DYNAMICS OF PIECEWISE INCREASING CONTRACTIONS

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Abstract. Let $0 < \lambda < 1$ and $I_1 = [a_0, a_1), \ldots, I_k = [a_{k-1}, a_k)$ be a partition of the interval $I = [0, 1)$ into $k \geq 1$ subintervals. Let $f : I \to I$ be a map where each restriction $f|_{I_i}$ is an increasing λ-Lipschitz function for $i = 1, \ldots, k$. We prove that any piecewise increasing contraction f admits at most k periodic orbits, where the upper bound is sharp. Our second result concerns piecewise λ-affine maps. Let $b_1, \ldots, b_k$ be real numbers. Let $F_\lambda : I \to \mathbb{R}$ be a family of piecewise λ-affine functions, where each restriction $F_\lambda|_{I_i}(x) = \lambda x + b_i$. Under a generic assumption on the parameters $a_1, \ldots, a_{k-1}, b_1, \ldots, b_k$ which define F_λ, we prove that, up to a zero Hausdorff dimension set of slopes $0 < \lambda < 1$, the ω-limit set of the piecewise λ-affine map $f_\lambda : x \in I \to F_\lambda(x)$ (mod 1) at every point equals a periodic orbit and there exist at most k periodic orbits.

1. Introduction

Let $I = [0, 1)$ and $f : I \to I$ be an interval map which is continuous up to finitely many points and right continuous at every discontinuity point. We call f a *piecewise increasing contraction*, if there exists $0 < \lambda < 1$ such that on every domain D of continuity of f, the restriction map $f|_D$ is increasing and λ-Lipschitz[1]. Throughout the paper *increasing* means strictly increasing.

Let $x \in I$ and denote by $\omega(f, x)$ the ω-limit set of f at the point x and

$$\omega(f) = \bigcup_{x \in I} \omega(f, x).$$

We say that f is *asymptotically periodic* if, for every $x \in I$, $\omega(f, x)$ equals a periodic orbit and $\omega(f)$ consists of finitely many periodic orbits.

The study of the dynamics of interval piecewise contractions has attracted the attention of many authors, in particular see [1, 2, 4, 7, 9, 11, 12, 13, 14, 15]. The motivation of our first theorem comes mainly from [15] which shows that generically piecewise contractions are asymptotically periodic. We recall that in [14], the authors prove that any injective interval piecewise contraction which has n discontinuities admits at most $n + 1$ periodic orbits and this upper bound is sharp. In this paper our first goal is to present classes of piecewise increasing contractions, not necessarily injective, which are asymptotically periodic and to prove an upper bound for their number of periodic orbits.

Theorem 1. *Let f be a piecewise increasing contraction with n discontinuity points. Then f has at most $n+1-\ell$ periodic orbits where ℓ is the number of discontinuity points whose image under f equals zero.*

We call attention that our approach to prove Theorem 1 is elementary and f is not assumed to be asymptotically periodic. We also recall that the result obtained in [15] for an upper bound of the number of periodic orbits concerns generic piecewise contractions, thus it can not be applied to prove Theorem 1.

Next we state a couple of corollaries of Theorem 1. First, Corollary 2 that generalizes previous results and also [11, Theorem 10] to any number of discontinuities, and

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[1] $|f(x) - f(y)| \leq \lambda |x - y|$ holds for any $x, y \in D$.

Corollary 3 which improves the upper bound obtained in [15, Theorem 1.1] in the case of positive slope. Theorem 1 and its corollaries are proved in Section 2.

Identifying the circle $\mathbb{R} \setminus \mathbb{Z}$ with the interval I through the canonical bijection $I \hookrightarrow \mathbb{R} \to \mathbb{R} \setminus \mathbb{Z}$, we may see any piecewise orientation-preserving contraction circle map as a piecewise increasing contraction.

Corollary 2. *Let f be a circle map which is a piecewise orientation-preserving contraction. Assume that f has k points of discontinuity on the circle $\mathbb{R} \setminus \mathbb{Z}$ and f is right continuous at those points. Then f has at most k periodic orbits and this upper bound is sharp.*

In this article we are also interested in the dynamics of piecewise λ-affine maps. Given $k \geq 1$, a function $F : I \to \mathbb{R}$ is called a *k-interval piecewise λ-affine function*, if there exist sequences of real numbers $a_0 = 0 < a_1 < \ldots < a_{k-1} < a_k = 1$ and $b_1, \ldots, b_k$ such that

$$F(x) = \lambda x + b_i \in \mathbb{R}, \ \forall\, 1 \leq i \leq k \text{ and } \forall\, x \in [a_{i-1}, a_i). \tag{1}$$

We call $f : x \in I \mapsto F(x) \pmod 1$ a *piecewise λ-affine map*. For this special case, we prove the following result.

Corollary 3. *Let $0 < \lambda < 1$ and let $F\colon I \to \mathbb{R}$ be a k-interval piecewise λ-affine function* (1)*. Then, the map $f = F \pmod 1$ has at most k periodic orbits.*

In [15] the authors considered a family of piecewise λ-affine maps, where $0 < |\lambda| < 1$ is fixed. Precisely, let $0 < |\lambda| < 1$ and F be a k-interval piecewise λ-affine function given by (1). Let δ be a real parameter and we define a parametrized family of piecewise λ-affine maps denoted by $f_\delta : I \to I$ using the following set up:

$$f_\delta : x \in I \mapsto F(x) + \delta \pmod 1.$$

It is proved in [15] that, for Lebesgue almost every δ, the map f_δ is asymptotically periodic and has at most $k + 1$ periodic orbits. However, the upper bound $k + 1$ can be achieved only when the slope λ takes a negative value.

In this paper another set up will be analyzed. Let $0 < \lambda < 1$ and F be a k-interval piecewise λ-affine function given by (1). We fix the real k-tuples $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ which appear in (1), then we will vary the value of the slope λ of F on the interval $(0, 1)$. Our goal is to measure the set of slopes $0 < \lambda < 1$ such that the associated piecewise λ-affine map f is not asymptotically periodic.

Let $\boldsymbol{a} = (a_1, \ldots, a_k)$ and $\boldsymbol{b} = (b_1, \ldots, b_k)$ be fixed k-tuples of real parameters as described above. We say that the k-tuples $\boldsymbol{a}$ and $\boldsymbol{b}$ are *$\mathbb{Z}$-independent*, if

$$a_i - b_j \notin \mathbb{Z}, \quad \forall\, (i, j) \in \{1, \ldots, k\}^2. \tag{2}$$

Remark that the condition (2) is a generic property.

