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Sturm-Liouville hypergroups without the compactness axiom

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Abstract

We establish a positive product formula for the solutions of the Sturm-Liouville equation $\ell(u) = \lambda u$, where ℓ belongs to a general class which includes singular and degenerate Sturm-Liouville operators. Our technique relies on a positivity theorem for possibly degenerate hyperbolic Cauchy problems and on a regularization method which makes use of the properties of the diffusion semigroup generated by the Sturm-Liouville operator.

We show that the product formula gives rise to a convolution algebra structure on the space of finite measures, and we discuss whether this structure satisfies the basic axioms of the theory of hypergroups. We introduce the notion of a degenerate hypergroup of full support and improve the known existence theorems for Sturm-Liouville hypergroups.

Keywords: Product formula, hypergroup, generalized convolution, Sturm-Liouville spectral theory, degenerate hyperbolic equation.

1 Introduction

A hypergroup is a generalized convolution operator $*$ on the space $\mathcal{M}_{\mathbb{C}}(K)$ of finite complex measures on an underlying space K which preserves the subset of probability measures on K and gives rise to a structure of Banach algebra with unit on $\mathcal{M}_{\mathbb{C}}(K)$. In the most common axiomatic definition of hypergroup, introduced by Jewett in [16], the convolution is also required to satisfy axioms of continuity and compactness of support; the compactness axiom requires, in particular, that the convolution of Dirac measures is a measure of compact support. An extensive theory of (probabilistic) harmonic analysis has been developed in the context of hypergroups, see the monographs [4, 3] and references therein.

Starting from the seminal works of Delsarte [8] and Levitan [20] on generalized translation operators, the development of the theory of hypergroups was largely motivated by the study of Sturm-Liouville differential operators on an interval (a, b) of the real line. The key idea here is the following: it is well known that the eigenfunction expansion of a Sturm-Liouville operator, say, of the form

$$\ell = -\frac{1}{r}\frac{d}{dx}\Big(p\,\frac{d}{dx}\Big), \qquad a < x < b$$

gives rise (under certain conditions) to an integral transform $(\mathcal{F}h)(\lambda) := \int_a^b h(x)\, w_\lambda(x)\, r(x)dx$ $(\lambda \in \mathbb{R})$ which is an isometry between L_2-spaces; here $\{w_\lambda\}$ is a family of solutions of the Sturm-Liouville equation

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$\ell(u) = \lambda u$. Now, if the functions w_λ are bounded and satisfy $w_\lambda(a) = 1$, then one may extend the transformation $\mathcal{F}$ to measures $\mu \in \mathcal{M}_{\mathbb{C}}[a,b)$ by defining

$$(\mathcal{F}\mu)(\lambda) \equiv \widehat{\mu}(\lambda) := \int_a^b w_\lambda(x)\, \mu(dx), \tag{1.1}$$

and then it is natural to ask: *does there exist a (generalized) convolution operator $*$ which is trivialized by the transformation (1.1), in the sense that the property $\widehat{\mu * \nu} = \widehat{\mu} \cdot \widehat{\nu}$ holds for all $\mu, \nu \in \mathcal{M}_{\mathbb{C}}[a,b)$?* If μ and ν are taken to be Dirac measures at the points $x, y \in [a,b)$, then the trivialization property reads

$$w_\lambda(x)\, w_\lambda(y) = \int_{[a,b)} w_\lambda \, d\boldsymbol{\nu}_{x,y} \tag{1.2}$$

where $\boldsymbol{\nu}_{x,y} = \delta_x * \delta_y$. The construction of generalized convolutions is therefore closely related to the problem of existence of a so-called *product formula* for the solutions of the Sturm-Liouville equation $\ell(u) = \lambda u$; in this problem, the goal is to determine a family $\{\boldsymbol{\nu}_{x,y}\} \subset \mathcal{M}_{\mathbb{C}}[a,b)$ such that (1.2) holds. For the hypergroup axioms to hold we actually need that the $\boldsymbol{\nu}_{x,y}$ are probability measures; in this case we say that (1.2) is a *hypergroup-like product formula*.

The Bessel operator $-\frac{d}{dx^2} - \frac{2\alpha+1}{x}\frac{d}{dx}$ and the Jacobi operator $-\frac{d}{dx^2} - [(2\alpha+1)\coth x + (2\beta+1)\tanh x]\frac{d}{dx}$ are standard examples of Sturm-Liouville operators on the half-line $[0,\infty)$ for which the kernel of the associated Sturm-Liouville integral transform (the Hankel and Jacobi transform, respectively) admits a hypergroup-like product formula where the measures $\boldsymbol{\nu}_{x,y}$ have been computed in closed form (see [14] and [18], respectively). Much more generally, it was shown by Zeuner [31] that any Sturm-Liouville operator of the form $-\frac{1}{A}\frac{d}{dx}\big(A\frac{d}{dx}\big)$, where $A \in \mathrm{C}^1(0,\infty)$ is positive, increasing and satisfies suitable assumptions (see Subsection 5.2 below) also admits a hypergroup-like product formula; in general, the measures $\boldsymbol{\nu}_{x,y}$ of the product formula (1.2) are not known in closed form, but their existence can be proved using a positivity property of hyperbolic partial differential equations.

A general property of the Sturm-Liouville operators considered by Zeuner is that the support $\mathrm{supp}(\boldsymbol{\nu}_{x,y})$ of the measures in the product formula is contained in $[|x-y|, x+y]$; in particular, $\mathrm{supp}(\boldsymbol{\nu}_{x,y})$ is compact, as required by the usual hypergroup axioms. However, the situation is quite different for the Whittaker convolution, generated by the normalized Whittaker differential operator $x^2\frac{d^2}{dx^2} + (1+2(1-\alpha)x)\frac{d}{dx}$ on the half-line $[0,\infty)$. In fact, in this case the measures in the product formula, whose closed form expression was recently determined by the authors in [25, 26], satisfy $\mathrm{supp}(\boldsymbol{\nu}_{x,y}) = [0,\infty)$ for all $x, y > 0$. But it turns out that the probability-preserving and continuity axioms are satisfied by the Whittaker convolution, and therefore it is still possible to develop harmonic analysis on the measure algebra $\big(\mathcal{M}_{\mathbb{C}}[0,\infty), *\big)$ (see [25, 26]). Moreover, one can show that the Whittaker operator restricted to any interval $[c,\infty)$, $c > 0$, can be reduced by a change of variable to an operator belonging to the class introduced by Zeuner, and therefore determines a convolution satisfying the compact support axiom. Hence it is natural to interpret the measure algebra associated to the Whittaker convolution as a degenerate hypergroup and to wonder if it is possible to construct degenerate hypergroup structures for other Sturm-Liouville operators. The use of the term "degenerate" is further justifies by the fact that in the limit $c = 0$ the hyperbolic Cauchy problem associated with ℓ (defined in Subsection 4.1) becomes parabolically degenerate at the initial line.

