^p\left(1+\binom{p}{p-1}\right)+...\end{array}\right]\\
&\quad +t^2\left[\begin{array}{c} f_2(r)-3rf_3(r)+6r^2f_4(r)+...\\ +\binom{p}{p-2}(-r)^{p-2}f_p(r)+f_{p+1}(r)(-r)^{p-1}\left(\binom{p}{p-1}+\binom{p}{p-2}\right)+...\end{array}\right]\\
&\quad +t^3\left[\begin{array}{c} f_3(r)-4rf_4(r)+...\\ +\binom{p}{p-3}(-r)^{p-3}f_p(r)+f_{p+1}(r)(-r)^{p-2}\left(\binom{p}{p-2}+\binom{p}{p-3}\right)+...\end{array}\right]\\
&\quad +...+t^{p-2}\left[\binom{p}{2}(-r)^2f_p(r)+f_{p+1}(r)(-r)^3\left(\binom{p}{3}+\binom{p}{2}\right)+...\right]\\
&\quad +t^{p-1}\left[\binom{p}{1}(-r)f_p(r)+f_{p+1}(r)(-r)^2\left(\binom{p}{2}+\binom{p}{1}\right)+...\right]\\
&\quad +t^p\left[f_p(r)+f_{p+1}(r)(-r)\left(\binom{p}{1}+1\right)+...\right]+...
\end{aligned}$$

Define $g(z)=\sum_{j\geqslant 0}(-1)^j f_j(r)\,z^j$, where $g(r)=\sum_{j\geqslant 0} f_j(r)(-r)^j$ represents

$$\begin{array}{c} f_0(r)+f_1(r)(-r)+f_2(r)(-r)^2+f_3(r)(-r)^3+...\\ +f_p(r)(-r)^p+f_{p+1}(r)(-r)^{p+1}+...\end{array}$$

Claim 16 $t\left[\begin{array}{c} f_1(r)-2rf_2(r)+3r^2f_3(r)-4r^3f_4(r)+...\\ +\binom{p}{p-1}(-r)^{p-1}f_p(r)+f_{p+1}(r)(-r)^p\left(1+\binom{p}{p-1}\right)+...\end{array}\right]$ *can be written as*

$$-tg'(z)\mid_{z=r}.$$

In fact $g'(z)=\sum_{j\geqslant 0}(-1)^{j+1}(j+1)\,f_{j+1}(r)\,z^j$. So

$$\begin{aligned}
-tg'(z)_{z=r} &= t\left[\sum_{j\geqslant 0}(-z)^j(j+1)\,f_{j+1}(r)\right]_{z=r} =\\
&= t\left[f_1(r)-2rf_2(r)+3r^2f_3(r)+...+p(-r)^{p-1}f_p(r)+...\right].
\end{aligned}$$

Claim 17 $t^2\left[\begin{array}{c} f_2(r)-3rf_3(r)+6r^2f_4(r)+...\\ +\binom{p}{p-2}(-r)^{p-2}f_p(r)+f_{p+1}(r)(-r)^{p-1}\left(\binom{p}{p-1}+\binom{p}{p-2}\right)+...\end{array}\right]$

can be written as

$$\frac{t^2}{2!}g''(z)\mid_{z=r}.$$

In fact $g''(z)=\sum\limits_{j\geqslant 0}(-1)^{j+2}(j+1)(j+2)f_{j+2}(r)z^j$. So

$$\begin{aligned}\frac{t^2}{2!}g''(z)_{z=r} &= \frac{t^2}{2!}\left[\sum_{j\geqslant 0}(-z)^j(j+1)(j+2)f_{j+2}(r)\right]_{z=r} = \\ &= t^2\left[f_2(r)-3rf_3(r)+6r^2f_4(r)+...+\binom{p}{2}(-r)^{p-2}f_p(r)+...\right].\end{aligned}$$

Repeating the process,

Claim 18 $t^n\left[f_n(r)+f_{n+1}(r)(-r)\left(\binom{n}{1}+1\right)+...\right]$
can be written as

$$(-1)^n\frac{t^n}{n!}g^{(n)}(z)\mid_{z=r}$$

where

$$g^{(n)}(z)=\sum_{j\geqslant 0}(-1)^{j+n}(j+1)(j+2)...(j+n)f_{j+n}(r)z^j.$$

We prove this fact by induction on n.
Proof: As we have already seen
$g'(z)=\sum\limits_{j\geqslant 0}(-1)^{j+1}(j+1)f_{j+1}(r)z^j$, so the case $n=1$ is verified.
Assuming
$g^{(n-1)}(z)=\sum\limits_{j\geqslant 0}(-1)^{j+n-1}(j+1)(j+2)...(j+n-1)f_{j+n-1}(r)z^j$,
we want to prove that
$g^{(n)}(z)=\sum\limits_{j\geqslant 0}(-1)^{j+n}(j+1)(j+2)...(j+n)f_{j+n}(r)z^j$.
We have

$$\begin{aligned}g^{(n)}(z) &= \frac{d}{dz}\left(\sum_{j\geqslant 0}(-1)^{j+n-1}(j+1)(j+2)...(j+n-1)f_{j+n-1}(r)z^j\right) = \\ &= \sum_{j\geqslant 0}(-1)^{j+n}(j+1)(j+2)...(j+n)f_{j+n}(r)z^j.\end{aligned}$$

