Reinsurance of multiple risks with generic dependence structures

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Abstract

We consider the optimal reinsurance problem from the point of view of a direct insurer owning several dependent risks, assuming a maximal expected utility criterion and independent negotiation of reinsurance for each risk. Without any particular hypothesis on the dependency structure, we show that optimal treaties exist in a class of independent randomized contracts. We derive optimality conditions and show that under mild assumptions the optimal contracts are of classical (non-randomized) type. A specific form of the optimality conditions applies in that case. We illustrate the results with some numerical examples.

Keywords: Reinsurance, Dependent Risks, Premium Calculation Principles, Expected Utility, Randomized reinsurance treaties

1 Introduction

Reinsurance constitutes a risk mitigation strategy and an essential tool in risk management. By transferring part of the risk to the reinsurer, the cedent company seeks a trade-off between profit, which is reduced by the reinsurance premium, and safety, which is increased by the reduction in exposure to the underlying risk. A large amount of works can be found in literature concerning optimal reinsurance strategies, as this problem has been for long considered in the actuarial community. The first works date as far back as the 60s, with the seminal paper of Borch [Borch, 1960]. It has been the subject of active research not only in the field of actuarial science, where the risk transfer contract between two insurance companies is usually analysed within a one-period setting, see for instance [Gajek and Zagrodny, 2004, Cai and Wei, 2012, Chi et al., 2017, Albrecher and Cani, 2019, Hu and Wang, 2019], but also in the financial mathematical context, within a dynamical framework usually in conjunction with investment strategies, see for instance [Cani and Thonhauser, 2016, Gu et al., 2018]. We refer to [Albrecher et al., 2017] for a comprehensive overview of the literature on optimal reinsurance.

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In this work we consider the optimal reinsurance problem within a one-period setting and from the cedent's perspective, the interest of the reinsurer being enclosed in the calculation principles considered for the reinsurance premium. The goal is the maximization of an expected concave utility of the insurer's wealth, which in the case of the exponential utility is known to be closely related with its adjustment coefficient [Guerra and Centeno, 2008].

In most works regarding optimal reinsurance, independence is assumed. Indeed, for many years dependence has not been considered in the study of optimal risk transfer, possibly due to its complexity. More recently, several works can be found accounting for dependencies, prompt by the need for real, robust and reliable quantitative risk analysis. One of the first works including the effects of dependence when investigating optimal forms or risk transfer is [Centeno, 2005]. There, dependencies of two classes of insurance businesses, through the number of claims, are included by means of a bivariate Poisson. Other authors have considered the optimal reinsurance problem under dependence between claim numbers, such as [Zhang et al., 2015] in a one-period setting and [Bi et al., 2016] in a dynamic setting. In [Cheung et al., 2014b], the authors do not assume any particular dependence structure, as they argue it is often difficult to determine it. They propose instead to use a minimax optimal reinsurance decision formulation, in which the worst-case scenario is first identified. In [Cai and Wei, 2012], positive dependencies in the individual risk are considered by means of the stochastic ordering.

Due to the analytical complexity of problems including dependencies, many works propose numerical frameworks [Bai et al., 2013, Asimit et al., 2017, Asimit et al., 2018, Zhang et al., 2016]. It is also worth mentioning the empirical approach proposed in [Tan and Weng, 2014] and [Sun et al., 2017], where reinsurance models are formulated based on data observation, without explicitly assuming the distribution of the underlying risks, which allows for the use of programming procedures to obtain an optimal solution.

Stop-loss was found to be optimal in several works which consider the expected value premium principle and various optimality criteria, both in the independent [Borch, 1960, Gajek and Zagrodny, 2004, Cheung et al., 2014a] and in the dependent cases [Cai and Wei, 2012, Zhang et al., 2015].

Most authors include various constraints on the type of reinsurance contracts being considered. For example, in [Borch, 1960, Gajek and Zagrodny, 2004, Zhang et al., 2015] the optimal solution is sought among combinations of quota-share and stop-loss treaties. A very common constraint, included in most references above, is that the ceded risk to the reinsurance is a decreasing function of the underlying risk. This is imposed to avoid solutions allowing for moral hazard.

In this work we will impose no further constraints besides that the ceded risk should be positive and should not exceed the underlying risk. We hope that by characterizing the solutions of the problem free of constraints we contribute to a better understanding of the necessity, role and impact of any constraints that might be introduced. The analysis of the impact of the "no moral hazard" constraint in the optimal solution will be the subject of future work.

Most literature on this topic considers forms or reinsurance of a deterministic nature, in the sense that for a given risk X, solutions are seek in a set of functions of X defining for each value of loss how much of that loss is ceded to the reinsurer. This is the traditional and intuitive way of formulating the problem. However, it has been shown that randomized treaties may be preferable to deterministic ones. In [Gajek and Zagrodny, 2004] the author observes that randomized reinsurance contracts may lead to lower ruin probability than deterministic ones, when an upper constraint on the reinsurance premium is imposed. The randomization improvement decreases as the probability mass of every atom decreases. The randomized decision can also be eliminated by adjusting the upper limit on the price of reinsurance. In [Albrecher and Cani, 2019] the authors explore the potential of randomizing reinsurance treaties to obtain optimal reinsurance strategies minimizing VaR under the expected value premium principle. They provide a possible interpretation for the randomization of the reinsurance treaty as the default risk of the reinsurer, since the indemnity may not be paid with a certain probability. It is also argued the advantage of randomized treaties concerning moral hazard issues, as due to randomness it is unclear *a priori* who will have to pay the claim. The authors study the possibility of implementing additional randomness in the settlement of risk transfer and show that randomizing the classical stop-loss can be beneficial for the insurer. In [Guerra and Centeno, 2012], randomization of the reinsurance treaties is used as a mathematical tool to find treaties minimizing several quantile risk measures when premia are calculated by a coherent risk measure. In the present work, randomized contracts will also serve as a mathematical tool to prove existence of the optimal contract when the underlying risks are dependent through an arbitrary joint distribution. To the best of our knowledge, this is the first time randomized treaties are analysed under dependencies.

In this work we consider the optimal reinsurance problem of n dependent risks. By risk we mean the aggregate claims of a line of business, a portfolio of policies or a policy. The dependence structure is arbitrary, defined through a generic joint distribution function. We assume that reinsurance is negotiated separately (independently) for each risk, and each premium is calculated by a (possibly different) function of moments of the ceded risk. The cedent's criterion is the maximization of the expected value of a concave utility function of the overall retained risk, net of reinsurance premia. We introduce the class of independent randomized strategies and show that it contains an optimal strategy. Optimality conditions are obtained, and it is shown that under mild conditions, the optimal strategy is deterministic. We then consider the particular cases of the expected value and variance type premium principles. We show that, without any constraints on the optimal treaty, the stop-loss is not optimal in general. In particular, the monotonicity constraint on the ceded risk considered in [Cai and Wei, 2012] is an active constraint.

This paper is organized as follows. In Section 2, we formulate the optimization problem. In Section 3 the class of randomized reinsurance strategies is introduced together with the proper probability spaces, and the existence result is proved. In Section 4, we provide necessary optimality conditions. In Section 5 it is shown that under very general conditions the optimal treaty is deterministic. The special cases of the expected-value and variance related premiums are developed. We conclude the paper in Section 6 with some numerical examples and concluding remarks.

2 The optimization problem

We consider a portfolio of $n \geq 2$ risks. Let X_i, $i = 1, 2, ..., n$ denote the aggregate value of claims placed under the i-th risk on a given period of time (say, one year). $X = (X_1, X_2, ..., X_n)$ is a non-negative random vector with joint probability law μ_X.

The direct insurer acquires a reinsurance policy for each risk separately. Each of these policies is a measurable function Z_i such that

$$\Pr\{0 \leq Z_i(X_i) \leq X_i\} = 1. \quad (1)$$

For each $i = 1, 2, ..., n$, let $\mathcal{Z}_i$ denote the set of measurable functions satisfying (1), and let $\mathcal{Z} = \prod_{i=1}^{n} \mathcal{Z}_i$, the cartesian product of all $\mathcal{Z}_i$.

For each risk, the corresponding policy is priced by a functional $P_i : \mathcal{Z}_i \mapsto [0, +\infty]$, depending only on the probability law of $Z_i(X_i)$. In this paper, we assume that these premia calculation principles are of type

$$P_i(Z_i) = \Psi_i\left(\mathbb{E}Z_i, \mathbb{E}Z_i^2, \ldots, \mathbb{E}Z_i^{k_i}\right), \tag{2}$$

where $\Psi_i : [0, +\infty[^{k_i} \mapsto \mathbb{R}$, $i = 1, 2, \ldots, n$ are continuous functions.

The insurer's net profit after reinsurance is

$$L_Z = c - \sum_{i=1}^{n} \left(P_i\left(Z_i\right) + X_i - Z_i\left(X_i\right)\right), \tag{3}$$

where c is the portfolio's aggregate premium income, net of non-claim refunding expenses. Thus, L_Z is a random variable taking values in the interval $] - \infty, c]$.

We assume that the insurer aims to choose a reinsurance strategy $Z = (Z_1, Z_2, \ldots, Z_n) \in \mathcal{Z}$ maximizing the expected utility of net profit, i.e., maximizing the functional

$$\rho(Z) = \mathbb{E}U(L_Z) \qquad Z \in \mathcal{Z}, \tag{4}$$

where $U :] - \infty, c] \mapsto \mathbb{R}$ is a concave nondecreasing function.

3 Existence of optimal reinsurance strategy

Under the formulation above, existence of an optimal reinsurance strategy in the class $\mathcal{Z}$ can not in general be guaranteed. However, existence in the larger class of *random treaties*, defined below, can easily be proved. Our approach to the problem outlined in the previous section will be to obtain optimality conditions for random treaties and then discuss conditions under which such conditions can only be satisfied by classical treaties of class $\mathcal{Z}$.

3.1 Random treaties

First, let us introduce some notation. For any array $x = (x_1, x_2, \ldots, x_n)$, we will use the usual notation x_i to denote the i^{th} element of x, and the notation $x_{[i]}$ to denote the array of $n-1$ elements obtained from x by deleting the element x_i, i.e.

$$x_{[i]} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

With the notation above, random treaties are defined as follows:

Definition 3.1 *A $\mathbb{R}^n$-valued random variable $Z = (Z_1, Z_2, \ldots, Z_n)$ is said to be a vector of (independent) random treaties or a randomized strategy if the following conditions hold for $i = 1, 2, \ldots, n$:*

1. $\Pr\{0 \leq Z_i \leq X_i\} = 1$*;*
2. *The random variable Z_i is conditionally independent of the random vector $\left(X_{[i]}, Z_{[i]}\right)$, given X_i.*

The second condition in Definition 3.1 enforces the assumption that reinsurance is acquired separately for different risks: given the value of X_i, the value of claims and refunds on other risks have no bearing on the value of the refund Z_i. However, Z_i, Z_j are not, in general, independent random variables, due to dependency between X_i, X_j.

Reinsurance strategies of class $\mathcal{Z}$ discussed in Section 2 are called *deterministic strategies* to distinguish them from randomized strategies defined above. Notice that for any deterministic strategy $Z \in \mathcal{Z}$, the random vector $(Z_1(X_1), Z_2(X_2), \ldots, Z_n(X_n))$ satisfies Definition 3.1. Thus, the class of deterministic strategies is a subset of the class of randomized strategies.

If expressions (2), (3), (4), are well defined for every deterministic strategy, then they are also well defined for every vector of random treaties. Thus, we may consider the problem of maximizing (4) over all vectors of independent random treaties.

3.2 Spaces of probability laws

Since (4) depends only of the probability law of the random vector (X, Z), we may discuss optimization in terms of probability laws instead of the random vectors inducing such laws. To this purpose, we will frame our argument in canonical spaces, i.e., we will consider the underlying measurable space to be $\mathbb{R}^{2n}$ provided with its Borel σ-algebra, $\mathcal{B}_{\mathbb{R}^{2n}}$. Random variables are Borel-measurable functions $\varphi : \mathbb{R}^{2n} \mapsto \mathbb{R}$, and the space of Borel probability measures $\nu : \mathcal{B}_{\mathbb{R}^{2n}} \mapsto [0,1]$ is denoted by $\mathcal{P}$. Expectations with respect to a particular probability law $\nu \in \mathcal{P}$ are

$$\mathbb{E}^{\nu}\varphi = \int_{\mathbb{R}^k} \varphi(u)\, \nu(du),$$

provided the integral exists. The space $\mathcal{P}$ is provided with its weak topology, that is, a sequence $\{\nu_j \in \mathcal{P}\}_{j\in\mathbb{N}}$ is said to converge to $\nu \in \mathcal{P}$ if and only if

$$\lim_{j\mapsto\infty} \mathbb{E}^{\nu_j}\varphi = \mathbb{E}^{\nu}\varphi$$

for every continuous bounded φ.