In this paper, our purpose is to introduce a new technique for proving the existence of a hypergroup-like product formula for Sturm-Liouville operators whose associated hyperbolic Cauchy problem is possibly parabolically degenerate at the initial line. Our technique is based on a regularization method which we now briefly sketch. The inversion formula for the integral transform $\mathcal{F}$ generated by ℓ provides a formal candidate for the measure $\boldsymbol{\nu}_{x,y}$, namely the inverse transform $\mathcal{F}^{-1}[w_{(\cdot)}(x)\, w_{(\cdot)}(y)]$. However, the inversion integral is, in general, divergent. To get around this, the idea is to consider instead the regularized inverse transform $\mathcal{F}^{-1}[e^{-t(\cdot)} w_{(\cdot)}(x)\, w_{(\cdot)}(y)]$, where $t > 0$, and to prove that the presence of the exponential term ensures the convergence of the inversion formula. It will then be seen that the measure $\boldsymbol{\nu}_{x,y}$ can be recovered from the measures $\boldsymbol{\nu}_{t,x,y}$ of the product formula for $e^{-t\lambda} w_\lambda(x)\, w_\lambda(y)$ as the weak limit as $t \downarrow 0$. This weak convergence argument relies on the nontrivial fact that the $\boldsymbol{\nu}_{t,x,y}$ (and therefore also $\boldsymbol{\nu}_{x,y}$) are probability measures; to justify this, we use a partial differential equation approach based on the maximum principle for hyperbolic equations.

Since we deal with hyperbolic Cauchy problems which may be parabolically degenerate, the classical theory of hyperbolic problems in two variables is, in general, not applicable. To overcome this, we use the spectral theory of Sturm-Liouville operators to deduce existence, uniqueness and positivity results for a general class of possibly degenerate Cauchy problems. This class also includes many (uniformly) hyperbolic equations with singularities which fall outside the scope of the classical methods. In such singular cases it will be seen that the uniform hyperbolicity yields a product formula where the $\boldsymbol{\nu}_{x,y}$ have compact support and the resulting generalized convolution operator satisfies all the hypergroup axioms, leading to an existence theorem for Sturm-Liouville hypergroups which generalizes previous results in the literature. On the other hand, as we will see, in the presence of parabolic degeneracy the product formula is such that the measures $\boldsymbol{\nu}_{x,y}$ are supported on the full interval $[a, b)$. This allows us to interpret the Whittaker convolution as a particular case of a general family of degenerate Sturm-Liouville hypergroups of full support; this is relevant because, to the best of our knowledge, no full support convolution structures generated by Sturm-Liouville operators other than the Whittaker operator were known to exist prior to this work.

The paper is organized as follows. Section 2 collects some preliminary facts about the solutions of Sturm-Liouville boundary value problems and the related eigenfunction expansions. In Section 3 we prove that the kernel $w_\lambda(x)$ of the integral transform generated by ℓ can be written as the Fourier transform of a probability measure, thereby generalizing a result which is known to hold for Sturm-Liouville hypergroups; this so-called Laplace-type representation is later used in the proof of the product formula. The proof of the hypergroup-like product formula for the functions $w_\lambda(x)$ is given in Section 4. In Section 5 we show, using the properties of the transformation (1.1), that the convolution determined by the product formula is continuous in the weak topology and yields a positivity-preserving Banach algebra structure on $\mathcal{M}_\mathbb{C}[a, b)$; we then study the support of the convolution of Dirac measures in the nondegenerate and degenerate cases, and relate our results with the axioms of hypergroups.

2 Preliminaries

The following notations will be used throughout the paper. For a subset $E \subset \mathbb{R}^d$, $\mathrm{C}(E)$ is the space of continuous complex-valued functions on E; $\mathrm{C_b}(E)$, $\mathrm{C_0}(E)$ and $\mathrm{C_c}(E)$ are, respectively, its subspaces of bounded continuous functions, of continuous functions vanishing at infinity and of continuous functions with compact support; $\mathrm{C}^k(E)$ stands for the subspace of k times continuously differentiable functions. $\mathrm{B_b}(E)$ is the space of complex-valued bounded and Borel measurable functions. The corresponding spaces of real-valued functions are denoted by $\mathrm{C}(E, \mathbb{R})$, $\mathrm{C_b}(E, \mathbb{R})$, etc. For a given measure μ on E, $L_2(E; \mu)$ denotes the Lebesgue space of complex-valued square-integrable functions with respect to μ. The restriction of a function $f : E \longrightarrow \mathbb{C}$ to a subset $B \subset E$ is denoted by $f|_B$. The space of probability (respectively, finite positive, finite complex) Borel measures on E will be denoted by $\mathcal{P}(E)$ (respectively, $\mathcal{M}_+(E)$, $\mathcal{M}_\mathbb{C}(E)$). The total variation of $\mu \in \mathcal{M}_\mathbb{C}(E)$ is denoted by $\|\mu\|$, and δ_x denotes the Dirac measure at a point x.

In all that follows we consider a Sturm-Liouville differential expression of the form

$$\ell = -\frac{1}{r}\frac{d}{dx}\Big(p\,\frac{d}{dx}\Big), \qquad x \in (a, b) \tag{2.1}$$

($-\infty \leq a < b \leq \infty$), where p and r are (real-valued) coefficients such that $p(x), r(x) > 0$ for all $x \in (a, b)$ and p, p', r and r' are locally absolutely continuous on (a, b). Concerning the behavior of the coefficients at the boundary $x = a$, we will always assume that the boundary condition

$$\int_a^c \int_y^c \frac{dx}{p(x)}\, r(y) dy < \infty \tag{2.2}$$

(where $c \in (a, b)$ is an arbitrary point) is satisfied.