The case $\lambda = 0$ is not interesting and the case $-1 < \lambda < 0$ can be treated similarly by adapting the arguments of this paper.

Our main theorem is the following:

Theorem 4. *Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be $\mathbb{Z}$-independent k-tuples as in* (2)*, where $k \geq 1$. Let F_λ be the family of k-interval piecewise λ-affine functions defined by $\boldsymbol{a}$ and $\boldsymbol{b}$, where $0 < \lambda < 1$, and let $f_\lambda : x \in I \mapsto F_\lambda(x) \pmod 1$ be the corresponding family of piecewise λ-affine maps. Then the set*

$$\{\lambda \in (0, 1)\colon f_\lambda \text{ is not asymptotically periodic}\}$$

has Hausdorff dimension zero.

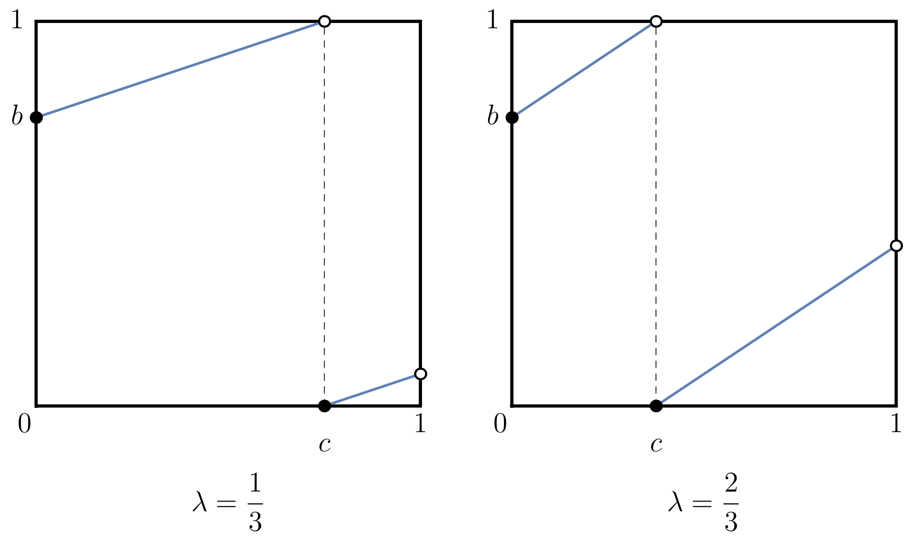

FIGURE 1. Plot of $R_{\lambda,b}$ for two distinct values of λ.

This theorem will be proved in Section 5.

A concrete example to which our result applies is to the so-called contracted rotations (see Figure 1). According to [11], a *contracted rotation* is a 2-interval piecewise λ-affine interval contraction defined in the following way: Let λ and b be two real numbers such that $0 < 1 - \lambda < b < 1$. Set $c = \dfrac{1-b}{\lambda}$. A contracted rotation is the map $R_{\lambda,b} : I \to I$ given by the splitted formula

$$R_{\lambda,b}(x) = \begin{cases} \lambda x + b, & \text{if} \quad 0 \le x < c, \\ \lambda x + b - 1, & \text{if} \quad c \le x < 1. \end{cases} \tag{3}$$

Notice that $R_{\lambda,b}(x) = \lambda x + b \pmod 1$ for every $x \in I$. Set $\boldsymbol{a} = (1)$ and $\boldsymbol{b} = (b)$ which satisfy (2), thus the contracted rotation is a piecewise λ-affine map which is defined by $\mathbb{Z}$-independent $\boldsymbol{a}$ and $\boldsymbol{b}$.

As we know (see [3],[11]), any contracted rotation $R_{\lambda,b}\colon I \to I$ admits a rotation number $0 < \rho(R_{\lambda,b}) < 1$. In Figure 2, it is described the regions in the triangle Δ formed by the parameters (λ, b), i.e.,

$$\Delta = \{(\lambda, b) : 0 < 1 - \lambda < b < 1\},$$

where the rotation number $\rho(R_{\lambda,b})$ takes a precise rational value.

Concerning the exceptional set

$$\mathcal{E} = \{(\lambda, b) \in \Delta\colon \rho(R_{\lambda,b}) \text{ is irrational}\},$$

the following is already known: It was proved in [11] that, once $0 < \lambda < 1$ is fixed, the Hausdorff dimension of the set

$$\{b \in (0,1) : (\lambda, b) \in \mathcal{E}\}$$

equals zero and, in [9], the authors showed that the Hausdorff dimension of the exceptional set $\mathcal{E}$ equals one. The proofs of these two results use the following proposition which describes the relation between the parameters λ and b when the rotation number takes a rational value (see [6],[3],[11]):

Proposition 5. *Let $0 < \lambda < 1$, then the rotation number of the map $R_{\lambda,b}$ takes the rational value $\dfrac{p}{q}$, where $1 \le p < q$ are relatively prime, if, and only if, b belongs to the*

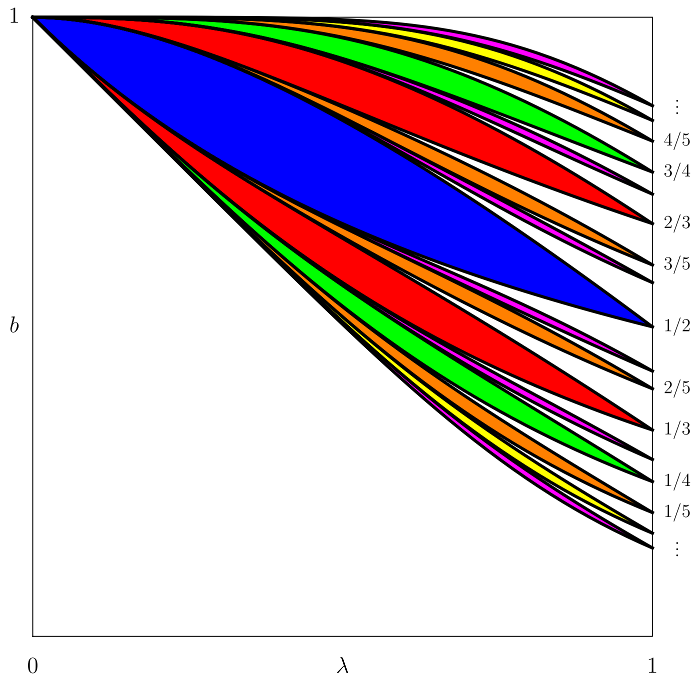

FIGURE 2. Rational tongues: regions where $\rho(R_{\lambda,b})$ takes a rational value in the interval $(0,1)$.

interval

$$\frac{1-\lambda}{1-\lambda^q} S\left(\lambda, \frac{p}{q}\right) \leq b \leq \frac{1-\lambda}{1-\lambda^q}\left(S\left(\lambda, \frac{p}{q}\right) + \lambda^{q-1} - \lambda^q\right),$$

where $S\left(\lambda, \frac{1}{2}\right) = 1$ *and* $S\left(\lambda, \frac{p}{q}\right) = 1 + \sum_{k=1}^{q-2}\left(\left[(k+1)\frac{p}{q}\right] - \left[k\frac{p}{q}\right]\right)\lambda^k$ *when* $q > 2$.