Let $\mathcal{P}_X$ be the set of all $\nu \in \mathcal{P}$ such that the marginal probability

$$\nu_X(A) = \nu\left(A \times \mathbb{R}^n\right) \qquad A \in \mathcal{B}_{\mathbb{R}^n}$$

is equal to μ_X, the joint probability law of the claim amounts, introduced in Section 2. We write $\nu \in \mathcal{H}_X$ if and only if ν is the probability law of a random vector (X, Z) where X is the vector of claim amounts and Z is some randomized strategy in the sense of Definition 3.1. It is clear that $\mathcal{H}_X \subset \mathcal{P}_X \subset \mathcal{P}$.

Coordinates in $\mathbb{R}^{2n}$ are indicated as $(x, z) = (x_1, \ldots, x_n, z_1, \ldots, z_n)$. For a given $\nu \in \mathcal{P}$, ν_X, ν_Z indicate the marginal probability laws:

$$\nu_X(A) = \nu\left(A \times \mathbb{R}^n\right), \quad \nu_Z(A) = \nu\left(\mathbb{R}^n \times A\right) \qquad A \in \mathcal{B}_{\mathbb{R}^n}.$$

Similar notation is used for other marginal probabilities, like ν_{X_i}, $\nu_{(X_i,Z_i)}$, $\nu_{(X_{[i]},Z_{[i]})}$, etc..

Given $\nu \in \mathcal{P}$ having a strictly positive density function f, the conditional probability law of (say) $(X_{[i]}, Z_{[i]})$ given (X_i, Z_i) is the family of probability measures

$$\nu_{(X_{[i]},Z_{[i]})|(X_i,Z_i)}(A) = \frac{\int_A f(x,z)dx_{[i]}dz_{[i]}}{\int_{\mathbb{R}^{2n-2}} f(x,z)dx_{[i]}dz_{[i]}} \qquad \forall A \in \mathcal{B}_{\mathbb{R}^{2n-2}}.$$

Notice that, for fixed $A \in \mathcal{B}_{\mathbb{R}^{2n-2}}$, $\nu_{\left(X_{[i]},Z_{[i]}\right)|(X_i,Z_i)}(A)$ is a measurable function of (x_i, z_i) defined up to null subsets of $\mathbb{R}^2$.

In the following, we will need to consider probability laws which do not have a density function and/or do not have full support. Thus, we will consider regular conditional probability laws. For a given $\nu \in \mathcal{P}$, a mapping $\nu_{\left(X_{[i]},Z_{[i]}\right)|(X_i,Z_i)} = Q : \mathbb{R}^2 \times \mathcal{B}_{\mathbb{R}^{2n-2}} \mapsto [0,1]$ is a regular conditional probability law of $\left(X_{[i]}, Z_{[i]}\right)$ given (X_i, Z_i) if it satisfies the following conditions:

(i) For every $B \in \mathcal{B}_{\mathbb{R}^{2n-2}}$ (fixed), the map $(x_i, z_i) \mapsto Q((x_i, z_i), B)$ is measurable with respect to the Borel σ-algebra of $\mathbb{R}^2$;

(ii) There is a set $A \in \mathcal{B}_{\mathbb{R}^2}$ such that $\nu_{(X_i,Z_i)}(A) = 1$ and for each $(x_i, z_i) \in A$ (fixed) the map $B \mapsto Q((x_i, z_i), B)$ is a probability measure in $\mathcal{B}_{\mathbb{R}^{2n-2}}$;

(iii) For every $A \in \mathcal{B}_{\mathbb{R}^2}$, $B \in \mathcal{B}_{\mathbb{R}^{2n-2}}$,

$$\int_A Q((x_i, z_i), B)\, \nu_{(X_i,Z_i)}(d(x_i, z_i)) = \\ = \nu\left\{(x, z) \in \mathbb{R}^{2n} : (x_i, z_i) \in A,\ (x_{[i]}, z_{[i]}) \in B\right\}.$$

Since $\left(\mathbb{R}^{2n}, \mathcal{B}_{\mathbb{R}^{2n}}\right)$ is a standard measurable space, every $\nu \in \mathcal{P}$ admits a regular conditional probability law $\nu_{(X_{[i]},Z_{[i]})|(X_i,Z_i)}$ (see e.g. [Çinlar, 2011], Theorem IV.2.7).

Conditional expectation of a random variable $\varphi : \mathbb{R}^{2n} \mapsto \mathbb{R}$ is defined as

$$\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|(X_i,X_i)}\varphi = \int_{\mathbb{R}^{2n-2}} \varphi(x, z)\, Q(x_i, z_i, d(x_{[i]}, z_{[i]})),$$

provided the integral on the right-hand side is well defined for $\nu_{(X_i,Z_i)}$-almost every $(x_i, z_i) \in \mathbb{R}^2$. In that case, it is a measurable function of (x_i, z_i).

If conditioned and conditioning coordinates do not span the whole space $\mathbb{R}^{2n}$, then the conditional probability law is defined as above with respect to the subspace spanned by the coordinates concerned. For example $\nu_{(X_{[i]},Z_{[i]})|X_i} : \mathbb{R} \times \mathcal{B}_{\mathbb{R}^{2n-2}} \mapsto [0,1]$ is defined with respect to the marginal probability law $\nu_{(X,Z_{[i]})} : \mathcal{B}_{\mathbb{R}^{2n-1}} \mapsto [0,1]$ instead of the joint probability law ν. Conditional expectations of random variables $\varphi = \varphi(x, z_{[i]})$ are defined as above, but the conditional expectation operator $\varphi \mapsto \mathbb{E}^{\nu}_{\left(X_{[i]},Z_{[i]}\right)|X_i}\varphi$ acts on random variables depending on the full range of coordinates (x, z) by

$$\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|X_i}\varphi = \int_{\mathbb{R}^{2n-2}} \varphi(x, z)\, Q'(x_i, d(x_{[i]}, z_{[i]})).$$

This is a measurable function of $(x_i, z_i) \in \mathbb{R}^2$, while

$$\mathbb{E}^{\nu}_{X_{[i]}|X_i}\varphi = \int_{\mathbb{R}^{n-1}} \varphi(x, z) Q''(x_i, dx_{[i]})$$

is a function of $(x_i, z) \in \mathbb{R}^{1+n}$.

Recall that two σ-algebras $\mathcal{A}_1$, $\mathcal{A}_2$ are conditionally independent given a σ-algebra $\mathcal{B}$ if and only if $\Pr\left(A|\,\sigma(\mathcal{A}_1 \cup \mathcal{B})\right) = \Pr\left(A|\,\mathcal{B}\right)$ for every $A \in \mathcal{A}_2$ (see e.g. [Çinlar, 2011], Proposition IV.3.2). Taking Definition 3.1 into account, $\mathcal{H}_X$ can be characterized as the set of all $\nu \in \mathcal{P}$ satisfying the following conditions:

(i) $\nu_X = \mu_X$;

(ii) ν is concentrated on the set

$$\left\{(x,z) \in \mathbb{R}^{2n} : 0 \leq z_i \leq x_i \quad i = 1, 2, \ldots, n\right\};$$

(iii) $\nu_{(X_{[i]},Z_{[i]})|(X_i,Z_i)} = \nu_{(X_{[i]},Z_{[i]})|X_i}, \qquad i = 1, 2, \ldots, n,$
i.e., for every $A \in \mathcal{B}_{\mathbb{R}^{2n-2}}$, the function $(x_i, z_i) \mapsto \nu_{(X_{[i]},Z_{[i]})|(X_i,Z_i)}(x_i, z_i, A)$ depends only on x_i.

The premium calculation principles (2) can be extended to $\nu \in \mathcal{H}_X$ such that $\mathbb{E}^\nu Z_i^{k_i} < +\infty$ through the obvious expression

$$P_i(\nu) = \Psi_i\left(\mathbb{E}^\nu Z_i, \mathbb{E}^\nu Z_i^2, \ldots, \mathbb{E}^\nu Z_i^{k_i}\right). \tag{5}$$

Similarly, the net profit random variable is the function $L_\nu : \mathbb{R}^{2n} \mapsto \mathbb{R}$ defined as

$$L_\nu(x,z) = c - \sum_{i=1}^{n} \left(P_i(\nu) + x_i - z_i\right), \tag{6}$$

and the expected utility functional (4) can be extended to $\nu \in \mathcal{H}_X$ by

$$\rho(\nu) = \mathbb{E}^\nu U\left(L_\nu\right). \tag{7}$$

The following proposition provides an important reason to consider the space $\mathcal{H}_X$ instead of $\mathcal{Z}$.

Proposition 3.2 *$\mathcal{H}_X$ is a compact subset of $\mathcal{P}$.*

Proof. It follows immediately from the definition of $\mathcal{H}_X$ that

$$\nu\left([0,M]^{2n}\right) = \mu_X\left([0,M]^n\right),$$

for any $\nu \in \mathcal{H}_X$ and $M \in [0, +\infty[$. Since $\lim\limits_{M\to+\infty} \mu_X\left([0,M]^n\right) = 1$, this shows that $\mathcal{H}_X$ is uniformly tight. Hence, Prohorov's theorem (see e.g. [Billingsley, 1999]) states that $\mathcal{H}_X$ is a relatively compact subset of $\mathcal{P}$.

Fix $\nu \in \partial\mathcal{H}_X$, and pick a sequence $\{\nu_j \in \mathcal{H}_X\}_{j\in\mathbb{N}}$ converging weakly to ν. Notice that for every continuous bounded function $\alpha : \mathbb{R} \mapsto \mathbb{R}$,

$$\mathbb{E}^\nu \alpha(X_i) = \int_{\mathbb{R}} \alpha \, d\mu_{X_i} = \mathbb{E}^{\nu_j} \alpha(X_i) \qquad \forall j \in \mathbb{N},\ i = 1, 2, \ldots, n.$$

First, we claim that for any continuous bounded function $\varphi : \mathbb{R}^{2n-1} \mapsto \mathbb{R}$, the sequence $\left\{\mathbb{E}^{\nu_j}_{(X_{[i]},Z)|X_i} \varphi(X_{[i]}, Z)\right\}_{j\in\mathbb{N}}$ converges pointwise to $\mathbb{E}^\nu_{(X_{[i]},Z)|X_i} \varphi(X_{[i]}, Z)$ almost certainly with respect to μ_{X_i}. To see this, notice that for any continuous bounded function $\alpha : \mathbb{R} \mapsto \mathbb{R}$, we have

$$\mathbb{E}^\nu\left[\alpha(X_i)\left(\mathbb{E}^{\nu_j}_{(X_{[i]},Z)|X_i} \varphi(X_{[i]}, Z) - \mathbb{E}^\nu_{(X_{[i]},Z)|X_i} \varphi(X_{[i]}, Z)\right)\right] =$$

$$=\mathbb{E}^{\nu}\mathbb{E}^{\nu_j}_{(X_{[i]},Z)|X_i}[\alpha\varphi]-\mathbb{E}^{\nu}\mathbb{E}^{\nu}_{(X_{[i]},Z)|X_i}[\alpha\varphi]=\mathbb{E}^{\nu_j}[\alpha\varphi]-\mathbb{E}^{\nu}[\alpha\varphi].$$

Since $\alpha\varphi$ is continuous and bounded in $\mathbb{R}^{2n}$, weak convergence of ν_j implies that the difference above converges to zero. Since α is arbitrary, this implies that $\mathbb{E}^{\nu_j}_{(X_{[i]},Z)|X_i}\varphi$ converges pointwise to $\mathbb{E}^{\nu}_{(X_{[i]},Z)|X_i}\varphi$.