Some important properties of the solutions of the Sturm-Liouville equation $\ell(u) = \lambda u$ ($\lambda \in \mathbb{C}$) are given in the following three lemmas. The notation $u^{[1]} := pu'$ is used in the sequel.

Lemma 2.1. *For each $\lambda \in \mathbb{C}$, there exists a unique solution $w_\lambda(\cdot)$ of the boundary value problem*

$$\ell(w) = \lambda w \quad (a < x < b), \qquad w(a) = 1, \qquad w^{[1]}(a) = 0. \tag{2.3}$$

Moreover, $\lambda \mapsto w_\lambda(x)$ is, for each fixed x, an entire function of exponential type.

Proof. Let

$$\eta_0(x) = 1, \qquad \eta_j(x) = \int_a^x \bigl(\mathfrak{s}(x) - \mathfrak{s}(\xi)\bigr)\eta_{j-1}(\xi)r(\xi)d\xi \quad (j = 1, 2, \ldots).$$

Pick an arbitrary $\beta \in (a, b)$ and define $\mathcal{S}(x) = \int_a^x \bigl(\mathfrak{s}(\beta) - \mathfrak{s}(\xi)\bigr)r(\xi)d\xi$, where $\mathfrak{s}(x) := \int_c^x \frac{d\xi}{p(\xi)}$. From the boundary assumption (2.2) it follows that $0 \leq \mathcal{S}(x) \leq \mathcal{S}(\beta) < \infty$ for $x \in (a, \beta]$. Furthermore, it is easy to show (using induction) that $|\eta_j(x)| \leq \frac{1}{j!}(\mathcal{S}(x))^j$ for all j. Therefore, the function

$$w_\lambda(x) = \sum_{j=0}^{\infty} (-\lambda)^j \eta_j(x) \qquad (a < x \leq \beta,\ \lambda \in \mathbb{C})$$

is well-defined as an absolutely convergent series. The estimate

$$|w_\lambda(x)| \leq \sum_{j=0}^{\infty} |\lambda|^j \frac{(\mathcal{S}(x))^j}{j!} = e^{|\lambda|\mathcal{S}(x)} \leq e^{|\lambda|\mathcal{S}(\beta)} \qquad (a < x \leq \beta)$$

shows that $\lambda \mapsto w_\lambda(x)$ is entire and of exponential type. In addition, for $a < x \leq \beta$ we have

$$\begin{aligned}
1 - \lambda \int_a^x \frac{1}{p(y)} \int_a^y w_\lambda(\xi)\, r(\xi) d\xi\, dy &= 1 - \lambda \int_a^x (\mathfrak{s}(x) - \mathfrak{s}(\xi)) w_\lambda(\xi)\, r(\xi) d\xi \\
&= 1 - \lambda \int_a^x (\mathfrak{s}(x) - \mathfrak{s}(\xi)) \biggl(\sum_{j=0}^{\infty} (-\lambda)^j \eta_j(\xi)\biggr) r(\xi) d\xi \\
&= 1 + \sum_{j=0}^{\infty} (-\lambda)^{j+1} \int_a^x (\mathfrak{s}(x) - \mathfrak{s}(\xi)) \eta_j(\xi)\, r(\xi) d\xi \\
&= 1 + \sum_{j=0}^{\infty} (-\lambda)^{j+1} \eta_{j+1}(x) \ = \ w_\lambda(x),
\end{aligned}$$

i.e., $w_\lambda(x)$ satisfies

$$w_\lambda(x) = 1 - \lambda \int_a^x \frac{1}{p(y)} \int_a^y w_\lambda(\xi)\, r(\xi) d\xi\, dy$$

This integral equation is equivalent to (2.3), so the proof is complete. ☐

We note also that the following converse of Lemma 2.1 holds: *if $\int_a^c \int_y^c \frac{dx}{p(x)} r(y) dy = \infty$ (so that (2.2) fails to hold) then for $\lambda < 0$ there exists no solution of $\ell(w) = \lambda w$ satisfying the boundary conditions $w(a) = 1$ and $w^{[1]}(a) = 0$.*

Indeed, if the integral $\int_a^c \int_y^c \frac{dx}{p(x)} r(y) dy$ diverges (which means, according to the Feller boundary classification given in [15, Section 5.11], that a is an exit boundary or a natural boundary for the operator ℓ) then it follows from [15, Sections 5.13–5.14] that any solution w of $\ell(w) = \lambda w$ $(\lambda < 0)$ either satisfies $w(a) = 0$ or $w^{[1]}(a) = +\infty$, so in particular (2.3) cannot hold.

Lemma 2.2. *Let $\{a_m\}_{m \in \mathbb{N}}$ be a sequence $b > a_1 > a_2 > \ldots$ with $\lim a_m = a$. For $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, let $w_{\lambda,m}(x)$ be the unique solution of the boundary value problem*

$$\ell(w) = \lambda w \quad (a_m < x < b), \qquad w(a_m) = 1, \qquad w^{[1]}(a_m) = 0. \tag{2.4}$$

Then

$$\lim_{m \to \infty} w_{\lambda,m}(x) = w_\lambda(x) \quad \textit{pointwise for each } a < x < b \textit{ and } \lambda \in \mathbb{C}.$$

Proof. In the same way as in the proof of Lemma 2.1 we can check that the solution of (2.4) is given by

$$w_{\lambda,m}(x) = \sum_{j=0}^{\infty} (-\lambda)^j \eta_{j,m}(x) \qquad (a_m < x < b,\ \lambda \in \mathbb{C})$$

where $\eta_{0,m}(x) = 1$ and $\eta_{j,m}(x) = \int_{a_m}^x \bigl(\mathfrak{s}(x) - \mathfrak{s}(\xi)\bigr)\eta_{j-1,m}(\xi)r(\xi)d\xi$. As before we have $|\eta_{j,m}(x)| \leq \frac{1}{j!}(\mathcal{S}(x))^j$ for $a_m < x \leq \beta$ (where $\mathcal{S}$ is the function from the proof of Lemma 2.1). Using this estimate and induction on j, it is easy to see that $\eta_{j,m}(x) \to \eta_j(x)$ as $m \to \infty$ $(a < x \leq \beta,\ j = 0, 1, \ldots)$. Noting that the estimate on $|\eta_{j,m}(x)|$ allows us to take the limit under the summation sign, we conclude that $w_{\lambda,m}(x) \to w_\lambda(x)$ as $m \to \infty$ $(a < x \leq \beta)$. □