An equivalent claim about the parameters λ and b when the rotation number takes a rational value with $0 < b < 1$ fixed and λ varies is not available. Nevertheless, as a corollary of our main theorem we obtain the following result:

Corollary 6. *Let* $0 < b < 1$*, then the set*

$$\{\lambda \in (0,1) : (\lambda, b) \in \mathcal{E}\}$$

has Hausdorff dimension zero.

The paper is organized as follows. In Section 2, we present some notions and preliminary results which will be needed throughout the paper. In particular, we show in Proposition 9 that any piecewise increasing contraction has null entropy. As a corollary, we obtain that the ω-limit set of any piecewise increasing contraction has zero Hausdorff dimension. Section 2 concludes with the proofs of Theorem 1 and Corollaries 2 and 3. In Section 3, we recall the notion of invariant quasi-partition and, in Theorem 18, we establish a sufficient condition for a piecewise increasing contraction to be asymptotically periodic. Section 4 is devoted to piecewise λ-affine maps, with $0 < \lambda < 1$, there we apply results proved in previous sections to piecewise λ-affine maps. Finally the proof of Theorem 4 is left to Section 5.

2. Preliminary results

Throughout this section f is assumed to be a piecewise increasing contraction as defined in Section 1.

Definition 7. A point $x \in I$ is called a *singular point of* f if either $x = 0$ or x is a discontinuity point of f. Given $n \geq 1$, a point $x \in I$ is called a *singular point of* f^n if there exists $0 \leq j < n$ such that $f^j(x)$ is a singular point of f. We denote by S the set of singular points of f and by $S^{(n)}$ the set of singular points of f^n.

Notice that $S = S^{(1)}$ and

$$S^{(n)} = \bigcup_{j=0}^{n-1} f^{-j}(S), \quad \forall\, n \in \mathbb{N}.$$

Lemma 8. *Let $n \in \mathbb{N}$. Then the set $S^{(n)}$ is finite, f^n is right continuous and the restriction of f^n to each connected component of $I \setminus S^{(n)}$ is increasing and λ^n-Lipschitz.*

Proof. Notice that S is finite. Because f has finitely many increasing branches, the pre-image of any finite set is also finite, whence $f^{-j}(S)$ is finite for every $j \geq 0$. Thus, $S^{(n)}$ is finite. Right continuity of f^n is obvious. Now, let J be a connected component of $I \setminus S^{(n)}$. The sets $J, f(J), \ldots, f^{n-1}(J)$ do not contain singular points of f, otherwise J would not be a connected component of $I \setminus S^{(n)}$. Hence, for every $0 \leq j < n$, $f^j(J)$ is an interval and $f|_{f^j(J)}$ is increasing and λ-Lipschitz. This shows that $f^n|_J$ is increasing and λ^n-Lipschitz. □

Since S is finite, we can write $S = \{s_0, s_1, \ldots, s_{N-1}\}$ where $s_0 = 0 < s_1 < \cdots < s_{N-1} < 1$. Let $s_N = 1$ and consider the partition of I,

$$I_1 = [s_0, s_1), I_2 = [s_1, s_2), \ldots, I_N = [s_{N-1}, s_N).$$

On each interval I_j, where $1 \leq j \leq N$, the restriction map $f|_{I_j}$ is increasing and λ-Lipschitz. Moreover, f is right continuous at singular points.

Let $x \in I$, we denote the corresponding *itinerary* of x under f on the partition $\{I_j\}_{j=1}^N$ of the interval I by $(i_n)_{n=0}^\infty \in \{1, \ldots, N\}^{\mathbb{N}}$, where $\mathbb{N} = \{1, 2, \ldots\}$, it means that $f^n(x) \in I_{i_n}$ for every $n \geq 0$. A tuple $(i_0, i_1, \ldots, i_{n-1}) \in \{1, \ldots, N\}^n$ is called an *itinerary of order* n *of* f if it equals the first n entries of an itinerary of some point $x \in I$. We denote by $\mathcal{I}_n$ the set of all itineraries of order n of f.

Following [10], the *singular entropy* of f is defined as

$$h_{\text{sing}}(f) = \limsup_{n\to\infty} \frac{1}{n} \log \#\mathcal{I}_n,$$

where $\#\mathcal{I}_n$ denotes the cardinality of the set $\mathcal{I}_n$. By the general result [10, Theorem 2], we know that the singular entropy of any $\mathbb{R}^d$ non-expanding conformal piecewise affine map equals 0. In our present situation, we can write a rather simple proof that $h_{\text{sing}}(f) = 0$ using the fact that f is an interval map.

Proposition 9. *The singular entropy of f equals zero, i.e. $h_{sing}(f) = 0$.*

Proof. Given $\rho > 1$, let $m = \lceil \log 2/\log \rho \rceil$ and $\tau = \tau(m) > 0$ be the smallest distance between any two singular points of f^m. The points in any interval whose length is less than τ define at most two distinct itineraries of order m of f. Let $n_0 = \lceil \log \tau/\log \lambda \rceil$. Clearly, the length of $f_\lambda^n(W)$ is less than τ, for every connected component W of $I \setminus S_\lambda^{(n)}$ whenever $n \geq n_0$.

Set $\alpha_n := \#\mathcal{I}_n$. Therefore, $\alpha_{n+m} \leq 2\,\alpha_n$, for every $n \geq n_0$. So $\alpha_{n_0+im} \leq 2^i \alpha_{n_0}$, for every $i \geq 0$. Thus,

$$\alpha_n \leq 2^{\frac{n-n_0}{m}} \alpha_{n_0}, \text{ for every } n \geq n_0.$$

Taking into account the choice of m, we get $\alpha_n \leq C\rho^n$, for every $n \geq n_0$, where $C := 2^{-n_0/m}\alpha_{n_0}$. Hence, $\frac{1}{n}\log \alpha_n \leq \frac{1}{n}\log C + \log \rho$, which implies that $\lim_n \frac{1}{n}\log \alpha_n \leq \log \rho$. As $\rho > 1$ can be chosen arbitrarily close to 1, this shows that $h_{\text{sing}}(f) = 0$. □

As f has singular entropy zero, we obtain the following corollary from [5, Proposition 6.6].