Now, consider continuous bounded functions $\alpha:\mathbb{R}^2\mapsto\mathbb{R}$, $\varphi:\mathbb{R}^{2n-2}\mapsto\mathbb{R}$. Since $\nu_j\in\mathcal{H}_X$, we have:

$$\begin{aligned}
&\mathbb{E}^{\nu}\left[\alpha(X_i,Z_i)\left(\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|(X_i,Z_i)}\varphi(X_{[i]},Z_{[i]})-\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|X_i}\varphi(X_{[i]},Z_{[i]})\right)\right]=\\
=&\mathbb{E}^{\nu}\left[\alpha\left(\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|(X_i,Z_i)}\varphi-\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|X_i}\varphi\right)\right]-\\
&-\mathbb{E}^{\nu_j}\left[\alpha\left(\mathbb{E}^{\nu_j}_{(X_{[i]},Z_{[i]})|(X_i,Z_i)}\varphi-\mathbb{E}^{\nu_j}_{(X_{[i]},Z_{[i]})|X_i}\varphi\right)\right]=\\
=&\mathbb{E}^{\nu}\left[\alpha\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|(X_i,Z_i)}\varphi\right]-\mathbb{E}^{\nu}\left[\alpha\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|X_i}\varphi\right]-\\
&-\mathbb{E}^{\nu_j}\left[\alpha\mathbb{E}^{\nu_j}_{(X_{[i]},Z_{[i]})|(X_i,Z_i)}\varphi\right]+\mathbb{E}^{\nu_j}\left[\alpha\mathbb{E}^{\nu_j}_{(X_{[i]},Z_{[i]})|X_i}\varphi\right]=\\
=&\mathbb{E}^{\nu}[\alpha\varphi]-\mathbb{E}^{\nu_j}[\alpha\varphi]+\mathbb{E}^{\nu_j}\left[\mathbb{E}^{\nu_j}_{(X_{[i]},Z_{[i]})|X_i}\alpha\mathbb{E}^{\nu_j}_{(X_{[i]},Z_{[i]})|X_i}\varphi\right]-\\
&-\mathbb{E}^{\nu}\left[\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|X_i}\alpha\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|X_i}\varphi\right]=\\
=&\mathbb{E}^{\nu}[\alpha\varphi]-\mathbb{E}^{\nu_j}[\alpha\varphi]+\\
&+\mathbb{E}^{\nu}\left[\mathbb{E}^{\nu_j}_{(X_{[i]},Z_{[i]})|X_i}\alpha\mathbb{E}^{\nu_j}_{(X_{[i]},Z_{[i]})|X_i}\varphi-\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|X_i}\alpha\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|X_i}\varphi\right].
\end{aligned}$$

The sequence $\mathbb{E}^{\nu_j}_{(X_{[i]},Z_{[i]})|X_i}\alpha\mathbb{E}^{\nu_j}_{(X_{[i]},Z_{[i]})|X_i}\varphi$ is essentially bounded and converges pointwise to $\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|X_i}\alpha\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|X_i}\varphi$. Therefore, weak convergence of ν_j and Lebesgue's dominated convergence theorem guarantee that the right-hand side in the equality above goes to zero when j goes to infinity. This shows that

$$\mathbb{E}^{\nu}\left[\alpha\left(\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|(X_i,Z_i)}\varphi-\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|X_i}\varphi\right)\right]=0.$$

Since α is arbitrary, this implies that $\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|(X_i,Z_i)}\varphi=\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|X_i}\varphi$ almost certainly with respect to $\nu_{(X_i,Z_i)}$. Since φ is arbitrary, this implies $\nu_{(X_{[i]},Z_{[i]})|(X_i,Z_i)}=\nu_{(X_{[i]},Z_{[i]})|X_i}$ and therefore $\nu\in\mathcal{H}_X$. Thus, $\mathcal{H}_X$ is closed and therefore it is compact. ■

3.3 Existence of random maximizers

Now, we prove that the functional (7) admits a maximizer in the space $\mathcal{H}_X$, i.e., the optimal reinsurance problem admits a solution in the class of randomized strategies (Theorem 3.4). To this purpose, we will need the following result concerning moments of ceded risks.

Proposition 3.3 *If $\mathbb{E}X_i^k<\infty$, then the functional $\nu\mapsto\mathbb{E}^{\nu}Z_i^k$ is continuous in $\mathcal{H}_X$.*

Proof. Consider a sequence $\{\nu_j \in \mathcal{H}_X\}_{j\in\mathbb{N}}$, converging weakly to some $\nu \in \mathcal{H}_X$.

By weak convergence, for every $M < +\infty$ we have

$$\mathbb{E}^{\nu}\left(Z_i \wedge M\right)^k = \lim_{j\to\infty} \mathbb{E}^{\nu_j}\left(Z_i \wedge M\right)^k \leq \liminf_{j\to\infty} \mathbb{E}^{\nu_j} Z_i^k.$$

Since $\mathbb{E}^{\nu} Z_i^k = \lim_{M\to\infty} \mathbb{E}^{\nu}\left(Z_i \wedge M\right)^k$, this proves that

$$\mathbb{E}^{\nu} Z_i^k \leq \liminf_{j\to\infty} \mathbb{E}^{\nu_j} Z_i^k.$$

If $\mathbb{E}\left[X_i^k\right] < +\infty$, then $\lim_{M\to+\infty} \mathbb{E}^{\nu}\left[X_i^k \chi_{\{X_i>M\}}\right] = 0$. Since

$$\begin{aligned}
\limsup_{j\to\infty} \mathbb{E}^{\nu_j} Z_i^k \leq & \limsup_{j\to\infty} \mathbb{E}^{\nu_j}\left[\left(Z_i \wedge M\right)^k + X_i^k \chi_{\{X_i>M\}}\right] = \\
= & \lim_{j\to\infty} \mathbb{E}^{\nu_j}\left(Z_i \wedge M\right)^k + \mathbb{E}^{\nu}\left[X_i^k \chi_{\{X_i>M\}}\right] = \\
= & \mathbb{E}^{\nu}\left(Z_i \wedge M\right)^k + \mathbb{E}^{\nu}\left[X_i^k \chi_{\{X_i>M\}}\right],
\end{aligned}$$

we see that

$$\limsup_{j\to\infty} \mathbb{E}^{\nu_j} Z_i^k \leq \mathbb{E}^{\nu} Z_i^k.$$

■

The main result in this Section is the following.

Theorem 3.4 *If $\mathbb{E}X_i^{k_i} < \infty$ for $i = 1, 2, \ldots, n$, then the expected utility functional (7) is upper semicontinuous in $\mathcal{H}_X$ and therefore admits a maximizer.*

Proof. Suppose that $\mathbb{E}X_i^{k_i} < \infty$ for $i = 1, 2, \ldots, n$, and fix a sequence $\{\nu_j \in \mathcal{H}_X\}_{j\in\mathbb{R}}$ converging weakly to some $\nu \in \mathcal{H}_X$.

By the definition of $\mathcal{H}_X$,

$$\mathbb{E}^{\nu_j}\varphi(X) = \mathbb{E}^{\nu}\varphi(X) \qquad \forall j \in \mathbb{N},$$

for any measurable function $\varphi : \mathbb{R}^n \mapsto \mathbb{R}$ such that the expectation on the right-hand side is well defined (possibly infinite). Therefore,

$$\begin{aligned}
& \mathbb{E}^{\nu_j} U\left(c - \sum_{i=1}^{n}(P_i(\nu_j) + X_i - Z_i)\right) = \\
= & \mathbb{E}^{\nu}\mathbb{E}^{\nu_j}_{Z|X} U\left(c - \sum_{i=1}^{n}(P_i(\nu_j) + X_i - Z_i)\right).
\end{aligned}$$

By Proposition 3.3, the argument in the proof of Proposition 3.2 can be used to show that

$$\mathbb{E}^{\nu_j}_{Z|X} U\left(c - \sum_{i=1}^{n}(P_i(\nu_j) + X_i - Z_i)\right) \to \mathbb{E}^{\nu}_{Z|X} U\left(c - \sum_{i=1}^{n}(P_i(\nu) + X_i - Z_i)\right)$$

pointwise when $j \to \infty$. Further, since the utility function $U :]-\infty, c] \mapsto \mathbb{R}$ is concave nondecreasing, it is continuous and bounded above by $U(c) < +\infty$. Therefore, due to Fatou's Lemma:

$$\begin{aligned}
&\limsup_{j\to\infty} \mathbb{E}^{\nu_j} U\left(c - \sum_{i=1}^{n}(P_i(\nu_j) + X_i - Z_i)\right) \leq \\
\leq &\mathbb{E}^{\nu} \limsup_{j\to\infty} \mathbb{E}^{\nu_j}_{Z|X} U\left(c - \sum_{i=1}^{n}(P_i(\nu_j) + X_i - Z_i)\right) = \\
= &\mathbb{E}^{\nu} U\left(c - \sum_{i=1}^{n}(P_i(\nu) + X_i - Z_i)\right),
\end{aligned}$$

i.e., the functional (7) is upper semicontinuous in $\mathcal{H}_X$.

Existence of a maximizer follows by Weierstrass' theorem. ∎

4 Optimality conditions

For any $i \in \{1, 2, \ldots, n\}$, $(\hat{x}_i, \hat{z}_i) \in \mathbb{R}^2$, and $\varepsilon \geq 0$, let $B_{i,\varepsilon} = B_{i,\varepsilon}(\hat{x}_i, \hat{z}_i)$ denote the cylinder

$$B_{i,\varepsilon} = \left\{(x, z) \in \mathbb{R}^{2n} : (x_i - \hat{x}_i)^2 + (z_i - \hat{z}_i)^2 \leq \varepsilon^2\right\}$$

Fix $\nu \in \mathcal{H}_X$. We consider perturbations of ν, that is, probability laws $\tilde{\nu} = \nu^{i,\hat{x}_i,\hat{z}_i,\alpha,\varepsilon}$ defined as

$$\tilde{\nu}(A) = \nu\left(A \setminus B_{i,\varepsilon}\right) + \nu\left\{(x, z) \in B_{i,\varepsilon} : (x, z + \alpha e_i) \in A\right\}.$$

Notice that if $0 \leq \hat{z}_i < \hat{x}_i$ (resp., $0 < \hat{z}_i \leq \hat{x}_i$) and $0 \leq \alpha < \hat{x}_i - \hat{z}_i$ (resp., $\hat{z}_i - \hat{x}_i < \alpha \leq 0$), then $\tilde{\nu} \in \mathcal{H}_X$ for every sufficiently small $\varepsilon \geq 0$.

Before we proceed, we need to introduce a few lemmas.

Lemma 4.1 *If $\mathbb{E}X_i^k < +\infty$ then, for any $\nu \in H_X$:*

$$\left(\mathbb{E}^{\tilde{\nu}} - \mathbb{E}^{\nu}\right) Z_i^k = \alpha\nu\left(B_{i,\varepsilon}\right)\left(k\hat{z}_i^{k-1} + O(|\alpha|) + O(\varepsilon)\right), \tag{8}$$

when $\alpha \to 0$, $\varepsilon \to 0^+$.

Proof. It follows from the definition of $\tilde{\nu}$ that

$$\begin{aligned}
\left(\mathbb{E}^{\tilde{\nu}} - \mathbb{E}^{\nu}\right) Z_i^k = &\int_{B_{i,\varepsilon}} \left((z_i + \alpha)^k - z_i^k\right) d\nu = \int_{B_{i,\varepsilon}} \int_0^1 k(z_i + t\alpha)^{k-1}\alpha \, dtd\nu = \\
= &\alpha \int_{B_{i,\varepsilon}} \int_0^1 k\left(\hat{z}_i^{k-1} + (z_i + t\alpha)^{k-1} - z_i^{k-1} + z_i^{k-1} - \hat{z}_i^{k-1}\right) dtd\nu.
\end{aligned}$$

∎

Lemma 4.2 *For any $k \in \mathbb{N}$, let*

$$D_k = \left\{x \in]0, +\infty[^k : x_1 < x_2^{\frac{1}{2}} < x_3^{\frac{1}{3}} < \cdots < x_k^{\frac{1}{k}}\right\}.$$

For any non-negative random variable Y such that $\mathbb{E}Y^k < +\infty$:

1. $\left(\mathbb{E}Y, \mathbb{E}Y^2, \ldots, \mathbb{E}Y^k\right) \in D_k \cup \left\{x \in [0,+\infty[^k \colon x_1 = x_2^{\frac{1}{2}} = x_3^{\frac{1}{3}} = \cdots = x_k^{\frac{1}{k}}\right\}$.