Lemma 2.3. *If $x \mapsto p(x)r(x)$ is an increasing function, then the solution of* (2.3) *is bounded:*

$$|w_\lambda(x)| \leq 1 \qquad \textit{for all } a < x < b,\ \lambda \geq 0. \tag{2.5}$$

Proof. Let us start by assuming that $p(a)r(a) > 0$. For $\lambda = 0$ the result is trivial because $w_0(x) \equiv 1$. Fix $\lambda > 0$. Multiplying both sides of the differential equation $\ell(w_\lambda) = \lambda w_\lambda$ by $2w_\lambda^{[1]}$, we obtain $-\frac{1}{pr}[(w_\lambda^{[1]})^2]' = \lambda(w_\lambda^2)'$. Integrating the differential equation and then using integration by parts, we get

$$\begin{aligned}\lambda\bigl(1 - w_\lambda(x)^2\bigr) &= \int_a^x \frac{1}{p(\xi)r(\xi)} \bigl(w_\lambda^{[1]}(\xi)^2\bigr)' d\xi \\ &= \frac{w_\lambda^{[1]}(x)^2}{p(x)r(x)} + \int_a^x \bigl(p(\xi)r(\xi)\bigr)' \biggl(\frac{w_\lambda^{[1]}(\xi)}{p(\xi)r(\xi)}\biggr)^2 d\xi, \qquad a < x < b\end{aligned}$$

where we also used the fact that $w_\lambda^{[1]}(a) = 0$ and the assumption that $p(a)r(a) > 0$. The right hand side is nonnegative, because $x \mapsto p(x)r(x)$ is increasing and therefore $(p(\xi)r(\xi))' \geq 0$. Given that $\lambda > 0$, it follows that $1 - w_\lambda(x)^2 \geq 0$, so that $|w_\lambda(x)| \leq 1$.

If $p(a)r(a) = 0$, the above proof can be used to show that the solution of (2.4) is such that $|w_{\lambda,m}(x)| \leq 1$ for all $a < x < b$, $\lambda \geq 0$ and $m \in \mathbb{N}$; then Lemma 2.2 yields the desired result. □

Remark 2.4. We shall make extensive use of the fact that the differential expression (2.1) can be transformed into the standard form

$$\widetilde{\ell} = -\frac{1}{A}\frac{d}{d\xi}\Bigl(A\frac{d}{d\xi}\Bigr) = -\frac{d^2}{d\xi^2} - \frac{A'}{A}\frac{d}{d\xi}.$$

This is achieved by setting

$$A(\xi) := \sqrt{p(\gamma^{-1}(\xi))\, r(\gamma^{-1}(\xi))}, \tag{2.6}$$

where γ^{-1} is the inverse of the increasing function

$$\gamma(x) = \int_c^x \sqrt{\frac{r(y)}{p(y)}} dy,$$

$c \in (a, b)$ being a fixed point (if $\sqrt{\frac{r(y)}{p(y)}}$ is integrable near a, we may also take $c = a$). Indeed, it is straightforward to check that a given function $\omega_\lambda : (a, b) \to \mathbb{C}$ satisfies $\ell(\omega_\lambda) = \lambda\omega_\lambda$ if and only if $\widetilde{\omega}_\lambda(\xi) := \omega_\lambda(\gamma^{-1}(\xi))$ satisfies $\widetilde{\ell}(\widetilde{\omega}_\lambda) = \lambda\widetilde{\omega}_\lambda$. It is interesting to note that the assumption of the previous lemma ($x \mapsto p(x)r(x)$ is increasing) is equivalent to requiring that the first-order coefficient $\frac{A'}{A}$ of the transformed operator $\widetilde{\ell}$ is nonnegative.

As it is well-known, the spectral expansion of self-adjoint realizations of the differential operator (2.1) in the space $L_2\bigl((a, b); r(x)dx\bigr)$ give rise to a Sturm-Liouville type integral transform. The next proposition collects some basic facts from the theory of eigenfunction expansions of Sturm-Liouville operators. For brevity we write $L_2(r) := L_2\bigl((a, b); r(x)dx\bigr)$.

Proposition 2.5. *Suppose that b is (according to Feller's boundary classification) a natural boundary for the differential expression ℓ, that is, the coefficients of ℓ satisfy*

$$\int_c^b \int_y^b \frac{dx}{p(x)}\, r(y)dy = \int_c^b \int_c^y \frac{dx}{p(x)}\, r(y)dy = \infty.$$

Then the operator

$$\mathcal{L} : \mathcal{D}_{\mathcal{L}}^{(2)} \subset L_2(r) \longrightarrow L_2(r), \qquad \mathcal{L}u = \ell(u)$$

where

$$\mathcal{D}_{\mathcal{L}}^{(2)} := \Bigl\{ u \in L_2(r) \Bigm| u \text{ and } u' \text{ locally abs. continuous on } (a,b),\ \ell(u) \in L_2(r),\ \lim_{x\downarrow a} u^{[1]}(x) = 0 \Bigr\} \tag{2.7}$$

is self-adjoint. There exists a unique locally finite positive Borel measure $\boldsymbol{\rho}_{\mathcal{L}}$ on $\mathbb{R}$ such that the map $h \mapsto \mathcal{F}h$, where

$$(\mathcal{F}h)(\lambda) := \int_a^b h(x)\, w_\lambda(x)\, r(x) dx \qquad \bigl(h \in \mathrm{C_c}[a,b),\ \lambda \geq 0\bigr), \tag{2.8}$$

induces an isometric isomorphism $\mathcal{F} : L_2(r) \longrightarrow L_2(\mathbb{R}; \boldsymbol{\rho}_{\mathcal{L}})$ whose inverse is given by