Corollary 10. *The set $\omega(f)$ has Hausdorff dimension zero. In particular, $\omega(f)$ is a totally disconnected set.*

Remark 11. Note that the above claim is trivial whenever f is asymptotically periodic, as in this case $\omega(f)$ is a finite set.

Remark 12. In the case f is a contracted rotation with an irrational rotation number (3), it is proved in [9] that the closure of the limit set $C = \overline{\omega(f)}$ is a Cantor set. Moreover, in the particular case, where the slope of f equals $\lambda = \dfrac{1}{n}$, $n = 2, 3, \ldots$, it is showed in [4] that every point $x \in C$, but $x = 0$ or $x = 1$, is a transcendental number.

Let $\mathrm{Per}(f)$ denote the set of periodic points of f and $O(f, x)$ denote the forward orbit of x under f, i.e., $O(f, x) = \{f^n(x) : n \geq 0\}$. We introduce the following equivalence relation $\sim$ on the set of singular points of f:

$$\text{if } x, y \in S \text{ we write } x \sim y \text{ if and only if } \ \omega(f, x) = \omega(f, y). \tag{4}$$

Define the quotient set $A(f) := S/\sim$.

The following theorem gives an upper bound for the number of periodic orbits of f.

Theorem 13. *Assume Per(f) is a nonempty set, then there is a map*

$$\psi : Per(f) \to A(f)$$

with the following property: $\psi(x) = \psi(y)$ if, and only if, $O(f, x) = O(f, y)$, i.e. x and y belong to the same periodic orbit.

Proof. Let $x \in \mathrm{Per}(f)$ with period $p \in \mathbb{N}$. Since $S^{(p)}$ is finite (see Lemma 8), we can write $S^{(p)} = \{c_0, c_1, \ldots, c_{m-1}\}$ where $c_0 = 0 < c_1 < \cdots < c_{m-1} < 1$. Let $c_m = 1$ and consider the partition of I,

$$W_0 = [c_0, c_1), W_1 = [c_1, c_2), \ldots, W_{m-1} = [c_{m-1}, c_m).$$

There is a unique $\ell(x) \in \{0, \ldots, m-1\}$ such that $x \in W_{\ell(x)}$. Since $f^p(x) = x$ and $f^p|_{W_{\ell(x)}}$ is increasing and λ^p-Lipschitz (see Lemma 8), we conclude that $f^p(W_{\ell(x)}) \subset W_{\ell(x)}$ and $\omega(f, z) = O(f, x)$ for every $z \in W_{\ell(x)}$. In particular, $\omega(f, c_{\ell(x)}) = O(f, x)$. Because $c_{\ell(x)}$ is a singular point of f^p, we may define

$$n(x) := \min\{q \geq 0 \colon f^q(c_{\ell(x)}) \in S\}.$$

Notice that, $n(x) < p$. Let $\kappa(x) \in \{0, \ldots, N-1\}$ such that $s_{\kappa(x)} = f^{n(x)}(c_{\ell(x)})$. Finally, define

$$\psi : x \in \mathrm{Per}(f) \mapsto [s_{\kappa(x)}] \in A(f).$$

Now it is easy to see that $\psi(x) = \psi(y)$ for two periodic points $x, y \in \mathrm{Per}(f)$, if and only if $O(f, x) = O(f, y)$. Indeed, suppose that $s_{\kappa(x)} \sim s_{\kappa(y)}$ for two periodic points $x, y \in \mathrm{Per}(f)$. Then, $\omega(f, f^{n(x)}(c_{\ell(x)})) = \omega(f, f^{n(y)}(c_{\ell(y)}))$. Whence,

$$O(f, x) = \omega(f, c_{\ell(x)}) = \omega(f, f^{n(x)}(c_{\ell(x)})) = \omega(f, f^{n(y)}(c_{\ell(y)})) = \omega(f, c_{\ell(y)}) = O(f, y).$$

□

Remark 14. Theorem 13 implies that any piecewise increasing contraction f has at most $\#A(f)$ periodic orbits, where $\#A(f)$ denotes the cardinality of the set $A(f)$ which is at most the number of domains of continuity of f.

2.1. **Proof of Theorem 1.** The discontinuity points of f whose image equals zero belong to the equivalence class of zero (in the sense defined in (4)). This implies that $\#A(f) \leq n + 1 - \ell$ where $\ell = \#\{x \in S \setminus \{0\} \colon x \sim 0\}$. Therefore, by Theorem 13, f has at most $n + 1 - \ell$ periodic orbits.

2.2. **Proof of Corollary 2.** Identifying the circle $\mathbb{R} \setminus \mathbb{Z}$ with the interval I through the canonical bijection $I \hookrightarrow \mathbb{R} \to \mathbb{R} \setminus \mathbb{Z}$, we may conjugate f to an interval map $\tilde{f} \colon I \to I$ which is a piecewise increasing contraction. By further rotating the circle, we assume that 0 is a discontinuity of f. Whence, $\tilde{f}$ has $k - 1$ discontinuity points plus some possible extra discontinuity points whose image under $\tilde{f}$ equals zero. By Theorem 1, these new discontiuity points only decrease the number of periodic orbits. Therefore, $\tilde{f}$ has at most k periodic orbits, and the same is true for f.

2.3. **Proof of Corollary 3.** The piecewise λ-affine map $f : I \to I$ is a piecewise increasing contraction. Since F is a k-interval piecewise λ-affine function, f has $k - 1$ discontinuity points plus some extra discontinuity points due to the mod 1. These extra discontinuity points have zero image under f (since F at those points takes an integer value). Therefore, by Theorem 1, f has at most k periodic orbits.

3. Invariant quasi-partition

Now we recall the notion of *invariant quasi-partition* [15, Definition 2.7].

