Ruin probabilities for open automobile portfolios with a bonus-malus system based on claim counts

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Abstract

For a large motor insurance portfolio, on an open environment, we study the impact of experience rating in finite and continuous time ruin probabilities. We consider a model for calculating ruin probabilities applicable to large portfolios where a Markovian Bonus-Malus System (BMS), based on claim counts, is used for an automobile portfolio using the classical risk framework model. New challenges are brought when an open portfolio scenario is introduced. When compared to a classical BMS approach ruin probabilities may change significantly. By using a BMS of a Portuguese insurer, we illustrate and discuss the impact of the proposed formulation on the initial surplus required to target a given ruin probability. Under an open portfolio setup, we show that we may have a significant impact on capital requirements when compared to the classical BMS, by having a significant reduction on the initial surplus needed to maintain a fixed level of the ruin probability.

Keywords: Ruin probability; finite time ruin probability; bonus-malus; open portfolio; experience rating; capital requirement.

1 Introduction and Motivation

The main goal of this work is to calculate finite time ruin probabilities for large motor insurance portfolios where a Markovian Bonus-Malus System (briefly BMS) based on claim counts is put in place as experience rating. The paper by Afonso *et al.* (2017) shows a way to do this calculation/estimation in the presence of a classical BMS model. Our aim is to update their model to provide the implementation of an open BMS as we believe that the resulting ruin probabilities have a better or realistic representation for the business. The classical BMS model has implicitly the idea that the policies that may exit the portfolio in some period of time will be compensated by incoming ones. In the real world we do not necessarily have this behaviour. Indeed, in the very competitive motor insurance market, we assist great market movements among insurers, where insureds try often to get better deals, lower premia, and insurers try to increase their businesses. Besides, every insurer can build their own bonus scale. Often insureds are quite conservative and try to not deal or have too many different insurers when they buy several coverages, i.e., if an insured move a policy to another insurer they are likely to move the whole portfolio.

Furthermore, classical BMS model assumes the existence of fixed entry bonus class for the portfolio newcomers. Nowadays, this is not appropriate since insurance regulators provide insurers with the past record of a policyholder irrespective of previous insurers. This leads to consider that a portfolio newcomer (at least in theory) can enter at any bonus class in a new portfolio if he changes insurer at some time period. Better information results in better risk classification, then more appropriate premia to be charged and also a better evaluation or estimation of capital requirements for the insurers' business.

A first question is: Does these new ideas have an impact on ruin probabilities? Also: Do they lead to a real change in the ruin probability figures shown by Afonso *et al.* (2017), for instance? We believe they may. In fact, we know already that there is an effect on optimal scales, see Guerreiro *et al.* (2014). Our aim is wider as we intend to show that modelling an open portfolio may lead to a significant change in ruin probabilities, when compared to the classical BMS model ones. Also, they may contribute to a re-evaluation of capital requirements of an insurance company, once fixed a level for the ruin probability, whether in finite or infinite horizon. Furthermore, we are interested in evaluating the impact on existing optimal scales when applied to an open model, to premia and ruin probabilities. Bonus classes may be allowed to be less dispersed (bonuses not as high or maluses as low as in the classic formulation). Long run behaviour is also important as, in general, most of existing BMS tend to concentrate most of the insureds in higher bonus classes.

There aren't many authors calculating ruin probabilities in the presence of a BMS, certainly even less when we consider an open BMS formulation. Lemaire (1995) is clearly a classical reference for BMS, an important and more recent reference is Denuit *et al.* (2007). These only deal with the classical model and do not calculate ruin probabilities. Reference Afonso *et al.* (2017) particularly is concerned with finite time ruin probabilities for BMS in automobile insurance. There are several references about open BMS, however they are not devoted to ruin probability calculation, they mostly work bonus scales and model efficiency. Chosen examples are Andrade e Silva and Centeno (2001), Guerreiro *et al.* (2014) and Mahmoudvand and Aziznasiri (2014).

The manuscript is organized as follows. Next section is devoted to the presentation of the base model framework, including definitions, risk model and assumptions, BMS in open portfolios, scenarios and ruin probability formulae, procedures and estimation. Section 3 is devoted to numerical calculation of finite time ruin probabilities in the open model, considering the different scenarios. It also includes estimation and distribution fitting, policy allocation along time considering different cases and scenarios, capital requirements and result discussion. It finishes with some concluding remarks.

2 Basic framework

In this section, we summarise briefly our basic framework for the calculation of the probability of ruin in finite and continuous time. The base model was taken from Afonso *et al.* (2009) and it is already summarized in Afonso *et al.* (2017). As for the modeling of BMS in open portfolios we will follow the work by Guerreiro *et al.* (2014) with some developments obtained in Esquível *et al.* (2014).

2.1 Modelling the ruin probability

In this section, we start by introducing our base model, main definitions and notation, most of them retrieved from Afonso *et al.* (2009) and Afonso *et al.* (2017). We summarise the main definitions considered relevant for an easy reading flow. We will locally define and introduce some other definitions or notation where appropriate.

Consider a risk process over an n-year period. We denote by $S(t)$ the aggregate claim amount up to time t, with $S(0) = 0$, and by Y_i the aggregate claim amount in year i, so that $Y_i = S(i) - S(i-1)$. In the n-year period, $\{Y_i\}_{i=1}^{n}$ is a sequence of independent and identically distributed (briefly *i.i.d.*) random variables with common compound Poisson distribution, whose first three moments exist. Poisson parameter is denoted as λ. Let us also set $f(\cdot, s)$ as the probability density function (p.d.f.) of $S(s)$ for $0 < s \leq 1$.

Let P_i denote the total amount of premia charged in the portfolio in year i, which depends on the allocation of policies throughout the *bonus* levels in each year. The estimation of this allocation will differ significantly whether we consider a classical or an open BMS formulation, and may naturally impact the magnitude of corresponding ruin probabilities. The measure of these impacts is focused in this paper.

Let $U(t)$ denote the insurer's surplus at time t, $0 \leq t \leq n$. It is assumed that premia are received continuously at a constant rate throughout each year. The initial surplus, u $(= U(0))$, and the initial premium, P_1, are known. For each year i, the premium P_i , $i \geq 2$ and surplus level $U(i)$, $i \geq 1$ are random variables since they both depend on the claim experience in previous years and on the annual allocation of policies throughout the BMS classes, which will determine the *bonus* or *malus* to apply to each policy. Whenever we refer to a particular realization of these random variables, we use the corresponding lower case letters p_i and $u(i)$.

