Quasi-analytical solution of an investment problem with decreasing investment cost due to technological innovations

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Abstract

In this paper we address, in the context of real options, an investment problem with two sources of uncertainty: the price (reflected in the revenue of the firm) and the level of technology. The level of technology impacts in the investment cost, that decreases when there is a technology innovation. The price follows a geometric Brownian motion, whereas the technology innovations are driven by a Poisson process. As a consequence, the investment region may be attained in a continuous way (due to an increase of the price) or in a discontinuous way (due to a sudden decrease of the investment cost).

For this problem no analytical solution is known, and therefore we propose a quasi-analytical method

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to find an approximated solution that preserves the qualitative features of the exact solution. This method is based on a truncation procedure. We prove that the truncated solution converges to the solution of the original problem.

We provide results of the comparative statics for the investment thresholds. These results show interesting behaviors. In particular the investment may be postponed or anticipated with the intensity of the technology innovations and with the impact on the investment cost.

1 Introduction

The optimal time to undertake an investment opportunity has been an important research question for both economists and mathematicians, mainly since the pioneering works of Dixit and Pindyck [6] and Trigeorgis [30]. Over time, the models to solve these problems have become more complex, since researchers and practitioners intend to represent the economic reality in a more realistic way. As a consequence, both the number of sequential decisions and the number of sources of uncertainty in these models have increased.

One particular aspect that has been under the spots of real options literature is the impact of technology innovations, that may lead to significant changes in the revenue and costs of the companies. In the past few years, given a large number of breakthroughs innovations, some industries have seen their investment costs decrease over time. Therefore, nowadays companies understand better the value of technology innovations (Guney et al. [14]), as they have been realizing that such innovations may create incentives for an early investment due to the lower costs. Due to the importance of these innovations to companies and governments, this has been broadly reported in the media (some examples are presented in the next paragraphs) and discussed in the economic literature (see for instance Flor and Hansen [10] and Murto [23]).

For example, in the renewable energy sector we see a large impact of falling costs. In December 2018, the International Renewable Energy Agency (IRENA) mentioned that the solar photovoltaic module prices have fallen by around 80% since 2009, and the wind turbine prices have fallen around 30-40% [1]. In the same line, Hanno Schoktlish, CEO of Kaiserwetter argues that “the decreases in the cost of renewable energy... has

[1] The report entitled *Renewable Power: Climate-Safe Energy Competes on Cost Alone* is available at https://www.irena.org/-/media/Files/IRENA/Agency/Publication/2018/Dec/IRENA_COP24_costs_update_2018.pdf.

occurred for several reasons. These include technological improvements..."[2]. Also Hunt and Shuttleworth [17] report about decreasing investment cost in energy investment. According to these authors, as a result of studies sponsored by space programs, it was possible to build turbines much more efficient and smaller than before, reducing in a drastic way the optimal power plant size, with enormous cost reduction. Aside from technological innovations, other factors may lead to a sudden decrease in the investment cost. For instance, as a result of government interventions in key areas of the economy (as the renewable energies), as Deng et al. [20] analyze.

A recent global survey of IT and finance leaders by independent research firm Vanson Bourne showed that in manufacturing, 42% of the CEO's of 900 companies said that they have already reduced operational costs through innovation[3]. This impact is particularly important in high-tech companies, where the progress in technology takes advantage of other industries, as it is the case of pharmaceutical companies. For example, advances in technology related to biomarkers, as well as developments in the understanding of the human genome, have changed the cost structure for firms developing products targeting small patient populations[4]. Still in the health sector, research studies (see for instance Lee and Choi [19]) have demonstrated that investing in health IT in a hospital setting has potential benefits, that impact in the reduction of the cost, by increasing efficiency and productivity metrics.

We consider an investment model with two sources of uncertainty: the price (reflected in the revenue of the firm) and the level of technology, which impacts in the investment cost. The firm needs to optimize its investment decision by taking into account the random fluctuations of the revenue and the changing investment cost. Problems with two sources of uncertainty have been studied in the real options literature, as one can see, for example, in Alghalith [1], Dixit and Pindyck [6], McDonald and Siegel [22], Murto [23], Pennings and Sereno [29], and Zambujal-Oliveira and Duque [31], among others.

According to Bethuyne [2], *technology can improve the output of equipment while costs remain unchanged, or it can reduce costs, while leaving production unaffected.* In this paper, we assume that technology innova-

[2] Available at https://es.kaiserwetter.energy/speakers-corner/single-view/production-cost-of-renewables-undercuts-fossil-fuel-energy-nuclear-power-for-the-first-time/

[3] Available at https://www.industryweek.com/leadership/why-it-so-hard-invest-technology

[4] Available at https://www.nber.org/reporter/2018number3/garthwaite.html

tions impact the investment cost and not the price. Additionally, the innovation process (assumed exogenous to the company), is driven by a Poisson process. Then, the investment cost decreases with technology innovation by means of decrease jumps. Furthermore, the size of the downward jump in the investment cost is known beforehand. These assumptions are in common with most literature about real options models for technology innovations (see for instance, Farzin et al. [9], Doraszelski [7] and Hagspiel at al. [16]).

The discontinuity of the cost process can be observed for instance in the following example. A firm that develops apps or games for mobile devices often needs to buy a large amount of smartphones. The purchase of smartphones is an investment cost for the firm. For example, in the Apple's case, when a new generation of the iPhone is launched, the price of the current generation jumps down[5]. This means that the technology innovation has an impact in the price, and consequently in the firm's investment cost. Moreover, between consecutive launches of new versions, the price usually stays constant.

This type of assumption in the dynamics of the investment cost is also considered in other papers. For instance, Mauer and Ott [21] make similar assumption regarding the impact of technology progress in the optimal replacement policy, when a technological breakthrough lowers the initial maintenance and operation cost. The authors also assume that such breakthroughs follow a Poisson process with constant intensity. Cheevaprawatdomrong and Smith [3], also in the context of replacement problems, assume a simpler model, as they consider that *the costs of equipment acquisition as well as maintenance and operating costs drop by a constant factor after each time period.*

Our framework is very close to the one assumed by Murto [23], since we both study the timing of investment under effects of technological and revenue uncertainty. Additionally, we both assume that the revenue stream generated by the investment follows a geometric Brownian motion (GBM), and that the technology progress follows a Poisson process.

In his paper, Murto [23] states that, although the problem is well-posed, there is no analytical solution to the Hamilton-Jacobi-Bellman (HJB) equation that characterizes the value function. The reason of this difficulty lays in the fact that the stopping region may be attained continuously (due to an increase in the price) or discontinuously (as consequence of a downward jump in the investment cost). Then Murto [23]

[5]Available at https://www.macworld.co.uk/feature/iphone/best-time-buy-iphone-3656359/.

proposes an analytical solution only in the following particular cases: either the price process is deterministic (meaning that he assumes that there is no volatility in the GBM); or the technological progress is deterministic (leading to an exponential decline in the investment cost). Therefore, in the cases he studied, instead of having a problem with two sources of uncertainty (that would lead to an exercise boundary and not to a point), he transforms it in a problem with just one source of uncertainty, where the classic tools (including verification theorems) may be used.

A similar model, where an analytical solution is obtained when there are two sources of uncertainty, can be found in Nunes and Pimentel [24]. In this paper, the authors consider that both the revenue and the costs are jump-diffusion processes, where the jumps in the revenue are downward jumps and the jumps in the investment cost are upwards. The direction of these jumps is such that, contrary to the case that we analyze in the current paper, the stopping region is always attained through a continuous movement. This, combined with the fact that the value function is homogeneous of degree one (and therefore one may consider a change of variable, as proposed in Dixit and Pindyck [6]), leads to an optimal stopping time problem where an analytical solution can be found. There are other cases, outside of the context of real options, where we can reduce a two dimensional optimal stopping problem to one dimensional one. For instance, the optimal stopping of Bessel processes with integer index (Dubins et al. [8]) and the one-asset-for-other problem of the right to exchange one asset by another, used mostly in the context of American options (Gerber et Shiu [11]). We refer to Christensen et al. [4] for an updated review on optimal stopping of multidimensional diffusions.

Our contribution to the state of the art is to propose an approximation for the value function, and for the prices and levels of technology that trigger the investment decision. The approximation proposed in this paper is based on a truncation approach, and, for that reason, we call the approximated model by truncated problem. The truncated method was first addressed, in the field of real options, by Guerra et al. [12]. Using the results found for this approximation, we are able to provide insights about the original investment problem. Furthermore, we prove that (i) the thresholds of the truncated problem converge to the thresholds of the original model, and (ii) the solution of this problem converges to the solution of the original problem. The approach considered in our paper differs from other papers that also deal with the numerical analysis of an investment problem with two sources of uncertainty. We cite, for instance, Lang et al. [18], and references therein. Our approach relies on a probabilistic framework rather than a numerical scheme to solve differential equations, which allows us to gain an important economic intuition about the behavior of the model's solution. Besides the advantage of the interpretation of the results for the truncated problem, we are able to provide the solution and to prove analytically its optimality, using a suitable verification theorem, for a specific case. We also present a numerical illustration that highlights the behavior of the truncated value function.

Furthermore, we are also able to provide formal proofs regarding the behavior of the investment thresholds with some parameters. Indeed, we show that the classical results of real options regarding the impact of the drift and volatility of the price hold in our case (namely, that the investment threshold increases with the volatility but decreases with the drift). But when one considers other parameters, in particular the ones related with the technology innovation process, the results are not standard, since they show non-monotonicity.

The method proposed in this paper can be useful for other problems with the same features. For instance, it may be used to analyze the problem addressed by Nunes and Pimentel [24], but assuming now that jumps may also lead directly to an investment decision. In the same line, it can be used in the context of the problem addressed by Couto et al. [5], where it is assumed that the processes that model the uncertainty is a jump-diffusion process. This method can also be applied to the problem presented in Hagspiel et al. [15], where, similar to our case, the stopping region may be attained by a jump, in a discontinuous way.

The paper is organized as follows: in Section 2 we define the investment model, in Section 3 we present a truncated problem and we prove the convergence of its solution to the solution of the original one. In Section 4 we derive the solution to the truncated problem for a specific case, and we show a numerical illustration of the results. In Section 5 we present the comparative statics for the investment threshold with respect to the relevant parameters and in Section 6 the main conclusions of the paper are presented. Finally, there are three appendices: in the first one we provide the general expressions of the truncated problem, the second is where we present all the proofs and in third we provide some numerical results.

2 Problem set-up

In this paper we consider a monopolistic firm that has the option to make a singular and irreversible investment in a certain market producing a single good. We assume that the price of this product, P, evolves randomly in time according to the following geometric Brownian motion:

$$dP_t = \mu P_t dt + \sigma P_t dW_t, \quad \text{with } P_0 = p > 0,$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$.

The investment cost depends on the level of technology on the market: the higher the technology level, the lower the investment cost. The level of technology evolves according to a point process, due to technology innovations. The sequence of times between consecutive innovations is a sequence of independent and identically exponentially distributed random variables, with parameter $\lambda > 0$. Therefore, if we let $N = \{N_t : t \geq 0\}$, with $N_0 = n$ and N_t representing the number of technology innovations occurred until time t, then N is a Poisson process with rate λ. Moreover, we assume that the process N is independent of the process P.

In our framework, the technological progress impacts the costs of the firm but does not affect explicitly the output (see for instance a discussion about this topic provided by Bethuyne [2]). Therefore, the investment process $I = \{I_t : t \geq 0\}$ is intrinsically related with the process N by:

$$I_t = I\phi^{N_t}, \quad \text{with} \quad I_0 = I\phi^n,$$

where I_t represents the investment cost at time t and $\phi \in]0,1[$. Therefore, each time there is an innovation, the investment cost decreases by a deterministic factor ϕ. Although this constant discount factor (that is also considered by Murto [23]) can be seen as a less realistic assumption, such a simplification results in a model that can be mathematically tractable, that allows us to get analytical and qualitative results.

Assuming that the current levels of price and technology are, respectively, $p > 0$ and $n \in \mathbb{N}_0$, the value of the firm that undertakes the investment opportunity at time τ is given by the functional

$$J(p,n,\tau) = E_{p,n}\left[\int_\tau^{+\infty} e^{-rs} P_s ds - e^{-r\tau} I_\tau\right],$$

where $r > 0$ represents the instantaneous interest rate. Here, $E_{p,n}[\cdot]$ represents the expected value conditional to the information $P_0 = p$ and $N_0 = n$. We assume that $r > \mu$ in order to ensure that $E_{p,n}\left[\int_0^{+\infty} e^{-rs} P_s ds\right] < +\infty$. Additionally, using the strong Markov property of the GBM, and taking into account that

$$E_{p,n}\left[\int_0^{+\infty} e^{-rs} P_s ds\right] = \frac{p}{r-\mu}$$

it follows that

$$J(p,n,\tau) = E_{p,n}\left[e^{-r\tau}\left(\frac{P_\tau}{r-\mu} - I_\tau\right)\right] \equiv E_{p,n}\left[e^{-r\tau} g(P_\tau, N_\tau)\right], \tag{1}$$

with $g(p,n) = \frac{p}{r-\mu} - I\phi^n$. Therefore, throughout this paper, we intend to find the optimal investment time τ^* that maximizes the functional J. Equivalently, we intend to find the value function

$$V(p,n) = \sup_{\tau \geq 0} J(p,n,\tau). \tag{2}$$

Using standard arguments (see, for example, Oksendal and Sulem [27]), the value function V must satisfy the following HJB equations:

$$\min\{rv(p,n) - \mu p v'(p,n) - \frac{\sigma^2}{2} p^2 v''(p,n) - \lambda\left(v(p,n+1) - v(p,n)\right), v(p,n) - g(p,n)\} = 0 \tag{3}$$

for almost every $(p,n) \in \mathbb{R}^+ \times \mathbb{N}_0$, with v' and v'' being, respectively, the first and second derivatives of v w.r.t. p.

We note that the first term in the HJB equation (3) allows us to obtain the value function in the continuation region, whereas the second term, $v(p,n) - g(p,n) = 0$, gives us the perpetual value of investment. Moreover, the term $\lambda(v(p,n+1) - v(p,n))$ can be interpreted as follows: in an infinitesimal period of time, a new technology innovation will occur with probability λdt, and the firm will gain $v(p,n+1) - v(p,n)$ by waiting this infinitesimal period of time.

Trivial financial arguments lead us to guess that the optimal strategy is to invest for high levels of price and high levels of technology (and, consequently, lower investment cost). We also expect that, given a certain level of technology, n, the price that triggers the investment decision, p_n^*, should be larger than the threshold price, $p_{\tilde{n}}^*$, corresponding to a higher level of technology $(\tilde{n} > n)$[6]. Therefore, we expect an investment region of the form

$$S = \{(p,n) \in \mathbb{R}^+ \times \mathbb{N}_0 : p \geq p_n^*\}, \tag{4}$$

[6] $p_n^\star$ is indeed a function of n, but we use the subscript instead of $p^\star(n)$ to short the notation.

and an optimal investment time given by

$$\tau^* = \inf\{t \geq 0 \,:\, P_t \geq p^*_{N_t}\}, \tag{5}$$

where $n \mapsto p^*_n$ is a decreasing function of the technology level, here denoted by n. This means that the smaller the technological level, the larger the investment cost and thus, the price needs to be larger to trigger the investment decision. This is in accordance with the classical results in real options. For instance, Dixit and Pindyck [6], in Chapter 6, Section 5, present a model with price and cost following two independent GBM's, where they conclude that investment will occur with a price sufficiently high for a given investment cost, or an investment cost sufficiently low for a given price.

Proposition 1 *The investment region associated to the investment problem* (2) *is given by* (4) *and the optimal investment time is given by* (5)*. Additionally, the function $n \mapsto p^\star_n$ is decreasing.*

Regarding the value function in the continuation region, we need to solve the left-hand side of the HJB equation, meaning that we need to find the solution to the following differential-difference equation

$$(\lambda + r)v(p, n) - \mu p v'(p, n) - \frac{\sigma^2}{2} p^2 v''(p, n) = \lambda v(p, n + 1).$$

The non-homogeneous part of the previous equation, $\lambda v(p, n+1)$, may be itself solution of a similar equation, in case $(p, n+1)$ is in the continuation region, or may be equal to $g(p, n)$, in case it is in the stopping region. This leads to a difficulty: the solution in the state (p, n) depends on the solution of the same equation in the state $(p, n + 1)$. We note that similar difficulty can be found in Murto [23].