Definition 15. Let $f : I \to I$ be an interval map and m a positive integer. Let $\mathcal{P} = \{J_1, \ldots, J_m\}$ be a collection of m pairwise disjoint open subintervals of the interval I. We say that $\mathcal{P}$ is an *invariant quasi-partition of I under f*, if it satisfies the following properties:

(P1) $I \setminus \displaystyle\bigcup_{i=1}^{m} J_i$ contains at most finitely many points;

(P2) For every $\ell = 1, \ldots, m$, there is $1 \leq \tau(\ell) \leq m$ such that $f(J_\ell) \subset J_{\tau(\ell)}$.

Throughout the rest of this section we assume that $f \colon I \to I$ is a piecewise increasing contraction as defined in Section 1. Below we state a result [15, Lemma 2.8] which assures the existence of an invariant quasi-partition for f. For the convenience of the reader we include here a proof. Recall that S is the set of singular points of f.

Lemma 16. *If the set*

$$Q := \bigcup_{n \geq 0} f^{-n}(S) \tag{5}$$

is finite, then f admits an invariant quasi-partition.

Proof. Let $\mathcal{P} = \{J_\ell\}_{\ell=1}^{m}$ denote the finite collection of all connected components of $I \setminus Q$. Since Q is finite, $I \setminus \bigcup_{i=1}^{m} J_i$ contains at most finitely many points, thus $\mathcal{P}$ verifies (P1). In order to prove (P2), suppose by contradiction that $f(J_\ell) \cap Q \neq \emptyset$ for some $1 \leq \ell \leq k$. Then $J_\ell \cap f^{-1}(Q) \neq \emptyset$. Using the fact that $f^{-1}(Q) \subset Q$, we conclude that $J_\ell \cap Q \neq \emptyset$, which contradicts the definition of J_ℓ. □

In the following we use the notation $f(x^{\pm}) = \lim\limits_{y \to x^{\pm}} f(y)$.

Definition 17. A point $x \in S \cup \{1\}$ is called a *left periodic singular point of f* if there exists $n \in \mathbb{N}$ such that $f^n(x^-) = x$.

The following theorem is adapted from the results of [15].

Theorem 18. *Let f be a piecewise increasing contraction with no left periodic singular point. If Q is finite, then f is asymptotically periodic.*

Proof. Let $\mathcal{P} = \{J_\ell\}_{\ell=1}^m$ denote the invariant quasi-partition of f constructed in Lemma 16, i.e., $\mathcal{P}$ is the finite collection of the connected components of $I \setminus Q$. Notice that, since $S \subset Q$ (see (5)), for every $\ell \in \{1,\dots,m\}$ there is $\eta(\ell) \in \{1,\dots,N\}$ such that $J_\ell \subset I_{\eta(\ell)}$.

First we show that $\omega(f,x)$ is a periodic orbit for every $x \in I$. We consider two cases:

(i) If $x \in I \setminus Q$, then there is a sequence $(\ell_n)_{n=0}^\infty \in \{1,\dots,m\}^{\mathbb{N}_0}$ such that $f^n(x) \in J_{\ell_n}$ and $\ell_{n+1} = \tau(\ell_n)$, for every $n \geq 0$. Clearly, the sequence $(\ell_n)_{n=0}^\infty$ is eventually periodic, i.e., there exist $k \geq 0$ and $p \geq 1$ such that $\ell_{k+p} = \ell_k$. Therefore, $f^p(J_{\ell_k}) \subset J_{\ell_{k+p}} = J_{\ell_k}$. Let $J_{\ell_k} = (c,d)$ where $c, d \in Q \cup \{1\}$. Since $f^p|_{(c,d)}$ is λ^p-Lipschitz (Lemma 8), and extends to $[c,d]$ by uniform continuity, there exists $z \in [c,d]$ such that $\omega(f^p,x) = \{z\}$. We want to show that $f^p(z) = z$, whence z is a periodic point of f and $\omega(f,x) = O(f,z)$. We claim that $z \in [c,d)$, which implies that $f^p(z) = z$ because f^p is right continuous (Lemma 8). Now we prove the claim. Suppose, by contradiction, that $z = d$. Then $f^p(d^-) = d$. If $d = 1$, then 1 is a left periodic singular point of f, thus contradicting the fact that f has no left periodic singular point. So we may suppose that $d < 1$. Because $d \in Q$, we define

$$r = \min\{n \geq 0 \colon f^n(d) \in S\}.$$

Notice that $r < p$. Indeed, if $r \geq p$, then $d \notin S^{(p)}$, which means that f^p is continuous at d, i.e., $f^p(d) = f^p(d^-) = d$. Hence $O(f,d) \cap S = \emptyset$, contradicting the fact that $d \in Q$. Now take any increasing sequence $x_n \nearrow d$. We know that $f^p(x_n) \to d$. Since f^r is continuous at d, we obtain

$$\begin{aligned} f^r(d) = f^r(f^p(d^-)) &= f^r(\lim_{n\to\infty} f^p(x_n)) \\ &= \lim_{n\to\infty} f^r(f^p(x_n)) \\ &= \lim_{n\to\infty} f^p(f^r(x_n)) \\ &= f^p(f^r(d)^-). \end{aligned}$$

Because $f^r(d)$ is a singular point of f, we conclude that $f^r(d)$ is a left periodic singular point of f, which cannot exist by hypothesis. Hence, $\omega(f,x)$ is a periodic orbit.

(ii) If $x \in Q$, then either $f^n(x) \in Q$ for every $n \geq 0$, which implies that the orbit of x is finite, whence eventually periodic, or else there is $k \geq 1$ such that $f^k(x) \in I \setminus Q$. In the later case we reduce to the case (i). Therefore, $\omega(f,x)$ is a periodic orbit.

Finally, as $\mathcal{P}$ is finite, the map f has at most finitely many periodic orbits. This shows that f is asymptotically periodic.

□

Remark 19. The left periodic singular point hypothesis in the previous theorem cannot be removed as the following example shows. Let $f : I \to I$ be the piecewise increasing contraction,

$$f(x) = \begin{cases} \frac{x}{2} + \frac{1}{4}, & 0 \leq x < \frac{1}{2} \\ \frac{x}{2} - \frac{1}{4}, & \frac{1}{2} \leq x < 1 \end{cases}.$$

The map f admits an invariant quasi-partition since $Q = \{0, \frac{1}{2}\}$. Moreover, as $f(\frac{1}{2}^-) = \frac{1}{2}$ the discontinuity is a left periodic singular point of f. Therefore, Theorem 18 cannot be applied to this case. In fact, f is not asymptotically periodic as the ω-limit set of any point equals $\{\frac{1}{2}\}$ and the orbit of the discontinuity is not periodic.