The evolution of the surplus of an insurance company or portfolio, $U(t)$, for any time t, $0 \leq t \leq n$, is driven by the following equation [as previously defined in Afonso *et al.* (2009), Formula (2.1)]:

$$U(t) = u + \sum_{j=1}^{i-1} P_j + (t - i + 1)P_i - S(t)\,, \tag{2.1}$$

where i is the positive integer such that $t \in [i-1, i)$ and $\sum_{j=1}^{0} P_j = 0$, by convention.

We summarize the basic assumptions of our model formulation as follows:

1. The portfolio is homogeneous with respect to claim severities;
2. The portfolio is heterogeneous with respect to claim frequencies, following a mixed Poisson distribution;

3. We consider a homogeneous claim frequency in each *bonus* level, i.e., in level j the number of reported claims in one year is Poisson distributed with parameter λ_j , $j = 1, 2, \ldots, L$, where L is the number of bonus-malus levels or classes;

4. The portfolio is open for incoming and outgoing of policyholders.

We note that Assumptions 1-3 above are the same as in Afonso *et al.* (2017). From there Assumption 4 has changed from closed to open portfolios. Our main goal in this manuscript is to show the impact/change on ruin probabilities motivated by the change in Assumption 4.

As in Afonso *et al.* (2017) let $\psi(u, n)$ denote the probability of ruin in continuous time within a period of n years and $\psi(u(i-1), 1, u(i))$ be the approximation to the probability of ruin within year i, given the surplus $u(i-1)$ at the beginning of the year, $u(i) \geq 0$ the surplus at the end of the year and a rate of premium income p_i during the year.

Let $H(s) + \kappa\, s$ be a random variable with a translated Gamma distribution whose first three moments match those of $S(s)$. We denote the parameters of the translated Gamma as α, β and κ, respectively the shape, scale and translation parameters, $\alpha, \beta > 0$ and $\kappa \in \mathbb{R}$. Denoting $F_G(\cdot, s)$ the cumulative distribution function and $f_G(\cdot, s)$ the *p.d.f.* of $H(s)$, Afonso *et al.* (2009) show that, after obtaining parameters α, β and κ, the approximation to the ruin probability in year i as defined above is given by, their Formula (3.1):

$$\begin{aligned}\psi(u(i-1), 1, u(i)) \quad = \quad & \frac{\int_{s=0}^{1-u(i)/p_i} f_G(u(i-1) + (p_i - \kappa)s, s) \frac{u(i)}{(1-s)} f_G((p_i - \kappa)(1-s) - u(i), 1-s) ds}{f_G(u(i-1) + p_i - \kappa - u(i), 1)} \\ & + \frac{f_G(u(i-1) + (p_i - \kappa)(1 - \frac{u(i)}{p_i}), 1 - \frac{u(i)}{p_i}) F_G(-\kappa u(i)/p_i, u(i)/p_i)}{f_G(u(i-1) + p_i - \kappa - u(i), 1)}. \qquad (2.2)\end{aligned}$$

In this paper, the estimated probability of ruin for a finite time, say n, will be obtained using formula (2.2) inserted in a simulation procedure that is described in Subsection 2.3.

2.2 BMS for open portfolios

In this subsection, we introduce main results on BMS for open portfolios, we follow Guerreiro *et al.* (2014) and some developments from Esquível *et al.* (2014). After, we make our main assumptions followed by the portfolio evolution and limiting results.

For a BMS with transition rules based on the claim frequency only, see Lemaire (1995) and Denuit *et al.* (2007) for instance, the position of each policyholder in the BMS level or class in a given annual period is determined uniquely by the class of the preceding year and by the number of claims reported during that period. The classical approach considers the BMS as an application of a homogeneous Markov chain with L finite states and estimates the step by step evolution and corresponding long run behaviour using well known results for Markov chains, see the above references Lemaire (1995) and Denuit *et al.* (2007).

We innovate now by assuming that in each year new policies enter the portfolio and some policyholders may leave the portfolio, as they wish. Assume also that the exits from the portfolio do not need to be perfectly compensated by new entries. In our formulation, as in Guerreiro *et al.* (2014), in order to account for the possibility of a policyholder leave the portfolio we consider an extra absorbing state in the Markov chain. This represents the exit from the portfolio. We'll have then $L + 1$ BMS classes, where states $1, \ldots, L$ are transient and $L + 1$ is an absorbing state.

For a given claim frequency λ and a set of transition rules T, the corresponding transition block matrix is denoted as

$$\boldsymbol{P}_{T,\lambda} = \begin{bmatrix} \boldsymbol{K}_{T,\lambda} & \boldsymbol{q}_\lambda \\ \boldsymbol{0} & 1 \end{bmatrix},$$

where $\boldsymbol{K}_{T,\lambda}$ is a $(L \times L)$ matrix representing the one step transition probabilities among the BMS classes for those policies that remain in the portfolio, $\boldsymbol{q}_\lambda$ a column vector of conditional probabilities of an insured leaving the portfolio at the end of the time period, and $\boldsymbol{0}$ is a null row vector.

Commonly in classical BMS models, for a given set of transition rules, the probability of a randomly chosen policyholder, with a given claim frequency λ, move from class l to class j is given by

$$p_{T,\lambda}(l,j) = \sum_{k=0}^{\infty} p_k(\lambda)\, t_{lj}(k)\ , \quad l,j = 1,\dots,L\,,$$

where $p_k(\lambda)$ is the probability of an insured with claim frequency λ report k claims in one year, $t_{lj}(k) = 1$ if he reports k claims leading the policy to move from class l to class j, according to transition rules T and $t_{lj}(k) = 0$, otherwise.

In the open model formulation, we set that the transition probabilities among classes $1,\dots,L$ are given by

$$k_{T,\lambda}(l,j) = p_{T,\lambda}(l,j)\,(1 - q_\lambda(l))\ , \qquad l,j = 1,\dots,L. \tag{2.3}$$

Here, $k_{T,\lambda}(l,j)$ is the entry (l,j) of matrix $\boldsymbol{K}_{T,\lambda}$ and $q_\lambda(l)$ is the l-th element of vector $\boldsymbol{q}_\lambda$. We highlight that Equation (2.3) reflects that, in an open portfolio formulation, the probability of a policyholder to move from class l to class j depends on the claim frequency λ, the transition rules T, and the probability of exiting the company $q_\lambda(l)$, which may be different from class to class. In other words, a policyholder with claim frequency λ moves, at the end of the year, from class l to class j only if he doesn't exit the company.

