Dynamic complex hedging in additive markets *

José M. Corcuera †, João M. E. Guerra‡

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Abstract

In general, geometric additive models are incomplete and the perfect replication of derivatives, in the usual sense, is not possible.

In this paper we complete the market by introducing the so-called power-jump assets. Using a static hedging formula, in order to relate call options and power-jump assets, we show that this market can also be completed by considering portfolios with a continuum of call options with different strikes and the same maturity.

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1 Introduction

Hedging in incomplete markets is an unsolved problem. In these markets a perfect replication of a derivative is, in general, not possible . Still we could try a *superhedging* strategy but the cost of these strategies is in many situations too high. For instance in [10] one can see that the superhedging cost of a call option is the price of the underlying asset. Many solutions have been proposed by using *risk-minimizing* strategies, trying to minimize the effect of imperfect replication. We could use the quadratic loss function and this leads to the notion of *mean-variance hedging* as in [13] or we could require the strategy to hedge the derivative with a high probability and in an optimal way, that is investing the least amount of initial wealth. This idea leads to the *quantile hedging* as can be seen in [12]. Other authors as Balland [1], Carr and Madam [4] and Jacod and Protter [15], between others, have tried to replicate complex derivatives by using liquid and non redundant derivatives as vanilla options or have introduced new derivatives to complete the market as in [3]. Here we follow this latter approach. The context will be of a geometric additive model where the log returns evolve as process with independent increments not necessarily homogeneous, called also non-homogeneous Lévy processes [11]. These models are in general incomplete, in fact only the cases where the additive process is a

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†Universitat de Barcelona, Gran Via de les Corts Catalanes, 585, E-08007 Barcelona, Spain. E-mail: jmcorcuera@ub.edu

‡CEMAPRE, Rua do Quelhas 6, 1200-781 Lisboa, Portugal. E-mail: jguerra@iseg.utl.pt

Brownian motion or a Poisson process are complete models. A recent paper [5] shows that these kind of models, if we add the self-similarity condition, capture quite well the term behaviour of the option prices. These processes have also been used successfully to model bond markets, see [11].

The rest of the paper is organized as follows: in the next section we define the market model. In section 3 we define the so called power-jump assets that will allow us to complete the market This completness property is analyzed in section 4. In section 5 we obtain hedging formulas for different kinds of derivatives, in particular derivatives with payoffs $S_T^k, k \geq 2$. We will see in this section how we can invert the hedging formulas and express the power-jump assets in terms of these derivatives. Finally by using the static hedging in terms of calls, see for instance [4], we obtain formulas for dynamic complex hedging. The last section is devoted to treat the optimization problem with complex portfolios in the context of Lévy processes. Similar results could be obtained for the non-homogeneous case but this will be the aim of another paper.

2 The geometric additive model

In this paper we will consider a market model where the stock price process $S = \{S_t, t \in [0,T]\}$ is a geometric additive process and satisfies the stochastic differential equation

$$\frac{\mathrm{d}S_t}{S_{t-}} = \mathrm{d}Z_t, \qquad S_0 > 0, \tag{1}$$

where $Z = \{Z_t, t \in [0,T]\}$ is an additive process. Moreover, in our market we have a riskless asset or bond B, evolving as

$$B_t = \exp\left(\int_0^t r_s \mathrm{d}s\right), \tag{2}$$

where r_t is the deterministic spot interest rate at time t.

We begin by recalling the definition and the main properties of additive processes. The fundamental references about this subject are [16] and [21]. The theory of integration and stochastic differential equations for the kind of additive processes that we shall consider in this paper is the theory of integration for semimartingales and a comprehensive introduction to this subject can be found in [20]. It is well known that additive processes generalize Lévy processes by relaxing the stationarity condition on the increments. Indeed, a real-valued stochastic process $Z = \{Z_t, t \in [0,T]\}$, defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, is called an additive process if it is stochastically continuous, their increments are independent and $Z_0 = 0$ a.s. The filtration $\mathbb{F}$ is the natural filtration generated by the stock price process S completed with the $\mathbb{P}$-null sets $\mathcal{N}$, that is, $\mathbb{F} = \{\mathcal{F}_t, t \in [0,T]\} \cup \mathcal{N}$ where $\mathcal{F}_t = \sigma(S_s : 0 \leq s \leq t)$. An additive process Z always has a càdlàg modification (see [21], page 63) and in the rest of this paper, we will always assume that we are dealing with this càdlàg modification. Moreover, Z_t has an infinitely divisible distribution for all t, which is determined by its system of generating triplets (Γ_t, C_t, μ_t), where the Gaussian covariance C_t is a nonnegative and increasing continuous function, the location parameter Γ_t is a continuous function and the Lévy measure μ_t is an increasing (in t) positive measure on $\mathbb{R}$ such that $\mu_t(\{0\}) = 0$, $\mu_s(B) \to \mu_t(B)$

as $s \to t$ for all measurable sets $B \subset \{x : |x| \geq \varepsilon\}$, for some $\varepsilon > 0$, and

$$\int_{\mathbb{R}} \left(1 \wedge x^2\right) \mu_t(\mathrm{d}x) < \infty, \tag{3}$$

for all $t \in [0, T]$.

Additive processes satisfy the generalized version of the Lévy-Itô decomposition (see [21], page 120)

$$Z_t = X_t^1 + X_t^2,$$

where X^2 is the continuous part of the process and

$$X_t^1 = \lim_{\varepsilon \searrow 0} \int_{\{s\in(0,t],\varepsilon<|x|<1\}} x(Q(\mathrm{d}s, \mathrm{d}x) - \widetilde{\mu}(\mathrm{d}\,(s,x)) + \int_{\{s\in(0,t],|x|\geq 1\}} xQ(\mathrm{d}s, \mathrm{d}x),$$

where $Q\,(\mathrm{d}s, \mathrm{d}x)$ is a Poisson random measure on $[0,T] \times \mathbb{R}\backslash\{0\}$ with intensity measure $\widetilde{\mu}\,(\mathrm{d}\,(s,x))$ (this intensity measure is defined by $\widetilde{\mu}\,((0,t] \times B) = \mu_t(B)$ for all measurable $B \subset \mathcal{B}\,(\mathbb{R})$). The integral $\int_{\{s\in(0,t],\varepsilon<|x|<1\}} x(Q(\mathrm{d}s,\mathrm{d}x) - \widetilde{\mu}(\mathrm{d}\,(s,x))$ as zero mean and its convergence, when $\varepsilon \searrow 0$, is uniform in t on any bounded interval. The process X^1 is an additive process with a system of generating triplets given by $(0, 0, \mu_t)$ and the process X^2 is also an additive process with generating triplets $(\Gamma_t, C_t, 0)$.

Not all additive processes are semimartingales. For instance, a continuous deterministic process with unbounded variation is not a semimartingale. In this paper, we will consider only semimartingales, since we need to use stochastic calculus and Itô's formula. Therefore, we will work with a subclass of the additive processes - the set of natural additive processes, as they were defined by Sato in [22]. An additive process is natural if the location parameter Γ_t has bounded variation. Let us now define the concept of factoring for an additive process. A factoring is a pair $(\{\rho_t : t \in [0,T]\}, \sigma)$, where σ is a continuous (atomless) finite measure on $[0, T]$ (e.g., the Lebesgue measure) and $\{\rho_t : t \in [0, T]\}$ is a family of infinitely divisible distributions such that the characteristic function of Z_t is given by $\exp \int_0^t \log\left(\widehat{\rho}_s\,(u)\right) \sigma\,(\mathrm{d}s)$,where $\widehat{\rho}_s$ is the characteristic function of ρ_s. Let us denote the generating triplet of ρ_t by $\left(\gamma_t, c_t^2, \nu_t\right)$. Then (see [22], Theorem 2.6 and Lemma 2.7, page 216)

$$\begin{aligned} \Gamma_t &= \int_0^t \gamma_s \sigma\,(\mathrm{d}s)\,, \\ C_t &= \int_0^t c_s^2 \sigma\,(\mathrm{d}s)\,, \\ \mu_t\,(B) &= \int_0^t \nu_s\,(B)\,\sigma\,(\mathrm{d}s)\,, \forall B \in \mathcal{B}\,(\mathbb{R}) \end{aligned}$$

and the elements of the triplet $\left(\gamma_t, c_t^2, \nu_t\right)$ are called the local characteristics of the natural additive process. From (3), it is clear that the family of Lévy measures $\{\nu_t\}_{t\in[0,T]}$ satisfies

$$\int_0^T \left(\int_{\mathbb{R}} \left(1 \wedge x^2\right) \nu_t(\mathrm{d}x)\right) \sigma\,(\mathrm{d}s) < \infty,$$

for all $t \in [0, T]$. An additive process is natural if and only if a factoring exists and it is a semimartingale if and only if it is natural (see [22], Theorem

2.6, Proposition 2.10 and [16], Corollary 5.11, page 116). Therefore, we shall consider natural additive processes with local characteristics $\left(\gamma_t, c_t^2, \nu_t\right)$ and for these kind of processes, the generalized version of the Lévy-Itô decomposition reads

$$Z_t = \int_0^t c_s \mathrm{d}W_s + X_t, \tag{4}$$

where $W = \{W_t, t \in [0,T]\}$ is a standard Brownian motion and $X = \{X_t, t \in [0,T]\}$ is a jump process independent of W. Moreover, the jump part is given by

$$X_t = \int_{\{s\in(0,t],|x|<1\}} x\left(Q(\mathrm{d}s,\mathrm{d}x) - \nu_s(\mathrm{d}x)\mathrm{d}s\right) + \int_{\{s\in(0,t],|x|\geq 1\}} xQ(\mathrm{d}s,\mathrm{d}x) + \int_0^t \gamma_s \mathrm{d}s, \tag{5}$$

where $Q(\mathrm{d}t,\mathrm{d}x)$ is a Poisson random measure on $[0,T]\times\mathbb{R}\backslash\{0\}$ with intensity measure $\nu_t\,(\mathrm{d}x)\mathrm{d}t$ (we shall consider the case where $\sigma(\mathrm{d}t) = \mathrm{d}t$ is the Lebesgue measure, these processes are also referred in the literature as non-homogeneous Lévy processes [11]). The decomposition (4) implies that the process $Z = \{Z_t, t \in [0,T]\}$ is a semimartingale with quadratic variation

$$[Z,Z]_t = \int_0^t c_s^2 \mathrm{d}s + \sum_{s\in(0,t]} |\Delta Z_s|^2$$

We assume that the family of Lévy measures $\{\nu_t\}_{t\in[0,T]}$ satisfies, for some $\varepsilon > 0$ and $\lambda > 0$,

$$\sup_{t\in[0,T]} \int_{(-\varepsilon,\varepsilon)^c} \exp(\lambda|x|)\nu_t(\mathrm{d}x) < \infty. \tag{6}$$

As a consequence of this assumption, it is easy to show that

$$\int_{-\infty}^{+\infty} |x|^i\,\nu_t(\mathrm{d}x) < \infty$$

for all $i \geq 2$ and all $t \in [0,T]$. Hence, all the moments of Z_t and X_t exist and we can define the following functions:

$$m_i\,(t) := \int_{-\infty}^{+\infty} x^i \nu_t(\mathrm{d}x), \quad i \geq 2 \tag{7}$$

$$M_i(t) := \int_0^t m_i\,(s)\,\mathrm{d}s, \quad i \geq 2 \tag{8}$$

By Itô's formula for càdlàg semimartingales (see [20]), we obtain the solution of the stochastic differential equation (1), which is given by

$$S_t = S_0 \exp\left(Z_t - \frac{1}{2}\left(\int_0^t c_s^2 \mathrm{d}s\right)\right) \prod_{0<s\leq t} (1+\Delta Z_s)\exp(-\Delta Z_s). \tag{9}$$

In order to ensure that $S_t > 0$ for all $t \geq 0$ *a.s.*, we require that $\Delta Z_t > -1$ for all t. Hence, we shall assume that the family of Lévy measures $\{\nu_t\}_{t\in[0,T]}$ is supported on $(-1,+\infty)$.