$$(\mathcal{F}^{-1}\varphi)(x) = \int_{\mathbb{R}} \varphi(\lambda)\, w_\lambda(x)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda), \tag{2.9}$$

the convergence of the latter integral being understood with respect to the norm of $L_2(r)$. The spectral measure $\boldsymbol{\rho}_{\mathcal{L}}$ is supported on $[0,\infty)$. Moreover, the differential operator $\mathcal{L}$ is connected with the integral transform (2.8) via the identity

$$[\mathcal{F}(\mathcal{L}h)](\lambda) = \lambda \cdot (\mathcal{F}h)(\lambda), \qquad h \in \mathcal{D}_{\mathcal{L}}^{(2)} \tag{2.10}$$

and the domain $\mathcal{D}_{\mathcal{L}}^{(2)}$ defined by (2.7) can be written as

$$\mathcal{D}_{\mathcal{L}}^{(2)} = \Bigl\{ u \in L_2(r) \Bigm| \lambda \cdot (\mathcal{F}f)(\lambda) \in L_2\bigl([0,\infty); \boldsymbol{\rho}_{\mathcal{L}}\bigr) \Bigr\}.$$

Proof. The fact that $(\mathcal{L}, \mathcal{D}_{\mathcal{L}}^{(2)})$ is self-adjoint is well-known, see [23, 21]. The existence of a generalized Fourier transform associated with the operator $\mathcal{L}$ is a consequence of the standard Weyl-Titchmarsh-Kodaira theory of eigenfunction expansions of Sturm-Liouville operators (cf. [27, Section 3.1] and [29, Section 8]).

In the general case the eigenfunction expansion is written in terms of two linearly independent eigenfunctions and a 2×2 matrix measure. However, from the regular/entrance boundary assumption (2.2) it follows that the function $w_\lambda(x)$ is square-integrable near $x = 0$ with respect to the measure $r(x)dx$; moreover, by Lemma 2.1, $w_\lambda(x)$ is (for fixed x) an entire function of λ. Therefore, the possibility of writing the expansion in terms only of the eigenfunction $w_\lambda(x)$ follows from the results of [9, Sections 9 and 10]. ◻

The integral transform $(\mathcal{F}h)(\lambda) = \int_a^b h(x)\, w_\lambda(x)\, r(x) dx$ is the so-called *$\mathcal{L}$-transform.* It is often important to know whether the inversion integral for the $\mathcal{L}$-transform is absolutely convergent. A sufficient condition is provided by the following lemma:

Lemma 2.6. ***(a)*** *For each $\mu \in \mathbb{C} \setminus \mathbb{R}$, the integrals*

$$\int_{[0,\infty)} \frac{w_\lambda(x)\, w_\lambda(y)}{|\lambda - \mu|^2} \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) \qquad \text{and} \qquad \int_{[0,\infty)} \frac{w_\lambda^{[1]}(x)\, w_\lambda^{[1]}(y)}{|\lambda - \mu|^2} \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) \tag{2.11}$$

converge uniformly on compact squares in $(a,b)^2$.

(b) *If $h \in \mathcal{D}_{\mathcal{L}}^{(2)}$, then*

$$h(x) = \int_{[0,\infty)} (\mathcal{F}h)(\lambda)\, w_\lambda(x)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) \tag{2.12}$$

$$h^{[1]}(x) = \int_{[0,\infty)} (\mathcal{F}h)(\lambda)\, w_\lambda^{[1]}(x)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) \tag{2.13}$$

where the right-hand side integrals converge absolutely and uniformly on compact subsets of (a,b).

Proof. **(a)** By [9, Lemma 10.6] and [28, p. 229],

$$\int_{[0,\infty)} \frac{w_\lambda(x) w_\lambda(y)}{|\lambda-\mu|^2} \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) = \int_a^b G(x,\xi,\mu) G(y,\xi,\mu)\, r(\xi) d\xi = \frac{1}{\mathrm{Im}(\mu)} \mathrm{Im}\bigl(G(x,y,\mu)\bigr)$$

where $G(x,y,\mu)$ is the resolvent kernel (or Green function) of the operator $(\mathcal{L}, \mathcal{D}_{\mathcal{L}}^{(2)})$. Moreover, according to [9, Theorems 8.3 and 9.6], the resolvent kernel is given by

$$G(x,y,\mu) = \begin{cases} w_\mu(x)\vartheta_\mu(y), & x<y \\ w_\mu(y)\vartheta_\mu(x), & x \geq y \end{cases}$$

where $\vartheta_\lambda(\cdot)$ is a solution of $\ell(u) = \lambda u$ which is square-integrable near ∞ with respect to the measure $r(x)dx$ and verifies the identity $w_\lambda(x)\vartheta_\lambda^{[1]}(x) - w_\lambda^{[1]}(x)\vartheta_\lambda(x) \equiv 1$. It is easily seen (cf. [24, p. 125]) that the functions $\mathrm{Im}\bigl(G(x,y,\mu)\bigr)$ and $\partial_x^{[1]}\partial_y^{[1]}\mathrm{Im}\bigl(G(x,y,\mu)\bigr)$ are continuous in $0<x,y<\infty$. Essentially the same proof as that of [24, Corollary 3] now yields that

$$\int_{[0,\infty)} \frac{w_\lambda^{[1]}(x)\, w_\lambda^{[1]}(y)}{|\lambda-\mu|^2} \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) = \frac{1}{\mathrm{Im}(\mu)}\, \partial_x^{[1]}\partial_y^{[1]}\mathrm{Im}\bigl(G(x,y,\mu)\bigr)$$

and that the integrals (2.11) converge uniformly for x,y in compacts.

(b) By Proposition 2.5 and the classical theorem on differentiation under the integral sign for Riemann-Stieltjes integrals, to prove (2.12)–(2.13) it only remains to justify the absolute and uniform convergence of the integrals in the right-hand sides.