2. $\mathbb{E}Y = \left(\mathbb{E}Y^2\right)^{\frac{1}{2}} = \left(\mathbb{E}Y^3\right)^{\frac{1}{3}} = \ldots = \left(\mathbb{E}Y^k\right)^{\frac{1}{k}}$ *if and only if Y is degenerate, i.e., if and only if it takes a constant value almost surely.*

Proof. Pick i, j such that $1 \le i < j \le k$. Using Hölder's inequality, we obtain

$$\mathbb{E}Y^i \le \left(\mathbb{E}1^{\frac{j}{j-i}}\right)^{\frac{j-i}{j}} \left(\mathbb{E}Y^j\right)^{\frac{i}{j}} = \left(\mathbb{E}Y^j\right)^{\frac{i}{j}},$$

with the equality holding if and only if $\{1, Y^i\}$ are linearly dependent. ∎

Lemma 4.3 *Let P_i be the premium calculation principle (5), and suppose that Ψ_i is continuously differentiable in D_{k_i}. Then, for every $\nu \in \mathcal{H}_X$ such that ν_{Z_i} is not concentrated at one single point:*

$$P_i(\tilde{\nu}) - P_i(\nu) = \alpha\nu\left(B_{i,\varepsilon}\right)\left(\sum_{j=1}^{k_i} j\hat{z}_i^{j-1}\frac{\partial\Psi_i}{\partial u_j} + O\left(|\alpha|\right) + O(\varepsilon)\right), \tag{9}$$

when $\alpha \to 0$, $\varepsilon \to 0^+$. Here, the partial derivatives $\frac{\partial\Psi_i}{\partial u_j}$ are evaluated at the point $u = \left(\mathbb{E}^\nu Z_i, \mathbb{E}^\nu Z_i^2, \ldots, \mathbb{E}^\nu Z_i^{k_i}\right)$.

If $\nabla\Psi_i : D_{k_i} \mapsto \mathbb{R}^{k_i}$ can be extended by continuity to the set

$$\left\{x \in [0,+\infty[^{k_i} \colon x_1 = x_2^{\frac{1}{2}} = \ldots = x_{k_i}^{\frac{1}{k_i}}\right\},$$

then (9) holds for every $\nu \in \mathcal{H}_X$.

Proof. Due to Lemma 4.2, we have

$$\begin{aligned} &P_i(\tilde{\nu}) - P_i(\nu) = \\ &= \int_0^1 \frac{d}{dt}\Psi_i\left(\mathbb{E}^\nu Z_i + t(\mathbb{E}^{\tilde{\nu}} - \mathbb{E}^\nu)Z_i, \ldots, \mathbb{E}^\nu Z_i^{k_i} + t(\mathbb{E}^{\tilde{\nu}} - \mathbb{E}^\nu)Z_i^{k_i}\right) dt. \end{aligned}$$

Hence, the Lemma follows from Lemma 4.1. ∎

Now, we can formulate the main result in this section:

Theorem 4.4 *Let $U :]-\infty, c] \mapsto \mathbb{R}$ be continuously differentiable in $]-\infty, c[$, let $\nu \in \mathcal{H}_X$ be an optimal randomized strategy, and suppose that $\mathbb{E}^\nu U(L_\nu) > -\infty$. Fix $i \in \{1, 2, \ldots, n\}$ such that the function Ψ_i is continuous in $D_{k_i} \cup \left\{u \in [0,+\infty[^{k_i} \colon u_i = u_2^{\frac{1}{2}} = \ldots = u_{k_i}^{\frac{1}{k_i}}\right\}$, continuously differentiable in D_{k_i}, and suppose the following assumptions hold:*

(A1) *Either the gradient $\nabla\Psi_i : D_{k_i} \mapsto \mathbb{R}^{k_i}$ can be extended by continuity to the set $\left\{u \in [0,+\infty[^{k_i} \colon u_1 = u_2^{\frac{1}{2}} = \ldots = u_{k_i}^{\frac{1}{k_i}}\right\}$, or the marginal distribution ν_{Z_i} is not concentrated at a single point.*

(A2) *There is some $\delta > 0$ such that $\mathbb{E}U(L_\nu - \delta) > -\infty$.*

Then:

1. *The inequality*

$$\left\lfloor \mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|X_i} U'\left(L_\nu\right)\right\rfloor (\hat{x}_i, \hat{z}_i) \leq \sum_{j=1}^{k_i} j\hat{z}_i^{j-1} \frac{\partial \Psi_i}{\partial u_j} \mathbb{E}^{\nu} U'(L_\nu), \tag{10}$$

holds for any $(\hat{x}_i, \hat{z}_i) \in \mathbb{R}^2$ such that

$$0 \leq \hat{z}_i < \hat{x}_i, \qquad \nu(B_{i,\varepsilon}) > 0 \quad \forall \varepsilon > 0. \tag{11}$$

2. *The inequality*

$$\left\lceil \mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|X_i} U'(L_\nu)\right\rceil (\hat{x}_i, \hat{z}_i) \geq \sum_{j=1}^{k_i} j\hat{z}_i^{j-1} \frac{\partial \Psi_i}{\partial u_j} \mathbb{E}^{\nu} U'(L_\nu), \tag{12}$$

holds for any $(\hat{x}_i, \hat{z}_i) \in \mathbb{R}^2$ such that

$$0 < \hat{z}_i \leq \hat{x}_i, \qquad \nu(B_{i,\varepsilon}) > 0 \quad \forall \varepsilon > 0. \tag{13}$$

Here, $\lfloor\varphi\rfloor$ and $\lceil\varphi\rceil$ denote, respectively, the lower semicontinuous and the upper semicontinuous envelopes of the function φ. The partial derivatives $\frac{\partial \Psi_i}{\partial u_j}$ are evaluated at the point $u = \left(\mathbb{E}^{\nu} Z_i, \mathbb{E}^{\nu} Z_i^2, \ldots, \mathbb{E}^{\nu} Z_i^{k_i}\right)$.

Under assumption **(A1)** *alone (i.e.,* **(A2)** *may fail):*

3. *If there is some $(\hat{x}_i, \hat{z}_i) \in \mathbb{R}^2$ satisfying (11), such that $\sum_{j=1}^{k_i} j\hat{z}_i^{j-1} \frac{\partial \Psi_i}{\partial u_j} < 0$, then $\nu\left\{U(L_\nu) = U(c)\right\} = 1$.*

4. *Inequality (12) holds for every $(\hat{x}_i, \hat{z}_i) \in \mathbb{R}^2$ satisfying (13) such that $\sum_{j=1}^{k_i} j\hat{z}_i^{j-1} \frac{\partial \Psi_i}{\partial u_j} > 0$.*

Remark 4.5 *From the definition of $\mathcal{H}_X$, it follows that if ν_{Z_i} is concentrated in a single point a, then $\Pr\{X_i < a\} = 0$.*

Remark 4.6 *If U is a exponential utility function, then assumption* **(A2)** *holds for any $\nu \in \mathcal{H}_X$ such that $\mathbb{E}^{\nu} U(L_\nu) > -\infty$. Therefore, we may take assumption* **(A2)** *as granted whenever there is some constant $R > 0$ such that*

$$\liminf_{x \to -\infty} \left(e^{Rx} U(x)\right) > -\infty.$$

Remark 4.7 *Notice that $\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|X_i} U'(L_\nu)(\hat{x}_i, \hat{z}_i)$ is the (infinitesimal) variation of the expected utility due to an (infinitesimal) variation in the cover of risk i in the neighbourhood of $\hat{x}_i$, while $\sum_{j=1}^{k_i} j\hat{z}_i^{j-1} \frac{\partial \Psi_i}{\partial u_j}$ is the corresponding variation in the premium for risk i. Thus, conditions (10), (12) express a very natural economic trade-off between the (local) effect of changing the cover for the event of a particular level of risk i and the (global) effect on the expected utility due to the corresponding change in premium amount.*

Proof of Theorem 4.4. Notice that, for any $\nu \in \mathcal{H}_X$, $\alpha \in \mathbb{R}$, $\varepsilon \geq 0$:

$$\begin{aligned}\mathbb{E}^{\tilde{\nu}} U\left(L_{\tilde{\nu}}\right) =& \mathbb{E}^{\tilde{\nu}} U\left(L_\nu - \left(P_i(\tilde{\nu}) - P_i(\nu)\right)\right) = \\ &= \int_{B_{i,\varepsilon}} U\left(L_\nu - \left(P_i(\tilde{\nu}) - P_i(\nu)\right) + \alpha\right) d\nu + \\ &\quad + \int_{\mathbb{R}^{2n} \setminus B_{i,\varepsilon}} U\left(L_\nu - \left(P_i(\tilde{\nu}) - P_i(\nu)\right)\right) d\nu. \end{aligned} \tag{14}$$

Let $\nu \in \mathcal{H}_X$ be an optimal randomized strategy and suppose that assumption **(A1)** holds.

Suppose there is some $(\hat{x}_i, \hat{z}_i)$ satisfying (11) such that $\sum_{j=1}^{k_i} j \hat{z}_i^{j-1} \frac{\partial \Psi_i}{\partial u_j} < 0$. By Lemma 4.3, $P_i(\tilde{\nu}) - P_i(\nu) < 0$ holds for sufficiently small $\varepsilon > 0$, $\alpha > 0$. Therefore,

$$\left(L_\nu - \left(P_i(\tilde{\nu}) - P_i(\nu)\right) + \alpha\right) \chi_{B_{i,\varepsilon}} + \left(L_\nu - \left(P_i(\tilde{\nu}) - P_i(\nu)\right)\right) \chi_{B_{i,\varepsilon}^c} > L_\nu.$$

Thus, statement 3 must hold, due to optimality of ν and monotonicity of U.

Now, suppose that assumption **(A2)** also holds, and fix $\delta > 0$ such that $\mathbb{E}^\nu U\left(L_\nu - \delta\right) > -\infty$. Due to Lemma 4.3, $P_i(\tilde{\nu}) - P_i(\nu) < \delta$ whenever ε and α are sufficiently small. Further, optimality of ν and equality (14) imply

$$\begin{aligned} & \mathbb{E}^{\tilde{\nu}} U(L_{\tilde{\nu}}) \leq \mathbb{E}^\nu U(L_\nu) \Leftrightarrow \\ \Leftrightarrow & \int_{B_{i,\varepsilon}} U\left(L_\nu - \left(P_i(\tilde{\nu}) - P_i(\nu)\right) + \alpha\right) - U\left(L_\nu - \left(P_i(\tilde{\nu}) - P_i(\nu)\right)\right) d\nu \leq \\ & \leq \mathbb{E}^\nu \left(U\left(L_\nu\right) - U\left(L_\nu - \left(P_i(\tilde{\nu}) - P_i(\nu)\right)\right)\right) \Leftrightarrow \\ \Leftrightarrow & \alpha \int_{B_{i,\varepsilon}} \int_0^1 U'\left(L_\nu - \left(P_i(\tilde{\nu}) - P_i(\nu)\right) + t\alpha\right) dt \, d\nu \leq \\ & \leq \left(P_i(\tilde{\nu}) - P_i(\nu)\right) \mathbb{E}^\nu \int_0^1 U'\left(L_\nu - t\left(P_i(\tilde{\nu}) - P_i(\nu)\right)\right) dt. \end{aligned} \tag{15}$$

If $(\hat{x}_i, \hat{z}_i)$ satisfy (11) and $\alpha > 0$, then dividing both sides of (15) by $\alpha\nu(B_{i,\varepsilon})$, using Fubini's theorem and conditional independence of $\left(X_{[i]}, Z_{[i]}\right)$ and Z_i given X_i, we obtain

$$\begin{aligned} & \frac{1}{\nu(B_{i,\varepsilon})} \int_0^1 \int_{B_{i,\varepsilon}} \mathbb{E}^\nu_{\left(X_{[i]}, Z_{[i]}\right)|X_i} U'\left(L_\nu\right)\left(x_i, z_i - \left(P_i(\tilde{\nu}) - P_i(\nu)\right) + \alpha\right) d\nu \leq \\ \leq & \frac{P_i(\tilde{\nu}) - P_i(\nu)}{\alpha\nu(B_{i,\varepsilon})} \mathbb{E}^\nu \int_0^1 U'\left(L_\nu - t\left(P_i(\tilde{\nu}) - P_i(\nu)\right)\right) dt. \end{aligned} \tag{16}$$

This implies

$$\begin{aligned} & \inf_{|x_i - \hat{x}_i| + |z_i - \hat{z}_i| \leq \varepsilon + \alpha + P_i(\tilde{\nu}) - P_i(\nu)} \mathbb{E}^\nu_{\left(X_{[i]}, Z_{[i]}\right)|X_i} U'\left(L_\nu\right)\left(x_i, z_i\right) \leq \\ \leq & \frac{P_i(\tilde{\nu}) - P_i(\nu)}{\alpha\nu(B_{i,\varepsilon})} \mathbb{E}^\nu \int_0^1 U'\left(L_\nu - t\left(P_i(\tilde{\nu}) - P_i(\nu)\right)\right) dt. \end{aligned}$$

Therefore, taking successive limits when $\varepsilon \to 0^+$ and $\alpha \to 0^+$ and taking into account Lemma 4.3, one obtains (10).