The n-step transition matrix, $n = 1, 2, , \dots,$ is given by

$$\boldsymbol{P}^{(n)}_{T,\lambda} = \begin{bmatrix} \boldsymbol{K}^n_{T,\lambda} & \boldsymbol{q}_{n,\lambda} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

with $\boldsymbol{q}_{n,\lambda} = \sum_{j=0}^{n-1} \boldsymbol{K}^j_{T,\lambda}\, \boldsymbol{q}_\lambda$, $\boldsymbol{K}^j_{T,\lambda}$ is the j-th power of matrix $\boldsymbol{K}_{T,\lambda}$, which corresponds to the j-step transition probabilities for the policyholders remaining in the portfolio after these j years, and $\boldsymbol{P}^{(1)}_{T,\lambda} = \boldsymbol{P}_{T,\lambda}$.

Let us now state some assumptions regarding the evolution of the portfolio along time:

1. New policies arrive at the portfolio are done at the beginning of each time period;
2. New policies entering the portfolio are allocated to a BMS level according to the probability vector $\boldsymbol{c}_i = [c_i(l)]_{1\times L}$, $i \in \mathbb{N}$;
3. The number of new policies entering the portfolio at time period i are independent random variables with mean value ϑ_i , $i \in \mathbb{N}$, and are denoted as E_i;

Before further developments we would like to comment that:

(a) Assumption 2 above allows for the allocation of new policyholders into any of the BMS levels. This allows the insurer to possibly observe the past claim history of the policyholder in the previous insurer and allocate the contract to the corresponding risk level of reported claims. It also allows for the estimation of allocation probabilities.

(b) We note that, if we further assume a Poisson distribution for each random variable E_i, it allow us to obtain confidence intervals and/or hypothesis testing for relevant parameters of the model, as shown in Guerreiro *et al.* (2014) and Esquível *et al.* (2014), if real data is used.

(c) In order to evaluate and compare the ruin probabilities over different hypothesis on portfolio evolution, we consider two different models for the mean value ϑ_i , $i \in \mathbb{N}$, namely:

Scenario 1 - Exponential Model: Following Guerreiro *et al.* (2014), mean value ϑ_i, $i \in \mathbb{N}$, is modelled by

$$\vartheta_i = \tau\left(1 - e^{-\delta i}\right) \,,\ i \in \mathbb{N} \,,\ \tau, \delta \in \mathbb{R}^+. \tag{2.4}$$

Scenario 2 - Sigmoid Model: Following Esquível *et al.* (2014), mean value ϑ_i, $i \in \mathbb{N}$, is modeled by

$$\vartheta_i = \left(a + b\, e^{-\theta i}\right)^{-1} \,, \quad (a, b, \theta) \in \Theta \,,\ i \in \mathbb{N} \,, \tag{2.5}$$

with $\Theta = \left\{(a, b, \theta) : a \in \mathbb{R}^+, b, \theta \in \mathbb{R}, a + b\, e^{-\theta i} > 0 \,,\ i \in \mathbb{N}\right\}$.

We remark that both models may be comparable in long run horizons, setting $\tau = 1/a$, however they model different evolutions before reaching a limiting situation.

(d) Other functions for modelling new annual entries could be used, provided the conditions for convergence of the model, such as those proven by Esquível *et al.* (2014).

We now discuss the portfolio evolution and limiting results. First, we present a modification of Proposition 1 from Guerreiro *et al.* (2014) considering that the insurer already detains a pre-existing portfolio:

Proposition 1 *When the mean number of new policies entering the portfolio in period i is modelled by ϑ_i, $i \in \mathbb{N}$, and the insurer already has an existing portfolio with policies distributed over BMS classes according to the row vector $\boldsymbol{\vartheta}_0' = [\vartheta_0(j)], j = 1, \ldots L$, the row vector of expected number of policyholders in the BMS classes in time period i, for a given λ, denoted by $\boldsymbol{\vartheta}^+_{i,\lambda}$, will be given by:*

$$\boldsymbol{\vartheta}^{+\,'}_{i,\lambda} \;=\; \boldsymbol{\vartheta}_0'\, \boldsymbol{K}_\lambda^i + \sum_{k=1}^{i} \vartheta_k\, \boldsymbol{c}_k'\, \boldsymbol{K}_\lambda^{i-k} \,,\ \ i \in \mathbb{N} \tag{2.6}$$

Proof. It is straightforward, following the proof in Guerreiro *et al.* (2014). ■

From here some remarks are pointed out:

Remark 1 *Since* $\boldsymbol{K}_{T,\lambda}$ *corresponds to the transition matrix of the sub-set of transient states of the Markov chain, is known that, see (Ross, 1996, Section 4.3),* $\lim_{i\to\infty}\boldsymbol{K}^{i}_{T,\lambda}=\mathbf{0}$ *so, limit results only rely on the second part of equation (2.6).*

Remark 2 *The sum of all components of vector* $\boldsymbol{\vartheta}^{+}_{i,\lambda}$ *corresponds to the expected number of policies in the portfolio in year* i*, which varies overtime, and will obviously have an impact on collected premia.*

Remark 3 *When using the classical model to predict BMS evolution, the total number of policies in the portfolio is constant overtime and the randomness comes only from the distribution of policyholders among the bonus levels. In open portfolio formulation, both policyholder allocation and portfolio dimension vary overtime and need to be predicted.*

Using (2.6) and the previous remarks it is easily established that the proportion of policyholders, with claim frequency λ, belonging to class j in year i, is given by

$$\pi_{i,\lambda}(j)=\frac{\vartheta^{+}_{i,\lambda}(j)}{\sum_{j=1}^{L}\vartheta^{+}_{i,\lambda}(j)}\quad,\ j=1,\ldots,L\ ,\ i\in\mathbb{N}.$$

We also remark that, with an open portfolio approach, the asymptotic properties of a Markov chain do not apply. The existence of a long run distribution for the proportion of policyholders in each BMS class depends on the functional form for the mean number of new annual policies incoming the portfolio in year i, ϑ_i, $i\in\mathbb{N}$, which, for the cases of Scenarios 1 and 2, is assured by the general results proven in Esquível *et al.* (2014).

Given λ, the limiting state probability for a policyholder belonging to bonus class j is given by

$$\pi_{\infty,\lambda}(j)=\frac{\vartheta^{+}_{\infty,\lambda}(j)}{\sum_{j=1}^{L}\vartheta^{+}_{\infty,\lambda}(j)}\quad,\ j=1,\ldots,L\,,$$

with $\boldsymbol{\vartheta}^{+}_{\infty,\lambda}=\lim_{i\to+\infty}\boldsymbol{\vartheta}^{+}_{i,\lambda}$.