Recall from Proposition 2.5 that the condition $h \in \mathcal{D}_{\mathcal{L}}^{(2)}$ implies that $\mathcal{F}h \in L_2\bigl([0,\infty);\boldsymbol{\rho}_{\mathcal{L}}\bigr)$ and also $\lambda\,(\mathcal{F}h)(\lambda) \in L_2\bigl([0,\infty);\boldsymbol{\rho}_{\mathcal{L}}\bigr)$. As a consequence, we obtain

$$\begin{aligned} &\int_{[0,\infty)} \bigl|(\mathcal{F}h)(\lambda) w_\lambda(x)\bigr| \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) \\ &\qquad \leq \int_{[0,\infty)} \lambda \bigl|(\mathcal{F}h)(\lambda)\bigr| \left|\frac{w_\lambda(x)}{\lambda+i}\right| \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) + \int_{[0,\infty)} \bigl|(\mathcal{F}h)(\lambda)\bigr| \left|\frac{w_\lambda(x)}{\lambda+i}\right| \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) \\ &\qquad \leq \bigl(\|\lambda\,(\mathcal{F}h)(\lambda)\|_\rho + \|(\mathcal{F}h)(\lambda)\|_\rho\bigr) \left\|\frac{w_\lambda(x)}{\lambda+i}\right\|_\rho \\ &\qquad < \infty \end{aligned}$$

where $\|\cdot\|_\rho$ denotes the norm of the space $L_2\bigl(\mathbb{R};\boldsymbol{\rho}_{\mathcal{L}}\bigr)$, and similarly

$$\int_{[0,\infty)} \bigl|(\mathcal{F}h)(\lambda)\, w_\lambda^{[1]}(x)\bigr| \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) \leq \bigl(\|\lambda\,(\mathcal{F}h)(\lambda)\|_\rho + \|(\mathcal{F}h)(\lambda)\|_\rho\bigr) \left\|\frac{w_\lambda^{[1]}(x)}{\lambda+i}\right\|_\rho < \infty.$$

We know from part (a) that the integrals which define $\bigl\|\frac{w_\lambda(x)}{\lambda+i}\bigr\|_\rho$ and $\bigl\|\frac{w_\lambda^{[1]}(x)}{\lambda+i}\bigr\|_\rho$ converge uniformly, hence the integrals in (2.12)–(2.13) converge absolutely and uniformly for x in compact subsets. ☐

It is also useful to know that, according to a standard result from the theory of diffusion processes and semigroups which we state below, Sturm-Liouville differential expressions of the form (2.1) generate positivity-preserving contraction semigroups acting on the space of bounded continuous functions. We recall from [6] that, for a subset $E \subset \mathbb{R}^d$, a *Feller semigroup* $\{T_t\}_{t\geq 0}$ on $\mathrm{C}_0(E,\mathbb{R})$ is a strongly continuous, positivity-preserving contraction semigroup on $\mathrm{C}_0(E,\mathbb{R})$, and that a Feller semigroup is *conservative* if its extension to $\mathrm{B_b}(E)$ satisfies $T_t\mathbb{1} = \mathbb{1}$ (here $\mathbb{1}$ denotes the function identically equal to one).

Proposition 2.7. *Suppose that b is a natural boundary for the differential expression ℓ. Then the operator*

$$\mathcal{L}^{(0)} : \mathcal{D}_{\mathcal{L}}^{(0)} \subset \mathrm{C}_0([a,b),\mathbb{R}) \longrightarrow \mathrm{C}_0([a,b),\mathbb{R}), \qquad \mathcal{L}^{(0)}u = -\ell(u)$$

with domain

$$\mathcal{D}_{\mathcal{L}}^{(0)} = \bigl\{u \in \mathrm{C}_0([a,b),\mathbb{R}) \bigm| \ell(u) \in \mathrm{C}_0([a,b),\mathbb{R}),\, \lim_{x\downarrow a} u^{[1]}(x) = 0\bigr\}$$

is the generator of a conservative Feller semigroup $\{\mathcal{P}_t\}_{t\geq 0}$ on $\mathrm{C}_0([a,b),\mathbb{R})$. The semigroup admits the representation

$$(\mathcal{P}_t u)(x) = \int_a^b h(y)\, p(t,x,y)\, r(y)dy, \qquad \bigl(h \in \mathrm{B}_\mathrm{b}\bigl([a,b),\mathbb{R}\bigr),\ t>0,\ x\in(a,b)\bigr) \tag{2.14}$$

where the (nonnegative) transition kernel $p(t,x,y)$ is given by

$$p(t,x,y) = \int_{[0,\infty)} e^{-t\lambda}\, w_\lambda(x)\, w_\lambda(y)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) \qquad \bigl(t>0,\ x,y\in(a,b)\bigr)$$

with the integral converging absolutely and uniformly on compact squares of $(a,b)\times(a,b)$ for each fixed $t>0$. If $h \in L_2(r)\cap \mathrm{B}_\mathrm{b}\bigl([a,b),\mathbb{R}\bigr)$, then (2.14) can also be written as

$$(\mathcal{P}_t u)(x) = \int_{[0,\infty)} e^{-t\lambda} w_\lambda(x)\, (\mathcal{F}h)(\lambda)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) \qquad \bigl(t>0,\ x\in(a,b)\bigr) \tag{2.15}$$

where the integral converges with respect to the norm of $L_2(r)$.

Proof. The first assertion is proved in [12, Sections 4 and 6] (see also [22, Section II.5]). The claimed representation for the transition semigroup and kernel follows from [21, Sections 2–3]. ◻

3 Laplace-type representation

As mentioned in the introduction, the existence of a hypergroup-like product formula for the kernel of the $\mathcal{L}$-transform is strongly connected with the positivity of the associated Cauchy problem. We now introduce an assumption which, as we will see in Subsection 4.1, is sufficient for the Cauchy problem to be positivity preserving. Recall that the function A, defined in (2.6), is the coefficient associated with the transformation of ℓ into the standard form (Remark 2.4).

Assumption MP. We have $\gamma(b) = \int_c^b \sqrt{\frac{r(y)}{p(y)}}dy = \infty$, and there exists $\eta \in \mathrm{C}^1(\gamma(a),\infty)$ such that $\eta \geq 0$, $\boldsymbol{\phi}_\eta := \frac{A'}{A} - \eta \geq 0$, and the functions $\boldsymbol{\phi}_\eta$ and $\boldsymbol{\psi}_\eta := \frac{1}{2}\eta' - \frac{1}{4}\eta^2 + \frac{A'}{2A}\cdot\eta$ are both decreasing on $(\gamma(a),\infty)$.