Under assumption **(A2)**, inequality (12) can be proved by a similar argument. Just notice that for $\alpha < 0$ inequality in (16) holds with the opposite inequality sign, and use supremum instead of infimum in the last step.

To prove statement 4, notice that if $(\hat{x}_i, \hat{z}_i)$ satisfies (13) and $\sum_{j=1}^{k_i} j\hat{z}_i^{j-1}\frac{\partial \Psi_i}{\partial u_j} > 0$, then $P_i(\tilde{\nu}) - P_i(\nu) < 0$ for $\varepsilon > 0$ and $\alpha < 0$ close to zero. Hence, integrability of $U\left(L_\nu - (P_i(\tilde{\nu}) - P_i(\nu))\right)$ follows from monotonicity of U, and therefore inequality (12) does not depend on assumption **(A2)**. ■

In Section 5, we will present conditions guaranteeing that optimal treaties are deterministic. In view of those results, we present the following version of Theorem 4.4.

Theorem 4.8 *Suppose that U is continuously differentiable in $]-\infty, c[$, and let $\nu \in \mathcal{H}_X$ be an optimal randomized strategy. Fix $i \in \{1, 2, \ldots, n\}$ such that the function Ψ_i is continuous in $D_{k_i} \cup \left\{u \in [0, +\infty[^{k_i} : u_1 = u_2^{\frac{1}{2}} = \ldots = u_{k_i}^{\frac{1}{k_i}}\right\}$, continuously differentiable in D_{k_i}, and suppose that assumptions* **(A1)**, **(A2)** *of Theorem 4.4 hold.*

If the marginal distributon μ_{X_i} is absolutely continuous except possibly for an atom at $x_i = 0$ and the strategy prescribed by ν for the risk i is deterministic (i.e., $Z_i = Z_i(X_i)$), then the optimal treaty Z_i satisfies the conditions

$$\mathbb{E}^{\nu}_{(X_{[i]}, Z_{[i]})|X_i} U'(L_\nu)(x_i, Z_i(x_i)) \leq \sum_{j=1}^{k_i} \frac{\partial \Psi_i}{\partial u_j} j Z_i(x_i)^{j-1} \mathbb{E}^{\nu} U'(L_\nu)$$

for μ_{X_i}-a.e. $x_i > 0$ such that $Z_i(x_i) < x_i$;

$$\mathbb{E}^{\nu}_{(X_{[i]}, Z_{[i]})|X_i} U'(L_\nu)(x_i, Z_i(x_i)) \geq \sum_{j=1}^{k_i} \frac{\partial \Psi_i}{\partial u_j} j Z_i(x_i)^{j-1} \mathbb{E}^{\nu} U'(L_\nu)$$

for μ_{X_i}-a.e. $x_i > 0$ such that $Z_i(x_i) > 0$.

The partial derivatives $\frac{\partial \Psi_i}{\partial u_j}$ are evaluated at the point

$$u = \left(\mathbb{E}Z_i(X_i), \mathbb{E}Z_i(X_i)^2, \ldots, \mathbb{E}Z_i(X_i)^{k_i}\right).$$

Proof. The proof follows an argument similar to the proof of Theorem 4.4, with some adaptations.

The absolute continuity assumption on the marginal distribution μ_{X_i} means that there is a constant $\alpha \in [0, 1[$ and a non-negative function f_{X_i} such that

$$\mu_{X_i}(A) = \alpha\delta_0(A) + \int_A f_{X_i}(x_i)dx_i \qquad \forall A \in \mathcal{B}_{[0,+\infty[},$$

where δ_0 denotes the Dirac measure concentrated at $x_i = 0$.

Let Z_i be the optimal treaty for risk i, and $\nu \in \mathcal{H}_X$ be the optimal strategy for the total portfolio of risks. Thus,

$$\nu(A) = \int_{\mathbb{R}} \int_{\mathbb{R}^{2n-2}} \chi_A\left(x, z_{[i]}, Z_i(x_i)\right) \nu_{(X_{[i]}, Z_{[i]})|X_i}\left(x_i, d(x_{[i]}, z_{[i]})\right) \mu_{X_i}(dx_i)$$

for any $A \in \mathcal{B}_{\mathbb{R}^{2n}}$. For fixed $\hat{x}_i \in]0, +\infty[$, $\varepsilon > 0$, $\zeta \in \mathbb{Q} \cap [0, \hat{x}_i[$, let

$$\tilde{Z}_i(x_i) = \begin{cases} Z_i(x_i), & \text{for } x_i \notin]\hat{x}_i - \varepsilon, \hat{x}_i + \varepsilon[, \\ \zeta, & \text{for } x_i \in]\hat{x}_i - \varepsilon, \hat{x}_i + \varepsilon[, \end{cases}$$

and let $\tilde{\nu}$ be the corresponding measure in $\mathcal{B}_{\mathbb{R}^{2n}}$. Notice that $\tilde{\nu} \in \mathcal{H}_X$, provided ε is sufficiently small.

An argument similar to the proof of Lemma 4.3 shows that

$$\begin{aligned} P_i(\tilde{\nu}) - P_i(\nu) =& 2\varepsilon \left(\zeta - Z_i(\hat{x}_i)\right) \sum_{j=1}^{k_i} \frac{\partial \Psi_i}{\partial u_j} j Z_i(\hat{x}_i)^{j-1} f_{X_i}(\hat{x}_i) + \\ & + \varepsilon o\left(\zeta - Z_i(\hat{x}_i)\right) + o(\varepsilon), \end{aligned}$$

for every $\hat{x}_i$, a Lebesgue point of the functions $Z_i^j f_{X_i}$, $j = 0, 1, \ldots, k_i$. Thus,

$$\begin{aligned} & \mathbb{E}^{\tilde{\nu}} U\left(L_{\tilde{\nu}}\right) = \\ =& \int_{\mathbb{R} \setminus]\hat{x}_i - \varepsilon, \hat{x}_i + \varepsilon[} \int_{\mathbb{R}^{2n-2}} U\left(L_\nu - \left(P_i(\tilde{\nu}) - P_i(\nu)\right)\right) d\nu_{\left(X_{[i]}, Z_{[i]}\right)|X_i} d\mu_{X_i} + \\ & + \int_{\hat{x}_i - \varepsilon}^{\hat{x}_i + \varepsilon} \int_{\mathbb{R}^{2n-2}} U\left(L_\nu - \left(P_i(\tilde{\nu}) - P_i(\nu)\right) + \zeta - Z_i\right) d\nu_{\left(X_{[i]}, Z_{[i]}\right)|X_i} d\mu_{X_i} = \\ =& \mathbb{E}^\nu U\left(L_\nu - \left(P_i(\tilde{\nu}) - P_i(\nu)\right)\right) + \\ & + \int_{\hat{x}_i - \varepsilon}^{\hat{x}_i + \varepsilon} \int_{\mathbb{R}^{2n-2}} \left(U\left(L_\nu - \left(P_i(\tilde{\nu}) - P_i(\nu)\right) + \zeta - Z_i\right) - U\left(L_\nu - \left(P_i(\tilde{\nu}) - P_i(\nu)\right)\right)\right) \\ & \qquad d\nu_{\left(X_{[i]}, Z_{[i]}\right)|X_i} d\mu_{X_i}. \end{aligned}$$

Optimality of ν implies that $\mathbb{E}^{\tilde{\nu}} U\left(L_{\tilde{\nu}}\right) \leq \mathbb{E}^\nu U\left(L_\nu\right)$, that is

$$\begin{aligned} & \int_{\hat{x}_i - \varepsilon}^{\hat{x}_i + \varepsilon} \int_{\mathbb{R}^{2n-2}} \left(U\left(L_\nu - \left(P_i(\tilde{\nu}) - P_i(\nu)\right) + \zeta - Z_i\right) - U\left(L_\nu - \left(P_i(\tilde{\nu}) - P_i(\nu)\right)\right)\right) \\ & \qquad d\nu_{\left(X_{[i]}, Z_{[i]}\right)|X_i} d\mu_{X_i} \leq \\ \leq & \mathbb{E}^\nu \left(U\left(L_\nu - \left(P_i(\tilde{\nu}) - P_i(\nu)\right)\right) - U\left(L_\nu\right)\right). \end{aligned}$$

Since U is concave and $P_i(\tilde{\nu}) \to P_i(\nu)$ when $\varepsilon \to 0$, it follows that for any constant $\eta \in \mathbb{Q} \cap]0, +\infty[$, the inequality

$$\begin{aligned} & \int_{\hat{x}_i - \varepsilon}^{\hat{x}_i + \varepsilon} \mathbb{E}^\nu_{\left(X_{[i]}, Z_{[i]}\right)|X_i} \int_0^1 U'\left(L_\nu + \eta + t(\zeta - Z_i)\right) dt \left(\zeta - Z_i\right) d\mu_{X_i} \leq \\ \leq & \left(P_i(\tilde{\nu}) - P_i(\nu)\right) \mathbb{E}^\nu \int_0^1 U'\left(L_\nu - t\left(P_i(\tilde{\nu}) - P_i(\nu)\right)\right) dt \end{aligned}$$

holds for every sufficiently small $\varepsilon > 0$. Dividing both sides by ε, making $\varepsilon \to 0$, and using Lebesgue's dominated convergence theorem, we obtain

$$\mathbb{E}^\nu_{\left(X_{[i]}, Z_{[i]}\right)|X_i} \int_0^1 U'\left(L_\nu + \eta + t(\zeta - Z_i(\hat{x}_i))\right) dt \left(\zeta - Z_i(\hat{x}_i)\right) \leq$$

$$\leq (\zeta - Z_i(\hat{x}_i)) \sum_{j=1}^{k_i} \frac{\partial \Psi_i}{\partial u_j} j Z_i(\hat{x}_i)^{j-1} \mathbb{E}^\nu U'(L_\nu) + o(\zeta - Z_i(\hat{x}_i)),$$

for every $\hat{x}_i > 0$, a Lebesgue point of the functions $Z_i^j f_{X_i}$, $j = 0, 1, \ldots, k_i$ and $\mathbb{E}^\nu_{(X_{[i]}, Z_{[i]})|X_i} \int_0^1 U'(L_\nu + \eta + t(\zeta$ such that $f_{X_i}(\hat{x}_i) > 0$.

If $Z_i(\hat{x}_i) < \hat{x}_i$ (resp., $Z_i(\hat{x}_i) > 0$), then we can pick a sequence $\zeta_k \searrow Z_i(\hat{x}_i)$ (resp., $\zeta_k \nearrow Z_i(\hat{x}_i)$) and obtain

$$\mathbb{E}^\nu_{(X_{[i]}, Z_{[i]})|X_i} U'(L_\nu + \eta) \leq \sum_{j=1}^{k_i} \frac{\partial \Psi_i}{\partial u_j} j Z_i(\hat{x}_i)^{j-1} \mathbb{E}^\nu U'(L_\nu)$$

(resp., $\mathbb{E}^\nu_{(X_{[i]}, Z_{[i]})|X_i} U'(L_\nu + \eta) \leq \sum_{j=1}^{k_i} \frac{\partial \Psi_i}{\partial u_j} j Z_i(\hat{x}_i)^{j-1} \mathbb{E}^\nu U'(L_\nu)$). Since η is arbitrary, the result follows by Lebesgue's dominated convergence theorem. ■

5 Deterministic optimal treaties

In this Section we show that under mild conditions the optimal treaty is of the deterministic type.

5.1 Problems with deterministic optimal strategies

The main result in this section is the following theorem.

Theorem 5.1 *Suppose that U is differentiable in $]-\infty, c[$, let $\nu \in \mathcal{H}_X$ be an optimal randomized strategy, and suppose that assumption* **(A2)** *of Theorem 4.4 holds. Pick $i \in \{1, 2, \ldots, n\}$ such that the marginal distribution μ_{X_i} has no atoms except possibly at $x_i = 0$, and the function Ψ_i is continuous in $D_{k_i} \cup \left\{ u \in [0, +\infty[^{k_i} : u_1 = u_2^{\frac{1}{2}} = \ldots = u_{k_i}^{\frac{1}{k_i}} \right\}$, continuously differentiable in D_{k_i}. Let $\varphi_{i,\nu} : [0, +\infty[\mapsto \mathbb{R}$ be the function*

$$\varphi_{i,\nu}(t) = \mathbb{E}^\nu U'(L_\nu) \sum_{j=1}^{k_i} \frac{\partial \Psi_i}{\partial u_j} \left(\mathbb{E}^\nu Z_i, \mathbb{E}^\nu Z_i^2, \ldots, \mathbb{E}^\nu Z_i^{k_i} \right) t^j.$$

If the functions U, $-\varphi_{i,\nu}$ are concave, with at least one of them being strictly concave, then the strategy prescribed by ν for the risk i is deterministic.