For a given p and n in the continuation region, after a jump in the technological process, one of the following situations hold: (1) $(p, n + 1) \notin \mathcal{S}$, and, consequently, the decision is to continue postponing the investment or (2) $(p, n + 1) \in \mathcal{S}$, which means that it is optimal to invest immediately after the jump (and thus $v(p, n + 1) = g(p, n + 1)$). This means that a jump may lead directly to the investment region. As a consequence, the value of the firm at state (p, n) depends recursively on all the levels above $(n+1, n+2, \cdots)$.

In an attempt to solve such problem, Murto [23] reduces the dimension of the stochastic process, leading to different HJB equations, with the new equations depend only on a single variable: instead of (p, n) as state variable, he uses $\frac{p}{I\phi^n}$. This strategy was already proposed by Dixit and Pindyck [6]; Nunes and Pimentel

[24] also use the same idea to solve an investment problem with two sources of uncertainty, as well as Couto et al. [5]. But contrary to these works (as in both cases the investment region may be reached only by the diffusion component and not by a jump), this technique is not successful to solve the current problem, as the new equation has exactly the same difficulty in the continuation region.

In the next section we present a different strategy, which allows us to approximate the solution of the problem. Additionally, this technique is efficient to study the qualitative behavior of the solution when there are some changes in the parameters that characterize the uncertainty of the market.

3 The truncated problem

In this section, we introduce the truncated problem. By truncated problem we mean that we optimize the functional J, defined in (1), imposing an additional constraint. In this case, we assume that the stopping time is bounded by a random time.

To solve the truncated optimal stopping problem, we present a suitable verification theorem establishing the conditions that the solution to the system of HJB equations has to meet. Additionally, we prove that both the value function and the thresholds for the truncated problem converges to the ones of the original problem. Through Section 4 and Appendix A, we provide the value function, and we prove its optimality.

For a given $\overline{n} \in \mathbb{N}$, we let $\tau_{\overline{n}} = \inf\{t \geq 0 \ : N_t \geq \overline{n}\}$ be the random time representing the first moment the Poisson process is, at least, $\overline{n}$. Then, we define the following problem:

$$V^{\overline{n}}(p,n) = \sup_{0 \leq \tau \leq \tau_{\overline{n}}} J(p,n,\tau), \tag{6}$$

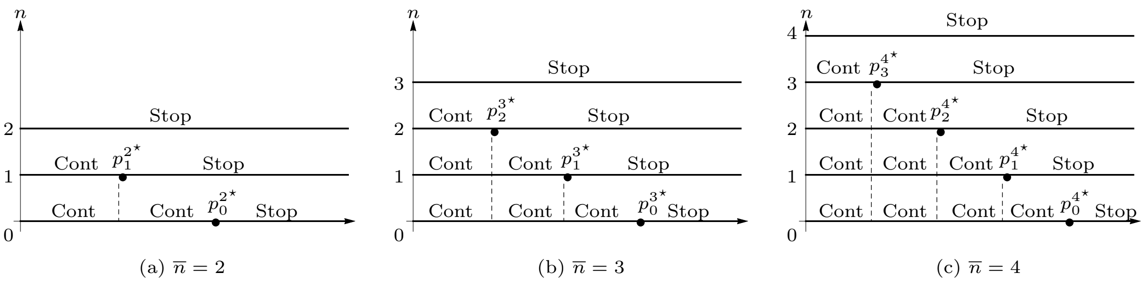
which is a *truncated* version of the one defined in (2) (in which the time horizon is infinite)[7].

We notice that in this formulation, if no decision is taken until the moment $\tau_{\overline{n}}$, then this time will be the optimal stopping time. When $\overline{n} \to +\infty$, then it follows trivially that $\tau_{\overline{n}} \nearrow +\infty$ and, as we show later, the solution of the truncated problem also converges to the solution of the original problem.

[7]We note that other truncations can also be defined. In this case we are truncating the time; we could, for instance, bound the number of jumps of the technology innovation process. In that case, we would end up with an absorbing state, meaning that the time would still be evolving but the state would remain fixed after attaining such (absorbing) state.

We notice that, *a priori*, the truncated problem does not have an economic interpretation as the original problem has. Indeed, using this truncation, when the state $\bar{n}$ is attained, the optimal decision is always to stop. If we kept the economic interpretation given by the original problem, this would mean that the investment could occur when the net present value is negative. Nevertheless, we will prove that the solution of the truncated problem converges to the solution of the original problem, which means that, at least for sufficiently large values of $\bar{n}$, the former can give us insights about the behavior of the latter.

In Figure 1 we illustrate our guess to the optimal strategy for the truncated problem, when $\overline{n} = 2, 3, 4$. In this figure we use the following notation: $p_n^{\overline{n}^\star}$ is the trigger value for the decision, when the maximum number of jumps is $\overline{n}$, and $n < \overline{n}$ jumps have already occurred. Due to the restriction on the stopping time, once the Poisson process reaches the boundary value $\overline{n}$, the optimal decision is to stop the processes. This means that by construction $p_{\overline{n}}^{\overline{n}^\star} = 0$. Moreover, as already discussed for the original case, assuming a fixed level $\bar{n}$, $p_n^{\overline{n}^\star}$ decreases with n.



(a) $\overline{n} = 2$ (b) $\overline{n} = 3$ (c) $\overline{n} = 4$

Figure 1: Continuing (Cont) and stopping (Stop) regions for different values of $\overline{n}$.

As we formally state below, for a fixed level $\overline{n}$, the value function $V^{\overline{n}}(p, n)$ is a solution of the following HJB equations:

$$\min\{(r+\lambda)v^{\overline{n}}(p,n) - \mu p(v^{\overline{n}})'(p,n) - \frac{\sigma^2}{2}p^2(v^{\overline{n}})''(p,n) - \lambda v^{\overline{n}}(p,n+1), v^{\overline{n}}(p,n) - g(p,n)\} = 0, \qquad (7)$$

with $n = 0, 1, 2, \cdots, \overline{n} - 1$. This amounts to solve a system of $\overline{n}$ HJB equations, with $v^{\overline{n}}(p, \overline{n}) = g(p, \overline{n})$, and $v^{\overline{n}}(p, n)$ being $C^2(]0, +\infty[\setminus\{p_0^{\bar{n}^*}, \cdots, p_{n-1}^{\bar{n}^*}\}) \cap C^1(\{p_0^{\bar{n}^*}, \cdots, p_{n-1}^{\bar{n}^*}\})$ (as a consequence of the usual smooth-fit conditions).

For each $\overline{n}$, we need to solve the following set of $\overline{n}$ ordinary differential equations:

$$(r+\lambda)v^{\overline{n}}(p,n)-\mu p(v^{\overline{n}})'(p,n)-\frac{\sigma^2}{2}p^2(v^{\overline{n}})''(p,n)=\lambda v^{\overline{n}}(p,n+1). \tag{8}$$

Note that Equation (8) needs to be solved using a backwards scheme. We want to find $v^{\overline{n}}(p,n)$, assuming that $v^{\overline{n}}(p,n+1)$ is already known. For instance, considering $\overline{n}=2$, by construction we have that $v^2(p,2)\equiv g(p,2)$, for $p>0$. Having $v^2(p,2)$, we solve (8) to find $v^2(p,1)$ in the continuation region. In this region we denote $v^2(p,1)$ by $v_0^2(p,1)$. This means that $v^2(p,1)$ is then as follows:

$$v^2(p,1)=\begin{cases} v_0^2(p,1), & 0<p<p_1^{2^\star} \\ g(p,1), & p\geq p_1^{2^\star} \end{cases}.$$

A similar argument can now be used to find $v^2(p,0)$ in the continuation region, for which we need $v^2(p,1)$, for $0<p<p_0^{2^\star}$. In this case, as Figure 1 illustrates, the expression for $v^2(p,1)$ depends on the interval of values of p that we are considering. There are two of such intervals, $(0,p_1^{2^\star})$ and $[p_1^{2^\star},p_0^{2^\star})$, and thus in the continuation region $v^2(p,1)$ is defined by two branches. Then (8) needs to be solved for each one of them. Therefore, the function $v^2(p,0)$ is defined as follows:

$$v^2(p,0)=\begin{cases} v_0^2(p,0), & 0<p<p_1^{2^\star} \\ v_1^2(p,0), & p_1^{2^\star}\leq p<p_0^{2^\star} \\ g(p,0), & p\geq p_0^{2^\star} \end{cases}.$$

Note that, since we are looking for a function $v^2(p,0)$ that is at least C^1, besides the usual value matching and smooth pasting conditions for the decision threshold $p_0^{2^\star}$, we also need to guarantee the same for $p_1^{2^\star}$.

Following a similar reasoning, for a general $\overline{n}$ and for a level n, the function $v^{\overline{n}}(p,n)$ is defined as follows:

$$v^{\overline{n}}(p,n) = \begin{cases} v_0^{\overline{n}}(p,n), & 0 < p < p_{\overline{n}-1}^{\overline{n}^\star} \\ v_1^{\overline{n}}(p,n), & p_{\overline{n}-1}^{\overline{n}^\star} \leq p < p_{\overline{n}-2}^{\overline{n}^\star} \\ v_2^{\overline{n}}(p,n), & p_{\overline{n}-2}^{\overline{n}^\star} \leq p < p_{\overline{n}-3}^{\overline{n}^\star} \\ \dots & \\ v_{\overline{n}-1-n}^{\overline{n}}(p,n), & p_{n+1}^{\overline{n}^\star} \leq p < p_{n}^{\overline{n}^\star} \\ g(p,n), & p \geq p_{n}^{\overline{n}^\star} \end{cases}.$$

A representation of the domain of the function $v^{\overline{n}}(p,n)$ is presented in Figure 2.

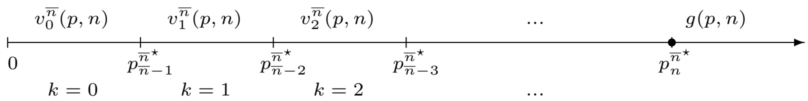


Figure 2: Representation of the domain of the function $v^{\overline{n}}(p,n)$.

One may notice that the function $v^{\overline{n}}(p,n)$ has different expressions in different intervals. This is because, in each interval of the type $[p_{\bar{n}-k}^{\bar{n}^*}, p_{\bar{n}-k-1}^{\bar{n}^*})$, $\bar{n}-n-k$ jumps are necessary to attain the stopping region.

As depicted in Figure 1, $\overline{n}$ represents the maximum number of jumps and n is the number of jumps already occurred. Additionally, k gives us information about the interval where the function $v_k^{\overline{n}}(p,n)$ is defined.

As in the previous case, we intend to find $v^{\overline{n}}(p,n)$ that is at least C^1, besides the usual value matching and smooth pasting conditions for the decision threshold $p_n^{\overline{n}^\star}$, we also need to guarantee the same conditions for all the thresholds from the levels above, summing in total $\overline{n}-n$ points.

The heuristic arguments used above to explain the idea behind the construction of the value function are usually formulated in a verification theorem. The set-up that we consider for this truncated problem can be seen as a particular case of the one presented in Oliveira and Perkowski [28] as in their case, the state process is represented by a continuous diffusion and a Markov chain. In our case, the Markov chain has a special structure. Indeed, given the truncation level $\overline{n}$, the Poisson process turns into a Markov chain with $\overline{n}+1$ states, and with non-zero transition probabilities only from state i to state $i+1$, with $i \in \{0,1,\dots,\overline{n}\}$. In view of this comment, we may then state a simpler version of the verification theorem presented in the mentioned work.

Theorem 1 *Let $D \subset \mathbb{R} \times \mathbb{N}$ be a finite set of points such that $v^{\bar{n}}(p,n)$ is C^2 in $\mathbb{R} \times \mathbb{N} \setminus D$ and C^1 in D. If $v^{\bar{n}}$ satisfies the system of HJB equations* (7) *and $\lim_{p \to 0^+} v^{\bar{n}}(p,n) < +\infty$, then the value function $V^{\bar{n}}$ defined in* (6) *verifies*

$$v^{\bar{n}}(p,n) = V^{\bar{n}}(p,n)$$

and the optimal stopping time is defined as $\tau^*_{\bar{n}} = \inf\{t \geq 0 : V^{\bar{n}}(P_t, N_t) = g(P_t, N_t)\}$.

Finally, we end this section with the following relevant result, that shows that the solution to the truncated optimal stopping problem (6) converges to the solution of the optimal stopping problem (2)[8].

Proposition 2 *The solution of the truncated problem* (6) *converges to the solution of the original one* (2)*, i.e.,*

$$\lim_{\overline{n} \to \infty} V^{\overline{n}}(p,n) = V(p,n) \qquad \forall (p,n) \in \mathbb{R}^+ \times \mathbb{N}_0.$$

Additionally, the sequence $\{p_n^{\bar{n}^*}\}_{\bar{n} \in \mathbb{N}}$ is such that

$$p_n^{\bar{n}^*} \to p_n^*$$

when $\bar{n} \to +\infty$, for all $n \in \mathbb{N}$.

In view of this result, we know that for sufficient large values of $\overline{n}$, the solution that we get for the truncated problem is arbitrarily close to the solution of the original problem. In particular, for sufficient large values of $\overline{n}$, the price thresholds $p_n^{\overline{n}^*}$ should be quite close to the price threshold $p_n^{(\overline{n}+1)^*}$. We notice that, by "sufficient large values of $\overline{n}$" we mean that $\overline{n}$ should be sufficiently away from the initial level technology n.

4 Approximated solution

Since we have already defined the truncated problem and presented a suitable Verification Theorem, in this section we will construct for a specific case the value function of the truncated problem and we will prove

[8] An alternative to our approach would be to show that the solution to the HJB equations associated to the truncated problem converges to the solution to the HJB equations associated to the original problem. Then we would need to use a suitable verification theorem to the original problem.

its optimality in light of Theorem 1.

As explained in the previous section, to define $v^{\overline{n}}(p,n)$, for each level n, we need to find $\overline{n}-n$ different particular solutions to the equation (8). The homogeneous differential equation associated to (8) is a Cauchy-Euler equation, whose solution is a function of the type $Ap^{d_1}+Bp^{d_2}$, with $A,B\in\mathbb{R}$, and d_1 and d_2 being the roots of the characteristic polynomial $\frac{\sigma^2}{2}d(d-1)+\mu d-(r+\lambda)$:

$$d_1=\frac{\left(\frac{\sigma^2}{2}-\mu\right)+\sqrt{\left(\frac{\sigma^2}{2}-\mu\right)^2+2\sigma^2(r+\lambda)}}{\sigma^2}>1 \text{ and } d_2=\frac{\left(\frac{\sigma^2}{2}-\mu\right)-\sqrt{\left(\frac{\sigma^2}{2}-\mu\right)^2+2\sigma^2(r+\lambda)}}{\sigma^2}<0.$$

Therefore, all branches of $v^{\overline{n}}(p,n)$ in the continuation region are given by the sum of $Ap^{d_1}+Bp^{d_2}$ with the particular solution, where A and B have to be found using the value matching and smooth pasting conditions (which implies, trivially, that for each branch we will have different expressions for these A and B). To guarantee the finiteness of the function, we have to set $B=0$ when $0<p<p_{n-1}^{\overline{n}^\star}$.