This assumption will be held throughout the remainder of the paper.

Having in mind the product formula that we shall establish for a general Sturm-Liouville operator satisfying Assumption MP, in this section we prove the related fact that the kernel $w_\lambda(x)$ of the $\mathcal{L}$-transform admits a representation as the Laplace transform of a subprobability measure. We start by stating a basic property which holds for all Sturm-Liouville operators (2.1) satisfying this assumption:

Lemma 3.2. *The function $\frac{A'}{A}$ is nonnegative, and there exists a finite limit $\sigma := \lim_{\xi\to\infty} \frac{A'(\xi)}{2A(\xi)} \in [0,\infty)$.*

Proof. See [31, Section 2]. ◻

The existence of a Laplace-type representation for the kernel of the $\mathcal{L}$-transform is already known to hold for a Sturm-Liouville operator of the form $-\frac{1}{A}\frac{d}{dx}\bigl(A\frac{d}{dx}\bigr)$ where the coefficient A satisfies the assumptions of the existence theorem of [31] for Sturm-Liouville hypergroups (see the discussion in Subsection 5.2). In particular, the following result is proved in [4, Theorem 3.5.58]:

Proposition 3.3. *Let $A \in \mathrm{C}^1[0,\infty)$ with $A(x) > 0$ for all $x \geq 0$. Suppose that there exists $\eta \in \mathrm{C}^1[0,\infty)$ such that $\eta \geq 0$, $\boldsymbol{\phi}_\eta \geq 0$ and the functions $\boldsymbol{\phi}_\eta$, $\boldsymbol{\psi}_\eta$ are both decreasing on $(0,\infty)$ ($\boldsymbol{\phi}_\eta$, $\boldsymbol{\psi}_\eta$ are defined as in Assumption MP). For $\lambda \in \mathbb{C}$, let $\boldsymbol{\omega}_\lambda(\cdot)$ be the unique solution of the boundary value problem*

$$-\frac{1}{\boldsymbol{A}}(\boldsymbol{A}\boldsymbol{\omega}')' = \lambda \boldsymbol{\omega} \quad (0 < x < \infty), \qquad \boldsymbol{\omega}(0) = 1, \qquad \boldsymbol{\omega}'(0) = 0.$$

Then for each $x \geq 0$ there exists a subprobability measure π_x on $\mathbb{R}$ such that

$$\boldsymbol{\omega}_{\tau^2+\sigma^2}(x) = \int_{\mathbb{R}} e^{i\tau s} \pi_x(ds) = \int_{\mathbb{R}} \cos(\tau s)\, \pi_x(ds) \qquad (\tau \in \mathbb{C})$$

where $\sigma = \lim_{\xi\to\infty} \frac{\boldsymbol{A}'(\xi)}{2\boldsymbol{A}(\xi)}$.

The following theorem generalizes the proposition above to the class of operators ℓ of the form (2.1) satisfying (2.2) and Assumption MP:

Theorem 3.4 (Laplace-type representation)**.** *For each $x \in [a,b)$ there exists a subprobability measure ν_x on $\mathbb{R}$ such that*

$$w_{\tau^2+\sigma^2}(x) = \int_{\mathbb{R}} e^{i\tau s} \nu_x(ds) = \int_{\mathbb{R}} \cos(\tau s)\, \nu_x(ds) \qquad (\tau \in \mathbb{C}) \tag{3.1}$$

where $\sigma = \lim_{\xi\to\infty} \frac{\boldsymbol{A}'(\xi)}{2\boldsymbol{A}(\xi)}$. In particular, the boundedness property (2.5) extends to

$$|w_{\tau^2+\sigma^2}(x)| \leq 1 \quad \text{on the strip } |\mathrm{Im}(\tau)| \leq \sigma \quad (a \leq x < b). \tag{3.2}$$

Proof. For $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, let $w_{\lambda,m}$ be the solution of (2.4). The function $\widetilde{w}_{\lambda,m}(\xi) = w_{\lambda,m}(\gamma^{-1}(\xi))$ is the solution of

$$\widetilde{\ell}(u) = \lambda u \quad (\tilde{a}_m < \xi < \infty), \qquad u(\tilde{a}_m) = 1, \qquad u^{[1]}(\tilde{a}_m) = 0$$

where $\tilde{a}_m = \gamma(a_m)$. By Assumption MP, the function $\boldsymbol{A}(y) := A(y + \tilde{a}_m)$ satisfies the assumption of Proposition 3.3. It follows that for each $\xi > \tilde{a}_m$ there exists a subprobability measure $\pi_{\xi,m}$ such that

$$\widetilde{w}_{\tau^2+\sigma^2,m}(\xi) = \int_{\mathbb{R}} e^{i\tau s} \pi_{\xi,m}(ds) = \int_{\mathbb{R}} \cos(\tau s)\, \pi_{\xi,m}(ds) \qquad (\tau \in \mathbb{C}).$$

In particular, $\tau \mapsto \widetilde{w}_{\tau^2+\sigma^2,m}(\xi)$ $(\tau \in \mathbb{R})$ is the Fourier transform of the measure $\pi_{\xi,m}$. We know (from Lemma 2.2) that $\widetilde{w}_{\tau^2+\sigma^2,m}(\xi) \longrightarrow \widetilde{w}_{\tau^2+\sigma^2}(\xi) := w_{\tau^2+\sigma^2}(\gamma^{-1}(\xi))$ pointwise as $m \to \infty$, the limit function being continuous in τ (cf. Lemma 2.1). Applying the Lévy continuity theorem [1, Theorem 23.8], we conclude that $\widetilde{w}_{\tau^2+\sigma^2}(\xi)$ is the Fourier transform of a subprobability measure π_ξ and, in addition, the measures $\pi_{\xi,m}$ converge weakly to π_ξ as $m \to \infty$. Therefore, for $\xi > \gamma(a)$ we have

$$\widetilde{w}_{\tau^2+\sigma^2}(\xi) = \int_{\mathbb{R}} e^{i\tau s} \pi_\xi(ds) = \int_{\mathbb{R}} \cos(\tau s)\, \pi_\xi(ds) \qquad (\tau \in \mathbb{R}). \tag{3.3}$$