To express the heterogeneity of the portfolio with respect to the claim frequency, it is common to consider λ as an outcome of a positive random variable, say Λ, with distribution function denoted as $V_\Lambda(\cdot)$. As widely set in the BMS literature, the unconditional probability of an insured belonging to class j, after i steps, and the long run distribution, for a policyholder chosen at random from the portfolio, is assumed as the expectation with respect to Λ, respectively

$$\pi_i(j)=\int_0^\infty \pi_{i,\lambda}(j)\,dV(\lambda)\quad,\ j=1,\ldots,L\,, \tag{2.7}$$

and

$$\pi_\infty(j)=\int_0^\infty \pi_{\infty,\lambda}(j)\,dV(\lambda)\quad,\ j=1,\ldots,L\,.$$

With a similar procedure, the portfolio dimension in year i, measured by the number of policies, is denoted as $NPol_i$, is given by

$$NPol_i=\sum_{j=1}^{L}NPol_i(j)\quad,\ i\in\mathbb{N}\,, \tag{2.8}$$

with

$$NPol_i(j) = \int_0^\infty \vartheta^+_{i,\lambda}(j)\, dV(\lambda) \quad , \; j = 1, \ldots, L, \;\; i \in \mathbb{N}\,. \tag{2.9}$$

Due to the fact that the portfolio is open, $NPol_i$ changes overtime considering the new annual entries and exits.

The total amount of premia to be charged annually, for the set of policyholders in the portfolio, is not constant overtime since it depends on the allocation of policyholders among the *bonus* levels (in both classical and open portfolio formulations) and on the portfolio dimension (in an open portfolio model) and is given by the sum of total premia collected in each class. For a given year i and known involved quantities, total premium in the presence of a BMS can be computed using Formula (2.6) from Afonso *et al.* (2017):

$$P_i = (1+\xi) NPol_i \sum_{j=1}^{L} \mathbb{E}\left[S(1)\right] \pi_i(j)\, b_j \quad , \; i = 1, \ldots, n\,, \tag{2.10}$$

where $\xi > 0$ is the safety loading parameter and b_j corresponds to the relativity of level j, i.e., the proportion of *a priori* premium to apply in level j.

As in Afonso *et al.* (2017), we consider $\mathbb{E}\left[S(1)\right]$ to be dependent on Class j and for BMS based only on claim frequency there is an implicit assumption that average individual claim size is constant across BMS classes.

2.3 Simulation and calculation procedure

Our method for computing ruin probabilities uses real data (historical data) for parameter estimation and a mix of calculation and simulation (not necessarily by this order). The basic procedure, taken from Afonso *et al.* (2017), is appropriately updated to accommodate the open model formulation. It is indeed updated and extended. The model for ruin probabilities estimation is targeted for large portfolios and we need to obtain annual aggregate claims. The approximation by a translated Gamma distribution with parameters α, β and κ is suggested. This was introduced at the end of Subsection 2.1.

The procedure is summarised and itemised as follows:

1. Estimation of expected claim frequency λ_j , $j = 1, \ldots, L$.

   From historical data, estimate the mean claim frequency of bonus level j , $j = 1, \ldots, L$. In level j, the number of reported claims is Poisson distributed with parameter λ_j.

2. Estimation of the mean value of new annual policies ϑ_i , $i = 1, \ldots, n$.

   Estimate the mean number of new annual contracts arriving to the portfolio. One may use regression techniques, as in Guerreiro *et al.* (2014) and Esquível *et al.* (2014) or time series models, as in Esquível *et al.* (2017).

3. Estimation of allocation probabilities $c_i(j)$, $i = 1, \ldots, n$, $j = 1, \ldots, L$.

   Using maximum likelihood estimator (briefly MLE) $\hat{c}_i(j) = E_{ij}/E_i$, $i = 1, \ldots, n$, $j = 1, \ldots, L$, estimate the probability of a new contract to be allocated at level j in year i. E_i refers to the number of new contracts in year i and E_{ij} to the number of contracts that, in year i, were allocated to level j.

In particular, in stable portfolios (with respect to allocation probabilities), we may set $\boldsymbol{c}_i \equiv \boldsymbol{c}$ and, in this way,

$$\hat{c}(j) = \frac{\sum_{i=1}^{n} E_{ij}}{\sum_{i=1}^{n} E_i} \,. \tag{2.11}$$

4. Estimation of exit probabilities $q_\lambda(j)$, $j = 1, \ldots, L$.

   Estimate the exit probabilities for each BMS level j, using the equivalent MLE in (2.11) for the number of contracts exiting the insurer.

5. Estimation of the expected number of claims for the portfolio, in year i, $i = 1, \ldots, n$.

   The expected number of claims in the portfolio, for year i, is given by, see Formula (2.10),

$$\mathbb{E}[N_i] = NPol_i \sum_{j=1}^{L} \lambda_j \, \pi_i(j) \, , \; i = 1, \ldots, n \, . \tag{2.12}$$

   For comparison to the classical BMS, we remark that in open BMS formulation $\pi_i(j)$, $i = 1, \ldots, n$, $j = 1, \ldots, L$ and $NPol_i$ are obtained by (2.7) and (2.8), respectively. Whereas in the classical BMS model, $\pi_i(j)$ is obtained by (2.4) of Afonso *et al.* (2017) and $NPol$ does not depend on i, since the portfolio is assumed to be closed and, therefore, constant overtime. From a practical point of view, the differences between (2.12) and that of (2.14) in Afonso *et al.* (2017) rely on the portfolio evolution and policies distribution which will, naturally, have impact on the mean value of the expected number of reported claims.

6. Simulation of the aggregate claim amount for each year i, $\{Y_i\}_{i=1}^{n}$.

   Let Y_i be the aggregate claim amount in a given year i, assumed to have (approximately) a translated Gamma distribution. Calculate the parameters of the translated Gamma distribution, α_i, β_i, κ_i, for each year i, $i = \ldots, n$, that match the first three moments of Y_i, considering the results obtained in Step 5 above and historical data for claim amounts.

7. Estimation of the premium collected in each year i, P_i , $i = 1, \ldots, n$.

   For a given *bonus* scale $\mathbf{b} = (b_1, \ldots, b_L)$, estimate the total amount of premium collected in year i, using (2.10).