3 Power jump processes

Let us define the following "power jump" processes,

$$
\begin{aligned}
Z_t^{(1)} &= Z_t \\
Z_t^{(i)} &= \sum_{0<s\leq t} (\Delta Z_s)^i, \qquad i \geq 2,
\end{aligned}
\tag{10}
$$

where $\Delta Z_s = Z_s - Z_{s-}$ and

$$
\begin{aligned}
X_t^{(1)} &= X_t \\
X_t^{(i)} &= \sum_{0<s\leq t} (\Delta X_s)^i, \qquad i \geq 2.
\end{aligned}
\tag{11}
$$

The process $X^{(i)}$ is called the ith-power-jump process. Note that $X_t^{(i)} = Z_t^{(i)}$ if $i \geq 2$ and these processes are again natural additive processes. They have jumps at the same points as the original natural additive processes and the size of the jumps of $Z_t^{(i)}$ is equal to the size of the original jumps to the power i.

Clearly, if we recall (7)-(8), then the moments of the power-jump processes are given by (see [20], page 29)

$$
\begin{aligned}
\mathbb{E}\left[X_t^{(1)}\right] &= \int_0^t \gamma_s \mathrm{d}s + \int_0^t \int_{\{|x|\geq 1\}} x\nu_s(\mathrm{d}x)\,\mathrm{d}s := \int_0^t m_1(s)\,\mathrm{d}s := M_1(t), \\
\mathbb{E}\left[X_t^{(i)}\right] &= M_i(t), \qquad i \geq 2,
\end{aligned}
$$

where we have set $m_1(t) := \gamma_t + \int_{\{|x|\geq 1\}} x\nu_t(\mathrm{d}x)$ and $M_1(t) := \int_0^t m_1(s)\mathrm{d}s$.

Let us introduce the following compensated power-jump processes:

$$
Y_t^{(i)} = Z_t^{(i)} - \mathbb{E}\left[Z_t^{(i)}\right] = Z_t^{(i)} - M_i(t), \quad i \geq 1. \tag{12}
$$

These processes are martingales (the Teugels martingales) - see [1] and [18]. Following the orthonormalization procedure described in [18] and generalized in [1] for regular martingales with independent increments, we obtain the following sequence of strongly orthonormal martingales $\left\{H^{(i)},\ i \geq 1\right\}$, defined by

$$
H_t^{(i)} = \int_0^t c_{i,i}(s)\,\mathrm{d}Y_s^{(i)} + \int_0^t c_{i,i-1}(s)\,\mathrm{d}Y_s^{(i-1)} + \cdots + \int_0^t c_{i,1}(s)\,\mathrm{d}Y_s^{(1)}, \quad i \geq 1 \tag{13}
$$

The deterministic functions $c_{i,j}(\cdot)$ are are the coefficients of the orthonormalization of the following polynomials with time dependent coefficients,

$$
\left\{\mathbf{1}_{\{s<t\}}, \mathbf{1}_{\{s<t\}}x, \mathbf{1}_{\{s<t\}}x^2, \ldots, \mathbf{1}_{\{s<t\}}x^{i-1}\right\},
$$

with respect to the measure $\varphi_s(\mathrm{d}x)\mathrm{d}s = \left(x^2\nu_s(\mathrm{d}x) + c_s^2\delta(x)\,\mathrm{d}x\right)\mathrm{d}s$ defined in $[0,T] \times \mathbb{R}$ (see [1]), i.e., we consider the orthogonalization with respect to the inner product

$$
\langle p, q\rangle = \int_0^T \int_{-\infty}^{+\infty} p_s(x)q_s(x)\left(x^2\nu_s(\mathrm{d}x) + c_s^2\delta(x)\,\mathrm{d}x\right)\mathrm{d}s, \tag{14}
$$

where $p_t(x)$ and $q_t(x)$ are real polynomials with time dependent coefficients. The process $H^{(i)}$ is called the orthonormalized ith-power-jump-process.

The martingale representation property (MRP) obtained in [18] and generalized in [1] allows the representation of any square integrable $\mathbb{Q}$-martingale as an orthogonal sum of stochastic integrals with respect to the orthonormalized power-jump-processes $\left\{H^{(i)},\ i \geq 1\right\}$, i.e., any square integrable $\mathbb{Q}$-martingale $M = \{M_t, t \in [0,T]\}$, admits the representation

$$M_t = M_0 + \int_0^t h_s^{(1)} \mathrm{d}Z_s + \sum_{i=2}^{\infty} \int_0^t h_s^{(i)} \mathrm{d}H_s^{(i)}, \tag{15}$$

where the processes $h_s^{(i)}$, $i \geq 1$, are predictable and

$$\mathbb{E}\left[\int_0^t \sum_{i=1}^{\infty} \left|h_s^{(i)}\right|^2 \mathrm{d}s\right] < \infty.$$

4 Market completeness

Suppose that in our market exists at least one equivalent martingale measure $\mathbb{Q}$ such that Z remains a natural additive process under $\mathbb{Q}$. If these conditions hold we say that $\mathbb{Q}$ is structure preserving (see Theorem 3.2. in [6] and Theorem 2.1 in [8]). Under this risk neutral measure, the discounted stock price process $\tilde{S} = \{\tilde{S}_t = S_t/B_t, 0 \leq t \leq T\}$ and the process $\tilde{Z} = \{Z_t - \int_0^t r_s ds,\ 0 \leq t \leq T\}$ are both $\mathbb{Q}$-martingales. Obviously $\Delta\tilde{Z}_t = \Delta Z_t$ and $\tilde{Z}_t^{(i)} = Z_t^{(i)}$, $i \geq 2$. Under $\mathbb{Q}$, we construct the power-jump processes $\left\{Y^{(i)},\ i \geq 1\right\}$ and the orthonormalized ith-power-jump-processes $\left\{H^{(i)},\ i \geq 1\right\}$ based on $\tilde{Z}$. Note that for $i \geq 2$, $\widetilde{m}_i(t) = \int_{\mathbb{R}} x^i \tilde{\nu}_t(\mathrm{d}x)$, where $\{\widetilde{\nu}_t\}_{t\in[0,T]}$ is the family of Lévy measures of Z (and $\tilde{Z}$) under $\mathbb{Q}$ and we require these Lévy measures to satisfy (6).

In order to complete the market (which is, in the general case, incomplete) we introduce new artificial assets: the power-jump assets $\overline{Y}^{(i)} = \left\{\overline{Y_t}^{(i)},\ t \in [0,T]\right\}$, given by

$$\overline{Y_t}^{(i)} := B_t Y_t^{(i)}, \quad i \geq 2.$$

Alternatively, we can also introduce the orthonormalized power-jump assets $\overline{H}^{(i)} = \left\{\overline{H_t}^{(i)},\ t \in [0,T]\right\}$, where

$$\overline{H_t}^{(i)} := B_t H_t^{(i)}, \quad i \geq 2.$$

Clearly, the discounted power-jump assets $\left\{Y^{(i)},\ i \geq 1\right\}$ and the discounted orthonormalized power-jump assets $\left\{H^{(i)},\ i \geq 1\right\}$ are $\mathbb{Q}$-martingales.

Let us define the attainable contingent claims in our market. We say that a nonnegative random variable $X \in L^2(\Omega, \mathcal{F}_T, \mathbb{Q})$ is attainable contingent claim in $L^2(\mathbb{Q})$ if there exists a self-financing portfolio whose values, at time T, converges in $L^2(\mathbb{Q})$ to X. In our market, a portfolio $\pi = \{\pi^n : n \geq 1\}$ is a sequence of finite-dimensional predictable processes

$$\left\{\pi_t^n = \left(\alpha_t^n, \beta_t^n, \beta_t^{(2),n}, \beta_t^{(3),n}, \ldots, \beta_t^{(k_n),n}\right), 0 \leq t \leq T, \quad n \geq 2\right\},$$

where α_t^n represents the number of bonds at time t, β_t^n represents the number of stocks at time t, $\beta_t^{(i),n}$ is the number of the ith-power-jump assets $H^{(i)}$ and k_n is an integer which depends on n. The portfolio $\pi = \{\pi^n : n \geq 1\}$ is self-financing if π^n is a self-financing-portfolio for each finite n.

Definition 1 *Let us consider a market model $M_{\mathbb{Q}}$ where the traded assets are bonds with price process given by (2), a stock with dynamics given by (1) and the family of power-jump assets $\left\{\overline{Y}^{(i)},\ i \geq 2\right\}$. Assume that the additive process Z satisfies (6) and that exists at least one equivalent martingale measure $\mathbb{Q}$, which is also structure preserving. Then $M_{\mathbb{Q}}$ is called an enlarged additive market model.*

The next theorem states that $M_{\mathbb{Q}}$ is a complete market. The proof of this theorem is based on the martingale representation property (15) and was given in [9] for Lévy markets and then was generalized for the additive case in [1].

Theorem 2 *The enlarged additive market model $M_{\mathbb{Q}}$ is complete, in the sense that any square-integrable contingent claim $X \in L^2(\Omega, \mathcal{F}_T, \mathbb{Q})$ can be replicated in $L^2(\mathbb{Q})$.*

This completeness concept (in the L^2 sense) corresponds to the concept of quasi-completeness defined in [1], [2] and [17].

5 Hedging portfolios

In this section we obtain hedging formulas that allow us to compute explicitly the hedging portfolio which replicates a contingent claim X whose payoff is a function of the value, at maturity, of the stock price S, of an absolutely continuous process $V^1 = \left\{V_t^1, t \in [0,T]\right\}$ and of a jump process $V^2 = \left\{V_t^2, t \in [0,T]\right\}$, defined by

$$V_t^1 := \int_0^t l(S_s)\,\mathrm{d}s,$$
$$V_t^2 := \int_0^t \int_{-\infty}^{+\infty} g(x)\tilde{M}(\mathrm{d}s, \mathrm{d}x)$$

where $l(x)$ is a continuous function, g is a smooth function such that $g(0) = g'(0) = 0$, $\int_0^t \int_{-\infty}^{+\infty} |g(x)|\tilde{\nu}_s(\mathrm{d}x)\mathrm{d}s < \infty$ and $\tilde{M}(\mathrm{d}s, \mathrm{d}x) = Q(\mathrm{d}s, \mathrm{d}x) - \tilde{\nu}_s(\mathrm{d}x)\mathrm{d}s$ is the compensated Poisson Random measure. The reason why we assume processes of this particular form is that they allow us to include in our theory contingent claims so popular as the asian options and other related derivatives whose payoff depend on the paths of the risky asset in the period $[0,T]$. The jump process V_t^2 allow us to consider also the portfolio optimization problem, which will be discussed in detail in Section 6. The payoff is therefore a function of the form $f(S_T, V_T^1, V_T^2)$. Using the independence of $\frac{S_T}{S_t}$ and $V_T^2 - V_t^2$ with respect to $\mathcal{F}_t$, the price function of the contingent claim X, at time t, is given by

$$
\begin{aligned}
&\exp\left(-\int_t^T r_s \mathrm{d}s\right)\mathbb{E}_{\mathbb{Q}}\left[f(S_T,V_T^1,V_T^2)|\mathcal{F}_t\right]\\
&=\exp\left(-\int_t^T r_s \mathrm{d}s\right)\mathbb{E}_{\mathbb{Q}}\left[f\left(\frac{S_T}{S_t}S_t,\int_t^T l\left(\frac{S_s}{S_t}S_t\right)\mathrm{d}s+V_t^1,V_T^2-V_t^2+V_t^2\right)|\mathcal{F}_t\right]\\
&=\exp\left(-\int_t^T r_s \mathrm{d}s\right)\\
&\mathbb{E}_{\mathbb{Q}}\left[f\left(\frac{S_T}{S_t}x_1,\left(\int_t^T l\left(\frac{S_s}{S_t}x_1\right)\mathrm{d}s+x_2,V_T^2-V_t^2+x_3\right)\right)\right]_{x_1=S_t,x_2=V_t}\\
&:=F(t,S_t,V_t^1,V_t^2).
\end{aligned}
$$

As in Lévy market models (see [8] and [9]) or in the classical Black-Scholes model, the price function F must satisfy a partial differential integral equation (PIDE). Let us introduce the notation $x := (x_1, x_2, x_3)$, $D_0 := \frac{\partial}{\partial t}$, $D_k := \frac{\partial}{\partial x_k}$ and $D_1^i := \frac{\partial^i}{\partial x_1^i}$. The price function $F(t,x)$ is a solution of the PIDE

$$
\begin{aligned}
&D_0F(t,x)+l(x_1)D_2F(t,x)+r_tx_1D_1F(t,x)+\frac{1}{2}c_t^2x_1^2D_1^2F(t,x) && (16)\\
&-D_3F(t,x)\int_{\mathbb{R}}g(y)\tilde{\nu}_t(\mathrm{d}y)+\mathcal{D}F(t,x)=r_tF(t,x), && (17)\\
&F(T,x)=f(x),
\end{aligned}
$$

where

$$
\mathcal{D}F(t,x):=\int_{-\infty}^{+\infty}\left(F\left(t,x_1\left(1+y\right),x_2,x_3+g(y)\right)-F(t,x)-x_1yD_1F(t,x)\right)\tilde{\nu}_t(\mathrm{d}y).
$$

This equation is a straightforward generalization of the PIDE obtained in [8], Theorem 3.1, page 290.