The particular solution of (8) may not be trivial to get, because in most cases $v^{\overline{n}}(p,n+1)$ is itself solution of a similar differential equation. This means that for $n<\bar{n}-1$, the non-homogeneous part of (8) is also the solution of a Cauchy-Euler equation, with the same powers d_1 and d_2 as the homogeneous solution. Therefore, the particular solution to (8) will include logarithmic terms. Consequently, at level n we have to solve $\overline{n}-n-1$ equations of the following type:

$$ap^2v''(p,n)+bpv'(p,n)+cv(p,n)=Dp^\alpha(\ln p)^m,$$

for $b,c\in\mathbb{R}$, $a,D,\alpha\in\mathbb{R}\setminus\{0\}$ and $m\in\mathbb{N}_0$. This is the setting of Nunes et al. [25], and we refer to this paper for the construction of the particular solution of the differential equation (8).

In Section 4.1, we present the solution for the truncated problem when $\overline{n}=2$. This case explains and motivates the general case. The solution of the truncated problem for an arbitrary $\overline{n}$ can be found in Appendix A, as the expressions of the value function are quite evolved. Moreover, in Appendix A, Proposition 5 states the optimality of the candidate function proposed to be the value function. Its proof involves many auxiliary results and can be found in Appendix B.5.

4.1 The case $\overline{n} = 2$

When we fix $\overline{n}$, the value function at level $\overline{n}$ is, by construction, given by $g(p, \overline{n})$, i.e. $v^{\overline{n}}(p, \overline{n}) = g(p, \overline{n})$. Additionally, the differential equation used to find $v^{\overline{n}}(p, \overline{n}-1)$ is very similar to the one considered in a standard investment problem. Then, we are left with the derivation of $v^{\overline{n}}(p, i)$ for $i = 0, 1, \ldots, \overline{n}-2$.

For $\overline{n} = 2$, it follows that

$$v^2(p,2) = \frac{p}{r-\mu} - I\phi^2, \quad \text{for } p > 0 \tag{9}$$

$$v^2(p,1) = \begin{cases} A^2_{1,0,0} p^{d_1} + \lambda \left[\frac{p}{(r-\mu)(r+\lambda-\mu)} - \frac{I\phi^2}{r+\lambda} \right], & 0 < p < p_1^{2^\star} \\ \frac{p}{r-\mu} - I\phi, & p \geq p_1^{2^\star} \end{cases} \tag{10}$$

where (9) is the value of the investment in perpetuity, and (10) is the value function of a standard investment problem. In particular, the usual value matching and smooth pasting conditions lead to

$$p_1^{2^\star} = \frac{d_2 - 1}{d_2} I\phi \left[r + \lambda(1-\phi) \right], \tag{11}$$

$$A^2_{1,0,0} = \frac{(p_1^{2^\star})^{1-d_1}}{d_1(r+\lambda-\mu)}. \tag{12}$$

where, as previously introduced, $p_n^{\overline{n}^\star}$ is the investment threshold where the maximum number of technology innovations is $\overline{n}$ (corresponding to the truncated problem) and n is the number of innovations already occurred.

Once we find $v^2(p,1)$, we are now in position to derive $v^2(p,0)$. Note that when $n = 0$ and $\overline{n} = 2$, we need to take into account three possible regions for p: for $0 < p < p_1^{2^\star}$, we know that we will not invest for sure even if the next jump occurs; when $p_1^{2^\star} \leq p < p_0^{2^\star}$, then we will invest surely after the next jump; finally, for $p \geq p_0^{2^\star}$ we invest right away (see Figure 1). Using this reasoning and the results derived in Nunes et al. [25] to solve the differential-difference equations corresponding to the continuation region, we end up with the following result:

$$v^2(p,0) = \begin{cases} \left(A^2_{0,0,0} + A^2_{0,0,1} \ln p \right) p^{d_1} + \lambda^2 \left[\frac{p}{(r-\mu)(r+\lambda-\mu)^2} - \frac{I\phi^2}{(r+\lambda)^2} \right], & 0 < p < p_1^{2^\star} \\ A^2_{0,1,0} p^{d_1} + B^2_{0,1,0} p^{d_2} + \lambda \left[\frac{p}{(r-\mu)(r+\lambda-\mu)} - \frac{I\phi}{r+\lambda} \right], & p_1^{2^\star} \leq p < p_0^{2^\star} \\ \frac{p}{r-\mu} - I, & p \geq p_0^{2^\star}. \end{cases} \tag{13}$$

As previously stated, the truncation method that we propose here is driven by mathematical arguments. Nevertheless, in view of the convergence result presented in Proposition 2, it is still interesting to give an economic interpretation to the value function obtained in the truncated case, as we can understand the different movements in the processes that may lead to the stopping region and how these influence the value function.

The interpretation of (13) is as follows: first we note that the value function for the case $0 < p < p_1^{2^\star}$ can be re-written as

$$\left(A^2_{0,0,0} + A^2_{0,0,1} \ln p\right) p^{d_1} + \left(\frac{\lambda}{r+\lambda}\right)^2 \left[\frac{p(r+\lambda)^2}{(r-\mu)(r+\lambda-\mu)^2} - I\phi^2\right],$$

where the first part accounts for the fact that the investment may occur due to an increase of the price. The term involving $\ln p$ is related with the value of the option when a jump in the technology level happens but the price is not large enough to trigger the investment, and, therefore, we stay in the continuation region. The second part is the perpetual value of the investment undertaken right after the two technology innovations take place. The term $\frac{\lambda}{r+\lambda}$ is the stochastic discount factor under a Poisson process and the investment cost in this case is $I\phi^2$, as we need to wait for two jumps, meaning that the investment cost is reduced by a factor of ϕ^2.

The meaning of the value function when $p_1^{2^\star} \leq p < p_0^{2^\star}$ is similar: firstly, the term $\lambda \left[\frac{p}{(r-\mu)(r+\lambda-\mu)} - \frac{I\phi}{r+\lambda}\right]$ represents the value of investment after one innovation, and, secondly, the expression $A^2_{0,1,0}p^{d_1} + B^2_{0,1,0}p^{d_2}$ represents the value of continuing (waiting). In fact, the investment may occur because (i) the price increases and becomes greater than $p_0^{2^\star}$ or (ii) the price goes down for levels lower than $p_1^{2^\star}$, which explains the factor associated with the negative root of the quadratic equation. In that case the investment will only occur after two jumps.

Finally, for $p \geq p_0^{2^\star}$, the investment takes place, by construction of the truncated problem.

In order to derive the expression for the constant term $A^2_{0,0,1}$, we use the method of undetermined coefficients, leading to the following:

$$A^2_{0,0,1} = -2\frac{\lambda A^2_{1,0,0}}{\sigma^2(d_1 - d_2)}. \tag{14}$$

For the rest of the terms, we use value matching and smooth pasting conditions, which results in:

$$A^2_{0,0,0} = A^2_{0,1,0} + \frac{(p_1^{2^*})^{-d_1}}{d_1}\left[d_2 B^2_{0,1,0}(p_1^{2^*})^{d_2} - A^2_{0,0,1}\left[1 + d_1 \ln p_1^{2^*}\right](p_1^{2^*})^{d_1} + \frac{\lambda p_1^{2^*}}{(r+\lambda-\mu)^2}\right] \tag{15}$$

$$A^2_{0,1,0} = \frac{(p_0^{2^*})^{-d_1}}{d_1 - d_2}\left[(1-d_2)\frac{p_0^{2^*}}{r+\lambda-\mu} + d_2 I \frac{r+\lambda(1-\phi)}{r+\lambda}\right] \tag{16}$$

$$B^2_{0,1,0} = \frac{(p_0^{2^*})^{-d_2}}{d_1 - d_2}\left[(d_1-1)\frac{p_0^{2^*}}{r+\lambda-\mu} - d_1 I \frac{r+\lambda(1-\phi)}{r+\lambda}\right]. \tag{17}$$

The threshold level $p_0^{2^*}$ is the unique solution of the following equation, that satisfies $p_0^{2^*} > p_1^{2^*}$, as we prove in Lemma 1 in Appendix B.

$$(d_1 - d_2)B^2_{0,1,0}(p_1^{2^*})^{d_2} + A^2_{0,0,1}(p_1^{2^*})^{d_1} + \lambda\left[\frac{(d_1-1)p_1^{2^*}}{(r+\lambda-\mu)^2} - \frac{d_1 I\phi(r+\lambda(1-\phi))}{(r+\lambda)^2}\right] = 0.^9 \tag{18}$$

In view of these results, we have the following proposition regarding the value function and the optimal strategy. Its proof can be found in Appendix B.4.

Proposition 3 *Consider the truncated optimal stopping problem defined by* (6), *when* $\overline{n} = 2$. *Then, the value function,* V^2, *is such that, for each* $n = 0, 1, 2$, $V^2(p,n) = v^2(p,n)$, *for* $p > 0$, *with* $v^2(p,n)$ *defined by* (9), (10) *and* (13), *and the parameters* $A^2_{1,0,0}$, $A^2_{0,0,1}$, $A^2_{0,0,0}$, $A^2_{0,1,0}$ *and* $B^2_{0,1,0}$ *are given by* (12) *and* (14)-(17), *respectively. The threshold* $p_1^{2^*}$ *is given by the expression* (11) *and* $p_0^{2^*}$ *is the unique solution to the equation* (18) *verifying* $p_0^{2^*} > p_1^{2^*}$. *Additionally, the stopping region is* $\{(p,n) \in \mathbb{R}^+ \times \mathbb{N} : p \geq p_n^{2^*} \vee n \geq 2\}$ *and the optimal stopping time is* $\tau_2^* = \inf\{t \geq 0 : P_t \geq p_{N_t}^{2^*}\} \wedge \tau_2$, *where* τ_2 *is defined in Section 3* [10].

4.2 Numerical illustration

The quasi-analytical method proposed in this paper to solve the investment problem (2) is good enough if the approximated solution converges to the solution of the original optimal investment problem (which is true in light of Proposition 2, at least for $\overline{n}$ large enough). In practical terms, we want also to know the value $\overline{n}$ that, for each case, provides a sufficiently close approximation. Throughout this section we illustrate numerically that we can get a desired accuracy with acceptable values of $\overline{n}$, in the sense that it does not need to be very far from the initial value n.

[9] Note that $B_{0,1,0}$ is indeed a function of $p_0^{2^*}$.

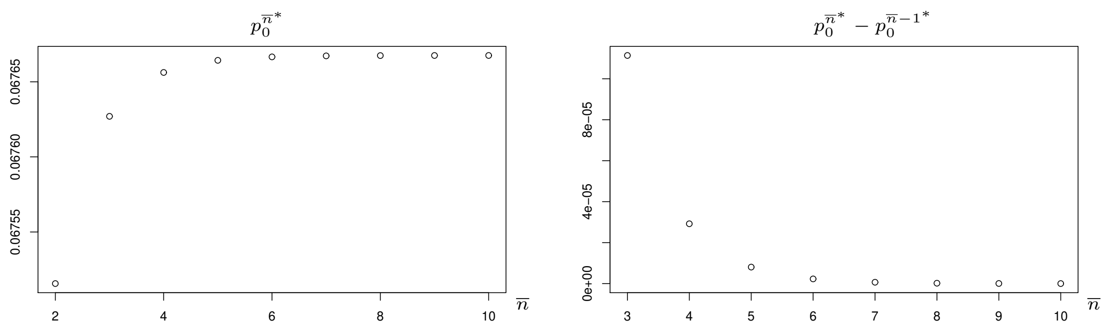
[10] The symbols $\vee$ and $\wedge$ are defined as follows: $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$.

Due to the Markovian nature of the technology innovation process and the fact that the investment cost decreases at a common ratio, ϕ, with the increase of the innovations, we may show that

$$E_{p,n}\left[e^{-r\tau}\left(\frac{P_\tau}{r-\mu}-I\phi^{N_\tau}\right)\right]=E_{p,0}\left[e^{-r\tau}\left(\frac{P_\tau}{r-\mu}-\tilde{I}\phi^{N_\tau}\right)\right],$$

where $\tilde{I}=I\phi^n$. Therefore, $V(p,n;I)=V(p,0;\tilde{I})$ and, consequently, the numerical results that we show for the case $n=0$ (in particular for $v^{\overline{n}}(p,0)$ and $p_0^{\overline{n}^*}$) play a fundamental role in our discussion about the convergence results of the truncated problem.

Henceforward, we consider the set of parameters $r=0.05, \sigma=0.1, \mu=0.03, \lambda=0.1, \phi=0.9$ and $I=1$[11] in order to proceed with our numerical illustration[12].



(a) Plot of $p_0^{\overline{n}^*}$, for $\overline{n}=2,\cdots,10$.

(b) Plot of $p_0^{\overline{n}^*}-p_0^{\overline{n}-1^*}$, for $\overline{n}=3,\cdots,10$.

Figure 3: Convergence of the thresholds.

In Figure 3 (a) we show that $p_0^{\overline{n}^*}$ is converging to a particular value that is approximately 0.06766755 according to Table 1. Additionally, from Proposition 2, we know that $\lim_{\overline{n}\to+\infty} p_0^{\overline{n}^*}=p_0^*$, where p_0^* is the threshold of the original model. Combining the arguments above, we may conclude that $p_0^*\simeq 0.06766755$. Furthermore, Figure 3 (b) depicts the speed of this convergence. Indeed, the function $p_0^{\overline{n}^*}-p_0^{\overline{n}-1^*}$ decreases with increasing $\overline{n}$, being almost 0 for $\overline{n}\geq 6$.

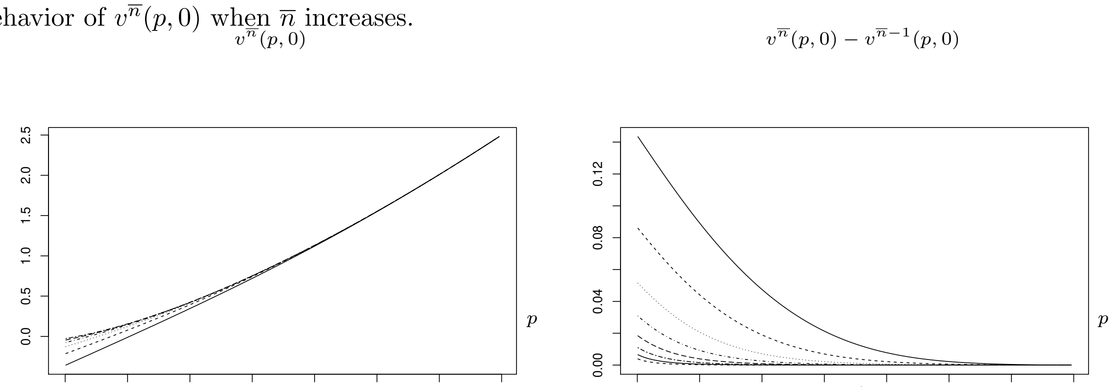
In Table 1 we present the investment thresholds $p_n^{\overline{n}^*}$ for $\overline{n}=2,\ldots,10$ and $n=0,\ldots,\overline{n}-1$. From this

[11]This set of values is also used as a baseline in the comparative statistics section.

[12]The expressions used to implement the numerical method can be found in Appendix A. Moreover, a sketch of the numerical implementation is available from the authors.

numerical illustration we conclude that for a fixed value n, the investment threshold $p_n^{\overline{n}^*}$ converges very fast with increasing $\overline{n}$. This suggests that one may get a good approximation of the solution of the original problem without the need to consider very large values of $\overline{n}$, which would imply a costly computation. We also present in Table 2 the differences $p_n^{\overline{n}^*} - p_n^{\overline{n}-1^*}$. From this table, one can easily conclude that the consecutive differences converge to zero in a very fast way. Moreover, the behavior is similar for all the initial values, n, considered in the table.

Although the thresholds, $p_n^{\overline{n}^\star}$, converge quite fast for a particular value $p_n^\star$, that is the threshold of the original model, the speed of the convergence of the function should also be verified. In Figure 4 we depict the behavior of $v^{\overline{n}}(p,0)$ when $\overline{n}$ increases.



(a) Plot of $v^{\overline{n}}(p,0)$, for $\overline{n} = 2, \cdots, 7$. (b) Plot of $v^{\overline{n}}(p,0) - v^{\overline{n}-1}(p,0)$, for $\overline{n} = 3, \cdots, 10$.

Figure 4: Convergence of the value functions. In (a) the functions appear in an increasing way in $\overline{n}$, whereas in (b) they appear in a decreasing way.