8. Estimation of the ruin probability in year n, $\psi(u, n)$.

   This step is performed as follows:

   (a) From the simulated values of $\{Y_i\}_{i=1}^{n}$, say $\{y_i\}_{i=1}^{n}$, calculate consecutively the surplus at the end of each year:

$$\begin{aligned} u(1) &= u + p_1 - y_1 \, , \text{ or} \\ u(i) &= u(i-1) + p_i - y_i \, , i = 2, \ldots, n \, . \end{aligned}$$

   (b) Denote as $\psi_m(u, n)$ the ruin probability in simulation (or run) number m. In run m:

- If $u(i) < 0$ for any i, $i = 1, 2, \ldots, n$, we set $\psi_m(u, n) = 1$ and start simulation $m + 1$, $m = 1, \ldots, M - 1$, where M is the number of runs for each path set;
- If $u(i) \geq 0$ for all $i, i = 1, 2, \ldots, n$, we calculate the approximation for run m $\psi_m(u(i-1), 1, u(i))$ using (2.2).

c) Calculate the finite time ruin probability estimate in run m, $\hat{\psi}_m(u, n)$, as follows:

$$\hat{\psi}_m(u, n) \quad = \quad 1 - \prod_{i=1}^{n} \left[1 - \hat{\psi}_m(u(i-1), 1, u(i)) \right] .$$

(d) The estimate for the continuous and finite time ruin probability, $\hat{\psi}(u, n)$, is set by the mean of the estimates obtained from each simulation, $\{\hat{\psi}_m(u, n)\}_{m=1}^{M}$.

Comparing to the classical formulation, we highlight that simulation Steps 1, 5, 6, 7 and 8 are performed for both classical and open BMS models, using the appropriate estimates involved. Steps 2-4 are only performed when adopting the open BMS model to evaluate ruin probability. This procedure also allows an easy calculation of the standard error of the estimate obtained. This suggests that this simulation procedure is general and has a wide range of applications.

3 Ruin probabilities in an open portfolio with a BMS

3.1 Data and distribution fitting

In this paper, and for comparison purposes, we use the same automobile portfolio and BMS, illustrated in Afonso *et al.* (2017). This allows us to compare their results directly with ours, and get clear conclusions. From there we retrieve:

- Insurer's commercial scale has $L = 18$ bonus-malus levels, with transition rules defined in Table 1 of Afonso *et al.* (2017), which establishes Level 10 as the entry class.
- Number of claims reported by a randomly chosen insured follows a mixed Poisson distribution, where the random parameter Λ follows an Inverse Gaussian distribution, with parameter estimates $\hat{\mu} = 0.082401$ and $\hat{\eta} = 0.130271$, according to data of their Table 2.
- In level j ($j = 1, \ldots, L$) the number of annual claims follow a Poisson distribution with parameter λ_j, estimated from data and illustrated in Table 3.1 below.
- The number of existing policies in each level j, at the evaluation date, $\boldsymbol{\vartheta}_0$, was known and is also presented in Table 3.1. In this paper this will be our starting point for the estimation of portfolio evolution considering the expected future number of incoming annual policies entering the portfolio as well as the expected number of annual exits.

Consider now the BMS model for open portfolios presented in Section 2.2. Every insured entering the portfolio will be allocated to one of the bonus malus levels. In Portugal, nowadays, the claim history of the insured is known to the insurer (it is made available by the control authority) so that an insured may be allocated to any class. The allocation and exiting probability estimates by class, $\hat{c}(j)$ and $\hat{q}(j)$ respectively, both estimated from data are

j	1	2	3	4	5	6
$\hat{\lambda}_j$	0.034516	0.072883	0.076425	0.080265	0.126855	0.135954
$\vartheta_0(j)$	174,173	109,113	42,736	29,134	23,730	4,241

j	7	8	9	10	11	12
$\hat{\lambda}_j$	0.148393	0.181802	0.195919	0.213730	0.237433	0.255984
$\vartheta_0(j)$	2,759	24,829	11,747	166	2,882	7,632

j	13	14	15	16	17	18
$\hat{\lambda}_j$	0.277505	0.301956	0.327931	0.358676	0.395719	0.441571
$\vartheta_0(j)$	250	710	2,256	2,643	1,304	2,183

Table 3.1: Number of existing policies and estimated Poisson parameter by class

j	1	2	3	4	5	6
$\hat{c}(j)$	0.265847	0.083959	0.037448	0.089331	0.063856	0.166594
$\hat{q}(j)$	0.046442	0.056989	0.056703	0.074157	0.070393	0.088040

j	7	8	9	10	11	12
$\hat{c}(j)$	0.109473	0.09595	0.039585	0.045002	0.001757	0.000939
$\hat{q}(j)$	0.100813	0.109777	0.147588	0.208660	0.380737	0.388989

j	13	14	15	16	17	18
$\hat{c}(j)$	0.000176	$2.93E{-}5$	$1.46E{-}5$	$1.39E{-}5$	$1.32E{-}5$	$1.26E{-}5$
$\hat{q}(j)$	0.397241	0.487619	0.497778	0.098462	0.087521	0.068072

Table 3.2: Allocation and annulment probability estimates per *bonus* class

shown in Table 3.2. Due to available data, annulment probabilities estimates were obtained not depending on λ.

Top left graph of Figure 2 gives a visual presentation of these probabilities. In particular we call the attention to: (i) The low proportion of insureds that are initially allocated to the entry level (Level 10). Only about 4.5% of new insureds are allocated to the entry class; (ii) the high proportion of allocations to Class 1. Note that about 26.6% of the new insureds enter the portfolio directly to the highest *bonus* level; (iii) the high magnitude of exiting probabilities for those insureds in Classes 11-15. This reflects that insureds in these classes seek for a better premium in another insurer. The very low figures of the exiting probabilities for insureds from Levels 16-18 when compared to the neighbouring lower classes, is certainly due to the fact that it is difficult for them to bargain a better premium in another insurer, since there is information disclosure among insurers regarding the reported claims. We believe that these observed patterns in allocation and exiting probabilities will have a significant impact on ruin probabilities.

As said earlier, in order to foresee the portfolio evolution, we considered different formulations to model the mean annual number of new policyholders: The Exponential and the Sigmoid models, see Equations (2.4) and (2.5), respectively. The portfolio starts with $442,490$ policies, distributed over the classes as shown in Table 3.1, at the evaluation date, $\vartheta_0(j)$. Regarding the insurer expectations and the insurance market in Portugal, we considered a long run target of $50,000$ incoming policies per year. Due to lack of real data that allowed us to fit a function to the insurer's historical data, we set $1/a = 50,000$, $b = 0.00025$ and $\theta = 0.55$ for Sigmoid Model and $\tau = 50,000$ and $\delta = 0.4$ for the Exponential Model. Parameters a and τ are related to long run target and the remainder of the parameters represent the evolution of new annual policies per year. In practice, real data should be observed, a model chosen appropriately and respective parameters should be estimated, as in Guerreiro *et al.* (2014) and Esquível *et al.* (2014). The estimate for the number of policies that in year i are allocated to each level j is obtained by $\hat{\boldsymbol{c}}\,\vartheta_i$, using $\hat{\boldsymbol{c}}$ from Table 3.2.