In order to simplify the notation, we set $V_t := \left(V_t^1, V_t^2\right)$. The explicit hedging portfolio which replicates a contingent claim $X = f\left(S_T, V_T\right)$, with a price function $F(t, S_t, V_t)$ satisfying some regularity conditions, is given in the following theorem.

Theorem 3 *Let us consider a contingent claim $X \in L^2\left(\Omega, \mathcal{F}_T, \mathbb{Q}\right)$ with payoff $X = f\left(S_T, V_T\right)$ and a price function $F(t, S_t, V_t)$ of class $C^{1,\infty,2,\infty}$. Consider the function $h : [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ given by*

$$
h(t,x,y):=F(t,x_1(1+y),x_2,x_3+g\left(y\right))-F(t,x)-x_1yD_1F(t,x). \tag{18}
$$

Assume that $h(t,x,y)$ is an analytic function in y for all $x \in \mathbb{R}^4$, $t \in [0,T]$ and that we have the Taylor series representation

$$
h\left(t,x,y\right)=\sum_{i=2}^{\infty}\frac{1}{i!}\frac{\partial^i h}{\partial y^i}\left(t,x,0\right)y^i, \tag{19}
$$

for all $y \in \mathbb{R}$.

Then X has a self-financing replicating portfolio, which is given, at time t, by:

$$\text{number of bonds} = \alpha_t = B_t^{-1}\left(F(t, S_{t-}, V_{t-}) - S_{t-} D_1 F(t, S_{t-}, V_{t-})\right) \tag{20}$$

$$\text{number of stocks} = \beta_t^{(1)} = D_1 F\left(t, S_{t-}, V_{t-}\right) \tag{21}$$

$$\text{number of shares of the } Y^i\text{-asset} = \beta_t^{(i)} = \frac{\frac{\partial^i}{\partial y^i} h(t, S_{t-}, V_{t-}, 0)}{i!}, \quad i = 2, 3, \dots . \tag{22}$$

Remark 4 *Theorem 3 is simply a version, for the additive case, of Theorem 3.2 in [8]. Note that if the payoff depends only on the stock at maturity then $F = F(t, S_t)$ and $h(t, x, y) = F(t, x(1 + y)) - F(t, x) - xyD_1F(t, x)$. In this case, we have*

$$\begin{aligned} h(t, x, 0) &= 0 \\ \frac{\partial}{\partial y} h(t, x, 0) &= 0 \\ \frac{\partial^n}{\partial y^n} h(t, x, 0) &= x^n D_1^n F(t, x), \quad n = 2, 3, \dots \end{aligned}$$

and (see [9])

$$\text{number of bonds} = \alpha_t = B_t^{-1}\left(F(t, S_{t-}) - S_{t-} D_1 F(t, S_{t-})\right) \tag{23}$$

$$\text{number of stocks} = \beta_t = D_1 F(t, S_{t-}) \tag{24}$$

$$\text{number of shares of } Y^i\text{-asset} = \beta_t^{(i)} = \frac{S_{t-}^i D_1^i F(t, S_{t-})}{i!}, \qquad i = 2, 3, \dots . \tag{25}$$

We want to have hedging formulas in terms of call options with the same maturity and different strikes. In order to pursue this goal, we begin by studying the relationship between the power-jump assets and the pricing functions of derivatives with payoff $X_{(k)} = S_T^k$, $k \geq 2$. The following Proposition gives us a representation formula for the discounted price function of these derivatives, in terms of the risky asset and the power-jump processes.

Proposition 5 *Denote by $\tilde{F}^{(k)}(t, S_t)$ the discounted price function of the contingent claims with payoff $X_{(k)} = S_T^k$, $k \geq 2$ and set $\tilde{F}^{(1)}(t, S_t) := \tilde{S}_t$. Then we have the representation formula:*

$$\tilde{F}^{(k)}(t, S_t) = \tilde{F}^{(k)}(0, S_0) + \sum_{i=1}^{k} \int_0^t \frac{F^{(k)}(s, S_{s-})}{B_s} \binom{k}{i} \mathrm{d}Y_s^{(i)}, \quad k \geq 1 \tag{26}$$

and its inverse representation

$$\mathrm{d}Y_t^{(i)} = (-1)^i B_t \left(\sum_{k=1}^{i} \binom{i}{k} (-1)^k \frac{1}{F^{(k)}(t, S_{t-})} \mathrm{d}\tilde{F}^{(k)}(t, S_t) \right), \quad i \geq 1. \tag{27}$$

Proof. Clearly, by the independence of $(S_T/S_t)^k$ with respect to $\mathcal{F}_t$, the price function of the derivatives $X_{(k)}$, is given by

$$F^{(k)}(t,S_t) = \exp\left(-\int_t^T r_s \mathrm{d}s\right)\mathbb{E}_{\mathbb{Q}}(S_T^k|\mathcal{F}_t)$$
$$= \exp\left(-\int_t^T r_s \mathrm{d}s\right) S_t^k \mathbb{E}_{\mathbb{Q}}\left(\left(\frac{S_T}{S_t}\right)^k\right) = \varphi^{(k)}(t,T)S_t^k,$$

where $\varphi^{(k)}(t,T)$ is a deterministic function. Applying Theorem 3 with $F(t,x) = \varphi^{(k)}(t,T)x^k$, we obtain

$$\mathrm{d}F^{(k)}(t,S_t) = r_t F^{(k)}(t,S_{t-})\left(1-k-\frac{1}{B_t}\sum_{i=2}^{k}\binom{k}{i}\overline{Y}_t^{(i)}\right)\mathrm{d}t \tag{28}$$

$$+\frac{kF^{(k)}(t,S_{t-})}{S_{t-}}\mathrm{d}S_t + \frac{1}{B_t}\sum_{i=2}^{k}\binom{k}{i}\mathrm{d}\overline{Y}_t^{(i)}. \tag{29}$$

Clearly, we have that

$$\mathrm{d}S_t = B_t \mathrm{d}\tilde{S}_t + r_t B_t \tilde{S}_{t-}\mathrm{d}t,$$

where $\widetilde{S} = \{\widetilde{S}_t := \exp\left(-\int_0^t r_s \mathrm{d}s\right) S_t, t \in [0,T]\}$ is the discounted stock price process. Moreover

$$\mathrm{d}\tilde{F}^{(k)}(t,S_t) = \frac{1}{B_t}\mathrm{d}F^{(k)}(t,S_t) - \frac{r_t}{B_t}F^{(k)}(t,S_t)\mathrm{d}t$$

and

$$\mathrm{d}\overline{Y}_t^{(i)} = B_t \mathrm{d}Y_t^{(i)} + r_t B_t Y_t^{(i)}\mathrm{d}t.$$

Replacing all these differentials in (28)-(29) and using the identity $Y_t^{(1)} = Z_t - \int_0^t r_s ds = \int_0^t \frac{B_s}{S_{s-}}\mathrm{d}\tilde{S}_s$, we obtain

$$\mathrm{d}\tilde{F}^{(k)}(t,S_t) = \frac{kF^{(k)}(t,S_{t-})}{S_{t-}}\mathrm{d}\tilde{S}_t + \frac{F^{(k)}(t,S_{t-})}{B_t}\sum_{i=2}^{k}\binom{k}{i}\mathrm{d}Y_t^{(i)}$$
$$= \sum_{i=1}^{k}\frac{F^{(k)}(t,S_{t-})}{B_t}\binom{k}{i}\mathrm{d}Y_t^{(i)}. \tag{30}$$

By a combinatorial argument, we now use (30) in order to obtain a representation formula for $Y_t^{(i)}$ in terms of the price functions $\tilde{F}^{(k)}(t,S_t)$, with $k \leq i$. Indeed, if $X_k = \sum_{i=1}^k \binom{k}{i} Y_i$ then $Y_i = \sum_{k=1}^i \binom{i}{k}(-1)^{i-k}X_k$ for any $k \geq 1$. Therefore, we obtain (27). ∎

Note that, since we are assuming that $\mathbb{Q}$ is an equivalent martingale measure, $\widetilde{S}$ and the processes $\{Y^{(i)},\ i \geq 1\}$ are $\mathbb{Q}$-martingales. Hence, by (30), $\tilde{F}^{(k)}(t,S_t)$ is also a $\mathbb{Q}$-martingale, as expected.

Consider the deterministic function $\varphi^{(k)}(t,T) = \exp\left(-\int_t^T r_s \mathrm{d}s\right)\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{S_T}{S_t}\right)^k\right]$.

Lemma 6 *We have that*

$$\varphi^{(k)}(t,T) = \exp\left\{\int_t^T\left(\frac{k(k-1)}{2}c_s^2 + (k-1)r_s + \sum_{i=2}^{k}\binom{k}{i}\widetilde{m}_i(s)\right)\mathrm{d}s\right\}. \tag{31}$$

Proof. Using the decompositions (4)-(5) and the stock price process (9) under the risk-neutral measure $\mathbb{Q}$, where Z is replaced by $\tilde{Z} = \{Z_t - \int_0^t r_s ds$, $0 \le t \le T\}$, we have that (see [8])

$$S_t^k = \exp\left(\int_0^t \left(kc_s \mathrm{d}\widetilde{W}_s - \frac{k}{2}c_s^2 + kr_s + \int_{\mathbb{R}} kx\left(\widetilde{Q}(\mathrm{d}s,\mathrm{d}x) - \widetilde{\nu}_s(\mathrm{d}x)\right)\right)\mathrm{d}s\right)$$
$$\prod_{0<s\le t}(1+\Delta\widetilde{X}_s)^k \exp(-k\Delta\widetilde{X}_s).$$

It is clear that

$$\exp\left(\int_0^t \left(\int_{\mathbb{R}} kx\left(\widetilde{Q}(\mathrm{d}s,\mathrm{d}x) - \widetilde{\nu}_s(\mathrm{d}x)\right)\right)\right)\prod_{0<s\le t}(1+\Delta X_s)^k \exp(-k\Delta X_s)$$
$$= \exp\left(\int_0^t \int_{\mathbb{R}} kx\left(\widetilde{Q}(\mathrm{d}s,\mathrm{d}x) - \widetilde{\nu}_s(\mathrm{d}x)\right)\mathrm{d}s\right)$$
$$\times \exp\left(-\int_{\mathbb{R}}\left(\log(1+x)^k - 1 - kx\right)\widetilde{\nu}_s(\mathrm{d}x)\mathrm{d}s\right)$$
$$\prod_{0<s\le t}(1+\Delta\widetilde{X}_s)^k \exp(-k\Delta\widetilde{X}_s)\exp\left(\int_0^t\int_{\mathbb{R}}\left(\log(1+x)^k - 1 - kx\right)\widetilde{\nu}_s(\mathrm{d}x)\mathrm{d}s\right)$$

and, by a simple generalization of Proposition 2.1 of [8], the process

$$U_t = \exp\int_0^t \int_{\mathbb{R}} kx\left(\widetilde{Q}(\mathrm{d}s,\mathrm{d}x) - \widetilde{\nu}_s(\mathrm{d}x)\right)\mathrm{d}s$$
$$\times \exp\left(-\int_{\mathbb{R}}\left(\log(1+x)^k - 1 - kx\right)\widetilde{\nu}_s(\mathrm{d}x)\mathrm{d}s\right)$$
$$\prod_{0<s\le t}(1+\Delta\widetilde{X}_s)^k \exp(-k\Delta\widetilde{X}_s)$$

is a local martingale (see also Lemma 3.1 in [6]). Indeed, by Itô's formula, we have

$$U_t = 1 + \int_0^t\int_{\mathbb{R}} U_{s-}\left((1+x)^k - 1\right)\left(\widetilde{Q}(\mathrm{d}s,\mathrm{d}x) - \widetilde{\nu}_s(\mathrm{d}x)\right)$$

and since $\widetilde{Q}((0,t],\Lambda) - \widetilde{\nu}_t(\Lambda)$ is a martingale, where $\Lambda \subset \mathbb{R}$ is measurable, then U_t is a local martingale. Moreover, using the exponential moment condition (6), simple processes and a monotone class argument, it is easy to show that

$$\mathbb{E}[U_t] = 1, \quad \forall t \in [0,T].$$

Then, using the independence of the jump process X_t and the Brownian motion W_t, we have

$$\varphi^{(k)}(t,T) = \exp\left(-\int_t^T r_s \mathrm{d}s\right)\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{S_T}{S_t}\right)^k\right]$$
$$= \exp\left\{\int_t^T\left(\frac{k(k-1)}{2}c_s^2 + (k-1)r_s + \int_{-\infty}^{+\infty}\left((1+x)^k - 1 - kx\right)\widetilde{\nu}_s(\mathrm{d}x)\right)\mathrm{d}s\right\}$$
$$= \exp\left\{\int_t^T\left(\frac{k(k-1)}{2}c_s^2 + (k-1)r_s + \sum_{i=2}^k \binom{k}{i}\widetilde{m}_i(s)\right)\mathrm{d}s\right\}.$$