In Figure 4 (a) we can see that the value functions $v^{\overline{n}}(p,0)$ with $\overline{n} = 5,6,7$ are close to each others. From Figure 4 (b) we conclude that the difference between the value functions for two consecutive values of $\overline{n}$ decreases significantly with increasing $\overline{n}$.

Finally, just for illustration purposes, we present in Figure 5 the value function considering $\overline{n} = 30$, mostly when p belongs to the continuation region. We note that, for this level, the value function is, as expected, always positive, which means that, for this parameters we can consider that $V(p,0) \approx v^{30}(p,0)$.

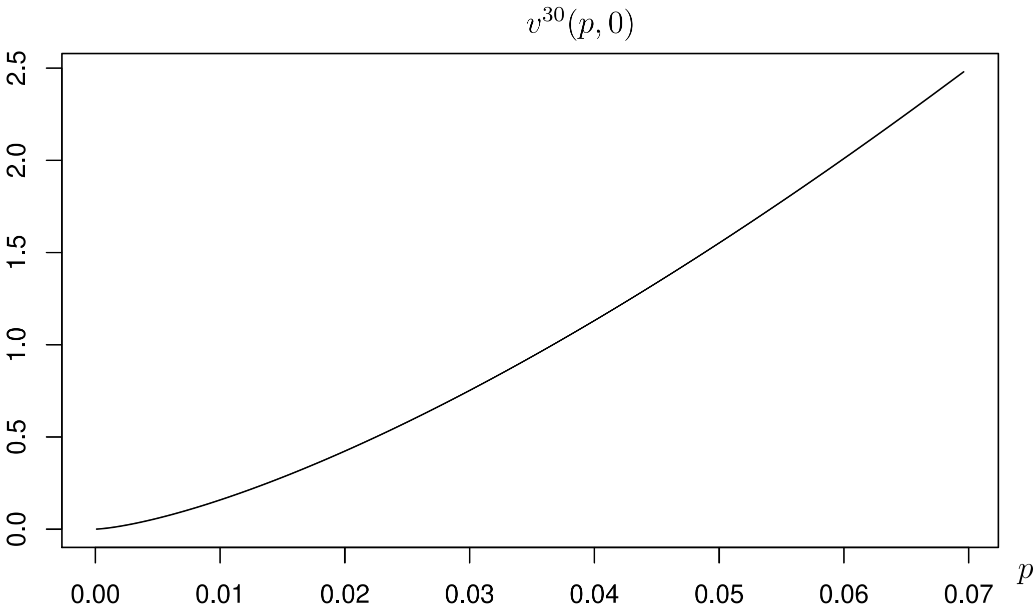


Figure 5: Plot of the function $v^{30}(p, 0)$.

5 Comparative Statics

In this section we provide some insights into the behavior of the investment thresholds with the parameters that influence each one of the uncertainties. It follows from the convergence of the truncated problem to the original one that the behavior of the thresholds is the same, independently of the particular $\overline{n}$ and n considered. As we can obtain a closed expression for $p_1^{2^\star}$ (defined in Equation (11)), we study analytically the influence of σ, μ, ϕ and λ in the investment decision for $\overline{n} = 2$ and $n = 1$. The results are presented in the next proposition.

Proposition 4 *The investment threshold $p_1^{2^\star}$ is increasing with σ and decreasing with μ. For ϕ, the behavior depends on the relationship between other parameters, as follows: if $r \geq \lambda$, then it increases with ϕ; in case $r < \lambda$, it increases with ϕ if $\phi < \frac{1}{2}\frac{r+\lambda}{\lambda}$ and decreases afterwards. Finally, $p_1^{2^\star}$ has monotonic behavior with*

λ, for "small values" of ϕ, and a non-monotonic behavior, for "large values" of ϕ.

We note that the results that we obtain for σ and μ agree with the standard case: increasing the volatility usually postpones the investment decision, whereas increasing the drift anticipates it, as we expect larger profits in the future. But the results for the investment parameters, ϕ and λ, are somehow unexpected.

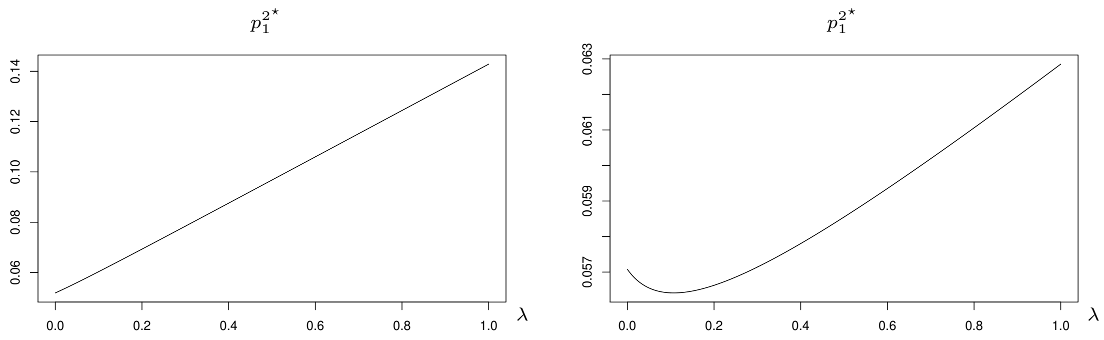
With increasing ϕ one may find two possible and opposite effects regarding the investment decision, which makes difficult *a priori* to predict if the investment threshold increases or decreases. On the one hand, when ϕ increases, the investment cost decreases less with innovation. Then the value of waiting for a new technological arrival should also decrease and, one might consider investing earlier than otherwise. On the other hand, for a fixed certain level of technology, increasing ϕ is equivalent to increase the investment cost which might result in a postponement of the investment (since the trigger price may increase). Accordingly, one may intuitively expect that the behavior of the trigger price should really depend on the arrival rate of a new technology. The result presented in Proposition 4 shows that, indeed, this is the case. When the intensity of new arrivals is larger than the interest rate and, simultaneously, ϕ is larger than $\frac{1}{2}\frac{r+\lambda}{\lambda}$, the investment decision is anticipated with increasing ϕ. Otherwise, the investment decision is postponed.

In fact, if the decrease in the investment cost is large (low ϕ) and the expected time until the next innovation is small (large λ), then the decision to invest should be postponed. This happens because as the expected time until the next arrival is small, the expected return of the investment during this period, if we would invest earlier, would be smaller than the decrease in the investment cost. But if the impact of the innovation is not so large, then it happens the other way around, i.e, it is more profitable for the company to invest earlier than to wait for the next innovation, which would lead to a small impact in the cost. Finally, we note that the function $\lambda \mapsto \frac{1}{2}\frac{r+\lambda}{\lambda}$ is decreasing. For $\lambda > r$, we have that

$$\lim_{\lambda \to r} \frac{1}{2}\frac{r+\lambda}{\lambda} = 1 \quad \text{and} \quad \lim_{\lambda \to +\infty} \frac{1}{2}\frac{r+\lambda}{\lambda} = \frac{1}{2},$$

which means that the investment decision is anticipated with increasing ϕ in an interval. In fact, the greater the λ, the larger the interval.

Regarding the influence of changing λ in the investment decision, it is possible to find two distinct behaviors, for different sets of parameters: (i) $p_1^{2^\star}$ is non-monotonic, in particular for large values of ϕ, which means low impact in the decrease of the investments costs due to technological innovations; (ii) $p_1^{2^\star}$ is monotonic for small values of ϕ. In Figure 5 we draw $p_1^{2^\star}$ as a function of λ for small and large values of ϕ, using as base-case parameters the ones mentioned in the previous section.



(a) For $\phi = 0.9$.

(b) For $\phi = 0.99$.

Figure 6: Investment threshold $p_1^{2^\star}$ as a function of λ.

These illustrations show that when the impact of technology innovations is significant in the reduction of the investment costs (corresponding to the case illustrated in the left hand panel of Figure 5), then the decision to invest should be postponed. This holds as increasing λ means that the expected time until next innovation decreases. Thus it is worthwhile to wait for a cheaper investment, because the value that could be accumulated during this period (if the investment would take place earlier) does not compensate the reduction in the investment cost. On the other hand, when the impact of the innovations in the investment cost is small (right hand panel of Figure 5) and the expected time until the next innovation is large (meaning small values of λ), there is no reason to wait for such innovation. Indeed, the firm expects to gain more starting producing than waiting for the cost reduction, therefore the decision to invest is anticipated. However, if the expected time until the next innovation is small (meaning large values of λ), then it pays back to invest latter, for similar reasons as the ones previously explained.

6 Conclusion

In this paper we propose a quasi-analytical method to solve an investment problem with two sources of uncertainty: the price, that follows a GBM, and the number of technology innovations, that is driven by a Poisson process.

The difficulty in the resolution of this investment problem comes from the fact that the investment region may be attained by an increase of either the level of technology available in the market or the price of the product. An interesting feature of the method developed in this paper is its flexibility and suitability to other problems with the same features.

The quasi-analytical method is based on the truncation of the stopping time. We prove that the approximated solution converges to the exact solution. Additionally, we illustrate that, from a numerical point of view, the convergence is quickly attained, which means that one can consider a reasonable level $\overline{n}$, in order to get a good approximation.

As a consequence of the convergence result, we know that the qualitative behavior of the threshold prices, $p_n^{\overline{n}^*}$, should be preserved for all $\overline{n} \in \mathbb{N}$ and $n < \overline{n}$, which allows us to provide an extensive sensitivity analysis. We prove that the standard results of real options still hold, notably the investment is postponed for increasing volatility and anticipated for increasing drift. Moreover, we find non-monotonic behaviors for the price threshold when one increases the impact of the innovation in the investment cost and the intensity of the technology innovations.

As possible research questions for future work, we mention two different directions. Firstly, one of the possible directions is to consider that the impact of technology innovations in the investment cost is not deterministic, but a random variable itself, following the comments of Murto [23]: "... The size of the investment cost reductions due to innovations would more realistically be random variables ..." Secondly, it would be interesting to consider a similar model, but modeling the innovation process as a GBM where the effect of technology on costs can possibly be described with a negative drift (see Bethuyne [2]).

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Appendix

A Approximated solution to the general case

In this section, we present the value function for the truncated optimal stopping problem (6) for $\overline{n} \in \mathbb{N}$. We use the same notation as before to denote the solution to the ODEs, $v^{\overline{n}}(p, n)$, and the thresholds price, $p_n^{\overline{n}^\star}$. We also borrow the arguments explained in the previous section to state that

$$v^{\overline{n}}(p, \overline{n}) = \frac{p}{r - \mu} - I\phi^{\overline{n}}, \quad \text{for all } p > 0 \equiv p_{\overline{n}}^{\overline{n}^\star},$$

which is the value function in the investment region. This accounts for the perpetual investment value when the maximum number of technologies innovations has occurred and therefore investment takes place right away.

In order to find $v^{\overline{n}}(p, n)$ in the continuation region and the corresponding threshold price, we need to follow a similar reasoning as the one presented for the case $\overline{n} = 2$. As before, this function is a solution of the ODE (8) and needs to be found backwards in n. But now the calculations are more cumbersome, as in order to find $v^{\overline{n}}(p, n)$ we have to study $\overline{n} - n$ regions in the continuation region (for instance, in the case $\overline{n} = 2$ and $n = 0$, we had already 2 different regions for p in the continuation region). The fact that we have the continuation region splitted in $\overline{n} - n + 1$ regions implies that we have also $\overline{n} - n$ different expressions for the value function. According to Verification Theorem 1, we are looking for a function $v^{\overline{n}}(p, n)$ that is C^2 almost everywhere and C^1 in a finite set of points. Therefore, at the thresholds, we have to check the usual smooth-fit conditions. We refer to Figure 1 for an illustration of this reasoning.

To take into account the different regions, we introduce further notation: we denote by $v_k^{\overline{n}}(p, n)$ the value function $v^{\overline{n}}(p, n)$ when $p_{\overline{n}-k}^{\overline{n}^\star} \leq p < p_{\overline{n}-1-k}^{\overline{n}^\star}$, with $k = 0, 1, 2, ..., \overline{n} - 1 - n$ (assuming that $p_{\bar{n}}^{\bar{n}^\star} \equiv 0$). Now we are in position to present the results for the value function:

$$v^{\overline{n}}(p,n)=\begin{cases} v_k^{\overline{n}}(p,n) & p_{\overline{n}-k}^{\overline{n}\star}\leq p<p_{\overline{n}-1-k}^{\overline{n}\star}, k=0,1,2,...,\overline{n}-1-n \\ \frac{p}{r-\mu}-I\phi^n & p\geq p_n^{\overline{n}\star}\end{cases}, \tag{19}$$

with $v_k^{\overline{n}}(p,n)$ given by:

$$\begin{aligned} v_k^{\overline{n}}(p,n) = & \sum_{j=0}^{\overline{n}-1-n-k}\left[A_{n,k,j}^{\overline{n}}\ (\ln p)^j\ p^{d_1}+B_{n,k,j}^{\overline{n}}\ (\ln p)^j\ p^{d_2}\right] \\ & +\lambda^{\overline{n}-n-k}\left[\frac{p}{(r-\mu)\,(r+\lambda-\mu)^{\overline{n}-n-k}}-\frac{I\phi^{\overline{n}-k}}{(r+\lambda)^{\overline{n}-n-k}}\right]. \end{aligned} \tag{20}$$

As in the case $\overline{n}=2$, this formula has a clear economical interpretation. The terms involving the parameters $A_{n,k,j}^{\bar{n}}$ and $B_{n,k,j}^{\bar{n}}$, for $j=0,1,\cdots,\bar{n}-1-n-k$, represent the option of investment due to a continuous movement after j jumps. The second term accounts for the perpetual value of investment due to $\bar{n}-n-k$ jumps, meaning, in particular, that the investment region is crossed due to the jumps. We note that $\bar{n}-n-k$ is the number of jumps in the technological innovation process that are needed in order to achieve the investment region, assuming that there are no movements in the price.