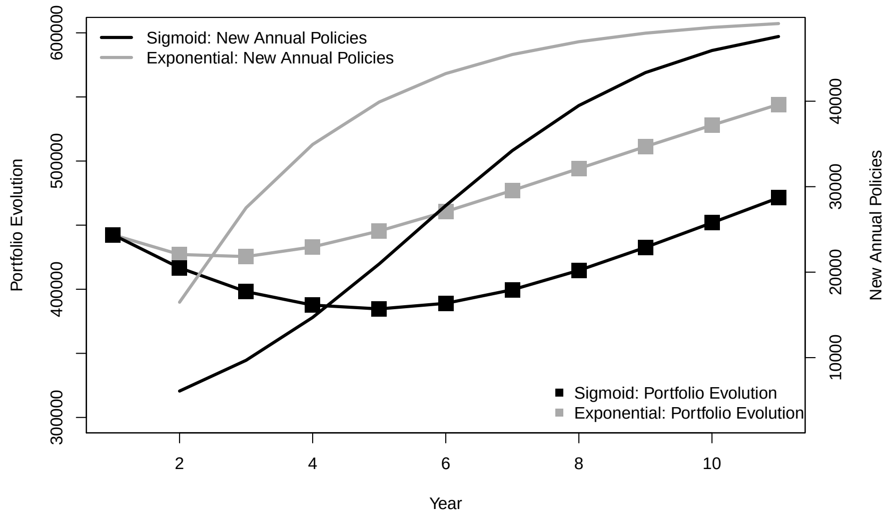

Figure 1: Evolution of the number of policies over time in the sigmoid and asymptotic models

Figure 1 shows the expected evolution of the portfolio and of the expected number of new policies, obtained from Equation (2.6) followed by (2.9), for both Exponential and Sigmoid models.

From the number of existing policies in each level, $\vartheta_0(j)$, illustrated in Table 3.1, we obtained the starting allocation of the policyholders among the bonus classes, this corresponds to column "Present" from Table 3.3. Columns 2-4 illustrates the stationary distribution of policies through the $L = 18$ levels of the BMS, $\pi_\infty(j)$, $j = 1, \ldots, L$. Column 2 contains the obtained figures for the classical BMS, Columns 3 and 4 those corresponding to the open BMS with Exponential and Sigmoid entries, respectively.

We highlight the differences between the classical and open model results. Due to the annulment probabilities, the higher levels are expected to have fewer policies in the open BMS formulation. Due to the possibility of allocation in different levels, we would expect that policies won't be so concentrated in the first level, when comparing to the classical formulation.

Considering the conclusions in Afonso *et al.* (2017) about ruin probabilities for different optimal scales, we chose to perform the calculations in this paper based on only two optimal scales, namely those proposed by Borgan *et al.* (1981) (denoted B) and the linearization of that proposed by Gilde and Sundt (1989) (denoted LB). The former was obtained with the same set of weights applied to a 20 year horizon used in Afonso *et al.* (2017), i.e., $w_{i+1} =$

j	Present	Classical BMS	Open BMS Exponential Entries	Sigmoid Entries
1	0.39362	0.73121	0.64235	0.63574
2	0.24659	0.04913	0.06251	0.07115
3	0.09658	0.05394	0.06683	0.08153
4	0.06584	0.05941	0.07689	0.07939
5	0.05363	0.01871	0.03986	0.04775
6	0.00958	0.01678	0.03818	0.02216
7	0.00623	0.01411	0.02860	0.02195
8	0.05611	0.00837	0.01764	0.01258
9	0.02655	0.00725	0.01098	0.01154
10	0.00037	0.00612	0.00729	0.01168
11	0.00651	0.00495	0.00295	0.00172
12	0.01725	0.00453	0.00195	0.00129
13	0.00056	0.00419	0.00119	0.00083
14	0.00160	0.00398	0.00066	0.00024
15	0.00510	0.00398	0.00064	0.00021
16	0.00597	0.00410	0.00050	0.00013
17	0.00295	0.00438	0.00045	0.00004
18	0.00493	0.00488	0.00053	0.00006

Table 3.3: Portfolio stationary distributions per classes

$w_i/1.05$, $i = 1, \dots, 20$. Both scales are illustrated in Table 3.4. In the same table we also show the commercial scale used by the insurer, denoted as C, and for comparison purposes the same scales obtained for the classical BMS approach.

From this table we see that the relativities derived from open model formulation are less dispersed, and bonuses and maluses are less extreme when compared to the classical model approach. The Sigmoid model even leads to a higher difference than the Exponential one. This fact, together with policy allocation through the levels, will expectedly have impact on ruin probabilities.

Regarding the aggregate claim severity, we retrieve the estimates from Afonso *et al.* (2017), as they were got from the history of the insurer's portfolio claim amounts: the aggregate claim mean is $1,766.31$, the variance is $71,097,953.5$ and the third central moment is $21,068,298,856,615$. These figures were then used to obtain estimates for parameters α, β and κ of the translated Gamma approximation.

3.2 Ruin probabilities

In the simulation procedure, summarized in Subsection 2.3, for the calculation of the ruin probability $\psi(u, n)$, we fixed the initial surplus at $u = 2,000,000$ and we used $M = 50,000$ runs with total average computation time of 75 minutes. All results were obtained in a PC with Intel® Xeon 10 cores and 32 GB of RAM. For the simulation process, as in Afonso *et al.* (2017), we used the loading coefficient of 80% and the same amount of aggregate premium $P^* = 1.8\,\mathbb{E}[S(1)] = 115,838,792$ calculated as if BMS was inexistent.