■

Let us now introduce the hedging formula for contingent claims in terms of call options with the same maturity and different strikes. This formula was obtained and proved in [4] and further exploited in [1]. Let $f(x)$ be a real function of class C^2 in $(0,\infty)$ and let $\tilde{C}_t(K) := \frac{1}{B_T}\mathbb{E}_{\mathbb{Q}}\left[(S_T-K)_+|\mathcal{F}_t\right]$ be the discounted price function of a call option with maturity T and strike K. Then

$$\mathbb{E}_{\mathbb{Q}}\left[B_T^{-1}f(S_T)|\mathcal{F}_t\right] = B_T^{-1}f(0) + f'(0)\tilde{S}_t + \int_0^\infty f''(K)\tilde{C}_t(K)\mathrm{d}K. \tag{32}$$

This formula provides a static hedge of the payoff $f(S_T)$.

The next theorem gives us the dynamic hedging formula in terms of call options and of the discounted real risky asset for the type of contingent claims considered in Theorem 3.

Theorem 7 *Let U be a contingent claim with payoff $U = g\left(S_T, V_T\right)$ and a price function $G(t,S_t,V_t)$ such that $G(t,x) \in C^{1,\infty,2,\infty}$ and*

$$h(t,x,y) := G(t,x_1(1+y),x_2,x_3+g\left(y\right)) - G(t,x) - x_1 y D_1 G(t,x).$$

is analytic in y for all $x \in \mathbb{R}^4$ and $t \in [0,T]$. Set

$$R(t,K) := \sum_{k=2}^{\infty} \frac{\frac{\partial^{k-1}}{\partial y^{k-1}}h(t,S_{t-},V_{t-},-1)}{(k-2)!\varphi^{(k)}(t,T)}\left(\frac{K}{S_{t-}}\right)^{k-2}. \tag{33}$$

and let us assume that

$$\int_0^\infty \sum_{k=2}^{\infty} \frac{\left|\frac{\partial^{k-1}}{\partial y^{k-1}}h(t,S_{t-},V_{t-},-1)\right|}{(k-2)!\varphi^{(k)}(t,T)}\left(\frac{K}{S_{t-}}\right)^{k-2} C_s(K)\mathrm{d}K < \infty. \tag{34}$$

Then, we have the representation

$$\sum_{i=2}^{\infty}\int_0^t \beta_s^{(i)}\mathrm{d}Y_s^{(i)} = \int_0^t\int_0^\infty \frac{B_s}{S_{s-}^2}R(s,K)\mathrm{d}\tilde{C}_s(K)\mathrm{d}K - \int_0^t \frac{B_s h(s,S_{s-},V_{s-},-1)}{S_{s-}}\mathrm{d}\tilde{S}_s \tag{35}$$

and the hedging portfolio, in terms of bonds, stocks and call options, is given by

$$\begin{aligned}
\alpha_t &= B_t^{-1}\left[G(t,S_{t-},V_{t-}) - S_{t-}D_1G(t,S_{t-},V_{t-})\right] \\
&\quad + B_t^{-1}\left[h(t,S_{t-},V_{t-},-1) - \int_0^\infty \frac{R(t,K)}{S_{t-}^2}C_{t-}(K)\mathrm{d}K\right] \\
\beta_t &= D_1G(t,S_{t-},V_{t-}) - \frac{h(t,S_{t-},V_{t-},-1)}{S_{t-}} \\
\beta_t^{(K)} &= \frac{R(t,K)}{S_{t-}^2},
\end{aligned}$$

where $\beta_t^{(K)}$ is the number of call options in the hedging portfolio, at time t, with strike K.

Proof. By application of (27), the value of the hedging portfolio in the first m discounted power-jump assets $Y_t^{(i)}$, $2 \le i \le m$, is given by

$$\sum_{i=2}^{m} \beta_t^{(i)} \mathrm{d}Y_t^{(i)} = B_t \sum_{i=2}^{m} \beta_t^{(i)} (-1)^i \left(\sum_{k=1}^{i} \binom{i}{k} (-1)^k \frac{1}{F^{(k)}(t, S_{t-})} \mathrm{d}\tilde{F}^{(k)}(t, S_t) \right).$$

Hence,

$$\sum_{i=2}^{m} \beta_t^{(i)} \mathrm{d}Y_t^{(i)} = B_t \sum_{k=1}^{m} \frac{(-1)^k}{F^{(k)}(t, S_{t-})} \left(\sum_{i=k\vee 2}^{m} \binom{i}{k} (-1)^i \beta_t^{(i)} \right) \mathrm{d}\tilde{F}^{(k)}(t, S_t).$$

Moreover, since

$$\beta_t^{(i)} = \frac{\frac{\partial^i}{\partial y^i} h(t, S_{t-}, V_{t-}, 0)}{i!},$$

we have

$$\sum_{i=2}^{m} \beta_t^{(i)} \mathrm{d}Y_t^{(i)}$$
$$= B_t \sum_{k=1}^{m} \frac{(-1)^k}{F^{(k)}(t, S_{t-})} \left(\sum_{i=k\vee 2}^{m} \binom{i}{k} (-1)^i \frac{\frac{\partial^i}{\partial y^i} h(t, S_{t-}, V_{t-}, 0)}{i!} \right) \mathrm{d}\tilde{F}^{(k)}(t, S_t).$$

The assumption (19) implies that the series $\sum_{i=2}^{\infty} \beta_t^{(i)} \mathrm{d}Y_t^{(i)}$ converges for every $\omega \in \Omega$, and therefore

$$\sum_{i=2}^{\infty} \beta_t^{(i)} \mathrm{d}Y_t^{(i)}$$
$$= \lim_{m\to\infty} B_t \left[\sum_{k=1}^{m} \frac{1}{k! F^{(k)}(t, S_{t-})} \left(\sum_{i=k\vee 2}^{m} (-1)^{i-k} \frac{\frac{\partial^i}{\partial y^i} h(t, S_{t-}, V_{t-}, 0)}{(i-k)!} \right) \mathrm{d}\tilde{F}^{(k)}(t, S_t) \right]. \tag{36}$$

Let us now consider the representation formula (32) with $f(x) = x^k$, $k \ge 2$. Then we have

$$\mathrm{d}\tilde{F}^{(k)}(t, S_t) = \int_0^{\infty} k(k-1) K^{k-2} \mathrm{d}\tilde{C}_t(K) \mathrm{d}K. \tag{37}$$

Hence

$$\sum_{i=2}^{\infty} \beta_t^{(i)} \mathrm{d}Y_t^{(i)} = \int_0^{\infty} \frac{B_t}{S_{t-}^2} R\,(t, K)\, \mathrm{d}\tilde{C}_t(K) \mathrm{d}K - \frac{B_t}{S_{t-}} h(t, S_{t-}, V_{t-}, -1) \mathrm{d}\tilde{S}_t. \tag{38}$$

Clearly, the series

$$R(t, K) = \sum_{k=2}^{\infty} \frac{\frac{\partial^{k-1}}{\partial y^{k-1}} h(t, S_{t-}, V_{t-}, -1)}{(k-2)! \varphi^{(k)}(t, T)} \left(\frac{K}{S_{t-}} \right)^{k-2}$$

is absolutely convergent for each t and each K since, by Lemma 6, we have that $\left|\varphi^{(2)}(t,T)\right| \leq \left|\varphi^{(k)}(t,T)\right|$ for all $t \in [0,T]$ and using the fact that $h(t,x,y)$ is an analytic function in y, we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left|\frac{\frac{\partial^{k-1}}{\partial y^{k-1}}h(t,S_{t-},V_{t-},-1)}{(k-2)!\varphi^{(k)}(t,T)}\right|\left|\frac{K}{S_{t-}}\right|^{k-2} \\
&\leq \left|\frac{1}{\varphi^{(2)}(t,T)}\right|\sum_{k=2}^{\infty}\left|\frac{\frac{\partial^{k-1}}{\partial y^{k-1}}h(t,S_{t-},V_{t-},-1)}{(k-2)!}\right|\left|\frac{K}{S_{t-}}\right|^{k-2} < \infty,
\end{aligned}$$

for all $t \in [0,T]$. Moreover, by assumption (34) we can apply Fubini's theorem to (38) in order to obtain

$$\sum_{i=2}^{\infty}\int_0^t \beta_s^{(i)}\mathrm{d}Y_s^{(i)} = \int_0^t\int_0^{\infty}\frac{B_s}{S_{s-}^2}R(s,K)\mathrm{d}\tilde{C}_s(K)\mathrm{d}K - \int_0^t \frac{B_s h(s,S_{s-},V_{s-},-1)}{S_{s-}}\mathrm{d}\tilde{S}_s.$$

The hedging portfolio is now a simple consequence of this representation and (20)-(22). ■

Note that replacing (37) in (27), we obtain

$$\mathrm{d}Y_t^{(i)} = (-1)^i B_t\left(\sum_{k=1}^{i}\binom{i}{k}(-1)^k\frac{1}{F^{(k)}(t,S_{t-})}\int_0^{\infty}k(k-1)K^{k-2}\mathrm{d}\tilde{C}_t(K)\mathrm{d}K\right),$$

which gives us the replication formula for the power-jump artificial assets in terms of call options with the same maturity T and with a continuum of strikes. The hedging formula (35) gives us the dynamic hedging portfolio in terms of call options and of the discounted stock, which is equivalent to the hedging portfolio in terms of the power-jump assets.