In order to completely define these functions, we use the following smooth-pasting conditions to calculate the constant terms and the thresholds,

$$v_k^{\overline{n}}(p_{\overline{n}-1-k}^{\overline{n}\star},n)=v_{k+1}^{\overline{n}}(p_{\overline{n}-1-k}^{\overline{n}\star},n)\quad\text{and}\quad(v_n^{\overline{n}})'(p_{\overline{n}-1-k}^{\overline{n}\star},n)=(v_{k+1}^{\overline{n}})'(p_{\overline{n}-1-k}^{\overline{n}\star},n),$$

for all $n=0,1,2,...,\overline{n}-1$ and $k=0,1,2,...,\overline{n}-n-1$. We start showing the expressions for the constants that are not multiplied by the logarithms.

$$\begin{aligned} A_{n,k,0}^{\overline{n}} = & A_{n,k+1,0}^{\overline{n}}+\frac{\left(p_{\overline{n}-1-k}^{\overline{n}\star}\right)^{-d_1}}{d_1-d_2}\left[\lambda^{\overline{n}-1-n-k}\left[(1-d_2)\,\frac{p_{\overline{n}-1-k}^{\overline{n}\star}}{(r+\lambda-\mu)^{\overline{n}-n-k}}+d_2I\phi^{\overline{n}-k-1}\ \frac{r+\lambda\,(1-\phi)}{(r+\lambda)^{\overline{n}-n-k}}\right]\right. \\ & +\left\{\left\{\sum_{j=1}^{\overline{n}-n-k-2}\left[\left(A_{n,k,j}^{\overline{n}}-A_{n,k+1,j}^{\overline{n}}\right)\left((d_2-d_1)\ln p_{\overline{n}-1-k}^{\overline{n}\star}-j\right)(\ln p_{\overline{n}-1-k}^{\overline{n}\star})^{j-1}(p_{\overline{n}-1-k}^{\overline{n}\star})^{d_1}\right.\right.\right. \\ & \left.\left.-\left(B_{n,k,j}^{\overline{n}}-B_{n,k+1,j}^{\overline{n}}\right)j(\ln p_{\overline{n}-1-k}^{\overline{n}\star})^{j-1}(p_{\overline{n}-1-k}^{\overline{n}\star})^{d_2}\right]\right\}\chi_{\{k\neq\overline{n}-n-2\}} \\ & +A_{n,k,\overline{n}-n-k-1}^{\overline{n}}\left((d_2-d_1)\ln p_{\overline{n}-1-k}^{\overline{n}\star}-(\overline{n}-n-k-1)\right)(\ln p_{\overline{n}-1-k}^{\overline{n}\star})^{\overline{n}-n-k-2}(p_{\overline{n}-1-k}^{\overline{n}\star})^{d_1} \\ & \left.\left.-(\overline{n}-n-k-1)B_{n,k,\overline{n}-n-k-1}^{\overline{n}}(\ln p_{\overline{n}-1-k}^{\overline{n}\star})^{\overline{n}-n-k-2}(p_{\overline{n}-1-k}^{\overline{n}\star})^{d_2}\right\}\chi_{\{k\neq\overline{n}-n-1\}}\right] \end{aligned} \tag{21}$$

and

$$
\begin{aligned}
B^{\overline{n}}_{n,k,0} = B^{\overline{n}}_{n,k+1,0} &+ \frac{\left(p^{\overline{n}\star}_{\overline{n}-1-k}\right)^{-d_2}}{d_1-d_2}\left[\lambda^{\overline{n}-1-n-k}\left[(d_1-1)\,\frac{p^{\overline{n}\star}_{\overline{n}-1-k}}{(r+\lambda-\mu)^{\overline{n}-n-k}} - d_1 I\phi^{\overline{n}-k-1}\,\frac{r+\lambda\,(1-\phi)}{(r+\lambda)^{\overline{n}-n-k}}\right]\right. \\
&+ \left\{\left\{\sum_{j=1}^{\overline{n}-n-k-2}\left[\left(B^{\overline{n}}_{n,k,j}-B^{\overline{n}}_{n,k+1,j}\right)\left((d_2-d_1)\ln p^{\overline{n}\star}_{\overline{n}-1-k}+j\right)(\ln p^{\overline{n}\star}_{\overline{n}-1-k})^{j-1}(p^{\overline{n}\star}_{\overline{n}-1-k})^{d_2}\right.\right.\right. \\
&+ \left.\left.\left(A^{\overline{n}}_{n,k,j}-A^{\overline{n}}_{n,k+1,j}\right) j(\ln p^{\overline{n}\star}_{\overline{n}-1-k})^{j-1}(p^{\overline{n}\star}_{\overline{n}-1-k})^{d_1}\right]\right\}\chi_{\{k\neq\overline{n}-2-n\}} \qquad (22) \\
&- B^{\overline{n}}_{n,k,\overline{n}-n-k-1}\left((d_1-d_2)(\ln p^{\overline{n}\star}_{\overline{n}-1-k})-(\overline{n}-n-k-1)\right)(\ln p^{\overline{n}\star}_{\overline{n}-1-k})^{\overline{n}-n-k-2}(p^{\overline{n}\star}_{\overline{n}-1-k})^{d_2} \\
&+ \left.\left.(\overline{n}-n-k-1)A^{\overline{n}}_{n,k,\overline{n}-n-k-1}(\ln p^{\overline{n}\star}_{\overline{n}-1-k})^{\overline{n}-n-k-2}(p^{\overline{n}\star}_{\overline{n}-1-k})^{d_1}\right\}\chi_{\{k<\overline{n}-n-1\}}\right],
\end{aligned}
$$

for $n=0,1,2,...,\overline{n}-2$ and $k=1,2,...,\overline{n}-1-n$, assuming that $A^{\overline{n}}_{n,\overline{n}-n,0}=0$ and $B^{\overline{n}}_{n,\overline{n}-n,0}=0$, and with χ_A denoting the indicator function of set A.

For $0\equiv p^{\overline{n}\star}_{\overline{n}} < p \leq p^{\overline{n}\star}_{\overline{n}-1}$ (which implies that $k=0$), the constants multiplied by p^{d_2} are all zero, i.e $B^{\bar{n}}_{n,0,j}=0$, for all $n=0,1,2,...,\overline{n}-1$ and $j=0,1,2,\cdots,\overline{n}-1-n$. In particular, $B^{\overline{n}}_{n,0,0}=0$, for $n=0,1,2,...,\overline{n}-1$. Furthermore, the constants multiplied by p^{d_1} take the form

$$
\begin{aligned}
A^{\overline{n}}_{n,0,0} =& A^{\overline{n}}_{n,1,0} + \frac{\left(p^{\overline{n}\star}_{\overline{n}-1}\right)^{-d_1}}{d_1}\left[\left\{\left\{\sum_{l=1}^{\overline{n}-2-n}\left[\left(A^{\bar{n}}_{n,1,l}-A^{\bar{n}}_{n,0,l}\right)\left[l+d_1\ln p^{\overline{n}\star}_{\overline{n}-1}\right]\left(\ln p^{\overline{n}\star}_{\overline{n}-1}\right)^{l-1}\left(p^{\overline{n}\star}_{\overline{n}-1}\right)^{d_1}\right.\right.\right. \\
&+ \left.\left.B^{\bar{n}}_{n,1,l}\left[l+d_2\ln p^{\bar{n}\star}_{\bar{n}-1}\right]\left(\ln p^{\bar{n}\star}_{\bar{n}-1}\right)^{l-1}\left(p^{\bar{n}\star}_{\bar{n}-1}\right)^{d_2}\right]\right\}\chi_{\{n\neq\bar{n}-2\}} + d_2 B^{\bar{n}}_{n,1,0}\left(p^{\bar{n}\star}_{\bar{n}-1}\right)^{d_2} \qquad (23) \\
&\left.\left.-A^{\bar{n}}_{n,0,\bar{n}-1-n}\left[(\overline{n}-1-n)+d_1\ln p^{\bar{n}\star}_{\bar{n}-1}\right]\left(\ln p^{\bar{n}\star}_{\bar{n}-1}\right)^{\overline{n}-2-n}\left(p^{\bar{n}\star}_{\bar{n}-1}\right)^{d_1}\right\}\chi_{\{n\neq\bar{n}-1\}} + \frac{\lambda^{\overline{n}-1-n}p^{\overline{n}\star}_{\overline{n}-1}}{(r+\lambda-\mu)^{\overline{n}-n}}\right],
\end{aligned}
$$

for $n=0,1,2,...,\overline{n}-1$. As above, this representation is correct when we assume that $A^{\overline{n}}_{\overline{n}-1,1,0}=0$. Indeed, in the case $n=\overline{n}-1$, we can get a simpler representation, which is

$$
A^{\overline{n}}_{\overline{n}-1,0,0} = \frac{\left(p^{\overline{n}\star}_{\overline{n}-1}\right)^{1-d_1}}{d_1\,(r+\lambda-\mu)}.
$$

As in the case $\overline{n}=2$, the remaining constants can be found by using the method of undetermined coefficients (see the proof of this result in Nunes et al. [25]), resulting in the following

$$
A^{\overline{n}}_{n,k,j} = -\frac{2\lambda}{\sigma^2}\sum_{l=j-1}^{\overline{n}-2-n-k}(-1)^{l+1-j}\,\frac{l!}{j!}\,\frac{A^{\overline{n}}_{n+1,k,l}}{(d_1-d_2)^{l+2-j}}, \qquad (24)
$$

$$
B^{\overline{n}}_{n,k,j} = -\frac{2\lambda}{\sigma^2}\sum_{l=j-1}^{\overline{n}-2-n-k}(-1)^{l+1-j}\,\frac{l!}{j!}\,\frac{B^{\overline{n}}_{n+1,k,l}}{(d_2-d_1)^{l+2-j}}, \qquad (25)
$$

for $n=0,1,2,...,\overline{n}-2$, $k=0,1,2,...,\overline{n}-2-n$ and $j=1,2,\cdots,\overline{n}-1-n-k$. As before, this representation is correct when we assume that $B^{\bar{n}}_{\bar{n}-1,0,0}=0$.

To finalize this section, we note that the thresholds $p_n^{\overline{n}^\star}$ are generally not possible to find explicitly. However, in light of the smooth-pasting conditions, we can define $p_n^{\overline{n}^\star}$, for each $n = 0, 1, 2, \cdots, \overline{n} - 1$, as the unique solution of the equation

$$
\begin{aligned}
0 = \Bigg\{ \Bigg\{ & \sum_{j=1}^{\overline{n}-2-n} \Big[(A_{n,0,j}^{\overline{n}} - A_{n,1,j}^{\overline{n}}) j (\ln p_{\overline{n}-1}^{\overline{n}^\star})^{j-1} (p_{\overline{n}-1}^{\overline{n}^\star})^{d_1} \\
& + B_{n,1,j}^{\overline{n}} \Big[(d_1 - d_2)(\ln p_{\overline{n}-1}^{\overline{n}^\star})^{j} - j (\ln p_{\overline{n}-1}^{\overline{n}^\star})^{j-1} \Big] (p_{\overline{n}-1}^{\overline{n}^\star})^{d_2} \Big] \Bigg\} \chi_{\{n \neq \overline{n}-2\}} \\
& + (d_1 - d_2) B_{n,1,0}^{\overline{n}} \left(p_{\overline{n}-1}^{\overline{n}^\star} \right)^{d_2} + A_{n,0,\overline{n}-1-n}^{\overline{n}} (\overline{n} - 1 - n)(\ln p_{\overline{n}-1}^{\overline{n}^\star})^{\overline{n}-2-n} (p_{\overline{n}-1}^{\overline{n}^\star})^{d_1} \Bigg\} \chi_{\{n \neq \overline{n}-1\}} \\
& + \lambda^{\overline{n}-n-1} \left[\frac{(d_1 - 1) p_{\overline{n}-1}^{\overline{n}^\star}}{(r + \lambda - \mu)^{\overline{n}-n}} - \frac{d_1 I \phi^{\overline{n}-1} (r + \lambda(1 - \phi))}{(r + \lambda)^{\overline{n}-n}} \right].
\end{aligned} \tag{26}
$$

As in the case $\overline{n} = 2$, we note that $B_{n,1,0}^{\overline{n}}$ is indeed a function of the threshold $p_n^{\overline{n}^\star}$. When $n = \bar{n} - 1$, an explicit solution to the previous equation is possible to obtain,

$$
p_{\overline{n}-1}^{\overline{n}^\star} = \frac{d_2 - 1}{d_2} I \phi^{\overline{n}-1} [r + \lambda(1 - \phi)].
$$

Proposition 5 *Consider the truncated optimal stopping problem defined by* (6)*, for a general* $\bar{n}$*. Then, the value function* $V^{\bar{n}}$ *is given by* (19)-(20)*, for all* $p > 0$ *and* $n = 0, 1, \cdots, \bar{n}$*, and the parameters* $A_{n,k,j}^{\bar{n}}$ *and* $B_{n,k,j}^{\bar{n}}$*, for* $n = 0, 1, \cdots, \bar{n}$*,* $k = 0, 1, \cdots, \bar{n} - 1 - n$ *and* $j = 0, 1, \cdots, \bar{n} - 1 - n - k$ *are defined by* (21)-(22)-(23)-(24)-(25)*. Additionally, the threshold* $p_n^{\bar{n}}$*, for* $n = 0, 1, \cdots, \bar{n} - 1$ *is the unique solution to the equation* (26)*.*

B Proofs

B.1 Proof of Proposition 1

Fix $n \in \mathbb{N}$. To prove that the stopping region is such as (4), one may start by computing

$$
\begin{aligned}
(r + \lambda) g(p, n) - \frac{\mu p}{r - \mu} - \lambda V(p, n + 1) &\leq (r + \lambda) g(p, n) - \frac{\mu p}{r - \mu} - \lambda g(p, n + 1) \\
&\leq p + \lambda I \phi^n (\phi - 1) - r I \phi^n
\end{aligned}
$$

Taking into account Propositions 3.3 and 3.4 in Oksendal and Sulem [27], it is never optimal to stop when

$$p + \lambda I\phi^n(\phi - 1) - rI\phi^n \leq 0 \Leftrightarrow p \leq I\phi^n(1-\phi) + rI\phi^n \equiv \tilde{p}_n,$$

which implies that there is $p_n^* \geq \tilde{p}_n$. Therefore, it is optimal to stop for levels of price and technology (p, n) such that $p \in (p_n^*, \bar{p}_n^*)$, with $\bar{p}_n^* \leq +\infty$. We want to prove that $\bar{p}_n^* = +\infty$ for all $n \in \mathbb{N}$ by using a contradiction argument.

Let us assume that there are (n, p) such that $\bar{p}_n^* = \bar{p}_m^* < +\infty$, for all $m > n$. To avoid misunderstandings, we write N_s^n to highlight that this is the Poisson process N_s with initial condition $N_0 = n$. We compare the following strategies: $\bar{\tau}_{\epsilon,n} = \inf\{t > 0 : P_t \leq \bar{p}^*_{N_t^n} - \epsilon\}$ and $\bar{\tau}_n = \inf\{t > 0 : P_t \leq \bar{p}^*_{N_t^n}\}$, for an initial condition $P_0 = p > \bar{p}_n^*$. We have that

$$\begin{aligned}\int_{\bar{\tau}_n}^{+\infty} e^{-rs} P_s ds - e^{-r\bar{\tau}_n} I\phi^{N^n_{\bar{\tau}_n}} - \left(\int_{\bar{\tau}_{\epsilon,n}}^{+\infty} e^{-rs} P_s ds - e^{-r\bar{\tau}_{\epsilon,n}} I\phi^{N^n_{\bar{\tau}_{\epsilon,n}}}\right)& \\ = -\int_{\bar{\tau}_{\epsilon,n}}^{\bar{\tau}_n} e^{-rs} P_s ds + e^{-r\bar{\tau}_{\epsilon,n}} I\phi^{N^n_{\bar{\tau}_{\epsilon,n}}} - e^{-r\bar{\tau}_n} I\phi^{N^n_{\tau^*}}.& \qquad (27)\end{aligned}$$

Given our construction, one may check that $\bar{\tau}_{\epsilon,n}(\omega) = \bar{\tau}_{\epsilon,n+1}(\omega)$ and $\bar{\tau}_n(\omega) = \bar{\tau}_{n+1}(\omega)$, for a fixed $P_0 = p$ and for a fixed ω in the sample space. Thus, Equation (27) can be written as

$$-\int_{\tau_{\epsilon,n+1}}^{\bar{\tau}_{n+1}} e^{-rs} P_s ds + e^{-r\tau_{\epsilon,n}} I\phi^{N^n_{\tau_{\epsilon,n}}} - e^{-r\bar{\tau}_n} I\phi^{N^n_{\bar{\tau}_n}} \qquad (28)$$

Since $\bar{p}_n^*$ is assumed to be the threshold that splits the continuation and stopping regions at any level m, with $m \geq n$, when the process P_s attains $\bar{p}_n^*$ the decisions of continuation or stopping are indifferent. Combining this fact with similar calculations to (28), we get that

$$\frac{1}{\epsilon} E\left[-\int_{\tau_{\epsilon,n+1}}^{\bar{\tau}_{n+1}} e^{-rs} P_s ds + e^{-r\tau_{\epsilon,n+1}} I\phi^{N^{n+1}_{\tau_{\epsilon,n+1}}} - e^{-r\bar{\tau}_{n+1}} I\phi^{N^{n+1}_{\bar{\tau}_{n+1}}}\right] \to 0.$$

Since for a fixed ω in the sample space and $P_0 = p > \bar{p}_n^*$ we have that $N^{n+1}_{\bar{\tau}_{n+1}} = N^n_{\bar{\tau}_n} + 1$, then for small ϵ Equation (27) becomes

$$E\left[-\left(e^{-r\tau_{\epsilon,n+1}} I\phi^{N^{n+1}_{\tau_{\epsilon,n+1}}} - e^{-r\tau_{n+1}} I\phi^{N^{n+1}_{\tau_{n+1}}}\right) + e^{-r\tau_{\epsilon,n+1}} I\phi^{N^{n+1}_{\tau_{\epsilon,n+1}}-1} - e^{-r\tau_{n+1}} I\phi^{N^{n+1}_{\tau_{n+1}}-1}\right] < 0.$$

As this argument does not depend on the level n, one may apply the same strategy as many times as necessary until finding a contradiction that $\bar{p}_n^* \leq p_n^*$. Therefore, the optimal strategy is $\tau^* = \inf\{t > 0 : P_t \geq p^*_{N_t}\}$.