In Table 3.5 we show the results for time horizons $t = 2, 5, 10$, focusing on the following

		Classical BMS		Open BMS - Exponential		Open BMS - Sigmoid	
j	C	B	LB	B	LB	B	LB
1	45	48.8	46.0	58.3	57.3	67.1	61.2
2	45	58.0	52.0	70.1	62.1	79.6	65.5
3	50	60.1	58.0	74.2	66.8	84.5	69.8
4	55	62.3	64.0	78.9	71.5	91.6	74.1
5	60	63.7	70.0	77.9	76.3	89.3	78.4
6	65	66.2	76.0	79.6	81.0	100.8	82.8
7	70	68.1	82.0	84.3	85.8	102.1	87.1
8	80	69.9	88.0	85.6	90.5	101.8	91.4
9	90	72.3	94.0	95.6	95.3	100.8	95.7
10	100	100.0	100.0	100.0	100.0	100.0	100.0
11	110	105.6	106.0	115.6	104.7	131.0	104.3
12	120	113.0	112.0	136.9	109.5	157.8	108.6
13	130	124.3	118.0	139.8	114.2	169.1	112.9
14	150	148.8	124.0	122.4	119.0	182.6	117.2
15	180	162.6	130.0	114.9	123.7	130.1	121.6
16	250	181.9	136.0	128.7	128.5	169.4	125.9
17	325	209.1	142.0	143.7	133.2	146.3	130.2
18	400	235.0	148.0	223.3	137.9	227.7	134.5

Table 3.4: *Bonus* scales (%)

cases:

Case I Classical Model - Optimal scales and portfolio evolution were obtained from classical BMS results, retrived from Afonso *et al.* (2017);

Case II Open Exponential Model - Optimal scales and portfolio evolution were obtained by open model formulation considering Scenario 1 for modelling new entries, according to Subsection 2.2;

Case III Open Sigmoid Model - Optimal scales and portfolio evolution were obtained by open model formulation considering Scenario 2 for modelling new entries, according to Subsection 2.2;

Case IV Combining Cases I and III - We show results when the insurer sets the optimal scale based on a classical BMS model but the portfolio evolution behaves according to the open model approach.

Focusing on these cases, we derived ruin probabilities for the last three. This allows us to:

(i) Compare results of Case I, presented in Afonso *et al.* (2017), with those obtained in this paper, following an open BMS perspective (in Cases II and III);

(ii) Analyse the impact of setting optimal scales using a closed model in an open portfolio (Case IV).

Looking at the results of Table 3.5 we highlight that:

- Comparing the four cases, we conclude that Cases II and III are the most favourable options in terms of ruin probability. This shows that if the insurer is dealing with an open portfolio, optimal scales should be estimated accordingly once this is the best case scenario in terms of short term ruin probabilities;

- Comparing the results of Cases I and IV, we don't observe a great reduction on ruin probabilities when we consider an open portfolio evolution instead of the closed one, although there is some reduction. This is easily justified with the distribution of policies throughout the bonus levels. We may conclude that Case I overestimates ruin probabilities. This shows that the inclusion of entries and exits on the model should not be neglected;

- There are no significant differences among ruin probabilities in time periods $t = 2, 5, 10$, this is similar for the closed model in Afonso *et al.* (2017). Again, this is due to the fact that if ruin happens it is likely to occur within the first two years (as we show in Table 3.6).

- We see no significant differences between Cases II and III in terms of ruin probabilities. From now on we will consider the Open Sigmoid Model only .

- Focusing on the commercial scale of the insurer, C, we can conclude that the scale shows to be inadequate as the corresponding ruin probabilities are much larger;

t	Case I Classical Model				Case II Open Exponential Model				Case III Open Sigmoid Model				Case IV Combining I and III	
	P^*	C	B	LB	P^*	C	B	LB	P^*	C	B	LB	B	LB
2	0	7.569	0.23300	1.18600	0	7.37875	0.00058	0.00272	0	7.40833	0.00007	0.00088	0.21620	1.12319
5	0	7.569	0.23300	1.18600	0	7.37875	0.00058	0.00272	0	7.40833	0.00007	0.00088	0.21620	1.12319
10	0	7.569	0.23300	1.18600	0	7.37875	0.00058	0.00272	0	7.40833	0.00007	0.00088	0.21620	1.12319

Table 3.5: Estimates for the probability of ruin, $\hat{\psi}(u = 2,000,000; t)(\%)$, for $t = 2, 5, 10$ for each BMS

i	Case I Classical Model	Case II Open Exponential Model				Case III Open Sigmoid Model				Case IV Combining I and III	
	LB	P^*	C	B	LB	P^*	C	B	LB	B	LB
0	1.17586	1.28E-05	7.23902	5.79E-04	2.72E-03	1.26E-05	7.23915	6.69E-05	8.76E-04	0.21605	1.11488
1	0.14271	0	2.71065	8.18E-32	5.28E-03	0	2.79345	5.06E-61	4.91E-03	0.11168	0
2	2.11E-13	0	9.77E-08	0	0	0	6.57E-07	0	0	1.14E-42	7.08E-23
3	0	0	1.74E-53	0	0	0	5.30E-38	0	0	0	0
...					...		...				...
10	0	0	0	0	0	0	0	0	0	0	0

Table 3.6: Average of the within the year ruin probabilities for each BMS (%)

	Scenario A		Scenario B		Scenario C	
j	$\hat{c}(j)$	$\hat{q}(j)$	$\hat{c}(j)$	$\hat{q}(j)$	$\hat{c}(j)$	$\hat{q}(j)$
1	0.00527	0.045628	0.265847	0.045628	0.265847	0.18699
2	0.007567	0.053246	0.083959	0.053246	0.083959	0.18699
3	0.018917	0.062136	0.037448	0.062136	0.037448	0.18699
4	0.037833	0.07251	0.089331	0.07251	0.089331	0.18699
5	0.063055	0.084616	0.063856	0.084616	0.063856	0.18699
6	0.090079	0.098743	0.166594	0.098743	0.166594	0.18699
7	0.112599	0.115229	0.109473	0.115229	0.109473	0.18699
8	0.125110	0.134467	0.09595	0.134467	0.095950	0.18699
9	0.125110	0.156918	0.039585	0.156918	0.039585	0.18699
10	0.113736	0.183116	0.045002	0.183116	0.045002	0.18699
11	0.09478	0.213689	0.001757	0.213689	0.001757	0.18699
12	0.072908	0.249366	0.000939	0.249366	0.000939	0.18699
13	0.052077	0.291000	0.000176	0.291000	0.000176	0.18699
14	0.034718	0.339584	0.000029	0.339584	0.000029	0.18699
15	0.021699	0.396281	0.000015	0.396281	0.000015	0.18699
16	0.012764	0.462443	0.000014	0.462443	0.000014	0.18699
17	0.007091	0.539651	0.000013	0.539651	0.000013	0.18699
18	0.004687	0.62975	0.000013	0.62975	0.000013	0.18699

Table 3.7: Allocation and exit probabilities for different scenarios

3.3 Changing allocations and annulment probabilities

To evaluate the impact of allocations and annulments on ruin probabilities, we considered the three scenarios for the allocation and exit probabilities that are shown in Table 3.7, working only with Open Sigmoid Model. Figure 2 illustrates the existing portfolio allocation and annulment probability estimates (top left graph) and gives visual presentation for the different scenarios.