Remark 8 *If the payoff depends only on the stock at maturity then $G = G(t,S_t)$, the representation formula in terms of call options of Theorem 7 is given by*

$$\begin{aligned}
&\sum_{i=2}^{\infty}\int_0^t \beta_s^{(i)}\mathrm{d}Y_s^{(i)} && (39)\\
&= \int_0^t\int_0^{\infty}\frac{B_s}{S_{s-}}\left(\sum_{k=2}^{\infty}\frac{D_1^{k-1}G(s,0)}{\varphi^{(k)}(s,T)\,(k-2)!}K^{k-2}\right)\mathrm{d}\tilde{C}_s(K)\mathrm{d}K && (40)\\
&- \int_0^t\left[\frac{B_s}{S_{s-}}\left(G\left(s,0\right) - G\left(s,S_{s-}\right) + S_{s-}D_1^1G(s,S_{s-})\right)\mathrm{d}\tilde{S}_s\right]. && (41)
\end{aligned}$$

Moreover, the hedging portfolio is given by

$$\begin{aligned}
\alpha_t &= B_t^{-1}\left[G(t,S_{t-},V_{t-}) - S_{t-}G(t,S_{t-},V_{t-}) + h(t,S_{t-},V_{t-},-1)\right] \\
&\quad - B_t^{-1}\sum_{k=2}^{\infty}\int_0^{\infty}\left(\frac{D_1^{k-1}G(t,0)}{S_{t-}\varphi^{(k)}(t,T)\,(k-2)!}K^{k-2}\right)C_{t-}(K)\mathrm{d}K,\\
\beta_t &= D_1G(t,S_{t-},V_{t-}) - \frac{G\left(t,0\right) - G\left(t,S_{t-}\right) + S_{t-}D_1^1G(t,S_{t-})}{S_{t-}},\\
\beta_t^{(K)} &= \frac{1}{S_{t-}}\left(\sum_{k=2}^{\infty}\frac{D_1^{k-1}G(t,0)}{\varphi^{(k)}(t,T)\,(k-2)!}K^{k-2}\right).
\end{aligned}$$

Remark 9 *We now discuss the relationship between the geometric Lévy model (or stochastic exponential model) and the usual exponential Lévy model. Let us assume that our stock price process $S = \{S_t, t \in [0,T]\}$ is given by the usual exponential:*

$$S_t = S_0 e^{\widetilde{Z}_t}, \qquad S_0 > 0, \tag{42}$$

where $\widetilde{Z} = \{\widetilde{Z}_t, t \in [0,T]\}$ is a Lévy process. The process S can also be modelled as a stochastic exponential of a Lévy process, which is defined as the solution of the linear SDE (1)(see [20]) and denoted by $S_t = S_0 \mathcal{E}(Z_t)$, where $Z = \{Z_t, t \in [0,T]\}$ is a Lévy process related to $\widetilde{Z}$. This relationship was rigorously described in [14] (Lemma A.8., pages 46-47), where the reader can find the proofs of the following properties.

1. *If $\widetilde{Z}$ is a Lévy process with characteristic triplet given by $\left(\widetilde{c}^2, \widetilde{\nu}, \widetilde{\gamma}\right)$ then the usual exponential $e^{\widetilde{Z}_t}$ is of the form $\mathcal{E}(Z_t)$ (stochastic exponential) for some Lévy process Z with characteristic triplet given by $\left(c^2, \nu, \gamma\right)$, where*

$$\begin{aligned}
\gamma &= \widetilde{\gamma} + \frac{\widetilde{c}^2}{2} + \int \left(\mathbf{1}_{\{|e^x - 1| \leq 1\}} (e^x - 1) - x \mathbf{1}_{\{|x| \leq 1\}}\right) \widetilde{\nu}(dx), \\
c^2 &= \widetilde{c}^2, \\
\nu(G) &= \int_{\mathbb{R}} \mathbf{1}_G (e^x - 1) \widetilde{\nu}(dx), \qquad G \in \mathcal{B}(\mathbb{R}).
\end{aligned}$$

   Note that the Lévy measure ν has support in $(-1, +\infty)$.

2. *If Z is a Lévy process with characteristic triplet given by $\left(c^2, \nu, \gamma\right)$ then the stochastic exponential $\mathcal{E}(Z_t)$ is of the form $e^{\widetilde{Z}_t}$ for some Lévy process $\widetilde{Z}$ with characteristic triplet given by $\left(\widetilde{c}^2, \widetilde{\nu}, \widetilde{\gamma}\right)$, where*

$$\begin{aligned}
\widetilde{\gamma} &= \gamma - \frac{\widetilde{c}^2}{2} + \int \left(\mathbf{1}_{\{|\log(1+x)| \leq 1\}} (\log(1+x)) - x \mathbf{1}_{\{|x| \leq 1\}}\right) \nu(dx), \\
\widetilde{c}^2 &= c^2, \\
\widetilde{\nu}(G) &= \int_{\mathbb{R}} \mathbf{1}_G (\log(1+x)) \nu(dx), \qquad G \in \mathcal{B}(\mathbb{R}).
\end{aligned}$$

   It is clear from (9) and (42) that

$$\begin{aligned}
\widetilde{S}_t &= \widetilde{S}_{t-} \exp\left(\Delta \widetilde{X}_t\right), \\
S_t &= S_{t-} (1 + \Delta X_t)
\end{aligned}$$

   where $\widetilde{X}$ is the jump part of $\widetilde{Z}$ and X is the jump part of Z.

Let us now consider the hedging portfolio of a contingent claim X when the stock dynamics is defined by (42). In this case,the pricing function $F(t,x)$ is a solution of the PIDE

$$\begin{aligned}
&D_0 F(t,x) + l(x_1) D_2 F(t,x) + r_t x_1 D_1 F(t,x) + \frac{1}{2} c_t^2 x_1^2 D_1^2 F(t,x) \\
&- D_3 F(t,x) \int_{\mathbb{R}} g(y) \widetilde{\nu}(\mathrm{d}y) + \mathcal{D} F(t,x) = r_t F(t,x), \\
&F(T,x) = f(x),
\end{aligned}$$

where

$$\mathcal{D}F\left(t,x\right):=\int_{-\infty}^{+\infty}h\left(t,x,y\right)\widetilde{\nu}_{t}\left(\mathrm{d}y\right)$$

and

$$h(t,x,y):=F\left(t,x_1e^y,x_2,x_3+g(y)\right)-F(t,x)-x_1\left(e^y-1\right)D_1F(t,x_1,x_2,x_3).$$

In the particular case where the contingent claim has a payoff which depends only on the stock at maturity, we have that $h(t,x,y)=F(t,xe^y)-F(t,x)-x\left(e^y-1\right)D_1F(t,x)$ *and*

$$\begin{aligned}
h(t,x,0)&=0,\\
\frac{\partial}{\partial y}h(t,x,0)&=0,\\
\frac{\partial^n}{\partial y^n}h(t,x,0)&=\sum_{k_1,k_2,\ldots,k_n,2\leq k\leq n}\frac{n!}{k_1!k_2!\cdots k_n!\left(2!\right)^{k_2}\left(3!\right)^{k_3}\cdots\left(n!\right)^{k_n}}D_1^{(k)}F(t,x)x^k,\\
n&=2,3,\ldots
\end{aligned}$$

where the sum is over all the partitions of n*, that is, over all the* n*-tuples* $(k_1,k_2,\ldots,k_n)$ *such that*

$$1k_1+2k_2+3k_3+\cdots+nk_n=n$$

and we use the notation

$$k:=k_1+k_2+\cdots+k_n$$

The hedging portfolio in the power-jump assets is then given by

$$\beta_t^{(i)}=\frac{\sum_{k_1,k_2,\ldots,k_n,2\leq k\leq i}\frac{n!}{k_1!k_2!\cdots k_n!(2!)^{k_2}(3!)^{k_3}\cdots(n!)^{k_n}}D_1^{(k)}F(t,S_{t-})S_{t-}^k}{i!},\qquad i=2,3,\ldots. \tag{43}$$

Remark 10 *If the payoff function* $X=g\left(S_T\right)$ *depends only on the stock at maturity and* $g(u)$ *is analytic in* u *then we can recover the static hedging (32) from the dynamic hedging (40)-(41). In order to show this, let us begin by using the independence of* $\frac{S_T}{S_t}$ *with respect to* $\mathcal{F}_t$ *to explore the relationship between the derivatives of the price function and the conditional expectation of the derivatives of* $g\left(u\right)$*. Indeed, we have that*

$$\begin{aligned}
G(t,S_t)&=\frac{B_t}{B_T}\mathbb{E}_{\mathbb{Q}}\left[g(S_T)|\mathcal{F}_t\right]\\
&=\frac{B_t}{B_T}\,\mathbb{E}_{\mathbb{Q}}\left[g\left(\frac{S_T}{S_t}x\right)\right]\Big|_{x=S_t}.
\end{aligned}$$

Hence

$$\begin{aligned}
D_1^iG(t,x)&=\frac{B_t}{B_T}\,\mathbb{E}_{\mathbb{Q}}\left[g^{(i)}\left(\frac{S_T}{S_t}x\right)\left(\frac{S_T}{S_t}\right)^i\right]\Big|_{x=S_t}\\
&=\frac{B_t}{B_T}x^{-i}\mathbb{E}_{\mathbb{Q}}\left[g^{(i)}\left(S_T\right)S_T^i|\mathcal{F}_t\right]
\end{aligned} \tag{44}$$

Now, recall the dynamic hedging formula (36) and the identity

$$\frac{\partial^i}{\partial y^i}h(t,x,0) = x^i D_1^i G(t,x).$$

Using (44), we have that

$$\begin{aligned}
&\sum_{i=k}^{\infty}\binom{i}{k}(-1)^{i-k}\frac{S_{t-}^i D_1^i G(t,S_{t-})}{i!} \\
&= \frac{B_t}{B_T}\sum_{i=k}^{\infty}\binom{i}{k}(-1)^{i-k}\frac{\mathbb{E}_{\mathbb{Q}}\left[g^{(i)}\left(S_T\right)S_T^i|\mathcal{F}_t\right]}{i!} \\
&= \frac{B_t}{B_T}\mathbb{E}_{\mathbb{Q}}\left[\sum_{i=k}^{\infty}\binom{i}{k}(-1)^{i-k}\frac{g^{(i)}\left(S_T\right)S_T^{i-k}S_T^k}{i!}|\mathcal{F}_t\right] \\
&= \frac{g^{(k)}\left(0\right)}{k!}F^{(k)}(t,S_{t-}),
\end{aligned} \tag{45}$$

Therefore, replacing (37) and (45) in (36), for $k \geq 2$, we obtain

$$\begin{aligned}
&B_t\sum_{k=2}^{\infty}\int_0^{\infty}\frac{(-1)^k}{F^{(k)}(t,S_{t-})}\left(\sum_{i=k}^{\infty}\binom{i}{k}(-1)^i\frac{S_{t-}^i D_1^i G(t,S_{t-})}{i!}\right)k(k-1)K^{k-2}\mathrm{d}\tilde{C}_t(K)\mathrm{d}K \\
&= B_t\sum_{k=2}^{\infty}\int_0^{\infty}\left(\frac{1}{F^{(k)}(t,S_{t-})}\right)\frac{g^{(k)}\left(0\right)}{k!}F^{(k)}(t,S_{t-})k(k-1)K^{k-2}\mathrm{d}\tilde{C}_t(K)\mathrm{d}K \\
&= B_t\int_0^{\infty}g''\left(K\right)\mathrm{d}\tilde{C}_t(K)\mathrm{d}K
\end{aligned} \tag{46}$$

and this is precisely the undiscounted integral term of the static hedging formula (32). Consider now the term $k = 1$ of the sum in the hedging formula (36). This term is given by

$$\begin{aligned}
&- B_t\left[\frac{1}{S_t}\left(\sum_{i=2}^{\infty}i(-1)^i\frac{S_{t-}^i D_1^i G(t,S_{t-})}{i!}\right)\mathrm{d}\tilde{S}_t\right] \\
&= B_t\left[\left(\sum_{i=0}^{\infty}(-1)^i\frac{S_{t-}^i D_1^{i+1}G(t,S_{t-})}{i!} - D_1 G\left(t,S_{t-}\right)\right)\mathrm{d}\tilde{S}_t\right] \\
&= B_t g'\left(0\right)\left(\frac{B_t}{B_T}\right)\frac{1}{S_{t-}}\mathbb{E}_{\mathbb{Q}}\left[S_T|\mathcal{F}_t\right]\mathrm{d}\tilde{S}_t - B_t D_1 G\left(t,S_{t-}\right)\mathrm{d}\tilde{S}_t \\
&= B_t g'\left(0\right)\mathrm{d}\tilde{S}_t - B_t D_1 G\left(t,S_{t-}\right)\mathrm{d}\tilde{S}_t
\end{aligned} \tag{47}$$

On the other hand, we have from (24) that the number of stocks of the hedging portfolio (or the delta hedging term) is

$$\beta_t^{(1)} = D_1 G\left(t,S_{t-}\right).$$

Considering the undiscounted version of this quantity and adding it to (47), we have

$$-B_t\left[\frac{1}{S_t}\left(\sum_{i=2}^{\infty}i(-1)^i\frac{S_{t-}^i D_1^i G(t,S_{t-})}{i!}\right)\mathrm{d}\tilde{S}_t\right] = B_t g'\left(0\right)\mathrm{d}\tilde{S}_t. \tag{48}$$

Note that the undiscounted delta hedging and the term $B_t D_1 G(t, S_{t-}) \mathrm{d}\tilde{S}_t$ cancel out. Finally, adding (46) and (48) we get the undiscounted static hedging formula (32) for the stocks and call options. Note that the static hedging in bonds term, which is given by $B_T^{-1} g(0)$, can be deduced from the dynamic hedging formula (23).