To prove that the function $n \to p_n^*$ is decreasing, one may use a similar argument to the one used above. Indeed, for a fixed ω in the sample space, and $P_0 = p < p_n^*$, if $\tau_{\epsilon,n} = \inf\{t > 0 : P_t \geq p^*_{N_t^n} - \epsilon\}$, we have that

$$E\left[\int_{\tau^*}^{+\infty} e^{-rs} P_s ds - e^{-r\tau^*} I \phi^{N_{\tau^*}^{n+1}} - \left(\int_{\tau_{\epsilon,n+1}}^{+\infty} e^{-rs} P_s ds - e^{-r\tau_{\epsilon,n+1}} I \phi^{N_{\tau_{\epsilon,n+1}}^{n+1}}\right)\right] < 0.$$

■

B.2 Proof of Theorem 1

Taking into account the proof of Theorem 5.1 in Oliveira and Perkowski [28], we only need to verify that $\{e^{-rP_\tau} v(P_\tau, N_\tau)\}$ is a uniformly integrable family of random variables. Indeed, one may notice that, for a fixed $\bar{n}, m \in \mathbb{N}_0$, we have

$$V^{\bar{n}}(p,n) \leq V^{\bar{n}+m}(p,n) \leq V(p,n), \quad \text{for all } n \in \mathbb{N}. \tag{29}$$

This follows from the fact that we are optimizing the functional J over a set of stopping times that verifies respectively

$$\{\tau : \tau \leq \tau_{\bar{n}}\} \subset \{\tau : \tau \leq \tau_{\bar{n}+m}\} \subset \{\tau : \tau < \infty\}.$$

Since $V(p,n) = g(p,n)$ for $p \geq p_n^*$, we have that $V^{\bar{n}}(p,n) = g(p,n)$, at least for $p \geq p_n^*$. Additionally, as the value function is continuous and $\lim_{p\to 0^+} v^{\bar{n}}(p,n) < +\infty$, there is a constant M such that $v^{\bar{n}}(p,n) \leq g(p,n) + M$. Therefore, for $f(x) = x^p$, with $p > 1$, we have that

$$\sup_{s\geq 0} E_{p,n}\left[f\left(e^{-rs} v_{p,n}^{\bar{n}}(P_s, N_s)\right)\right] \leq \sup_{s\geq 0} f\left(E_{p,n}\left[e^{-rs}\left(\frac{P_s}{r-u} - I\phi^{\bar{n}} + M\right)\right]\right) < +\infty,$$

for all $p > 0$ fixed, which finishes the proof in light of Definition C.2 and Theorem C.3 in Øksendal [26]. ■

B.3 Proof of Proposition 2

Notice that by definition $V(p,n) = \sup_{\tau\geq 0} J(p,n,\tau) \geq \sup_{\tau \geq 0} J(p,n,\tau\wedge\tau_{\overline{n}}) = V^{\overline{n}}(p,n)$, for any $\overline{n} \in \mathbb{N}$, which implies that $\limsup_{\overline{n}\to+\infty} V^{\overline{n}}(p,n) \leq V(p,n)$. In order to prove that $\liminf_{\overline{n}\to+\infty} V^{\overline{n}}(p,n) \geq V(p,n)$, we show that, for all stopping times $\tau \geq 0$, $\liminf_{\overline{n}\to+\infty} J(p,n,\tau\wedge\tau_{\overline{n}}) \geq J(p,n,\tau)$. Fix $\tau \geq 0$ and notice that

$$\begin{aligned} J(p,n,\tau) - J(p,n,\tau\wedge\tau_{\overline{n}}) &= E_{p,n}\left[-\int_{\tau\wedge\tau_{\overline{n}}}^{\tau} e^{-rs} P_s ds - e^{-r\tau} I_\tau + e^{-r(\tau\wedge\tau_{\overline{n}})} I_{\tau\wedge\tau_{\overline{n}}}\right] \\ &\leq E_{p,n}\left[-e^{-r\tau} I_\tau + e^{-r(\tau\wedge\tau_{\overline{n}})} I_{\tau\wedge\tau_{\overline{n}}}\right]. \end{aligned}$$

For any $\tau \geq 0$, $\tau \wedge \tau_{\overline{n}} \leq \tau$, $N_{\tau \wedge \tau_{\overline{n}}} \leq N_\tau$ and, consequently, $I \geq e^{-r(\tau \wedge \tau_{\overline{n}})} I_{\tau \wedge \tau_{\overline{n}}} \geq e^{-r\tau} I_\tau$ almost surely. By construction, $e^{-r(\tau \wedge \tau_{\overline{n}})} I_{\tau \wedge \tau_{\overline{n}}} \to e^{-r\tau} I_\tau$ almost surely, when $\overline{n} \to +\infty$. Thus, it follows from the dominated convergence theorem that

$$\lim_{\overline{n} \to +\infty} E_{p,n} \left[-e^{-r\tau} I_\tau + e^{-r(\tau \wedge \tau_{\overline{n}})} I_{\tau \wedge \tau_{\overline{n}}} \right] = 0,$$

which is sufficient to conclude the first part of the proof.

To prove the second result, we note that combining (29) and the fact that $V(p,n) = g(p,n)$, for $p \geq p_n^*$ and $V^{\overline{n}+m}(p,n) = g(p,n)$, for $p \geq p_n^{\overline{n}+m^*}$, it follows that

$$p_n^{\overline{n}^*} \leq p_n^{\overline{n}+m^*} \leq p_n^*.$$

Therefore, the sequence $\{p_n^{\bar{n}^*}\}_{\bar{n} \in \mathbb{N}}$ is increasing, bounded, and, consequently, convergent. This means that $p_n^{\bar{n}^*} \to \tilde{p}_n \leq p_n^*$.

Let us assume that $\tilde{p}_n < p_n^*$. Then the following happens: fix $p \in (\tilde{p}_n, p_n^*)$ and note that $V^{\bar{n}}(p,n) = g(p,n)$. Additionally, $V^{\bar{n}}(p,n) \to V(p,n) > g(p,n)$, when $\bar{n} \to +\infty$, which is a contradiction because choosing $\bar{n}$ large enough the strategy $\tau^{\bar{n}} = \inf\{t \geq 0 : P_t \geq p_{N_t}^{\bar{n}^*}\}$ is not optimal. As a conclusion, we have that $p_n^{\bar{n}^*} \to p_n^*$. ■

B.4 Proof of Proposition 3

In order to ease the proof of Proposition 3 we state an auxiliary result.

Lemma 1 *Consider the function*

$$\begin{aligned} f(p) =& A_{0,0,1}^2 (p_1^{2^*})^{d_1} + \left[(d_1 - 1) \frac{p}{r + \lambda - \mu} - d_1 I \frac{r + \lambda(1-\phi)}{r + \lambda} \right] \left(\frac{p}{p_1^{2^*}} \right)^{-d_2} \\ &+ \lambda \left[\frac{(d_1 - 1) p_1^{2^*}}{(r + \lambda - \mu)^2} - \frac{d_1 I \phi (r + \lambda(1-\phi))}{(r+\lambda)^2} \right], \end{aligned}$$

where $A_{0,0,1}^2$ and $p_1^{2^*}$ are defined, respectively, by (14) *and* (11)*, respectively. Then, the equation $f(p) = 0$ has a unique root, $p_0^{2^*}$, verifying $p_0^{2^*} > p_1^{2^*} > 0$.*

Proof of Lemma 1 Firstly, we note that

$$A_{0,0,1}^2 (p_1^{2^*})^{d_1} = -\frac{2\lambda}{\sigma^2 (d_1 - d_2)} \times \frac{p_1^{2^*}}{d_1 (r + \lambda - \mu)},$$

which, combined with the following reparametrization,

$$r+\lambda = -\frac{\sigma^2}{2}d_1 d_2 \quad \text{and} \quad \mu = \frac{\sigma^2}{2}(1-d_1-d_2),$$

allows us to make the calculations

$$\begin{aligned} f(0) &= A^2_{0,0,1}(p_1^{2^*})^{d_1} + \lambda\left[\frac{(d_1-1)p_1^{2^*}}{(r+\lambda-\mu)^2} - \frac{d_1 I\phi(r+\lambda(1-\phi))}{(r+\lambda)^2}\right] \\ &= -\frac{2\lambda p_1^{2^*}}{\sigma^2(r+\lambda-\mu)} \times \frac{1-d_1}{d_2(d_1-d_2)(1-d_2)} < 0. \end{aligned}$$

Taking into account that the derivative of the function f is given by

$$f'(p) = \left(\frac{p}{p_1^{2^*}}\right)^{-d_2}\left[\frac{(1-d_2)(d_1-1)}{r+\lambda-\mu} + d_1 d_2 I\frac{r+\lambda(1-\phi)}{r+\lambda}p^{-1}\right],$$

we conclude that the function f is decreasing for $p < p*$ and increasing for $p > p*$, where

$$p^* = d_1 d_2 I \times \frac{r+\lambda(1-\phi)}{r+\lambda} \times \frac{r+\lambda-\mu}{(d_2-1)(d_1-1)}.$$

Combining the previous information with the fact that $\lim\limits_{p\to+\infty} f(p) = +\infty$, we can conclude that there is a unique $p = p_0^{2^*}$ that satisfies the equation $f(p) = 0$.

To complete the proof, we note that $f(p_1^{2^*}) = f(0) + \left[(d_1-1)\frac{p_1^{2^*}}{r+\lambda-\mu} - d_1 I\frac{r+\lambda(1-\phi)}{r+\lambda}\right] < 0$, which follows in light of the facts that $f(0) < 0$ and

$$\left[(d_1-1)\frac{p_1^{2^*}}{r+\lambda-\mu} - d_1 I\frac{r+\lambda(1-\phi)}{r+\lambda}\right] = -\frac{2(\phi-1)}{d_2\sigma^2}I\left[r+\lambda(1-\phi)\right] < 0.\blacksquare$$

We start by noticing that $v^2(\cdot,2)$ satisfies the boundary condition of Equation (7). To complete this proof we need to verify that:

$$v^2(p,1) \text{ is such that } \min\{(r+\lambda)v^2(p,1) - \mu p(v^2)'(p,1) - \frac{\sigma^2}{2}p^2(v^2)''(p,1) - \lambda v^2(p,2), v^2(p,1) - g(p,1)\} = 0, \tag{30}$$

$$v^2(p,0) \text{ is such that } \min\{(r+\lambda)v^2(p,0) - \mu p(v^2)'(p,0) - \frac{\sigma^2}{2}p^2(v^2)''(p,0) - \lambda v^2(p,1), v^2(p,0) - g(p,0)\} = 0. \tag{31}$$

Proof of (30): By construction, the function $p \mapsto A^2_{1,0,0}p^{d_1} + \lambda\left[\frac{p}{(r-\mu)(r+\lambda-\mu)} - \frac{I\phi^2}{r+\lambda}\right]$ is a solution to the ODE $(r+\lambda)v^2(p,1) - \mu p(v^2)'(p,1) - \frac{\sigma^2}{2}p^2(v^2)''(p,1) - \lambda v^2(p,2) = 0$. Additionally, trivial calculations show that the smooth-pasting conditions

$$v^2(p_1^{2^*},1) = g(p_1^{2^*},1) \quad \text{and} \quad (v_1^2)'(p_1^{2^*},1) = g_1'(p_1^{2^*},1)$$

are satisfied if and only if $p_1^{2^*}$ and $A_{1,0,0}^2$ are given by (11) and (12). To finish this part of the proof, we still need to verify that the function $v^2(p,1)$ satisfies the inequalities

$$(r+\lambda)v^2(p,1) - \mu p(v^2)'(p,1) - \frac{\sigma^2}{2}p^2(v^2)''(p,1) - \lambda v^2(p,2) \geq 0, \quad \text{for all } p \geq p_1^{2^*} \tag{32}$$

$$v^2(p,1) - g(p,1) \geq 0, \quad \text{for all } 0 < p \leq p_1^{2^*}. \tag{33}$$

For $p \geq p_1^{2^*}$, we have $v^2(p,1) = \frac{p}{r-\mu} - I\phi$, which allow us to obtain

$$(r+\lambda)v^2(p,1) - \mu p(v^2)'(p,1) - \frac{\sigma^2}{2}p^2(v^2)''(p,1) - \lambda v^2(p,2) = p - I\phi(r+\lambda(1-\phi)).$$

Consequently, condition (32) is equivalent to $p \geq I\phi(r+\lambda(1-\phi))$, and therefore, the first inequality holds true because

$$p_1^{2^*} = \frac{d_2-1}{d_2}I\phi(r+\lambda(1-\phi)) \geq I\phi(r+\lambda(1-\phi)).$$

To prove the inequality (33), we use (12) to see that

$$v^2(p,1) - g(p,1) = \frac{1-d_1}{d_1} \times \frac{p}{r+\lambda-\mu} + I\phi\frac{r}{r+\lambda}.$$

It is now trivial to observe that the function $p \mapsto v^2(p,1) - g(p,1)$ is decreasing, which combined with the fact that $v^2(p_1^{2^*},1) - g(p_1^{2^*},1) = 0$ proves the inequality (33).

Proof of (31): It is a matter of calculations to see that the ODE $(r+\lambda)v^2(p,0) - \mu p(v^2)'(p,0) - \frac{\sigma^2}{2}p^2(v^2)''(p,0) - \lambda v^2(p,1) = 0$ is satisfied by the functions $p \mapsto \left(A_{0,0,0}^2 + A_{0,0,1}^2 \ln p\right)p^{d_1} + \lambda^2\left[\frac{p}{(r-\mu)(r+\lambda-\mu)^2} - \frac{I\phi^2}{(r+\lambda)^2}\right]$, with $A_{0,0,1}^2$ as in (14), when $p \leq p_1^{2^*}$ and $p \mapsto A_{0,1,0}^2 p^{d_1} + B_{0,1,0}^2 p^{d_2} + \lambda\left[\frac{p}{(r-\mu)(r+\lambda-\mu)} - \frac{I\phi}{r+\lambda}\right]$, when $p \geq p_1^{2^*}$. By using the C^1 continuity of the function $v^2(p,0)$ it follows that the parameters $A_{0,0,1}^2$, $A_{0,0,0}^2$, $A_{0,1,0}^2$ and $B_{0,1,0}^2$ are given by (14)-(17) while $p_0^{2^*}$ is such that (18) holds true and $p_1^{2^*}$ is given by (11). Additionally, in light of Lemma 1, $p_0^{2^*}$ is the unique solution to the equation $f(p) = 0$ and verify $p_0^{2^*} > p_1^{2^*} > 0$. To finish this part of the proof, we have to verify that the function $v^2(p,1)$ satisfies the inequalities

$$(r+\lambda)v^2(p,0) - \mu(r+\lambda)v^2(p,0) - \mu p(v^2)'(p,0) - \frac{\sigma^2}{2}p^2(v_0^2)''(p,0) - \lambda v^2(p,1) \geq 0, \quad \text{for all } p \geq p_0^{2^*} \tag{34}$$

$$v^2(p,0) - g(p,0) \geq 0, \quad \text{for all } 0 < p \leq p_0^{2^*}. \tag{35}$$

The proof of conditions (34)-(35) follows in light of a similar argument to the one used to prove the conditions in (32)-(33).