Hence, we can write

$$\begin{aligned}\sum_{j\geq 0}f_j(r)(t-r)^j &= g(r)-tg'(r)+\frac{t^2}{2!}g''(r)-\frac{t^3}{3!}g'''(r)+...+(-1)^p\frac{t^p}{p!}g^{(p)}(r)+... = \\ &= \sum_{j\geqslant 0}(-1)^j\frac{t^j}{j!}g^{(j)}(r)=\sum_{j\geqslant 0}(-1)^j\frac{t^j}{j!}\left(\sum_{i\geqslant 0}(-1)^{i+j}(i+1)...(i+j)f_{i+j}(r)r^i\right) = \\ &= \sum_{j\geqslant 0}\frac{t^j}{j!}\left(\sum_{i\geqslant 0}(-r)^i\frac{(i+j)!}{i!}f_{i+j}(r)\right)=\sum_{j\geqslant 0}\frac{t^j}{j!}\left(\sum_{i\geqslant 0}f_{i+j}(r)\frac{(-r)^i}{i!}(i+j)!\right),\end{aligned}$$

where we used the equality $(i+1)(i+2)(i+3)\ldots(i+j) = \dfrac{(i+j)!}{i!}$. ■

7. An Application

Suppose we want to study a population $P(t) = (x(t), y(t))$, where $x(t)$ denotes the average height and $y(t)$ the average weight. It was observed that $x(t)$ depends on the height of the previous generation through $x'(t) = Bx(t-r)$, where r is the size (in units of time) of a generation.

We can determine explicitly the behaviour of this variable regarding the fourth generation. This means that we want to compute $x_4(t)$, given $x(t) = C$ for $t \in [-r, 0]$, where C is the average height.

Using equation (16) we can determine it directly, without having to compute the height for the previous generations,

$$x_4(t) = \sum_{j\geq 0} w_j^4(r)\, t^j = C \sum_{j\geq 0} \frac{B^j}{j!} P_j^4(rB)\, t^j,$$

where $P_j^4(rB)$ are computed applying theorem 11. From this theorem, as

$$P_4^4(rB) = 1$$

$$P_3^4(rB) = 1 + \sum_{i=0}^{0} \frac{(-rB)^{i+1}}{(i+1)!}(i+3)^{i+1} = 1 + 3(-rB)$$

$$P_2^4(rB) = 1 + \sum_{i=0}^{1} \frac{(-rB)^{i+1}}{(i+1)!}(i+2)^{i+1} = 1 + 2(-rB) + \frac{3^2}{2!}(-rB)^2$$

$$P_1^4(rB) = 1 + \sum_{i=0}^{2} \frac{(-rB)^{i+1}}{(i+1)!}(i+1)^{i+1} = 1 + (-rB) + \frac{2^2}{2!}(-rB)^2 + \frac{3^3}{3!}(-rB)^3$$

$$P_0^4(rB) = 1 + \sum_{i=0}^{3} \frac{(-rB)^{i+1}}{(i+1)!} i^{i+1} = 1 + \frac{(-rB)^2}{2!} + \frac{2^3}{3!}(-rB)^3 + \frac{3^4}{4!}(-rB)^4,$$

for $j = 0, 1, 2, 3, 4$, then we have

$$\begin{aligned} x_4(t) &= C\left\{P_0^4(rB) + BP_1^4(rB)\,t + \frac{B^2}{2!}P_2^4(rB)\,t^2 + \frac{B^3}{3!}P_3^4(rB)\,t^3 + \frac{B^4}{4!}P_4^4(rB)\,t^4\right\} \\ &= C\left\{\begin{array}{c} 1 + \frac{(-rB)^2}{2!} + \frac{2^3}{3!}(-rB)^3 + \frac{3^4}{4!}(-rB)^4 + Bt\left(1 - rB + \frac{2^2}{2!}(-rB)^2 + \frac{3^3}{3!}(-rB)^3\right) + \\ + \frac{B^2}{2!}t^2\left(1 - 2rB + \frac{3^2}{2!}(-rB)^2\right) + \frac{B^3}{3!}t^3(1 - 3rB) + \frac{B^4}{4!}t^4 \end{array}\right\}. \end{aligned}$$

8. Conclusion and Future Directions

We have developed a new method for computing the analytical solution to the simplest delay differential equation. Future directions of research include: extending this method to a much wider class of DDEs, as $x'(t) = Ax(t) + Bx(t-$

r), and finding the relationship between our solution of equation (1), which uses the polynomials $P_j^n(rB)$, and the alternative solution

$$x(t) = Ce^{\frac{W(rB)}{r}t}$$

in terms of the *Lambert W-function.*

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