Until now, we have considered contingent claims with a price function $G(t, x)$ satisfying the analytic assumptions in Theorem 7. These regularity assumptions are rather strong and we would like to obtain hedging formulas for more general contingent claims. In order to get such formulas, we will consider the discounted orthonormalized power-jump processes $\{H^{(i)}, i \geq 2\}$. Recall the orthonormalization coefficients from the orthonormalization procedure which leads to (13) and consider the orthonormal real polynomials $p_t^{(i)}(x)$, $i \geq 1$, with these time dependent coefficients:

$$p_t^{(i)} := c_{i,i}(t) x^i + c_{i,i-1}(t) x^{i-1} + \cdots + c_{i,1}(t) x, \quad i \geq 1$$

Let us begin by stating an important Lemma, which is a simple generalization of Lemma 5 in [19].

Lemma 11 *Let $f : [0, T] \times \mathbb{R}^4 \to \mathbb{R}$ be a measurable function such that*

$$\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} |f(s, x, y)|^2 \tilde{\nu}_s(\mathrm{d}y)\mathrm{d}s\right] < \infty.$$

Then we have

$$\begin{aligned}\sum_{0<s\leq t} f(s, S_{s-}, V_{s-}, \Delta X_s) = &\sum_{i=1}^{\infty} \int_0^t \left\langle f(s, x, \cdot), p_s^{(i)}(\cdot)\right\rangle_{L^2(\tilde{\nu}_s)} \mathrm{d}H_s^{(i)} \\ &+ \int_0^t \int_{\mathbb{R}} f(s, x, y)\, \tilde{\nu}_s(\mathrm{d}y)\mathrm{d}s.\end{aligned}$$

We now state and prove a theorem that will enable us to hedge a general contingent claim (in the L^2-sense) in terms of call options, bonds and the risky asset.

Theorem 12 *Let U be a contingent claim with payoff $U = g(S_T, V_T)$ and a price function $G(t, S_t, V_t)$ such that $G(t, x)$ is of class $C^{1,2,2,2}$ in $[0, T] \times \mathbb{R}^3$. Consider the function*

$$h(t, x, y) := G(t, x_1(1 + y), x_2, x_3 + g(y)) - G(t, x) - x_1 y D_1 G(t, x). \tag{49}$$

and assume that

$$\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} |h(s, x, y)|^2 \tilde{\nu}_s(\mathrm{d}y)\mathrm{d}s\right] < \infty.$$

Set

$$N^{(m)}(s, K) := \sum_{i=2}^{m} \sum_{k=2}^{i} \frac{(-1)^{i-k} \beta_s^{(i,m)} \binom{i}{k} k(k-1)}{\varphi^{(k)}(s, T)} \left(\frac{K}{S_{s-}}\right)^{k-2},$$

where $\beta_s^{(i)} := \int_{\mathbb{R}} h(s, S_{s-}, V_{s-}, y) p_s^{(i)}(y) \tilde{\nu}_s(\mathrm{d}y)$ *and* $\beta_s^{(i,m)} := \sum_{i=j}^{m} c_{i,j}(s) \beta_s^{(i)}$. *Then we have the representation formula (the series converge in the* L^2 *sense)*

$$
\begin{aligned}
&\sum_{i=1}^{\infty} \int_0^t \beta_s^{(i)} \mathrm{d}H_s^{(i)} \\
&= \lim_{m\to\infty} \left[\int_0^{\infty} \int_0^t \frac{B_s N^{(m)}(s,K)}{S_{s-}^2} \mathrm{d}\tilde{C}_s(K) \mathrm{d}K - \sum_{i=1}^{m} \int_0^t \frac{i(-1)^i \beta_s^{(i,m)} B_s}{S_{s-}} \mathrm{d}\widetilde{S}_s \right].
\end{aligned} \tag{50}
$$

Moreover, the hedging portfolio in terms of bonds, stocks and call options is given by

$$
\begin{aligned}
\alpha_t &= \frac{1}{B_t} \left[G(t, S_{t-}, V_{t-}) - S_{t-} D_1 G(t, S_{t-}, V_{t-}) \right] \\
&+ \frac{1}{B_t} \lim_{m\to\infty} \left[\sum_{i=1}^{m} \frac{i(-1)^i \beta_t^{(i,m)}}{S_{t-}} - \int_0^{\infty} \frac{N^{(m)}(t,K)}{S_{t-}^2} C_{t-}(K) \mathrm{d}K \right], \\
\beta_t &= D_1 G(t, S_{t-}, V_{t-}) - \lim_{m\to\infty} \sum_{i=1}^{m} \int_0^t \frac{i(-1)^i \beta_t^{(i,m)}}{S_{t-}}, \\
\beta_t^{(K)} &= \lim_{m\to\infty} \frac{N^{(m)}(t,K)}{S_{t-}^2}
\end{aligned}
$$

Proof. Applying Itô's formula to $G(t, S_t, V_t)$ we obtain

$$
\begin{aligned}
&G(t, S_t, V_t) = G(0, S_0, V_0) + \int_0^t D_1 G(s, S_{s-}, V_{s-}) \mathrm{d}S_s \\
&+ \int_0^t \left(D_0 G(s, S_{s-}, V_{s-}) + \frac{1}{2} c_s^2 S_{s-}^2 D_1^2 G(s, S_{s-}, V_{s-}) + l(S_{s-}) D_2 G(s, S_{s-}, V_{s-}) \right) \mathrm{d}s \\
&+ \sum_{0<s\le t} \left[G(s, S_s, V_{s-}) - G(s, S_{s-}, V_{s-}) - D_1 G(s, S_{s-}, V_{s-}) \Delta S_s \right]
\end{aligned}
$$

Clearly, $\Delta S_s = S_{s-} \Delta X_s$ and therefore

$$
\begin{aligned}
&\sum_{0<s\le t} \left[G(s, S_s, V_{s-}) - G(s, S_{s-}, V_{s-}) - S_{s-} \Delta X_s D_2 G(s, S_{s-}, V_{s-}) \right] \\
&= h(s, S_{s-},, V_{s-}, \Delta X_s).
\end{aligned}
$$

If we now use the PIDE (16)-(17), then we have that

$$
\begin{aligned}
&G(t, S_t, V_t) = G(0, S_0, V_0) + \int_0^t D_1 G(s, S_{s-}, V_{s-}) \mathrm{d}S_s \\
&+ \int_0^t \frac{(G(s, S_{s-}, V_{s-}) - S_{s-} D_1 G(s, S_{s-}, V_{s-}))}{B_s} \mathrm{d}B_s \\
&+ \sum_{0<s\le t} h(s, S_{s-}, V_{s-}, \Delta X_s) - \int_0^t \int_{\mathbb{R}} h(s, S_{s-}, V_{s-}, y) \tilde{\nu}_s(\mathrm{d}y) \mathrm{d}s,
\end{aligned} \tag{51}
$$

and this gives us the representation of the hedging portfolio in terms of bonds and stocks.

Moreover, note that

$$\begin{aligned}M_t &:= \sum_{0<s\le t} h(s, S_{s-}, V_{s-}, \Delta X_s) - \int_0^t \int_{\mathbb{R}} h\left(s, S_{s-}, V_{s-}, y\right) \tilde{\nu}_s(\mathrm{d}y)\mathrm{d}s \\ &= \sum_{0<s\le t} h(s, S_{s-}, V_{s-}, \Delta X_s) - \mathbb{E}(\sum_{0<s\le t} h(s, S_{s-}, V_{s-}, \Delta X_s))\end{aligned}$$

is a square-integrable $\mathbb{Q}$-martingale and, by Lemma 11, we have that

$$M_t = \sum_{i=1}^{\infty} \int_0^t \beta_s^{(i)} \mathrm{d}H_s^{(i)}$$

where

$$\beta_s^{(i)} = \int_{\mathbb{R}} f(s, S_{s-}, V_{s-}, y) p_s^{(i)}(y) \tilde{\nu}_s(\mathrm{d}y).$$

and

$$\mathbb{E}\left[\int_0^t \sum_{i=1}^{\infty} \left|\beta_s^{(i)}\right|^2 \mathrm{d}s\right] < \infty.$$

By (13), we have

$$\beta_s^{(i)} \mathrm{d}H_s^{(i)} = \beta_s^{(i)} \sum_{j=1}^{i} c_{i,j}\left(s\right) \mathrm{d}Y_s^{(j)}$$

In general, we can write

$$\begin{aligned}\sum_{i=1}^{\infty} \int_0^t \beta_s^{(i)} \mathrm{d}H_s^{(i)} &= \lim_m \sum_{i=1}^{m} \int_0^t \beta_s^{(i)} \mathrm{d}H_t^{(i)} \\ &= \lim_m \sum_{j=1}^{m} \int_0^t \left(\sum_{i=j}^{m} c_{i,j}\left(s\right) \beta_s^{(i)}\right) \mathrm{d}Y_s^{(j)} = \lim_m \sum_{j=1}^{m} \int_0^t \beta_s^{(j,m)} \mathrm{d}Y_s^{(j)},\end{aligned}$$

where $\beta_s^{(j,m)} := \sum_{i=j}^{m} c_{i,j}\left(s\right) \beta_s^{(i)}$. Recalling (27), we obtain

$$\begin{aligned}\sum_{i=1}^{\infty} \int_0^t \beta_s^{(i)} \mathrm{d}H_s^{(i)} &= \lim_m \sum_{i=1}^{m} \int_0^t \beta_s^{(i,m)} \mathrm{d}Y_s^{(i)} \\ &= \lim_m \sum_{i=1}^{m} \int_0^t B_s \beta_s^{(i,m)} (-1)^i \sum_{k=1}^{i} \binom{i}{k} (-1)^k \frac{1}{F^{(k)}(s, S_{s-})} \mathrm{d}\tilde{F}^{(k)}(s, S_s) \\ &= \lim_m \sum_{k=1}^{m} (-1)^k \int_0^t \frac{B_s}{F^{(k)}(s, S_{s-})} \sum_{i=k}^{m} \beta_s^{(i,m)} (-1)^i \binom{i}{k} \mathrm{d}\tilde{F}^{(k)}(s, S_s)\end{aligned}$$

The representation (37) yields

$$\begin{aligned}&\sum_{i=1}^{\infty} \int_0^t \beta_s^{(i)} \mathrm{d}H_s^{(i)} \\ &= \lim_m \left[\sum_{i=2}^{m} (-1)^i \int_0^{\infty} \int_0^t B_s \beta_s^{(i,m)} \sum_{k=2}^{i} (-1)^k \binom{i}{k} k(k-1) \frac{K^{k-2}}{\varphi^{(k)}(s, T) S_{s-}^k} \mathrm{d}\tilde{C}_s(K) \mathrm{d}K \right. \\ &\left. - \sum_{i=1}^{m} \int_0^t \frac{i(-1)^i \beta_s^{(i,m)} B_s}{S_{s-}} \mathrm{d}\widetilde{S}_s\right].\end{aligned}$$

Combining this representation with (51), we obtain the hedging portfolio. ■

We now present two concrete hedging examples for a call option and an asian option.