Finally, one can notice that by construction the functions $v^2(\cdot,1)$ is $C^2(]0,+\infty[\backslash\{p_1^{2^*}\})\cap C^1(\{p_1^{2^*}\})$, and $v^2(\cdot,0)$ is $C^2(]0,+\infty[\backslash\{p_0^{2^*},p_1^{2^*}\})\cap C^1(\{p_0^{2^*},p_1^{2^*}\})$. Indeed, both $v^2(\cdot,1)$ in $]0,+\infty[\backslash\{p_1^{2^*}\}$ and $v^2(\cdot,0)$ in $]0,+\infty[\backslash\{p_0^{2^*},p_1^{2^*}\}$ are classical solutions for an ODE, thus, they are C^2 in the correspondent domain. Moreover, in light of the smooth-fit conditions, the functions are C^1 at the thresholds.

Taking into account Theorem 1, the value function V^2 verifies $V^2(p,n)=v^2(p,n)$, the optimal stopping time τ_2^* is given by $\tau_2^*=\inf\{t\geq 0 \,:\, P_t\geq p_{N_t}^{2^*}\}$ and the stopping region is $\{(p,n)\,:\,p\geq p_n^{2^*}\vee n\geq 2\}$. ■

B.5 Proof of Proposition 5

To facilitate the understanding of the proof of Proposition 5, we will firstly state and prove three lemmas regarding the behavior of the solution of the HJB equation.

Lemma 2 *Equation* (26) *has a unique solution.*

Proof of Lemma 2 Fix $a\in]0,+\infty[$. From the general theory of ordinary differential equations, it is known that the initial value problem

$$\begin{cases} (r+\lambda)v^{\overline{n}}(p,n)-\mu p(v^{\overline{n}})'(p,n)-\frac{\sigma^2}{2}p^2(v^{\overline{n}})''(p,n)=\lambda v^{\overline{n}}(p,n+1) \\ v^{\overline{n}}(a)=g(a) \quad \text{and} \quad (v^{\overline{n}})'(a)=g'(a) \end{cases} \tag{36}$$

has a unique solution. Taking into account that d_1 and d_2 are real and distinct values, then any solution for a boundary problem like the previous one can be written as

$$v^{\overline{n}}(p,n)=Ap^{d_1}+Bp^{d_2}+v_h(p,n).$$

In our case, v_h that is the particular solution to this equation can be defined as

$$\begin{aligned} v_h(p,n)=&\sum_{j=1}^{\overline{n}-1-n-k}\left[A_{n,k,j}^{\overline{n}}\;(\ln p)^j\;p^{d_1}+B_{n,k,j}^{\overline{n}}\;(\ln p)^j\;p^{d_2}\right] \\ &+\lambda^{\overline{n}-n-k}\left[\frac{p}{(r-\mu)\,(r+\lambda-\mu)^{\overline{n}-n-k}}-\frac{I\phi^{\overline{n}-k}}{(r+\lambda)^{\overline{n}-n-k}}\right],\quad \text{for}\quad p_{\overline{n}-k}^{\overline{n}^\star}\leq p<p_{\overline{n}-1-k}^{\overline{n}^\star}, \end{aligned}$$

where $p_n^{\overline{n}^\star}$ should be replace by a. Assuming that $v^{\overline{n}}(p,n+1)$ is already defined, in the interval $0<p<p_{\overline{n}-1}^{\overline{n}^\star}$ we get

$$v_h(p,n)=\sum_{j=1}^{\overline{n}-1-n}\left[A_{n,0,j}^{\overline{n}}\right]\;(\ln p)^j\;p^{d_1}+\lambda^{\overline{n}-n}\left[\frac{p}{(r-\mu)\,(r+\lambda-\mu)^{\overline{n}-n}}-\frac{I\phi^{\overline{n}}}{(r+\lambda)^{\overline{n}-n}}\right].$$

Then, from the general theory of ordinary differential equations, we can state that the system

$$v^{\overline{n}}(a) = g(a,n) \quad \text{and} \quad (v^{\overline{n}})'(a) = g'(a,n)$$

has a unique solution. We note that the parameters A and $B$are piecewise functions of p, since the function $v^{\overline{n}}(p,n+1)$ is also a piecewise function. Assuming that the function $v^{\overline{n}}_{(p,n+1)}$ is already defined, the parameters A and B that solve the system of equations can be defined in light of the results presented in Appendix A. In fact, replacing $p_n^{\overline{n}^*}$ by a, the parameters A and B are such that $A = A^{\overline{n}}_{n,k,0}$ and $B = B^{\overline{n}}_{n,k,0}$ in the intervals $p^{\overline{n}^*}_{\overline{n}-k} \leq p \leq p^{\overline{n}^*}_{\overline{n}-1-k}$, for $k = 1, 2, \cdots, \overline{n}-1-n$, where $A^{\overline{n}}_{n,k,0}$ and $B^{\overline{n}}_{n,k,0}$ are defined by (21) and (22). In the interval $0 < p \leq p_{\overline{n}-1}$, we have $A = A^{\overline{n}}_{n,0,0}$ and $B = \tilde{B}^{\overline{n}}_{n,0,0}$, where $A^{\overline{n}}_{n,0,0}$ is defined as in (23) and $\tilde{B}^{\overline{n}}_{n,0,0}$ is such that

$$\begin{aligned}
\tilde{B}^{\overline{n}}_{n,0,0} = B^{\overline{n}}_{n,1,0} + \frac{\left(p^{\overline{n}^\star}_{\overline{n}-1}\right)^{-d_2}}{d_1-d_2} & \left[\left\{ \left\{ \sum_{j=1}^{\overline{n}-2-n} \left[(A^{\overline{n}}_{n,0,j} - A^{\overline{n}}_{n,1,j}) j (\ln p^{\overline{n}^\star}_{\overline{n}-1})^{j-1} (p^{\overline{n}^\star}_{\overline{n}-1})^{d_1} \right.\right.\right.\right. \\
& + B^{\overline{n}}_{n,1,j} \left[(d_1-d_2)(\ln p^{\overline{n}^\star}_{\overline{n}-1})^j - j(\ln p^{\overline{n}^\star}_{\overline{n}-1})^{j-1} \right] (p^{\overline{n}^\star}_{\overline{n}-1})^{d_2} \Bigg\} \chi_{\{n \neq \overline{n}-2\}} \\
& + A^{\overline{n}}_{n,0,\overline{n}-1-n} (\overline{n}-1-n)(\ln p^{\overline{n}^\star}_{\overline{n}-1})^{\overline{n}-2-n} (p^{\overline{n}^\star}_{\overline{n}-1})^{d_1} \Bigg\} \chi_{\{n \neq \overline{n}-1\}} \\
& + \lambda^{\overline{n}-n-1} \left[\frac{(d_1-1) p^{\overline{n}^\star}_{\overline{n}-1}}{(r+\lambda-\mu)^{\overline{n}-n}} - \frac{d_1 I \phi^{\overline{n}-1}(r+\lambda(1-\phi))}{(r+\lambda)^{\overline{n}-n}} \right] \Bigg] .
\end{aligned} \tag{37}$$

Nevertheless, due to the growth conditions required by the verification theorem, our solution must be such that $\lim_{p\to 0^+} v^{\overline{n}}(p,n) < +\infty$,. Therefore, we have to find a such that $\tilde{B}^{\overline{n}}_{n,0,0} = 0$ in the interval $0 < p < p^{\overline{n}}_{\overline{n}-1}$. Looking at the expression of $\tilde{B}^{\overline{n}}_{n,0,0}$, one can check that $\tilde{B}^{\overline{n}}_{n,0,0}$ is increasing with respect to a if

$$(d_1-1)\frac{a}{r+\lambda-\mu} - d_1 I \phi^n \frac{r+\lambda(1-\phi)}{r+\lambda}$$

is an increasing function of a, which is a triviality. Moreover, since

$$\lim_{a\to+\infty} \frac{a^{-d_2}}{d_1-d_2}\left[(d_1-1)\frac{a}{r+\lambda-\mu} - d_1 I\phi^n \frac{r+\lambda(1-\phi)}{r+\lambda}\right] = +\infty,$$

we only have to check that $\lim_{a\to 0^+} \tilde{B}^{\overline{n}}_{n,0,0} = -\infty$, to guarantee the existence of a unique a such that $\tilde{B}^{\overline{n}}_{n,0,0}$. In light of the comments above, we have that

$$\lim_{a\to 0^+} \tilde{B}^{\overline{n}}_{n,0,0} < 0 \Leftrightarrow \text{ there is } a \text{ such that } \lim_{p\to 0^+} v^{\overline{n}}(p,n) = -\infty$$

According to Guerra et al. [13], the integral representation to the solution of the boundary problem (36)

$$v^{\overline{n}}(p,n) = \frac{d_1\left(\frac{p}{a}\right)^{d_1} - d_2\left(\frac{p}{a}\right)^{d_2}}{d_1-d_2}g(a,n) + \frac{\left(\frac{p}{a}\right)^{d_1} - \left(\frac{p}{a}\right)^{d_2}}{d_1-d_2}ag'(a,n)$$
$$+\frac{2\lambda}{(d_1-d_2)\sigma^2}\int_p^a \frac{\left(\frac{p}{s}\right)^{d_1} - \left(\frac{p}{s}\right)^{d_2}}{s}v^{\overline{n}}(s,n+1)ds.$$

Taking into account that,

$$\lim_{p\to 0^+} \frac{d_1\left(\frac{p}{a}\right)^{d_1} - d_2\left(\frac{p}{a}\right)^{d_2}}{d_1-d_2}g(a,n) + \frac{\left(\frac{p}{a}\right)^{d_1} - \left(\frac{p}{a}\right)^{d_2}}{d_1-d_2}ag'(a,n)$$
$$= \lim_{p\to 0^+}\left(\frac{-d_2}{a^{d_2}(d_1-d_2)}g(a,n) - \frac{a}{a^{d_2}(d_1-d_2)}g'(a,n)\right)p^{p_2}$$

and

$$\lim_{p\to 0^+}\frac{2\lambda}{(d_1-d_2)\sigma^2}\int_p^a \frac{\left(\frac{p}{s}\right)^{d_1} - \left(\frac{p}{s}\right)^{d_2}}{s}v^{\overline{n}}(s,n+1)ds = \lim_{p\to 0^+} -\frac{2\lambda}{(d_1-d_2)\sigma^2}p^{d_2}\int_0^a s^{-d_2-1}v^{\overline{n}}(s,n+1)ds,$$

we can conclude that

$$\lim_{p\to 0^+} v^{\overline{n}}(p,n) = \lim_{p\to 0^+}\frac{p^{d_2}}{d_1-d_2}\left(\frac{-d_2}{a^{-d_2}}g(a,n) - \frac{a}{a^{-d_2}}g'(a,n) - \frac{2\lambda}{\sigma^2}\int_0^a s^{-d_2-1}v^{\overline{n}}(s,n+1)ds\right).$$

Since we can pick a as small as we need and $\lim_{a\to 0^+}\int_0^a s^{-d_2-1}v^{\overline{n}}(s,n+1)ds = 0$ we get $\lim_{p\to 0^+} v^{\overline{n}}(p,n) = -\infty$. This allows us to say that there is a unique point $a = p_n^{\overline{n}}$ that satisfies $\tilde{B}^{\overline{n}}_{n,0,0}$. In light of the definition of $\tilde{B}^{\overline{n}}_{n,0,0}$, given by (37), we get existence and uniqueness of solution to Equation (26). ■

Lemma 3 *Let $v^{\overline{n}}$ be the function defined as in* (19)*. Then, the function $v^{\overline{n}}$ is increasing.*

Proof of Lemma 3 We start by noticing that the function $v^{\overline{n}}(\cdot,n)$ is increasing for $p \geq p_n^{\overline{n}^*}$ because $v^{\overline{n}}(p,n) = \frac{p}{r-\mu} - I\phi^n$. For $p \leq p_n^{\overline{n}^*}$, the function $v^{\overline{n}}(p,n)$ satisfies the boundary problem (36) as discussed in the proof of Lemma 2. For this case, we prove by induction that, for a fixed $\bar{n}$, the function $p \to v^{\bar{n}}(p,n)$ is increasing for each $n \leq \bar{n}$. For $n = \bar{n}$ the function $p \to v^{\bar{n}}(p,\bar{n})$ is trivially increasing because $v^{\bar{n}}(p,\bar{n}) = g(p,\bar{n})$. Now our induction hypothesis is that if $v^{\bar{n}}(p,n+1)$ is increasing then $v^{\bar{n}}(p,n)$ is also increasing (note that the induction process, in this case, is applied in a backward way). The induction hypothesis can be proven by contradiction.

Let us assume that, indeed, $v^{\bar{n}}(p,n+1)$ is increasing but $v^{\bar{n}}(p,n)$ is non-monotonic. We know that there is at least p_n^* that satisfies the conditions $v^{\bar{n}}(p_n^*,n) = g(p_n^*,n)$ and $(v^{\bar{n}})'(p_n^*,n) = g'(p_n^*,n)$. Then, at this point

$(v^{\bar{n}})'(p_n^*, n) > 0$, which implies that there is $\epsilon > 0$ such that $v^{\bar{n}}(\cdot, n)$ is increasing in the interval $(p_n^* - \epsilon, p_n^*)$. Therefore, if $v^{\bar{n}}(p, n)$ is not monotonic, then there is $p_1 = \min\{0 < p < p_n^* : (v^{\bar{n}})'(p, n) = 0\ ,\ (v^{\bar{n}})''(p, n) > 0\}$. Consequently, one gets that

$$(r+\lambda)v^{\overline{n}}(p_1, n) - \frac{\sigma^2}{2}p^2(v^{\overline{n}})''(p_1, n) = \lambda v^{\overline{n}}(p_1, n+1) < (r+\lambda)v^{\overline{n}}(p_1, n). \tag{38}$$

Looking to the expression of $v^{\overline{n}}(p, n)$, one may notice that $\lim_{p\to 0^+}(v^{\overline{n}})'(p, n) > 0$. Then, there is $p_2 < p_1$ such that $p_2 = \max\{0 < p < p_1 : (v^{\bar{n}})'(p, n) = 0\ ,\ (v^{\bar{n}})''(p, n) < 0\}$. This implies that

$$(r+\lambda)v^{\overline{n}}(p_2, n) - \frac{\sigma^2}{2}p^2(v^{\overline{n}})''(p_2, n) = \lambda v^{\overline{n}}(p_2, n+1) > (r+\lambda)v^{\overline{n}}(p_2, n). \tag{39}$$

Combining the inequalities (38) and (39), with the fact

$$\lim_{p\to 0^+}(r+\lambda)(v^{\overline{n}})(p, n) - \lambda v^{\overline{n}}(p, n+1) = 0,$$

one may conclude that $v^{\overline{n}}(p, n+1)$ is not monotonic in p, which is a contradiction. This proves the induction hypothesis and the statement that $v^{\overline{n}+1}(\cdot, n)$ is increasing in the interval $(0, p_n^{\overline{n}^*})$. ■

Lemma 4 *Let $v^{\overline{n}}$ be the function defined as in (19). Then, the function $v^{\overline{n}}(\cdot, n)$ is convex almost everywhere in the interval $(0, p_n^{\overline{n}^*})$, for $n = 0, \cdots, \overline{n} - 1$.*

Proof of Lemma 4 Proving the convexity of the function $v^{\overline{n}}(\cdot, n)$ in the interval $(0, p_n^{\overline{n}^*})$ is equivalent to prove that the function $(v^{\overline{n}})'(\cdot, n)$ is increasing. Since the function $v^{\overline{n}}(p, n)$ satisfies the boundary problem (36), then the function $(v^{\overline{n}})'(p, n)$ satisfies almost everywhere the second order ordinary differential equation

$$(r+\lambda-\mu)w^{\overline{n}}(p, n) - (\mu+\sigma^2)p(w^{\overline{n}})'(p, n) - \frac{\sigma^2}{2}p^2(w^{\overline{n}})''(p, n) = \lambda w^{\overline{n}}(p, n+1). \tag{40}$$

Additionally, it is known that the following boundary conditions are also verified:

$$\lim_{p=0^+} w^{\overline{n}}(p, n) = \frac{\lambda^{\overline{n}-n}}{(r-\mu)(r+\lambda-\mu)^{\overline{n}-n}} \quad \text{and} \quad w^{\overline{n}}(p_n^{\overline{n}^*}, n) = \frac{1}{r-\mu}. \tag{41}$$

From (41), one can also conclude that

$$\lim_{p\to 0^+}(r+\lambda-\mu)w^{\overline{n}}(p, n) - \lambda w^{\overline{n}\prime}(p, n+1) = 0 \tag{42}$$

Therefore, we can prove by induction that, for a fixed $\overline{n}$, the function $p \to (v^{\bar{n}})'(p, n)$ is increasing for each $n = 0, \cdots, \overline{n}-1$. For $n = \overline{n}-1$ the function $p \to (v^{\overline{n}})(p, \overline{n})$ is trivially convex because $v^{\overline{n}}(p, \overline{n}) = A^{\overline{n}}_{\overline{n}-1,0,0}p^{d_1} +$

$\lambda\left[\frac{p}{(r-\mu)(r+\lambda-\mu)} - \frac{I\phi^{\overline{n}}}{(r+\lambda)}\right]$, with $A^{\overline{n}}_{\overline{n}-1,0,0} > 0$. Now, our induction hypothesis is that if $v^{\bar{n}}(p, n+1)$ is convex for $p < p^{\overline{n}^*}_{n+1}$, then $v^{\overline{n}}(p, n)$ is also convex when $p < p^{\overline{n}^*}_{n}$. As in the proof of the previous lemma the induction process is applied in a backward way and the induction hypothesis can be proven by contradiction.