4. Piecewise λ-affine contractions

Given $k \in \mathbb{N}$, let $F_\lambda : I \to \mathbb{R}$ be the k-interval piecewise λ-affine function defined by the tuples $\boldsymbol{a} = (a_1, \ldots, a_k)$ and $\boldsymbol{b} = (b_1, \ldots, b_k)$ (see (1)). Let $f_\lambda : I \to I$ be the piecewise λ-affine map defined by $f_\lambda = F_\lambda \pmod 1$.

Throughout this section $f_\lambda \colon I \to I$ is a piecewise λ-affine map as defined above. We will fix the tuples $\boldsymbol{a}$ and $\boldsymbol{b}$ and let λ vary in $(0,1)$. Notice that f_λ is a piecewise increasing contraction as defined in Section 2.

Recall that the tuples $\boldsymbol{a}$ and $\boldsymbol{b}$ are called $\mathbb{Z}$-independent if and only if

$$a_i - b_j \notin \mathbb{Z}, \quad \forall\, (i,j) \in \{1, \ldots, k\}^2.$$

Following the terminology in Section 2, we denote the set of singular points of f_λ by S_λ, and, given $n \in \mathbb{N}$, we denote the set of singular points of f_λ^n by $S_\lambda^{(n)}$. We also denote by N_λ the number of connected components of $I \setminus S_\lambda^{(1)}$. Notice that $S_\lambda = S_\lambda^{(1)}$. As in Section 2, the set of singular points of f_λ defines a collection of intervals $I_j = [s_{j-1}, s_j)$, $1 \le j \le N_\lambda$, which forms a partition of I. On each interval I_j, where $1 \le j \le N_\lambda$, according to (1) the map f_λ takes the expression

$$f_\lambda(x) = \varphi_j(x), \quad \forall x \in I_j, \tag{6}$$

with $\varphi_j(x) := \lambda x + \delta_j$, where $\delta_j = \beta_j + p_j$ for some $p_j \in \mathbb{Z}$ and $\beta_j \in \{b_1, \ldots, b_k\}$. Similarly to N_λ, the parameters δ_j defining φ_j may vary with λ. Indeed, the set of such λ's where both N_λ and the parameters δ_j change value are contained in the set

$$\mathcal{V} := \{\lambda \in (0,1) \colon \exists\, 1 \le j \le k \ \text{ such that } F_\lambda(a_j^-) \in \mathbb{Z} \text{ or } F_\lambda(a_j^+) \in \mathbb{Z}\}. \tag{7}$$

Notice that, $\mathcal{V}$ is a finite set and both N_λ and the parameters δ_j remain constant as λ varies inside each connected component of $(0,1) \setminus \mathcal{V}$.

A crucial step in the proof of Theorem 4 is to estimate the growth of a larger set of itineraries which contains all itineraries of nearby maps f_λ, for this purpose we need to exclude possible bifurcations of singular points.

Definition 20. A *singular connection of f_λ of order $n \ge 1$* is an n-tuple $(i_0, i_1, \ldots, i_{n-1}) \in \{1, \ldots, N_\lambda\}^n$ such that

$$\varphi_{i_{n-1}} \circ \cdots \circ \varphi_{i_0}(\{a_0, a_1, \ldots, a_k\}) \cap \{a_0, a_1, \ldots, a_k\} \neq \emptyset.$$

We say that f_λ *has a singular connection* if it has a singular connection of some order $n \ge 1$.

Notice that, if f_λ has a left periodic singular point (see Definition 17), then it has a singular connection.

Lemma 21. *Let f_λ be a family of piecewise λ-affine maps defined by $\mathbb{Z}$-independent tuples $\boldsymbol{a}$ and $\boldsymbol{b}$. Then the set*

$$\{\lambda \in (0,1) \colon f_\lambda \text{ has a singular connection}\}$$

is at most countable.

Proof. Let J be a connected component of $(0,1) \setminus \mathcal{V}$ (see (7)). Since $\mathcal{V}$ is finite there are at most a finite number of connected components. As previously discussed, N_λ and the parameters δ_j defining φ_j in (6) remain constant for every $\lambda \in J$.

Now, given $\lambda \in J$, f_λ has a singular connection of order of $n \ge 1$ if there exist $\omega = (i_0, \ldots, i_{n-1}) \in \{1, \ldots, N_\lambda\}^n$ and $x, y \in \{a_0, a_1, \ldots, a_k\}$ such that

$$y = \lambda^n x + \delta_{i_{n-1}} + \delta_{i_{n-2}}\lambda + \cdots + \lambda^{n-1}\delta_{i_0}. \tag{8}$$

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are $\mathbb{Z}$-independent, we have $y \neq \delta_{i_{n-1}}$. Consequently, the polynomial

$$Q_{x,y,\omega}(\lambda) = y - \left(\lambda^n x + \delta_{i_{n-1}} + \delta_{i_{n-2}}\lambda + \cdots + \lambda^{n-1}\delta_{i_0}\right)$$

is not identically zero. Thus, it has at most finitely many roots. Therefore, up to countable many λ's in J, equation (8) does not hold for every $x, y \in \{a_0, a_1, \ldots, a_k\}$ and ω of any order. This shows that, up to many countable λ's in the interval $(0,1)$, f_λ has no singular connection.

□

Let J be a connected component of $(0,1) \setminus \mathcal{V}$ (see (7)). Notice that $\mathcal{V}$ is finite. Given $\nu \in J$ and $0 < \varepsilon < 1$, we define the open interval

$$J_\varepsilon(\nu) := J \cap (\nu - \varepsilon, \nu + \varepsilon). \tag{9}$$

Recall that $\mathcal{I}_n$ is the set of itineraries of order n of f_λ (see Section 2). Of course, it depends on the choice of λ and we will write $\mathcal{I}_n(\lambda)$ to stress its dependency. Denote by $\mathcal{I}_n^\varepsilon = \mathcal{I}_n^\varepsilon(\nu)$ the union of all $\mathcal{I}_n(\lambda)$ over $\lambda \in J_\varepsilon(\nu)$.