Proof. Pick $\nu \in \mathcal{H}_X$, and suppose that the strategy it prescribes for risk i is not deterministic. Then, the marginal distribution ν_{Z_i} is not concentrated at a single point and there are $\hat{x}_i, \hat{z}_i, \tilde{z}_i$, with $0 \leq \hat{z}_i < \tilde{z}_i \leq \hat{x}_i$, such that

$$\nu(B_{i,\varepsilon}(\hat{x}_i, \hat{z}_i)) > 0, \quad \nu(B_{i,\varepsilon}(\hat{x}_i, \tilde{z}_i)) > 0, \qquad \forall \varepsilon > 0.$$

Fix $\hat{x}_i, \hat{z}_i, \tilde{z}_i$ as above and pick constants $\alpha, \beta > 0$ such that $\alpha + \beta < \tilde{z}_i - \hat{z}_i$, and therefore

$$0 \leq \hat{z}_i < \hat{z}_i + \alpha < \tilde{z}_i - \beta < \tilde{z}_i \leq \hat{x}_i.$$

Let $\hat{A}, \tilde{A}$ be open cylinders

$$\hat{A} = \left\{(x,z) \in \mathbb{R}^{2n} : (x_i, z_i) \in \hat{A}_i\right\}, \qquad \tilde{A} = \left\{(x,z) \in \mathbb{R}^{2n} : (x_i, z_i) \in \tilde{A}_i\right\},$$

with

$$\begin{aligned} \hat{A}_i &\subset \left\{(x_i,z_i) \in \mathbb{R}^2 : (x_i - \hat{x}_i)^2 + (z_i - \hat{z}_i)^2 < \varepsilon^2\right\} \\ \tilde{A}_i &\subset \left\{(x_i,z_i) \in \mathbb{R}^2 : (x_i - \hat{x}_i)^2 + (z_i - \tilde{z}_i)^2 < \varepsilon^2\right\}, \end{aligned}$$

and let $\tilde{\nu}$ be the measure

$$\begin{aligned} \tilde{\nu}(B) =& \nu\left(B \setminus (\hat{A} \cup \tilde{A})\right) + \nu\left\{(x,z) \in \hat{A} : (x, z + \alpha e_i) \in B\right\} + \\ &+ \nu\left\{(x,z) \in \tilde{A} : (x, z - \beta e_i) \in B\right\}. \end{aligned}$$

For ε sufficiently small, $\tilde{\nu}$ is an element of $\mathcal{H}_X$. The argument used to prove Lemma 4.1 shows that

$$\begin{aligned} &\left(\mathbb{E}^{\tilde{\nu}} - \mathbb{E}^{\nu}\right) Z_i^k = \\ =& \nu(\hat{A})\left((\hat{z}_i + \alpha)^k - \hat{z}_i^k + O(\varepsilon)\right) + \nu(\tilde{A})\left((\tilde{z}_i - \beta)^k - \tilde{z}_i^k + O(\varepsilon)\right), \end{aligned}$$

for any $k \in \mathbb{N}$ such that $\mathbb{E}Z_i^k < +\infty$. Therefore, the argument used to prove Lemma 4.3 shows that

$$\begin{aligned} P_i(\tilde{\nu}) - P_i(\nu) =& \nu(\hat{A}) \sum_{j=1}^{k_i} \frac{\partial \Psi_i}{\partial u_j} \times \left((\hat{z}_i + \alpha)^j - \hat{z}_i^j + O(\varepsilon)\right) + \\ &+ \nu(\tilde{A}) \sum_{j=1}^{k_i} \frac{\partial \Psi_i}{\partial u_j} \times \left((\tilde{z}_i - \beta)^j - \tilde{z}_i^j + O(\varepsilon)\right) + \\ &+ o\left(\nu(\hat{A}) + \nu(\tilde{A})\right), \end{aligned} \tag{17}$$

where the partial derivatives $\frac{\partial \Psi_i}{\partial u_j}$ are evaluated at the point $u = \left(\mathbb{E}^{\nu} Z_i, \mathbb{E}^{\nu} Z_i^2, \ldots, \mathbb{E}^{\nu} Z_i^{k_i}\right)$. It follows that

$$\begin{aligned} &\mathbb{E}^{\tilde{\nu}} U\left(L_{\tilde{\nu}}\right) - \mathbb{E}^{\nu} U\left(L_{\nu}\right) = \\ =& \int_{\hat{A}} U\left(L_{\nu} - (P_i(\tilde{\nu}) - P_i(\nu)) + \alpha\right) d\nu + \int_{\tilde{A}} U\left(L_{\nu} - (P_i(\tilde{\nu}) - P_i(\nu)) - \beta\right) d\nu + \\ &+ \int_{\mathbb{R}^{2n} \setminus (\hat{A} \cup \tilde{A})} U\left(L_{\nu} - (P_i(\tilde{\nu}) - P_i(\nu))\right) d\nu - \int_{\mathbb{R}^{2n}} U\left(L_{\nu}\right) d\nu. \end{aligned}$$

Under assumption **(A2)** and differentiability of U, this is

$$\begin{aligned} &\mathbb{E}^{\tilde{\nu}} U\left(L_{\tilde{\nu}}\right) - \mathbb{E}^{\nu} U\left(L_{\nu}\right) = \\ =& \alpha \int_{\hat{A}} \int_0^1 U'\left(L_{\nu} - (P_i(\tilde{\nu}) - P_i(\nu)) + t\alpha\right) dt\, d\nu - \\ &- \beta \int_{\tilde{A}} \int_0^1 U'\left(L_{\nu} - (P_i(\tilde{\nu}) - P_i(\nu)) - t\beta\right) dt\, d\nu - \end{aligned}$$

$$- (P_i(\tilde{\nu}) - P_i(\nu)) \int_{\mathbb{R}^{2n}} \int_0^1 U'\left(L_\nu - t(P_i(\tilde{\nu}) - P_i(\nu))\right) dt\, d\nu.$$

Taking into account concavity of U and estimate (17), this implies

$$\begin{aligned}
&\mathbb{E}^{\tilde{\nu}} U\left(L_{\tilde{\nu}}\right) - \mathbb{E}^{\nu} U\left(L_\nu\right) \geq \\
\geq&\alpha \int_{\hat{A}} U'\left(L_\nu + \alpha + O(\nu(\hat{A}) + \nu(\tilde{A}))\right) d\nu - \\
&- \beta \int_{\tilde{A}} U'\left(L_\nu - \beta + O(\nu(\hat{A}) + \nu(\tilde{A}))\right) d\nu - \\
&- (P_i(\tilde{\nu}) - P_i(\nu)) \int_{\mathbb{R}^{2n}} U'\left(L_\nu + O(\nu(\hat{A}) + \nu(\tilde{A}))\right) d\nu \geq \\
\geq&\alpha \int_{\hat{A}} \mathbb{E}^{\nu}_{(X_{[i]}, Z_{[i]})|X_i} U'\left(L_\nu\right)(\hat{x}_i, \hat{z}_i + \alpha + O(\varepsilon))\, d\nu - \\
&- \beta \int_{\tilde{A}} \mathbb{E}^{\nu}_{(X_{[i]}, Z_{[i]})|X_i} U'\left(L_\nu\right)(\hat{x}_i, \tilde{z}_i - \beta + O(\varepsilon))\, d\nu - \\
&- (P_i(\tilde{\nu}) - P_i(\nu)) \int_{\mathbb{R}^{2n}} U'\left(L_\nu + O(\nu(\hat{A}) + \nu(\tilde{A}))\right) d\nu = \\
=&\alpha\nu(\hat{A}) E^{\nu}_{(X_{[i]}, Z_{[i]})|X_i} U'\left(L_\nu\right)(\hat{x}_i, \hat{z}_i + \alpha + O(\varepsilon)) - \\
&- \beta\nu(\tilde{A}) \mathbb{E}^{\nu}_{(X_{[i]}, Z_{[i]})|X_i} U'\left(L_\nu\right)(\hat{x}_i, \tilde{z}_i - \beta + O(\varepsilon)) - \\
&- \nu(\hat{A}) \sum_{j=1}^{k_i} \frac{\partial \Psi_i}{\partial u_j} \left((\hat{z}_i + \alpha)^j - \hat{z}_i^j + O(\varepsilon)\right) \mathbb{E}^{\nu} U'\left(L_\nu\right) - \\
&- \nu(\tilde{A}) \sum_{j=1}^{k_i} \frac{\partial \Psi_i}{\partial u_j} \left((\tilde{z}_i - \beta)^j - \tilde{z}_i^j + O(\varepsilon)\right) \mathbb{E}^{\nu} U'\left(L_\nu\right) + o\left(\nu(\hat{A}) + \nu(\tilde{A})\right).
\end{aligned}$$

Since the distribution of X_i has no atoms except possibly 0 and $\hat{x}_i > 0$, for every $\varepsilon > 0$ there is a pair of cylinders $\hat{A}_\varepsilon$, $\tilde{A}_\varepsilon$ such that:

1. $\hat{A}_\varepsilon = B_{i,\varepsilon}(\hat{x}_i, \hat{z}_i)$, or $\tilde{A}_\varepsilon = B_{i,\varepsilon}(\hat{x}_i, \tilde{z}_i)$;
2. $\alpha\nu\left(\hat{A}_\varepsilon\right) = \beta\nu\left(\tilde{A}_\varepsilon\right)$.

For such pairs of cylinders, the estimate above becomes

$$\begin{aligned}
&\mathbb{E}^{\tilde{\nu}} U\left(L_{\tilde{\nu}}\right) - \mathbb{E}^{\nu} U\left(L_\nu\right) \geq \\
\geq&\alpha\nu\left(\hat{A}_\varepsilon\right) \Bigg(E^{\nu}_{(X_{[i]}, Z_{[i]})|X_i} U'\left(L_\nu\right)(\hat{x}_i, \hat{z}_i + \alpha + O(\varepsilon)) - \\
&\qquad - \mathbb{E}^{\nu}_{(X_{[i]}, Z_{[i]})|X_i} U'\left(L_\nu\right)(\hat{x}_i, \tilde{z}_i - \beta + O(\varepsilon)) + \\
&\qquad + \mathbb{E}^{\nu} U'\left(L_\nu\right) \sum_{j=1}^{k_i} \frac{\partial \Psi_i}{\partial u_j} \left(\frac{(\tilde{z}_i - \beta)^j - \tilde{z}_i^j}{-\beta} + O(\varepsilon)\right) - \\
&\qquad - \mathbb{E}^{\nu} U'\left(L_\nu\right) \sum_{j=1}^{k_i} \frac{\partial \Psi_i}{\partial u_j} \left(\frac{(\hat{z}_i + \alpha)^j - \hat{z}_i^j}{\alpha} + O(\varepsilon)\right) \Bigg) + o\left(\nu\left(\hat{A}_\varepsilon\right)\right) =
\end{aligned}$$

$$
\begin{aligned}
=&\alpha\nu\left(\hat{A}_\varepsilon\right)\Bigg(E^{\nu}_{\left(X_{[i]},Z_{[i]}\right)|X_i}U'\left(L_\nu\right)\left(\hat{x}_i,\hat{z}_i+\alpha+O(\varepsilon)\right)-\\
&\qquad -\mathbb{E}^{\nu}_{\left(X_{[i]},Z_{[i]}\right)|X_i}U'\left(L_\nu\right)\left(\hat{x}_i,\tilde{z}_i-\beta+O(\varepsilon)\right)+\\
&\qquad +\frac{\varphi_{i,\nu}(\tilde{z}_i-\beta)-\varphi_{i,\nu}(\tilde{z}_i)}{-\beta}-\frac{\varphi_{i,\nu}(\hat{z}_i+\alpha)-\varphi_{i,\nu}(\hat{z}_i)}{\alpha}+O(\varepsilon)\Bigg)+\\
&+o\left(\nu\left(\hat{A}_\varepsilon\right)\right).
\end{aligned}
$$

Since U is concave,

$$
\begin{aligned}
&E^{\nu}_{\left(X_{[i]},Z_{[i]}\right)|X_i}U'\left(L_\nu\right)\left(\hat{x}_i,\hat{z}_i+\alpha+O(\varepsilon)\right)\geq\\
\geq&\mathbb{E}^{\nu}_{\left(X_{[i]},Z_{[i]}\right)|X_i}U'\left(L_\nu\right)\left(\hat{x}_i,\tilde{z}_i-\beta+O(\varepsilon)\right)
\end{aligned}
$$

holds for sufficiently small ε, with strict inequality if U is strictly concave. In addition, since $\varphi_{i,\nu}$ is convex,

$$
\frac{\varphi_{i,\nu}(\tilde{z}_i-\beta)-\varphi_{i,\nu}(\tilde{z}_i)}{-\beta}\geq\frac{\varphi_{i,\nu}(\hat{z}_i+\alpha)-\varphi_{i,\nu}(\hat{z}_i)}{\alpha}
$$

holds, and the inequality is strict if $\varphi_{i,\nu}$ is strictly convex.