Let us assume that $(v^{\overline{n}})'(p, n)$ is not increasing. Taking into account (41), we know that $\lim_{p=0^+}(v^{\overline{n}})'(p, n) < (v^{\overline{n}})'(p^{\overline{n}^*}_{n}, n)$, then there is an interval $I \subset]0, p^{\overline{n}^*}_{n}[$ such that $(v^{\overline{n}})'(\cdot, n)$ is increasing.Therefore, one of the following situations may happen:

(a) there is $p_1 \in]0, p^{\overline{n}^*}_{n}[$ such that $p_1 = \arg\max_{p\in]0,p^{\overline{n}^*}_{n}[}(v^{\overline{n}})'(p, n)$ and $(v^{\overline{n}})'(p_1, n) \geq \dfrac{1}{r-\mu}$

(b) there is $p_2 \in]0, p^{\overline{n}^*}_{n}[$ such that $p_2 = \arg\min_{p\in]0,p^{\overline{n}^*}_{n}[}(v^{\overline{n}})'(p, n)$ and $(v^{\overline{n}})'(p_2, n) \leq \dfrac{\lambda^{\overline{n}-n}}{(r-\mu)(r+\lambda-\mu)^{\overline{n}-n}}$

(c) there are $p_1 < p_2 < p^{\overline{n}^*}_{n}$ such that $p_1 = \arg\max_{p\in]0,p^{\overline{n}^*}_{n}[}(v^{\overline{n}})'(p, n)$, $p_2 = \arg\min_{p\in]0,p^{\overline{n}^*}_{n}[}(v^{\overline{n}})'(p, n)$

$$(v^{\overline{n}})'(p_1, n) < \frac{1}{r-\mu} \text{ and } (v^{\overline{n}})'(p_2, n) > \frac{\lambda^{\overline{n}-n}}{(r-\mu)(r+\lambda-\mu)^{\overline{n}-n}}$$

Situation (a): At the point p_1, it is known that $(w^{\overline{n}})'(p_1, n) = 0$ and $(w^{\overline{n}})''(p_1, n) < 0$. Therefore, the ordinary differential equation (40), at the point p_1, takes the form

$$(r+\lambda-\mu)w^{\overline{n}}(p_1, n) - \frac{\sigma^2}{2}p_1^2(w^{\overline{n}})''(p_1, n) = \lambda(v^{\overline{n}})'(p_1, n+1) > (r+\lambda-\mu)w^{\overline{n}}(p_1, n) > \frac{r+\lambda-\mu}{r-\mu}. \quad (43)$$

Combining Equations (42) and (43), with the fact that

$$\frac{\lambda}{r-\mu} = \lambda(v^{\overline{n}})'(p^{\overline{n}^*}_{n+1}, n+1) < (r+\lambda-\mu)w^{\overline{n}}(p^{\overline{n}^*}_{n}, n) = \frac{r+\lambda-\mu}{r-\mu},$$

we can conclude that $(v^{\overline{n}})'(\cdot, n+1)$ has a maximum such that $\lambda(v^{\overline{n}})'(p_1, n+1) > \frac{r+\lambda-\mu}{r-\mu}$, which is a contradiction.

Situation (b): At the point p_2, it is known that $(w^{\overline{n}})'(p_2, n) = 0$ and $(w^{\overline{n}})''(p_2, n) > 0$. Therefore, the ordinary differential equation (40), at the point p_2, is the following

$$\begin{aligned}(r+\lambda-\mu)w^{\overline{n}}(p_2, n) - \frac{\sigma^2}{2}p_2^2(w^{\overline{n}})''(p_2, n) = \lambda(v^{\overline{n}})'(p_2, n+1) < (r+\lambda-\mu)w^{\overline{n}}(p_2, n) \\ \leq \frac{\lambda^{\overline{n}-n}}{(r-\mu)(r+\lambda-\mu)^{\overline{n}-n-1}}.\end{aligned} \quad (44)$$

Combining Equations (41), (42) and (44), with the fact that

$$\lambda(v^{\overline{n}})'(p^{\overline{n}^*}_{n+1}, n+1) = \frac{\lambda^{\overline{n}-n}}{(r-\mu)(r+\lambda-\mu)^{\overline{n}-n-1}}$$

we can conclude that $(v^{\overline{n}})'(\cdot, n+1)$ has a minimum, which is a contradiction.

Situation (c) can be proved by using the same type of arguments used in situations (a) and (b). In this case, the contradiction is obtained noticing that at the function $(v^{\overline{n}})'(\cdot, n+1)$ has at least a maximum in the interval (p_1, p_2). ■

To complete the proof of Proposition 5, we start by proving the following conditions:

$$(r+\lambda)v^{\overline{n}}(p,n) - \mu p (v^{\overline{n}})'(p,n) - \frac{\sigma^2}{2}p^2(v^{\overline{n}})''(p,n) - \lambda v^{\overline{n}}(p,n+1) \geq 0, \quad \text{for all } p \geq p_n^{\overline{n}^*} \tag{45}$$

$$v^{\overline{n}}(p,n) - g(p,n) \geq 0, \quad \text{for all } 0 < p \leq p_n^{\overline{n}^*} \tag{46}$$

Proof of (45)*:* We start by noticing that, for $p \geq p_n^{\overline{n}^*}$

$$(r+\lambda)v^{\overline{n}}(p,n) - \mu p (v^{\overline{n}})'(p,n) - \frac{\sigma^2}{2}p^2(v^{\overline{n}})''(p,n) - \lambda v^{\overline{n}}(p,n+1) = p - I\phi^n(r+\lambda(1-\phi)).$$

Let us assume that $p_n^{\overline{n}^*}$ is such that $p_n^{\overline{n}^*} - I\phi^n(r+\lambda(1-\phi)) < 0$. Then, taking into account the smooth-paste conditions we have that

$$p_n^{\overline{n}^*} - I\phi^n(r+\lambda(1-\phi)) - \frac{\sigma^2}{2}(p_n^{\overline{n}^*})^2(v^{\overline{n}})''(p_n^{\overline{n}^*}, n) = 0,$$

which is a contradiction, because, in that case, there is $\epsilon > 0$ such that $(v^{\overline{n}})''(p,n) < 0$ for $p \in (p_n^{\overline{n}^*} - \epsilon, p_n^{\overline{n}^*}]$.

Proof of (46)*:* The inequality (46) follows trivially from the convexity of the function $v^{\overline{n}}(\cdot, n)$ and the fact that the curve $v^{\overline{n}}(\cdot, n)$ is tangent to $g(\cdot, n)$ at the point $p_n^{\overline{n}^*}$.

Finally, one can notice that by construction the functions $v^{\overline{n}}(\cdot, n)$ is $C^2(]0,+\infty[\backslash\{p_{\overline{n}-1}^{\overline{n}^*}, p_{\overline{n}-2}^{\overline{n}^*}, \cdots, p_n^{\overline{n}^*}\}) \cap C^1\left(\{p_{\overline{n}-1}^{\overline{n}^*}, p_{\overline{n}-2}^{\overline{n}^*}, \cdots, p_n^{\overline{n}^*}\}\right)$. Indeed, the function $v^{\overline{n}}(\cdot, n)$ in $]0,+\infty[\backslash\{p_{\overline{n}-1}^{\overline{n}^*}, p_{\overline{n}-2}^{\overline{n}^*}, \cdots, p_n^{\overline{n}^*}\}$ is a classical solution for an ODE, thus, it C^2 in the correspondent domain. Moreover, in light of the smooth-fit conditions, the functions are C^1 at the thresholds.

B.6 Proof of Proposition 4

We split the proof in three parts.

- *Monotony of $p_1^{2^*}$ regarding μ and σ:*

To prove this part, we notice that

$$\frac{\partial p_1^{2^*}}{\partial i} = d_2^{-2}\frac{\partial d_2}{\partial i} I\phi(r+\lambda(1-\phi))$$

with $i = \mu, \sigma^2$. The intended result follows taking into account that

$$\frac{\partial p_1^{2^*}}{\partial \mu} = \frac{-1}{\sigma^2}\left(1 + \frac{\frac{\sigma^2}{2} - \mu}{\sqrt{\left(\frac{\sigma^2}{2} - \mu\right)^2 + 2\sigma^2(r+\lambda)}}\right) < 0$$

and

$$\frac{\partial p_1^{2^*}}{\partial \sigma^2} = \frac{2\mu^2 + 2\sigma^2(r+\lambda) + \mu\left(-\sigma^2 + \sqrt{(\sigma^2 - 2\mu)^2 + 8\sigma^2(r+\lambda)}\right)}{\sigma^4\sqrt{(\sigma^2 - 2\mu)^2 + 8\sigma^2(r+\lambda)}} > 0.$$

The second inequality follows in light of the condition $r + \lambda > \mu$.

- *Monotony of $p_1^{2^*}$ regarding ϕ:*

It is a matter of calculations to see that

$$\frac{\partial p_1^{2^*}}{\partial \phi} = \frac{d_2 - 1}{d_2} I(r + \lambda(1 - 2\phi)).$$

Therefore, $\frac{\partial p_1^{2^*}}{\partial \phi} > 0$ if and only if $\phi < \frac{r+\lambda}{2\lambda}$. Additionally, we notice that $\frac{r+\lambda}{2\lambda} \Leftrightarrow r \geq \lambda \geq 1$, which concludes this part of the prof.

- *Monotony of $p_1^{2^*}$ regarding λ:*

After some calculations, one can obtain that

$$\frac{\partial p_1^{2\star}}{\partial \lambda} = I\phi d_2^{-2}\left[\frac{\partial d_2}{\partial \lambda}(r + \lambda(1-\phi)) + (d_2^2 - d_2)(1-\phi)\right].$$

Additionally, one can easily see that, for every $0 \leq \phi < 1$,

$$\lim_{\lambda \to \infty} \frac{\partial d_2}{\partial \lambda}(r + \lambda(1-\phi)) + (d_2^2 - d_2)(1-\phi) = +\infty,$$

and, for $\phi = 1$, $\frac{\partial p_1^{2\star}}{\partial \lambda} = I d_2^{-2} \frac{\partial d_2}{\partial \lambda} r < 0$. Therefore, taking into account that $p_1^{2^*}$ is a continuous function in ϕ, we conclude that for large values of ϕ (at least in a neighborhood of 1), the sign of the derivative $\frac{\partial p_1^{2\star}}{\partial \lambda}$ must change from negative to positive. Then, for such values of ϕ, the function $p_1^{2\star}$ is non-monotonic with respect to λ.

Moreover, for $\phi = 0$, some trivial calculations lead us to

$$\begin{aligned}
\frac{\partial d_2}{\partial \lambda}(r + \lambda(1-\phi)) + (d_2^2 - d_2)(1-\phi)|_{\phi=0} &= \frac{\partial d_2}{\partial \lambda}(r+\lambda) + (d_2^2 - d_2) \\
&= d_2^2\left[\frac{\sigma^2}{2} + \sqrt{\left(\frac{\sigma^2}{2} - \mu\right)^2 + 2\sigma^2(r+\lambda)}\right],
\end{aligned}$$

which is always positive. Thus, for small values of ϕ (at least in a neighborhood of 0), the sign of the derivative $\frac{\partial p_1^{2\star}}{\partial \lambda}$ must be always positive. Then, for such values of ϕ, the function $p_1^{2\star}$ increases with respect to λ. ■

C Tables

	$\bar{n}=2$	$\bar{n}=3$	$\bar{n}=4$	$\bar{n}=5$	$\bar{n}=6$	$\bar{n}=7$	$\bar{n}=8$	$\bar{n}=9$	$\bar{n}=10$
$p_0^{\bar{n}\star}$	0.06751563	0.06762701	0.06765623	0.06766430	0.06766660	0.06766727	0.06766747	0.06766753	0.06766755
$p_1^{\bar{n}\star}$	0.06033744	0.06076407	0.06086431	0.06089061	0.06089787	0.06089994	0.06090054	0.06090072	0.06090078
$p_2^{\bar{n}\star}$		0.05430369	0.05468766	0.05477788	0.05480155	0.05480808	0.05480994	0.05481049	0.05481065
$p_3^{\bar{n}\star}$			0.04887332	0.04921890	0.04930009	0.04932139	0.04932727	0.04932895	0.04932944
$p_4^{\bar{n}\star}$				0.04398599	0.04429701	0.04437008	0.04438925	0.04439454	0.04439605
$p_5^{\bar{n}\star}$					0.03958739	0.03986731	0.03993307	0.03995033	0.03995509
$p_6^{\bar{n}\star}$						0.03562865	0.03588058	0.03593977	0.03595529
$p_7^{\bar{n}\star}$							0.03206579	0.03229252	0.03234579
$p_8^{\bar{n}\star}$								0.02885921	0.02906327
$p_9^{\bar{n}\star}$									0.02597329

Table 1: $p_n^{\overline{n}^*}$ for $\overline{n} = 2 \cdots 10$ and $n = 0, \cdots, \overline{n} - 1$.

	$\bar{n}=3$	$\bar{n}=4$	$\bar{n}=5$	$\bar{n}=6$	$\bar{n}=7$	$\bar{n}=8$	$\bar{n}=9$	$\bar{n}=10$
$p_0^{\bar{n}\star}p_0^{\bar{n}-1\star}$	0.00011137	0.00002922	0.00000807	0.00000230	0.00000067	0.00000020	0.00000006	0.00000002
$p_1^{\bar{n}\star}p_1^{\bar{n}-1\star}$	0.00042664	0.00010024	0.00002630	0.00000726	0.00000207	0.00000060	0.00000018	0.00000005
$p_2^{\bar{n}\star}p_2^{\bar{n}-1\star}$		0.00038397	0.00009021	0.00002367	0.00000653	0.00000186	0.00000054	0.00000016
$p_3^{\bar{n}\star}p_3^{\bar{n}-1\star}$			0.00034558	0.00008119	0.00002130	0.00000588	0.00000168	0.00000049
$p_4^{\bar{n}\star}p_4^{\bar{n}-1\star}$				0.00031102	0.00007307	0.00001917	0.00000529	0.00000151
$p_5^{\bar{n}\star}p_5^{\bar{n}-1\star}$					0.00027992	0.00006577	0.00001725	0.00000476
$p_6^{\bar{n}\star}p_6^{\bar{n}-1\star}$						0.00025192	0.00005919	0.00001553
$p_7^{\bar{n}\star}p_7^{\bar{n}-1\star}$							0.00022673	0.00005327
$p_8^{\bar{n}\star}p_8^{\bar{n}-1\star}$								0.00020406

Table 2: $p_n^{\overline{n}^*}p_n^{\overline{n}-1^*}$ for $\overline{n}=3\cdots 10$ and $n=0,\cdots,\overline{n}-2$.

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