Lemma 22. *Let $\nu \in J$ and f_ν be a piecewise ν-affine map with no singular connection. Then*

$$\lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \frac{1}{n} \log \#\mathcal{I}_n^\varepsilon(\nu) = 0.$$

Proof. The proof of this lemma is an adaptation of the proof of Proposition 9. We outline the main steps. As in that proof, given $\rho > 1$, let $m = \lceil \log 2/\log \rho \rceil$ and $\tau = \tau(m) > 0$ be the smallest distance between any two singular points of f_ν^m. Because f_ν has no singular connection, there is an $\varepsilon > 0$ such that the points in any interval whose length is less than $\tau/2$ define at most two distinct itineraries of order m of f_λ for every $\lambda \in J_\varepsilon(\nu)$. Moreover, choosing ε smaller if necessary, we can guarantee that the set of itineraries of order m of f_λ for any $\lambda \in J_\varepsilon(\nu)$ coincides with the set of itineraries of order m of f_ν. Now we choose $n_0 \in \mathbb{N}$, depending on m and ε, large enough such that for every $n \geq n_0$ the length of $f_\lambda^n(W)$ is less than $\tau/2$, for every $\lambda \in J_\varepsilon(\nu)$, and every connected component W of $I \setminus S_\lambda^{(n)}$. Set $\alpha_n := \#\mathcal{I}_n^\varepsilon$. The rest of the proof follows the same lines as the proof of Proposition 9. So we conclude that $\lim_n \frac{1}{n} \log \alpha_n \leq \log \rho$ for every $\varepsilon > 0$ sufficiently small. As $\rho > 1$ can be chosen arbitrarily close to 1, this proves the lemma. □

Given $n \in \mathbb{N}$ and $\omega = (i_0, \ldots, i_{n-1}) \in \{1, \ldots, N_\lambda\}^n$, we define the polynomial

$$H_\omega(\lambda) := \varphi_{i_{n-1}} \circ \cdots \circ \varphi_{i_0}(0) = \sum_{j=0}^{n-1} \lambda^j \delta_{i_{n-1-j}},$$

where the parameters δ_j remain constant for every $\lambda \in J_\varepsilon(\nu)$ (see (7) and (9)). Notice that $H_\omega(\lambda) + \lambda^n x = f_\lambda^n(x)$ for any $x \in I$, where ω is the corresponding itinerary of order n of f_λ. Also define

$$\Omega_\varepsilon(\lambda) := \bigcap_{m \geq 1} \overline{\bigcup_{n \geq m} \bigcup_{\omega \in \mathcal{I}_n^\varepsilon} \{H_\omega(\lambda)\}}. \tag{10}$$

Lemma 23. *Let $\lambda \in J_\varepsilon(\nu)$ as in (9) and $n \in \mathbb{N}$. Then the set $\Omega_\varepsilon(\lambda)$ can be covered by finitely many intervals of length $6\lambda^n$ which are centered at the points $H_\omega(\lambda)$, where $\omega \in \mathcal{I}_n^\varepsilon$.*

Proof. Given $y \in \Omega_\epsilon(\lambda)$, there are $n_k \nearrow \infty$ and $\omega_k \in \mathcal{I}_{n_k}^\varepsilon$ such that $y_k := H_{\omega_k}(\lambda) \to y$, as $k \to \infty$. Take k sufficiently large such that $n_k \geq n$ and $|y - y_k| \leq \lambda^n$. Denote by $[\omega_k]$ the last n entries of ω_k. Then $|y_k - H_{[\omega_k]}(\lambda)| = |H_{\omega_k}(\lambda) - H_{[\omega_k]}(\lambda)| \leq 2\lambda^n$ which gives $|y - H_{[\omega_k]}(\lambda)| \leq 3\lambda^n$ by the triangle inequality. □

Recall the set Q defined in Lemma 16. Because Q varies with λ, we shall denote it by Q_λ. The next result gives a sufficient condition for the set Q_λ to be finite whenever $\lambda \in J_\varepsilon(\nu)$ (see (9)). Recall the definition of $\Omega_\varepsilon(\lambda)$ in (10).

Lemma 24. *If*

$$\Omega_\varepsilon(\lambda) \cap \{a_0, a_1, \ldots, a_{k-1}\} = \emptyset,$$

then the set Q_λ is finite.

Proof. Since $\Omega_\varepsilon(\lambda) \cap \{a_0, a_1, \ldots, a_{k-1}\} = \emptyset$, there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$\min_{0 \le i < k} |H_\omega(\lambda) - a_i| \ge \delta, \quad \forall\, n \ge n_0,\, \omega \in \mathcal{I}_n^\varepsilon. \tag{11}$$

Let $n \ge n_1 := \max\{n_0, \lceil \log\delta / \log\lambda \rceil\}$ and suppose that there is $x \in I$ such that $f_\lambda^n(x) \in S_\lambda$ but $f_\lambda^m(x) \notin S_\lambda$ for every $0 \le m < n$, i.e., x is a singular point of f_λ^{n+1} but not of a lower iterate of f_λ. We have two cases:

(i) If $f_\lambda^n(x) \in \{a_0, a_1, \ldots, a_{k-1}\}$, then $|H_\omega(\lambda) - f_\lambda^n(x)| \le \lambda^n < \delta$, where ω is the itinerary of order n associated to x. Thus, $H_\omega(\lambda)$ is δ-close to $\{a_0, a_1, \ldots, a_{k-1}\}$ which contradicts (11).

(ii) If $f_\lambda^n(x) \in S_\lambda \setminus \{a_0, a_1, \ldots, a_{k-1}\}$, then $f_\lambda^{n+1}(x) = 0$. Let ω be the itinerary of order $n+1$ associated to x, then

$$H_\omega(\lambda) = |H_\omega(\lambda) - f_\lambda^{n+1}(x)| \le \lambda^{n+1} < \delta,$$

thus $H_\omega(\lambda)$ is δ-close to $a_0 = 0$ which contradicts (11).

Both cases contradict (11). Thus, for $n \ge n_1$ no new singular point of f_λ^n is created, i.e., f_λ^n and $f_\lambda^{n_1}$ have the same singular points for every $n \ge n_1$. Therefore, the set $\bigcup_{n\ge0} f_\lambda^{-n}(S_\lambda)$ is finite. ☐

5. Proof of Theorem 4

In this section we prove Theorem 4. We will use the following metric Łojasiewicz-type inequality [8, Theorem 4.6].