Regarding allocation probabilities, Scenarios B and C keep the portfolio estimates and in Scenario A we give higher weights for mid level classes. Concerning the exit probabilities, Scenarios A and B put high probability annulments in aggravated classes. This implies that fewer insureds will be paying *malus* premia, and this will reduce the total premium collected in each year. In Scenario C we have leveled the probability annulments. It reflects a portfolio where insureds decide to leave regardless the premium paid. Total premium collected is higher than in Scenarios A and B.

In Table 3.8 we can observe that allocation and exit probabilities clearly have impact in the ruin probabilities estimates. Scenario A shows to be the best scenario according to the resulting ruin probabilities (first line replicates Case III from Table 3.5). This shows that the insurer should estimate carefully the allocation and exit probabilities in order to obtain an accurate estimate for ruin probability.

3.4 Capital requirements and ruin probabilities

From the observation of Case III in Table 3.5 (similar to Case II) we can infer that if we consider roughly a goal of 1% for the ruin probability (for instance $\hat{\psi}(u, 10) \approx 1\%$) we may

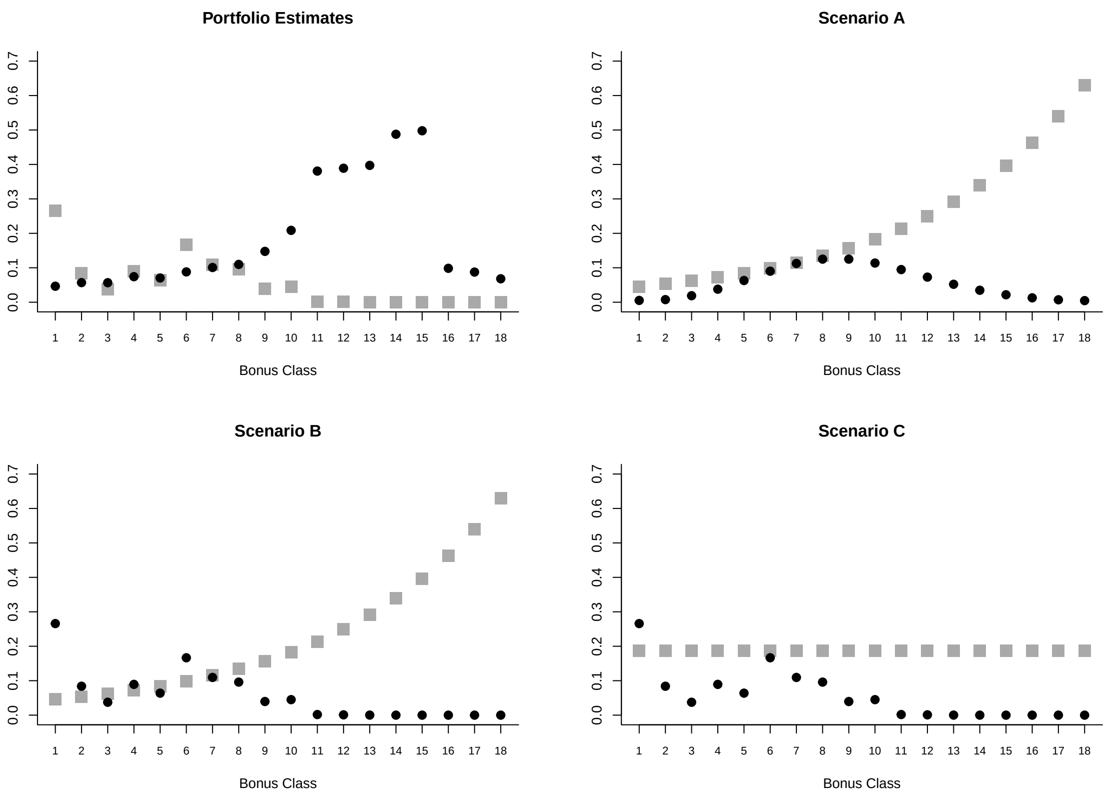

Figure 2: Allocation (•) and exit (■) probabilities per bonus level

	B	LB
Case III	0.00007	0.00088
Scenario A	0.00005	0.00010
Scenario B	0.00058	0.00096
Scenario C	0.00014	0.00020

Table 3.8: Ruin probability estimates, $\hat{\psi}(u = 2\,000\,000; t = 2)$ (in %)

	P_0	C	B	LB
Case I	0.14247	42.75240	10.82196	20.29612
Case III	0.14164	42.45439	0.35658	**1.18617**
Case IV			10.67296	20.0926
Case III, Scenario A			0.31819	**0.44008**
Case III, Scenario B			0.99725	1.23793
Case III, Scenario C			0.50822	**0.61055**

Table 3.9: Ruin probability estimates, $\hat{\psi}(u = 650,000; t = 2)$ (in %), for each BMS

greatly reduce the amount of the initial reserve u. This can be regarded as an advantage in terms of capital requirements for the insurer. To illustrate this point, focusing on LB optimal scale, we considered several values for the initial reserve. To achieve our goal we set $u = 650,000$ and obtained around 1% ruin probability in two years for Case III. We recall that in order to obtain equivalent ruin probability the classical model would need $u = 2,000,000$. We show the results, in percentage, for $\hat{\psi}(u = 650,000; t = 2)$ in Table 3.9. We see there is a great reduction in the ruin probability for Case III with Scenarios A and C. This highlights the importance of a good estimation and monitoring of the allocation and exit probabilities. The ruin probabilities for Scale B are lower than those corresponding in LB. This is due to the smoothing of scale LB, as seen in Table 3.4.

For $t = 5$ and $t = 10$, ruin probability $\hat{\psi}(u = 650,000; t)$ has a similar behaviour as those in Table 3.5, explained mainly by the results of Table 3.6.

Following the results of Table 3.9 we can conclude that setting optimal bonus scales according to open model formulation reduces ruin probabilities. Regardless of scenario of allocation and exit probabilities, we may also reduce the capital requirement.

3.5 Some Final remarks

Calculating ruin probabilities by using an open portfolio formulation as we did in this manuscript appears to be more realistic than the classical BMS model, where it is induced that incoming policies are compensated by annulments. We know that the motor insurance market can be very competitive and market movements may result in significant change in a portfolio composition. Subsequently, due to the use of a BMS where levels can have very different premia, it may lead to a significant change in the portfolio's financial movements, as well as risk composition leading to a significant change in figures for ruin probabilities. Since nowadays regulators provide insurers with past information of policyholder behaviour, the motor insurance business is more transparent. That should result into a better classification and allocation of risks to the appropriate *bonus* class, resulting into a ruin probability reduction. However, there are dangers that may not be evaluated properly: Since the market is very competitive insurers may be tempted to attract clients by offering them a higher bonuses, or less penalties, although inappropriately, in a fast increasing business environment.

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