Remark 13 *Consider a call option struck at K_* in an additive market with bond price process is B_t. Its price function is given by*

$$\begin{aligned} G(t,S_t) &= \frac{B_t}{B_T}\mathbb{E}_{\mathbb{Q}}\left[(S_T-K_*)_+\,|\mathcal{F}_t\right] \\ &= \frac{B_t}{B_T}S_t\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{S_T}{S_t}-\frac{K_*}{x}\right)_+\right]_{x=S_t} \\ &= \frac{B_t}{B_T}S_t\psi\left(t,x\right)|_{x=S_t}, \end{aligned}$$

where $\psi(t,x) := \mathbb{E}_{\mathbb{Q}}\left[\left(\frac{S_T}{S_t}-\frac{K_*}{x}\right)_+\right]$. The price function $G(t,x) = \frac{B_t}{B_T}x\psi\left(t,x\right)$ must satisfy the PIDE (16)-(17) and therefore

$$\begin{aligned} &\frac{\partial}{\partial t}\psi\left(t,x\right) - rx\frac{\partial}{\partial x}\psi(t,x) + \frac{c_t^2}{2}x^2\frac{\partial^2}{\partial x^2}\psi(t,x) + r\psi(t,x) \\ &+\int_{-1}^{\infty}\left((1+y)\left(\psi\left(t,\frac{x}{1+y}\right)-\psi\left(t,x\right)\right)+yx\frac{\partial}{\partial x}\psi\left(t,x\right)\right)\tilde{\nu}_t(\mathrm{d}y) = 0, \\ &\psi(T,x) = (x-K_*)_+\,. \end{aligned}$$

Assume that $G(t,x) = e^{-r(T-t)}x\psi\left(t,x\right)$ is analytic in x for all $x > 0$ and $t \in [0,T]$. By Theorem 7, the portfolio in the power-jump assets $Y^{(i)}$, $i \geq 2$ can be represented by

$$\sum_{i=2}^{\infty}\int_0^t \beta_s^{(i)}\mathrm{d}Y_s^{(i)} = \int_0^{\infty}\int_0^t \frac{B_s}{S_{s-}^2}R(s,K)\mathrm{d}\tilde{C}_s(K)\mathrm{d}K - \int_0^t \frac{B_s h(s,S_{s-},V_{s-},-1)}{S_{s-}}\mathrm{d}\tilde{S}_s$$

We can approximate the call option payoff $g(u) = (u-K_*)_+$ by a regularizing sequence of analytic functions $g_n\left(u\right)$ in such a way that $g_n'\left(u\right)$ converges to the Heaviside function $\mathcal{H}\left(u-K_*\right)$ and $g_n''\left(u\right)$ converges to the Dirac delta function $\delta\left(u-K_*\right)$, in the sense of distributions, when $n \to \infty$. Therefore, by Remark 10, it is clear that

$$\begin{aligned} &\sum_{i=2}^{\infty}\int_0^t \beta_s^{(i)}\mathrm{d}Y_s^{(i)} = \\ &= B_t\int_0^{\infty} g_n''\left(K\right)\mathrm{d}\tilde{C}_t(K)\mathrm{d}K \underset{n\to\infty}{\longrightarrow} B_t\int_0^{\infty}\delta\left(K-K_*\right)\mathrm{d}\tilde{C}_t(K)\mathrm{d}K \\ &= B_t\mathrm{d}\tilde{C}_t(K_*). \end{aligned}$$

Moreover, $g(0) = g'(0) = 0$ and the hedging portfolio is simply given by the call option with maturity T and strike K_*, as expected.

Example 14 *Consider an Asian option struck at K, that is an option with payoff*

$$X = \left(\frac{1}{T}\int_0^T S_u\mathrm{d}u - K\right)_+.$$

in an additive market. Then, the price process is

$$G(t,S_t,V_t)=\frac{B_t}{B_T}\mathbb{E}_{\mathbb{Q}}\left[X|\mathcal{F}_t\right],$$

where $V_t:=\frac{1}{T}\int_0^t S_u\mathrm{d}u$ *and* $X=(V_T-K)_+$. *In fact, we have*

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[X|\mathcal{F}_t\right]&=\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T S_u\mathrm{d}u-K\right)_+\middle|\mathcal{F}_t\right]\\&=S_t\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{1}{T}\int_t^T\frac{S_u}{S_t}\mathrm{d}u+x\right)_+\right]_{x=\frac{V_t-K}{S_t}}\\&=S_t\phi\left(t,\frac{U_t}{S_t}\right),\end{aligned}$$

where $U_t:=V_t-K$ *and* $\phi(t,x):=\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{1}{T}\int_t^T\frac{S_u}{S_t}\mathrm{d}u+x\right)_+\right]$ *is a deterministic function. Hence,*

$$G(t,S_t,V_t)=\frac{B_t}{B_T}S_t\phi\left(t,\frac{U_t}{S_t}\right).$$

In order to obtain this price function we can solve the PIDE (16)-(17), i. e.,

$$\begin{aligned}&D_0G(t,x_1,x_2)+\frac{1}{T}x_1D_2G(t,x_1,x_2)+r_tx_1D_1G(t,x_1,x_2)+\frac{1}{2}c_t^2x_1^2D_1^2G(t,x_1,x_2)\\&+\mathcal{D}G(t,x_1,x_2)=r_tG(t,x_1,x_2),\\&G(T,x_1,x_2)=(x_2-K)_+.\end{aligned}$$

In terms of the function $\phi(t,x)$, *the PIDE can be written as*

$$\begin{aligned}&\frac{\partial}{\partial t}\phi(t,x)+\left(\frac{1}{T}-r_tx\right)\frac{\partial}{\partial x}\phi(t,x)+\frac{c_t^2x^2}{2}\frac{\partial^2}{\partial x^2}\phi(t,x)+r_t\phi(t,x)\\&+\int_{-\infty}^{\infty}\left((1+y)\left(\phi\left(t,\frac{x}{1+y}\right)-\phi(t,x)\right)+yx\frac{\partial}{\partial x}\phi(t,x)\right)\tilde{\nu}_t(\mathrm{d}y)=0,\\&\phi(T,x)=x_+.\end{aligned}$$

Let $G(t,x_1,x_2)=\left(\frac{B_t}{B_T}\right)x_1\phi\left(t,\frac{x_2-K}{x_1}\right)$ *be the price function of the asian option. Then, by Theorem3, the hedging portfolio in terms of the power-jump assets is given by*

$$\begin{aligned}\alpha_t&=B_T^{-1}\left(S_{t-}\phi\left(t,\frac{U_{t-}}{S_{t-}}\right)-S_{t-}\phi\left(t,\frac{U_{t-}}{S_{t-}}\right)+U_{t-}\frac{\partial}{\partial x}\phi\left(t,\frac{U_{t-}}{S_{t-}}\right)\right)\\\beta_t^{(1)}&=\frac{B_t}{B_T}\left[\phi\left(t,\frac{U_{t-}}{S_{t-}}\right)-\left(\frac{U_{t-}}{S_{t-}}\right)\frac{\partial}{\partial x}\phi\left(t,\frac{U_{t-}}{S_{t-}}\right)\right]\\\beta_t^{(i)}&=\frac{B_t}{B_T}\left[\frac{S_{t-}^i\left.\frac{\partial^i}{\partial x_1^i}\left(x_1\phi\left(t,\frac{U_t}{x_1}\right)\right)\right|_{x_1=S_{t-}}}{i!}\right],\qquad i=2,3,\ldots.\end{aligned}$$

On the other hand, using Theorem 7, the portfolio in the power-jump assets $Y^{(i)}$, $i \geq 2$, is equivalent to the following portfolio in terms of discounted call options and discounted stocks:

$$\sum_{i=2}^{\infty}\int_0^t \beta_s^{(i)}\mathrm{d}Y_s^{(i)} = \int_0^{\infty}\int_0^t \frac{B_s}{S_{s-}^2}R(s,K)\mathrm{d}\tilde{C}_s(K)\mathrm{d}K - \int_0^t \frac{B_s^2 U_{s-}\frac{\partial}{\partial x}\phi\left(t,\frac{U_{t-}}{S_{t-}}\right)}{B_T}\mathrm{d}\tilde{S}_s,$$

where $R(t,K)$ is given by (33) and

$$\begin{aligned} h(t,S_{t-},V_{t-},y) &= \left(\frac{B_t}{B_T}\right)S_{t-}(1+y)\left(\phi\left(t,\frac{U_{t-}}{S_{t-}(1+y)}\right)-\phi\left(t,\frac{U_{t-}}{S_{t-}}\right)\right) \\ &\quad + \left(\frac{B_t}{B_T}\right)yU_{t-}\frac{\partial}{\partial x}\phi\left(t,\frac{U_{t-}}{S_{t-}}\right). \end{aligned}$$

6 Optimal portfolios

Let us now consider the problem of portfolio optimization in a Lévy market driven by the Lévy process $Z = \{Z_t, t \in [0,T]\}$ with Lévy triplet (γ, c^2, ν). In order to simplify the notation, we shall consider that the riskless asset has the simple dynamics

$$B_t = e^{rt},$$

where r is a constant.

Let $\mathbb{Q}$ be an equivalent martingale measure which is structure preserving. We would like to have a characterization of *structure preserving,* $\mathbb{P}$-equivalent martingale measures $\mathbb{Q}$, under which Z remains a natural additive process and the discounted price process $\tilde{S} = \{\tilde{S}_t = \frac{S_t}{B_t}, 0 \leq t \leq T\}$ is an $\{\mathcal{F}_t\}$-martingale.

We have the following result (see Theorems 33.1 and 3.32 in [21])

Theorem 15 *Let $Z = \{Z_t, 0 \leq t \leq T\}$ be a Lévy process with Lévy triplet (γ, c^2, ν) under some probability measure $\mathbb{Q}$. Then, the following conditions are equivalent:*

(a) There is a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$, such that Z is a $\mathbb{Q}$-Lévy process with triplet $(\widetilde{\gamma}, \widetilde{c}^2, \widetilde{\nu})$.

(b) We have for all $t \in [0,T]$:

(i) $\tilde{\nu}(\mathrm{d}x) = H(x)\nu(\mathrm{d}x)$ *for some Borel function* $H : \mathbb{R} \to \mathbb{R}^+$

(ii) $\widetilde{\gamma} = \gamma + \int_{-\infty}^{+\infty} x\mathbf{1}_{\{|x|<1\}}(H(x)-1)\nu(\mathrm{d}x) + Gc^2$ *for some real number G.*

(iii) $\widetilde{c} = c$.

(iv) $\int_{-\infty}^{+\infty}\left(1-\sqrt{H(x)}\right)^2\nu(\mathrm{d}x) < \infty$.

If the previous equivalent assumptions hold, then the density process $\xi := \left\{\xi_t = \frac{\mathrm{d}\mathbb{Q}_t}{\mathrm{d}\mathbb{P}_t}\right\}_{t\in[0,T]}$ is given by

$$\begin{aligned} \xi_t = \exp\Bigg(& cGW_t - \frac{1}{2}c^2G^2t + \int_{-\infty}^{+\infty}\log H(x)\left(Q((0,t],\mathrm{d}x) - t\nu(\mathrm{d}x)\right) \\ & - t\int_{-\infty}^{+\infty}(H(x)-1-\log H(x))\nu(\mathrm{d}x)\Bigg) \end{aligned} \tag{52}$$

with $\mathbb{E}_{\mathbb{P}}[\xi_t] = 1$.

The previous theorem implies that the process $\widetilde{W} = \{\widetilde{W}_t, 0 \leq t \leq T\}$ defined by

$$\widetilde{W}_t = W_t - cGt \tag{53}$$

is a Brownian motion under $\mathbb{Q}$ and the process X is a Lévy process with Doob-Meyer decomposition (with respect to $\mathbb{Q}$)

$$X_t = \widetilde{L}_t + t\left(a + \int_{-\infty}^{+\infty} x(H(x)-1)\nu(\mathrm{d}x)\right), \tag{54}$$

where $\widetilde{L} = \{\widetilde{L}_t\}_{t\in[0,T]}$ is a $\mathbb{Q}$-martingale.