Lemma 25. *Let $0 \le a < b < 1$ and $r \ge 9$. There exist $0 < \theta \le 1$ and $0 < \epsilon_0 < 1$ such that if $0 < \epsilon < \epsilon_0$ and $p(x)$ is a polynomial of the form*

$$p(x) = 1 + c_1 x + c_2 x^2 + \cdots + c_n x^n, \quad c_i \in [-r, r], \quad n \in \mathbb{N},$$

then the following holds:

$$\mathrm{Leb}(\{x \in [a,b] \colon |p(x)| < \epsilon\}) \le C\epsilon^\theta,$$

where Leb *means Lebesgue measure and*

$$C = \frac{2^{5+\frac{3}{\theta}}(1+r)^{\frac{2}{\theta}} \deg(p)^{2+\frac{2}{\theta}}(2+\frac{1}{\theta})}{\epsilon_0^2}.$$

5.1. **Proof of Theorem 4.** Let f_λ be a family of piecewise λ-affine maps defined by $\mathbb{Z}$-independent k-tuples $\boldsymbol{a}$ and $\boldsymbol{b}$. We want to show that the set

$$\{\lambda \in (0,1) \colon f_\lambda \text{ is not asymptotically periodic}\}$$

has zero Hausdorff dimension. Denote by E the set of $\lambda \in (0,1)$ such that f_λ has a singular connection. By Lemma 21, the exceptional set E is at most countable. Notice that, f_λ has no left periodic singular point for every $\lambda \in (0,1) \setminus E$. Thus, according to Theorem 18, it is enough to show that $Z := \Lambda \setminus E$ has zero Hausdorff dimension, where

$$\Lambda := \{\lambda \in (0,1) \colon Q_\lambda \text{ is not finite}\}.$$

Let J be a connected component of $(0,1)\setminus\mathcal{V}$ (see (7)). Let $\nu\in J\setminus E$, $0<\varepsilon<1-\nu$ and $J_\varepsilon(\nu)$ be the interval defined in (9). Recall that the parameters δ_j in (6) remain constant for every $\lambda\in J_\varepsilon(\nu)$. By Lemma 24,

$$Z_\varepsilon(\nu):=Z\cap J_\varepsilon(\nu)\subset\{\lambda\in J_\varepsilon(\nu)\colon \Omega_\varepsilon(\lambda)\cap\{a_0,a_1,\ldots,a_{k-1}\}\neq\emptyset\}.$$

According to Lemma 23, the set $\Omega_\varepsilon(\lambda)$ can be covered by $\#\mathcal{I}_n^\varepsilon$ intervals of length $6\lambda^n$, thus

$$Z_\varepsilon(\nu)\subset\bigcup_{\omega\in\mathcal{I}_n^\varepsilon}\bigcup_{j=0}^{k-1}\{\lambda\in J_\varepsilon(\nu)\colon |H_\omega(\lambda)-a_j|\le 6(\nu+\varepsilon)^n\}.$$

For any $0\le j<k$ and $\omega\in\mathcal{I}_n^\varepsilon$, the polynomial $H_\omega(\lambda)-a_j\in\mathbb{R}[\lambda]$ is not identically zero and has degree $\le n-1$. Indeed, because f_λ is defined by $\mathbb{Z}$-independent tuples $\boldsymbol{a}$ and $\boldsymbol{b}$, we can write

$$\begin{aligned}H_\omega(\lambda)-a_j&=\delta_{i_{n-1}}+\delta_{i_{n-2}}\lambda+\cdots+\delta_{i_0}\lambda^{n-1}-a_j\\&=(\delta_{i_{n-1}}-a_j)\left(1+\delta'_{i_{n-2}}\lambda+\cdots+\delta'_{i_0}\lambda^{n-1}\right),\end{aligned}$$

where $\delta'_{i_m}:=\delta_{i_m}/(\delta_{i_{n-1}}-a_j)$ for $m=0,\ldots,n-2$ and

$$0<|\delta'_{i_m}|\le\frac{1}{|\delta_{i_{n-1}}-a_j|}\le\frac{1}{\min\limits_{1\le i,j\le k}\min\limits_{p\in\mathbb{Z}}|p+b_i-a_j|}=:r<\infty.$$

Taking $n_0\ge 1$ sufficiently large, we can apply Lemma 25 with $\epsilon=6r(\nu+\varepsilon)^n$ and get the following estimate which holds for every $n\ge n_0$,

$$\operatorname{Leb}\left(\{\lambda\in J_\varepsilon(\nu)\colon |H_\omega(\lambda)-a_j|\le 6(\nu+\varepsilon)^n\}\right)\le Cn^{2+\frac{2}{\theta}}(\nu+\varepsilon)^{\theta n},$$

where $0<\theta\le 1$, and $C>0$ is a constant independent of ω and n.

Therefore, the set $\{\lambda\in J_\varepsilon(\nu)\colon |H_\omega(\lambda)-a_j|\le 6(\nu+\varepsilon)^n\}$ can be covered by the union of n intervals $U_{\omega,j}^{(1)},\ldots,U_{\omega,j}^{(n)}$ of length less than $\eta_n:=Cn^{2+\frac{2}{\theta}}(\nu+\varepsilon)^{\theta n}$, i.e.,

$$\{\lambda\in J_\varepsilon(\nu)\colon |H_\omega(\lambda)-a_j|\le 6(\nu+\varepsilon)^n\}\subset\bigcup_{i=1}^n U_{\omega,j}^{(i)}\quad\text{and}\quad \operatorname{diam}(U_{\omega,j}^{(i)})<\eta_n.$$

Notice that $\lim_{n\to\infty}\eta_n=0$. Thus, for every $0<\sigma\le 1$, we have

$$\begin{aligned}\mathcal{H}^\sigma_{\eta_n}(Z_\varepsilon(\nu))&=\inf\left\{\sum_i\operatorname{diam}(U_i)^\sigma\colon Z_\varepsilon(\nu)\subset\bigcup_i U_i,\ \operatorname{diam}(U_i)<\eta_n\right\}\\&\le\sum_{\omega,j,i}\operatorname{diam}\left(U_{\omega,j}^{(i)}\right)^\sigma\\&<\sum_{\omega\in\mathcal{I}_n^\varepsilon}\sum_{j=0}^{k-1}\sum_{i=1}^n\eta_n^\sigma\\&=C^\sigma(\#\mathcal{I}_n^\varepsilon)\,k\,n^{1+\sigma(2+\frac{2}{\theta})}(\nu+\varepsilon)^{\sigma\theta n}.\end{aligned}$$

By Lemma 22, we have that $\lim_{n\to\infty}\mathcal{H}^\sigma_{\eta_n}(Z_\varepsilon(\nu))=0$, whence $\mathcal{H}^\sigma(Z_\varepsilon(\nu))=0$ for every $0<\sigma\le 1$. Thus, $Z_\varepsilon(\nu)$ has Hausdorff dimension equal to zero. Because Z is a countable union of the family of sets $\{Z_{\varepsilon_i}(\nu_i)\}_i$ each with zero Hausdorff dimension, we conclude that Z also has zero Hausdorff dimension.

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