It follows that for sufficiently small $\varepsilon>0$, $\mathbb{E}^{\tilde{\nu}}U\left(L_{\tilde{\nu}}\right)>\mathbb{E}^{\nu}U\left(L_\nu\right)$, and therefore ν is not optimal. ■

The following corollary gives conditions that are easier to check than the conditions in Theorem 5.1.

Corollary 5.2 *Suppose that U is differentiable and concave in $]-\infty,c[$, and let $\nu\in\mathcal{H}_X$ be an optimal randomized strategy. Suppose that assumption* **(A2)** *of Theorem 4.4 holds, and the marginal distribution μ_{X_i} has no atoms except, possibly, at $x_i=0$.*

Suppose that the reinsurance premium for risk $i\in\{1,2,\ldots,n\}$ is computed by a function Ψ_i of the moments of order up to $k_i\in\mathbb{N}$, continuous in $D_{k_i}\cup\left\{u\in[0,+\infty[^{k_i}\colon u_1=u_2^{\frac{1}{2}}=\ldots=u_{k_i}^{\frac{1}{k_i}}\right\}$, continuously differentiable in D_{k_i}, such that

$$
\frac{\partial\Psi_i}{\partial u_j}(u)\geq 0\qquad\forall j\geq 2,\ u\in D_{k_i}.
$$

If at least one of the following conditions holds:

1. *The functions $\frac{\partial\Psi_i}{\partial u_j}$, $j=1,2,\ldots,k_i$, have at most one common zero in D_{k_i}, and $\mathbb{E}^{\nu}U'(L_\nu)>0$;*
2. *U is strictly concave in $]-\infty,c]$;*

then the optimal reinsurance for risk i is deterministic.

Proof. Under the assumptions of the corollary, the functions U, $-\varphi_{i,\nu}$ are concave with at least one of them being strictly concave. Thus, the result follows immediately from Theorem 5.1. ■

5.2 The expected value principle

From Theorem 4.8 and Corollary 5.2, it is easy to check that under mild conditions, the optimal stategy concerning a risk for which the reinsurance premium is computed by the expected value principle is deterministic, and derive the corresponding optimality condition. More precisely, we obtain the following corollary:

Corollary 5.3 *Suppose that U is continuously differentiable and concave in $]-\infty, c[$, let $\nu \in \mathcal{H}_X$ be an optimal randomized strategy, and suppose that assumption* **(A2)** *of Theorem 4.4 holds.*

If the reinsurance premium for risk $i \in \{1, 2, \ldots, n\}$ is computed by the expected value principle (i.e., $\Psi_i(u) = (1+\theta_i)u$), and the marginal distribution μ_{X_i} is absolutely continuous except, possibly, for an atom at $x_i = 0$, then the optimal reinsurance for risk i is a deterministic function $Z_i \in \mathcal{Z}_i$, such that

$$
\begin{aligned}
&E^{\nu}_{(X_{[i]},Z_{[i]})|X_i} U'(L_\nu)(x_i, 0) \leq (1+\theta_i)\mathbb{E}^{\nu} U'(L_\nu) && \text{if } Z_i(x_i) = 0, \\
&E^{\nu}_{(X_{[i]},Z_{[i]})|X_i} U'(L_\nu)(x_i, Z_i(x_i)) = (1+\theta_i)\mathbb{E}^{\nu} U'(L_\nu) && \text{if } 0 < Z_i(x_i) < x_i, \\
&E^{\nu}_{(X_{[i]},Z_{[i]})|X_i} U'(L_\nu)(x_i, x_i) \geq (1+\theta_i)\mathbb{E}^{\nu} U'(L_\nu) && \text{if } Z_i(x_i) = x_i,
\end{aligned}
$$

holds for μ_{X_i}-almost every x_i.

Proof. The fact that the optimal reinsurance strategy for risk i is deterministic follows immediately from Corollary 5.2. Thus, the optimality conditions follow from Theorem 4.8. ■

5.3 The stop-loss is not optimal

It is known [Cai and Wei, 2012] that if all reinsurance premia are calculated by expected value principia, and the risks are positively dependent in stochastic order, then stop-loss treaties $Z_i(x) = \max(0, x - M_i)$ are optimal among the class of all deterministic treaties such that the retained risks $X_i - Z_i(X_i)$ are increasing functions of X_i.

We will now show that stop-loss treaties are in general not optimal in the wider class of measurable deterministic treaties, even in the setting above. Thus, the monotonicity constraint on the retained risk is typically an active constraint.

Consider two risks, X_1 and X_2, with absolutely continuous (marginal) distributions with support $[0, +\infty[$, each being reinsured through a stop-loss treaty, priced by an expected value premium principle:

$$
Z_i(x) = \max(0, x - M_i), \quad P(Z_i) = (1+\theta_i)\mathbb{E}Z_i(X_i), \qquad i = 1, 2,
$$

and consider an exponential utility function

$$
U(x) = -e^{-Rx}.
$$

By Corollary 5.3, if such a strategy is optimal, then the conditions

$$
\begin{cases}
(1+\theta_i)E\left[e^{R(X_1-Z_1+X_2-Z_2)}\right] \geqslant E\left[e^{R(X_1-Z_1+X_2-Z_2)}|X_i = x_i\right] & x_i \leqslant M_i \\
(1+\theta_i)E\left[e^{R(X_1-Z_1+X_2-Z_2)}\right] = E\left[e^{R(X_1-Z_1+X_2-Z_2)}|X_i = x_i\right] & x_i \geqslant M_i
\end{cases}
$$

hold for μ_{X_i}-almost every x_i, $i = 1, 2$. Thus, for $i = 1$ (and similarly for $i = 2$), we have

$$(1 + \beta_1)E\left[e^{R(X_1 - Z_1 + X_2 - Z_2)}\right] = e^{RM_1}E\left[e^{R(X_2 - Z_2)}|X_1 = x_1\right], \tag{18}$$

for μ_{X_1}-almost every $x_1 \geq M_1$. If X_1 and X_2 are dependent through a given copula $C(u, v)$, then

$$\begin{aligned} E\left[e^{R(X_2-Z_2)}|X_1 = x_1\right] &= \\ &= e^{RM_2} - \int_0^{M_2} \left(e^{RM_2} - e^{Rx_2}\right) \frac{\partial^2 C}{\partial u \partial v}(u, F_2(x_2)) f_2(x_2)\, dx_2, \end{aligned}$$

which, from (18), must be constant for $u \in [F(M_1), 1]$, implying that:

$$\int_0^{M_2} \left(e^{RM_2} - e^{Rx_2}\right) \frac{\partial^3 C}{\partial u^2 \partial v}(u, F_2(x_2)) f_2(x_2)\; dx_2 = 0, \quad \forall u \in [F_1(M_1), 1]. \tag{19}$$

Consider now that the risks are dependent through a copula:

$$C(u, v) = uv\,(1 + (u-1)(v-1)\alpha)\,, \qquad \text{with } \alpha \in]0, 1]. \tag{20}$$

Since

$$\frac{\partial^2 C}{\partial u^2} = 2v(v-1)\alpha \leq 0, \quad \frac{\partial^2 C}{\partial v^2} = 2u(u-1)\alpha \leq 0, \qquad \forall (u, v) \in [0, 1]^2,$$

the risks X_1, X_2 are positively dependent in stochastic ordering. Further, $\frac{\partial^3 C}{\partial u^2 \partial v} = 2(2v-1)\alpha$ and, the derivative with respect to M_2 of the left-hand side of (19) becomes

$$\begin{aligned} \frac{\partial}{\partial M_2} \int_0^{M_2} \left(e^{RM_2} - e^{Rx_2}\right) \frac{\partial^3 C}{\partial u^2 \partial v}(u, F_2(x_2)) f_2(x_2)\; dx_2 &= \\ &= Re^{RM_2} 2\alpha F_2(M_2)\,[F_2(M_2) - 1] < 0. \end{aligned}$$

Hence, $M_2 = 0$ is the unique value of M_2 satisfying (19), i.e., stop-loss is optimal only if it is optimal to cede the totality of the risk.

5.4 Variance-related principles

If the reinsurance premium is computed by a variance-related premium calculation principle, then the optimal treaty for that risk is deterministic and the optimality conditions are as follows.

Corollary 5.4 *Suppose that U is differentiable and concave in $]-\infty, c[$, and let $\nu \in \mathcal{H}_X$ be an optimal randomized strategy, and suppose that assumption* **(A2)** *of Theorem 4.4 holds*

If the marginal distribution μ_{X_i} is absolutely continuous except possibly for an atom at $x_i = 0$, and the reinsurance premium for risk $i \in \{1, 2, \ldots, n\}$ is computed by a variance-related principle

$$P_i(\nu) = \mathbb{E}^\nu Z_i + g\left(\mathbb{E}^\nu Z_i^2 - (\mathbb{E}^\nu Z_i)^2\right),$$

with g continuous in $[0, +\infty[$, continuously differentiable in $]0, +\infty[$, monotonically increasing, then the optimal reinsurance for risk i is a deterministic function $Z_i \in \mathcal{Z}_i$ such that

$$\mathbb{E}^\nu_{(X_{[i]}, Z_{[i]})|X_i} U'(L_\nu)(x_i, 0) \leq \left(1 - 2\mathbb{E}Z_i g'(\mathrm{Var}(Z_i))\right) \mathbb{E}^\nu U'(L_\nu) \qquad \textit{if } Z_i(x_i) = 0,$$

$$\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|X_i} U'(L_\nu)(x_i, Z_i(x_i)) = \left(1 + 2\left(Z_i(x_i) - \mathbb{E}Z_i\right) g'(\mathrm{Var}(Z_i))\right) \mathbb{E}^{\nu} U'(L_\nu) \quad \text{if } 0 < Z_i(x_i) < x_i,$$
$$\mathbb{E}^{\nu}_{(X_{[i]},Z_{[i]})|X_i} U'(L_\nu)(x_i, x_i) \geq \left(1 + 2\left(x_i - \mathbb{E}Z_i\right) g'(\mathrm{Var}(Z_i))\right) \mathbb{E}^{\nu} U'(L_\nu) \qquad \text{if } Z_i(x_i) = x_i$$

holds for μ_{X_i}*-almost every* x_i*.*

Proof. Under the assumptions of the Corollary,

$$\Psi_i(u_1, u_2) = u_1 + g(u_2 - u_1^2)$$

Is continuous in $D_2 \cup \left\{(u_1, u_2) \in [0, +\infty[^2 : u_1 = u_2^{\frac{1}{2}}\right\}$, continuously differentiable in D_2, and

$$\frac{\partial \Psi_i}{\partial u_1}(u_1, u_2) = 1 - 2u_1 g'\left(u_2 - u_1^2\right), \qquad \frac{\partial \Psi_i}{\partial u_2}(u_1, u_2) = g'\left(u_2 - u_1^2\right).$$

By assumption, $\frac{\partial \Psi_i}{\partial u_2} \geq 0$, and clearly $\frac{\partial \Psi_i}{\partial u_1}, \frac{\partial \Psi_i}{\partial u_2}$ have no common zeros. Thus, Corollary 5.2 guarantees that the optimal strategy for risk i is deterministic, and the optimality conditions follow from Theorem 4.8. ■

Notice that the assumptions of Corollary 5.4 (and a fortiori, Theorem 5.1) include cases where the premium calculation principle is not a convex functional in the space of deterministic strategies. For details, see the characterization of convex variance related principia in [Guerra and Centeno, 2010], Proposition 1.