The discounted price process $\tilde{S}$ can be written as

$$\begin{aligned}\tilde{S}_t = S_0 \exp\left(c\tilde{W}_t + \tilde{L}_t + t\left(a - r + c^2 G - \frac{c^2}{2}\right)\right) \\ \times \exp\left(t\int_{-\infty}^{+\infty} x(H(x)-1)\nu(\mathrm{d}x)\right) \prod_{0<s\leq t}(1+\Delta\tilde{L}_s)\exp(-\Delta\tilde{L}_s).\end{aligned}$$

The process

$$\exp\left(c\tilde{W}_t + \tilde{L}_t - \frac{c^2 t}{2}\right) \prod_{0<s\leq t}(1+\Delta\tilde{L}_s)\exp(-\Delta\tilde{L}_s)$$

is a martingale (see Proposition 2.1 in [8]). Then, a necessary and sufficient condition for $\tilde{S}$ to be a $\mathbb{Q}$-martingale is the existence of G and $H(x)$, for which the process ξ is a positive martingale and such that

$$c^2 G + a - r + \int_{-\infty}^{+\infty} x(H(x)-1)\nu(\mathrm{d}x) = 0. \tag{55}$$

Hence, by (53), (54) and (55), we have

$$Z_t = c\tilde{W}_t + \tilde{L}_t + rt$$

and therefore $Z_t - rt$ is a $\mathbb{Q}$-martingale. Moreover, the dynamics of $\widetilde{S}$ under $\mathbb{Q}$ is given by

$$\widetilde{S}_t = S_0 \exp\left(c\tilde{W}_t + \tilde{L}_t - \frac{c^2 t}{2}\right) \prod_{0<s\leq t}(1+\Delta\tilde{L}_s)\exp(-\Delta\tilde{L}_s) \tag{56}$$

or

$$\begin{aligned}\mathrm{d}\widetilde{S}_t &= c_t\widetilde{S}_{t-}\mathrm{d}\tilde{W}_t + \widetilde{S}_{t-}\mathrm{d}\tilde{L}_t \\ &= c_t\widetilde{S}_{t-}\mathrm{d}\tilde{W}_t + \widetilde{S}_{t-}\int_{-\infty}^{+\infty} xM(\mathrm{d}t,\mathrm{d}x) - \widetilde{S}_{t-}\int_{-\infty}^{+\infty} x(H(x)-1)\nu(\mathrm{d}x)\mathrm{d}t.\end{aligned}$$

We now describe the portfolio optimization problem. Given an initial wealth w_0 and an utility function U we want to find the optimal terminal wealth $\mathcal{W}_T$, that is, the value of $\mathcal{W}_T$ that maximizes $\mathbb{E}_{\mathbb{P}}(U(\mathcal{W}_T))$. Let $\mathbb{Q}$ be an equivalent martingale measure which is structure preserving and let us assume that the random variable $\mathcal{W}_T \in L^1(\Omega, \mathcal{F}_T, \mathbb{Q})$ can be strongly replicated in our enlarged market $M_{\mathbb{Q}}$ (see section 4 in [8]). Then the initial wealth is given by $w_0 = \mathbb{E}_{\mathbb{Q}}\left(\frac{\mathcal{W}_T}{B_T}\right)$. Therefore, we will consider the optimization problem

$$\underset{\mathcal{W}_T \in L^1(\mathbb{Q})}{\arg\max} \left\{ \mathbb{E}_{\mathbb{P}}(U(\mathcal{W}_T)) : \mathbb{E}_{\mathbb{Q}}\left(\frac{\mathcal{W}_T}{B_T}\right) = w_0 \right\}.$$

The Lagrangian for this optimization problem is given by

$$\mathbb{E}_{\mathbb{P}}(U(\mathcal{W}_T)) - \lambda \mathbb{E}_{\mathbb{Q}}\left(\frac{\mathcal{W}_T}{B_T} - w_0\right) = \mathbb{E}_{\mathbb{P}}\left(U(\mathcal{W}_T) - \lambda_T\left(\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}_T}\frac{\mathcal{W}_T}{B_T} - w_0\right)\right).$$

Then, the optimal terminal wealth is

$$\mathcal{W}_T = (U')^{-1}\left(\frac{\lambda_T}{B_T}\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}_T}\right),$$

where λ_T is the solution of the equation

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{B_T}(U')^{-1}\left(\frac{\lambda_T}{B_T}\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}_T}\right)\right] = w_0. \tag{57}$$

In [8] (pages 296-297) it is shown that

$$\mathcal{W}_T = (U')^{-1}\left(m(T)\, S_T^G e^{V_T}\right),$$

where

$$m(t) := \frac{\lambda_t}{B_t} S_0^{-G} \exp\left(-\frac{1}{2}G^2c^2t - G\left(a + b - \frac{c^2}{2}\right)t \right.$$
$$\left. + t\int_{-\infty}^{+\infty} ((\log H(x) - G\log(1+x))H(x) - H(x) + 1 + Gx)\,\nu(\mathrm{d}x)\right)$$

and

$$V_t = \int_{-\infty}^{+\infty} g(x)\tilde{M}((0,t],\mathrm{d}x),$$

with

$$g(x) := \log H(x) - G\log(1+x).$$

The real number G and the positive Borel function H must satisfy the equation (55).

In order to replicate the optimal wealth W_T, we need to know its price process

$$E_{\mathbb{Q}}\left[\frac{B_t}{B_T}\mathcal{W}_T|\mathcal{F}_t\right] = E_{\mathbb{Q}}\left[\frac{B_t}{B_T}(U')^{-1}\left(\frac{\lambda_T}{B_T}\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}_T}\right)|\mathcal{F}_t\right]$$

and this depends on the utility function considered. Suppose that the utility function satisfies

$$(U')^{-1}(xy) = a_1(x)(U')^{-1}(y) + a_2(x), \text{ for any } x, y \in \mathbb{R},$$

for certain C^∞ functions $a_1(x), a_2(x)$. Then (see [8], page 298)

$$E_Q\left[\frac{B_t}{B_T}\mathcal{W}_T|\mathcal{F}_t\right] = \phi(t,T)\mathcal{W}_t + \chi(t,T),$$

for certain deterministic functions $\phi(t,T)$ and $\chi(t,T)$. On the other hand

$$(U')^{-1}(xy) = a_1(x)(U')^{-1}(y) + a_2(x), \text{ for any } x,y \in \mathbb{R},$$

for certain C^∞ functions $a_1(x), a_2(x)$ if and only if $\frac{U'(x)}{U''(x)} = ax+b$, for any $x \in \text{dom}(U)$ and for some constants $a,b \in \mathbb{R}$ (see Lemma 4.1. in [8]). These kind of utility functions include the logarithm and the HARA utilities as particular cases. So, if $\frac{U'(x)}{U''(x)} = ax+b$ then $E_Q\left[\frac{B_t}{B_T}\mathcal{W}_T|\mathcal{F}_t\right] = \phi(t,T)\mathcal{W}_t + \chi(t,T)$ and we are able to characterize the equivalent martingale measures $\mathbb{Q}$ such that the wealth dynamics associated to the "jump" part of the original Lévy process vanishes in such a way that the optimization problem is solved using only portfolios of bonds and stocks (see [8] for a rigorous presentation of this topic).

Example 16 *Consider the logarithm utility function $U_l(x) := \log x$. Then, by solving (57), we have*

$$\mathcal{W}_T = w_0 B_T \frac{\mathrm{d}\mathbb{P}_T}{\mathrm{d}\mathbb{Q}_T} = \left(m(T)S_T^G e^{V_T}\right)^{-1}.$$

Moreover, the price function of $\mathcal{W}_T$ at time t is given by

$$\begin{aligned} F(t,S_t,V_t) &= \mathbb{E}_{\mathbb{Q}}\left[\frac{B_t}{B_T}\mathcal{W}_T|\mathcal{F}_t\right] = w_0 B_t \mathbb{E}_{\mathbb{Q}}\left[\frac{\mathrm{d}P_T}{\mathrm{d}\mathbb{Q}_T}|\mathcal{F}_t\right] = w_0 B_t \frac{\mathrm{d}\mathbb{P}_t}{\mathrm{d}\mathbb{Q}_t} \\ &= \left(m(t)S_t^G e^{V_t}\right)^{-1} := \mathcal{W}_t \end{aligned}$$

It follows from Theorem 3 that the fraction of optimal wealth invested in stocks, at time t, is constant and is given by

$$\frac{\beta_t S_{t-}}{\mathcal{W}_{t-}} = -G$$

and the number of power-jump assets in the optimal portfolio, at time t, is

$$\beta_t^{(i)} = \frac{\mathcal{W}_{t-}}{i!}\frac{\partial^i}{\partial y^i}\left(\frac{1}{H(y)}\right)\Bigg|_{y=0}, \; i = 2,3,...$$

If we require the optimal portfolio to involve only bonds and stocks, we should consider an equivalent martingale measure $\mathbb{Q}$ such that

$$H(y) = \frac{1}{1-Gy}.$$

and where G satisfies the equation

$$c^2 G + a + b - r + G\int_{-\infty}^{+\infty} \frac{x^2}{1-Gx}\nu(\mathrm{d}x) = 0.$$

On the other and, if we consider an equivalent martingale measure $\mathbb{Q}$ such that $\frac{\partial^i}{\partial y^i}\left(\frac{1}{H(y)}\right)\Big|_{y=0} \neq 0$ for infinitely many values of i then, in order to obtain *the optimal portfolio, we can apply our hedging formulas given in Theorem 7. Hence, the optimal portfolio is given by*

$$\begin{aligned}
\alpha_t &= B_t^{-1}\left[\mathcal{W}_{t-} + \frac{G}{S_{t-}}\mathcal{W}_{t-} + \mathcal{W}_{t-}\left(\frac{1}{H(-1)} - 1 - G\right)\right] \\
&\quad - B_t^{-1}\int_0^\infty \frac{R(t,K)}{S_{t-}^2} C_{t-}(K)\mathrm{d}K \\
\beta_t &= -\frac{G}{S_{t-}}\mathcal{W}_{t-} - \frac{\mathcal{W}_{t-}}{S_{t-}}\left(\frac{1}{H(-1)} - 1 - G\right) \\
\beta_t^{(K)} &= \frac{R(t,K)}{S_{t-}^2},
\end{aligned}$$

where β_t^K gives the number of call options with strike K at instant t and

$$R(t,K) := \sum_{k=2}^{\infty} \frac{\mathcal{W}_{t-} \frac{\partial^{k-1}}{\partial y^{k-1}}\left(\frac{1}{H(y)}\right)\Big|_{y=-1}}{(k-2)!\varphi^{(k)}(t,T)} \left(\frac{K}{S_{t-}}\right)^{k-2}.$$

Example 17 *If we now consider the HARA utilities $U(x) = \frac{x^{1-p}}{1-p}$ with $p \in \mathbb{R}_+ \setminus \{0,1\}$ and apply Theorem 3, we obtain the optimal dynamic portfolio*

$$\begin{aligned}
\beta_t &= -\frac{G\mathcal{W}_{t-}}{pS_{t-}} \\
\beta_t^{(i)} &= \frac{\mathcal{W}_{t-}}{i!}\frac{\partial^i}{\partial y^i}\left(H(y)^{-\frac{1}{p}}\right)\Big|_{y=0}, \ i = 2,3,...
\end{aligned}$$

and we will have an optimal portfolio based in bonds and stocks (with no power-jump assets) if and only if

$$H(y) = \frac{1}{\left(1 - \frac{G}{p}y\right)^p},$$

where G satisfies

$$c^2G + a + b - r + \int_{-\infty}^{\infty} x\left(\frac{1}{\left(1 - \frac{G}{p}x\right)^p} - 1\right)\nu(\mathrm{d}x) = 0.$$

If $\mathbb{Q}$ is an equivalent martingale measure $\mathbb{Q}$ such that $\frac{\partial^i}{\partial y^i}\left(H(y)^{-\frac{1}{p}}\right)\Big|_{y=0} \neq 0$ for infinitely many values of i then, by the hedging formulas in Theorem 7, we obtain the optimal portfolio

$$\begin{aligned}
\alpha_t &= B_t^{-1}\left[\mathcal{W}_{t-} + \frac{G}{pS_{t-}}\mathcal{W}_{t-}^{\frac{1}{p}} + \mathcal{W}_{t-}\left((H(-1))^{-\frac{1}{p}} - 1 - \frac{G}{p}\right)\right] \\
&\quad - B_t^{-1}\int_0^\infty \frac{R(t,K)}{S_{t-}^2} C_{t-}(K)\mathrm{d}K \\
\beta_t &= -\frac{G}{pS_{t-}}\mathcal{W}_{t-}^{\frac{1}{p}} - \frac{\mathcal{W}_{t-}^{\frac{1}{p}}}{S_{t-}}\left((H(-1))^{-\frac{1}{p}} - 1 - \frac{G}{p}\right) \\
\beta_t^{(K)} &= \frac{R(t,K)}{S_{t-}^2},
\end{aligned}$$

where

$$R(t,K) := \sum_{k=2}^{\infty} \frac{\mathcal{W}_{t-}\ \frac{\partial^{k-1}}{\partial y^{k-1}} \left(H(y)\right)^{-1/p}\Big|_{y=-1}}{(k-2)!\varphi^{(k)}(t,T)} \left(\frac{K}{S_{t-}}\right)^{k-2}.$$

Similar results on the optimization problem can be obtained if we consider an additive market model, where the stock price satisfies the stochastic differential equation (1). This problem is studied in detail in [7].

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