6 Numerical illustration

We present here some numerical examples illustrating the theoretical results of this work. We will not go into detail on the numerical strategy for solving the optimization problem, which will be the subject of future work. It consists on the implementation of a direct minimization algorithm. The solution $Z_i(X_i)$ of the ceded risk is found at discretization points as the solution of the discretized minimization problem. The discretization points are chosen to be quantiles of each marginal distributions and linear interpolation is performed. We consider two risks, X_1 and X_2, with distribution functions given by $F_1(x) = 1 - e^{-x}$ and $F_2(x) = 1 - \left(\frac{4}{4+x}\right)^5$, respectively, such that $E[X_1] = E[X_2] = 1$ and $Var[X_1] = 1$ and $Var[X_2] = 5/3$. Three different dependence structures will be analysed, by means of copulas. We will consider three different premium calculation principles: (i) the expected value principle, in which case the loadings are chosen to be $\theta_1 = 0.3$ and $\theta_2 = 0.5$ for risks X_1 and X_2, respectively; (ii) the standard deviation principle, for which the premium loadings are $\theta_1 = \theta_2 = 0.5$. (ii) and the variance principle, for which the premium loadings are $\theta_1 = \theta_2 = 0.5$. The results are presented in comparison with the independence case.

Regarding the dependence structure, we will consider Frank's Copula, given by ([Denuit et al., 2005]):

$$C_\alpha(u_1, u_2) = -\frac{1}{\alpha} \log\left(1 + \frac{(e^{-\alpha u_1} - 1)(e^{-\alpha u_2} - 1)}{e^{-\alpha} - 1}\right).$$

This copula is known to have no upper nor lower tail dependence. When the copula parameter $\alpha = 0$, the random variables are independent, if $\alpha > 0$ there is a positive dependence and when $\alpha < 0$ the dependence is negative. We consider two cases: $\alpha = 10$, and $\alpha = -10$.

We also consider a copula, with no general expression, which includes positive and negative dependencies in different regions of the domain. The three copulas are represented in Figure 1. The numerical solutions of the optimal treaties are presented in Figures 2, 3, and 4.

In the case of positive dependence (Figure 2) we observe that for the expected value principle, the optimal treaty for risk X_1 (light tailed) is a decreasing function of the retained risk (notice that the plot represents the ceded risk). In this case, the optimal treaty under independence is the stop loss. Regarding the variance related principles, the optimal treaty under dependence has similar behaviour to that of the independent case, but the ceded risk is higher when positive dependencies are present.

When the risks are negatively dependent (Figure 3), the optimal ceded risk is always lower than the optimal ceded risk under independence, whatever the premium principle and for both risks. When the variance or the standard deviation principles are considered, the optimal ceded risk is a non-monotonic function. It is zero on neighbourhood of the expected value, being the minimum between a convex function and the claim amount.

In the case of more complex dependencies (Figure 4), including negative and positive dependencies along the domain, the optimal treaty exhibits abrupt swings in the amount of ceded risk. For the light-tailed risk (X_1), the optimal treaty cedes all the risk at small claim amounts and cedes no risk on the tail. For the heavy-tailed risk (X_2) it is optimal to cede all the risk at small claim amounts and at the tail, ceding no risk in a neighbourhood of the expected value. This pattern persists for all three premium principles being considered. This example shows that dependencies may lead to unexpected optimal treaties, specially when more intricate dependencies are at stake.

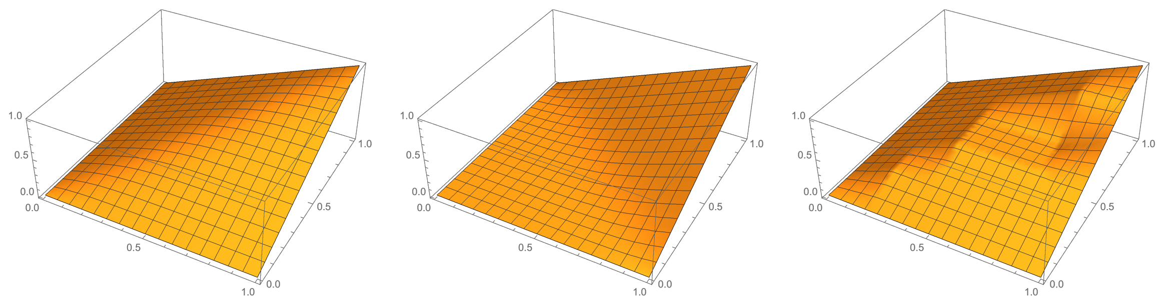

Figure 1: The three copulas considered. Left: Frank's copula with $\alpha = 10$; Middle: Frank's copula with $\alpha = -10$; Right: copula including positive and negative dependencies.

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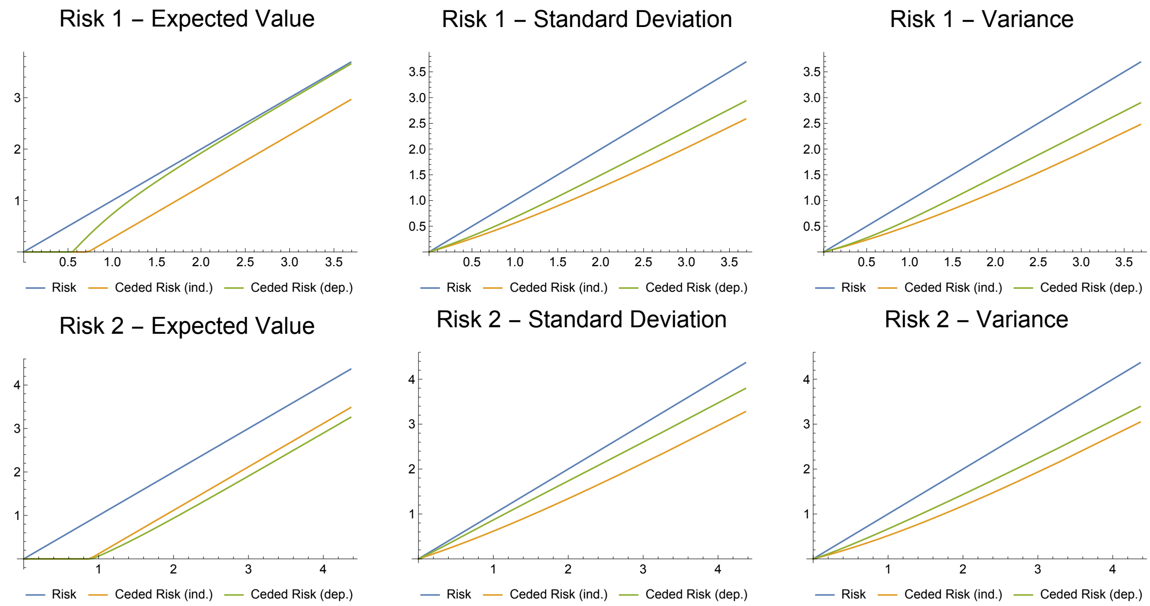

Figure 2: Optimal reinsurance treaties for risks X_1 and X_2 for the three premium calculation principles when the risks are dependent through Frank's copula with $\alpha = 10$. In blue is represented the underlying risk, in yellow the optimal solution (ceded risk) for the independent case and in green the optimal solution (ceded risk) in the dependent case.

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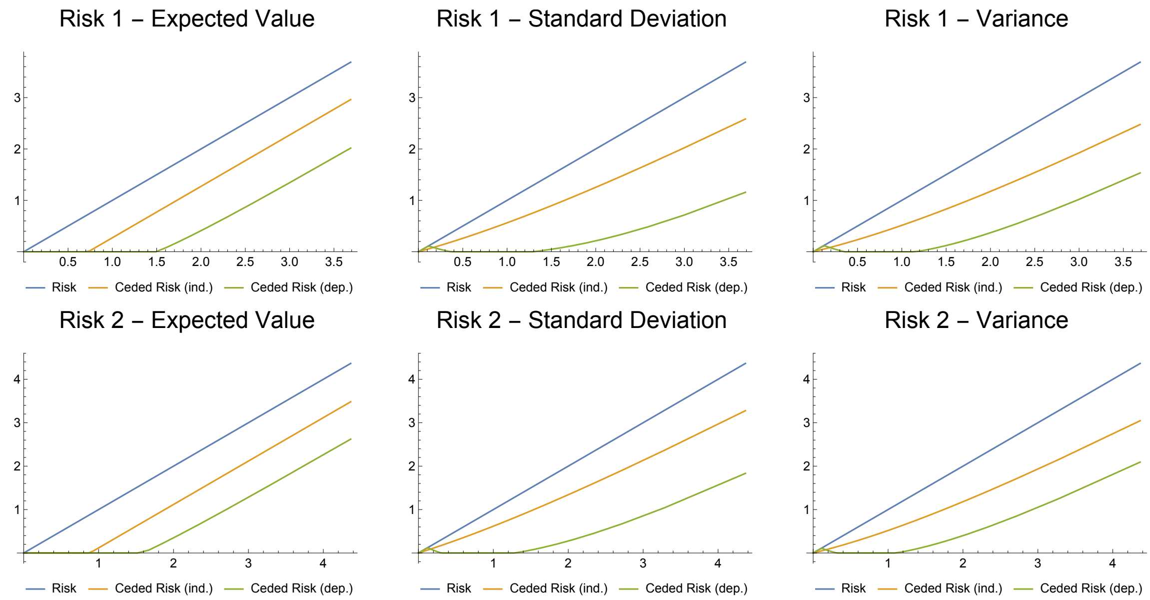

Figure 3: Optimal reinsurance treaties for risks X_1 and X_2 for the three premium calculation principles when the risks are dependent through Frank's copula with $\alpha = -10$. In blue is represented the underlying risk, in yellow the optimal solution (ceded risk) for the independent case and in green the optimal solution (ceded risk) in the dependent case.

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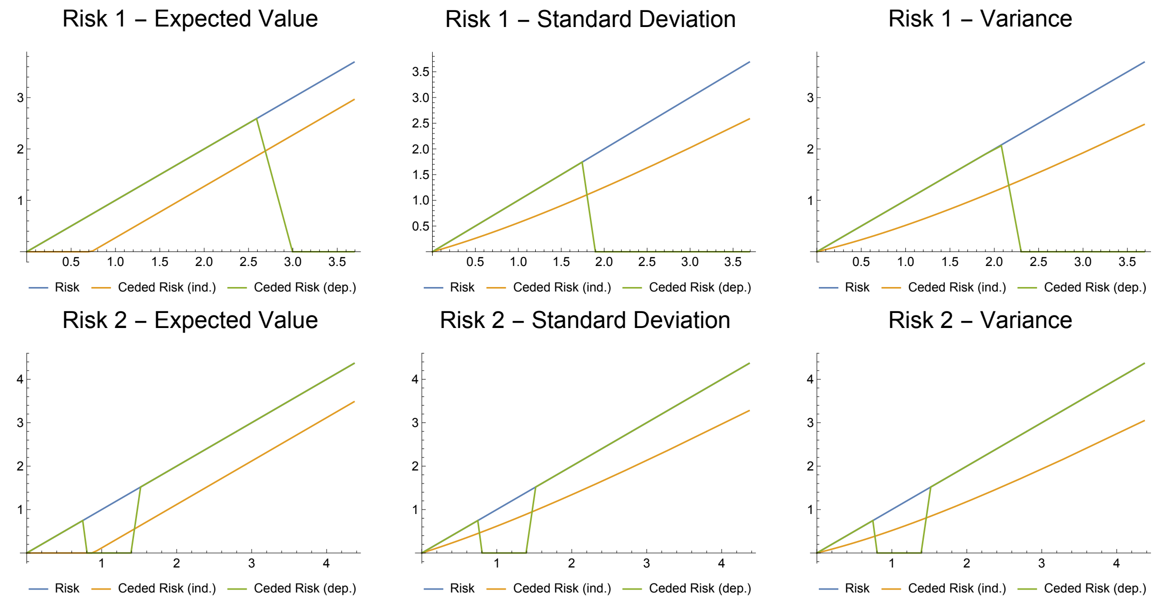

Figure 4: Optimal reinsurance treaties for risks X_1 and X_2 for the three premium calculation principles when the risks are dependent through a copula including positive and negative dependencies (see Figure 1). In blue is represented the underlying risk, in yellow the optimal solution (ceded risk) for the independent case and in green the optimal solution (ceded risk) in the dependent case.

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