In order to extend (3.3) to $\tau \in \mathbb{C}$, we let $0 \leq \phi_1 \leq \phi_2 \leq \ldots$ be functions with compact support such that $\phi_n \uparrow 1$ pointwise, and for fixed $\xi > \gamma(a)$, $\kappa > 0$ we compute

$$\begin{aligned}
\int_{\mathbb{R}} \cosh(\kappa s)\, \pi_\xi(ds) &= \lim_{n\to\infty} \int_{\mathbb{R}} \phi_n(s) \cosh(\kappa s)\, \pi_\xi(ds) \\
&= \lim_{n\to\infty} \lim_{m\to\infty} \int_{\mathbb{R}} \phi_n(s) \cosh(\kappa s)\, \pi_{\xi,m}(ds) \\
&\leq \lim_{m\to\infty} \int_{\mathbb{R}} \cosh(\kappa s)\, \pi_{\xi,m}(ds) \;=\; \lim_{m\to\infty} \widetilde{w}_{\sigma^2-\kappa^2,m}(\xi) \;=\; \widetilde{w}_{\sigma^2-\kappa^2}(\xi) < \infty
\end{aligned}$$

From this estimate we easily see that the right-hand side of (3.3) is an entire function of τ; therefore, by analytic continuation, (3.3) holds for all $\tau \in \mathbb{C}$. Setting $\nu_x = \pi_{\gamma(x)}$ gives (3.1).

Finally, if $|\mathrm{Im}(\tau)| \leq \sigma$ then

$$|w_{\tau^2+\sigma^2}(x)| \leq \int_{\mathbb{R}} |\cos(\tau s)| \nu_x(ds) \leq \int_{\mathbb{R}} \cosh(\sigma s)\, \nu_x(ds) = w_0(x) = 1$$

and therefore (3.2) is true. ∎

The rest of this section provides some additional properties of the solutions of $\ell(u) = \lambda u$ which will be needed later.

Proposition 3.5. *If $\lambda > \sigma^2$, then the equation $\ell(u) = \lambda u$ is oscillatory at b, that is, all solutions of $\ell(u) = \lambda u$ have infinitely many zeros clustering at b. Consequently, b is a natural boundary for ℓ.*

Proof. The results of [11, Lemma 3.7] on the asymptotic behavior of the solutions of the standardized equation $\widetilde{\ell}(u) = (\tau^2 + \sigma^2)u$ show that for $\tau > 0$ this equation has a linearly independent pair of solutions with infinitely many zeros clustering at infinity; hence any solution of $\widetilde{\ell}(u) = (\tau^2 + \sigma^2)u$ has this property (cf. [7, Section 8.1]). It immediately follows that the same is true for any solution of $\ell(u) = (\tau^2 + \sigma^2)u$ $(\tau > 0)$.

According to [21, p. 348], if $\widetilde{\ell}(u) = \lambda u$ is oscillatory at b for some $\lambda > 0$ then b is a natural (Feller) boundary for the operator ℓ, so the final assertion holds. □

Proposition 3.6. *The spectral measure from Proposition 2.5 is such that* $\operatorname{supp}(\boldsymbol{\rho}_{\mathcal{L}}) = [\sigma^2, \infty)$. *In addition, $\mathcal{L}$ has purely absolutely continuous spectrum in (σ^2, ∞).*

Proof. To show that the essential spectrum of $\mathcal{L}$ equals $[\sigma^2, \infty)$, we may assume that the differential expression ℓ is regular at the endpoint a: this is so because, by a well-known result [28, Theorem 9.11], the essential spectrum of $\mathcal{L}$ is the union of the essential spectrums of self-adjoint realizations of ℓ restricted to the intervals (a, c) and (c, b) (where $c \in (a, b)$), and because it is known from [23, Theorem 3.1] that the spectrum is purely discrete whenever there are no natural boundaries.

The equation $\ell(u) = \lambda u$ is clearly non-oscillatory at a; it is oscillatory at b for $\lambda > \sigma^2$ and (by the Laplace representation (3.1)) non-oscillatory at b for $\lambda < \sigma^2$. Hence it follows from [21, Theorem 2] that the essential spectrum of $\mathcal{L}$ is contained in $[\sigma^2, \infty)$. Now, the operator $\mathcal{L}$ is unitarily equivalent, via the Liouville transformation (see e.g. [10, Section 4.3] and [21, Section 4]), to a self-adjoint realization of the differential expression $-\frac{d^2}{d\xi^2} + \mathfrak{q}$, where

$$\mathfrak{q}(\xi) = \Bigl(\frac{A'(\xi)}{2A(\xi)}\Bigr)^2 + \Bigl(\frac{A'(\xi)}{2A(\xi)}\Bigr)' = \frac{1}{4}\phi_\eta^2(\xi) + \psi_\eta(\xi) + \frac{1}{2}\phi_\eta'(\xi), \qquad \xi \in (\gamma(a), \infty). \tag{3.4}$$

It follows from [31, Lemma 2.9 and Remark 2.12] that the function η in Assumption MP can be chosen such that $\lim_{\xi\to\infty} \phi_\eta(\xi) = 0$ and $\lim_{\xi\to\infty} \eta'(\xi) = 0$. Consequently, $\lim_{\xi\to\infty} \frac{1}{4}\phi_\eta^2(\xi) + \psi_\eta(\xi) = \sigma^2$. In turn, the fact that ϕ_η is positive and decreasing clearly implies that $\phi_\eta' \in L_1([c,\infty), d\xi)$ for $c > \gamma(a)$. Using [29, Theorem 15.3], we conclude that the spectrum of $\mathcal{L}$ is purely absolutely continuous on (σ^2, ∞) and the essential spectrum equals $[\sigma^2, \infty)$.

It remains to show that $\mathcal{L}$ has no eigenvalues on $[0, \sigma^2]$. Indeed, if we assume that $0 \leq \lambda_0 \leq \sigma^2$ is an eigenvalue of $\mathcal{L}$, then w_{λ_0} belongs to $\mathcal{D}_{\mathcal{L}}^{(2)}$ and therefore, by the Laplace representation (3.1), w_λ belongs to $\mathcal{D}_{\mathcal{L}}^{(2)}$ for all $\lambda \geq \sigma^2$; since the eigenvalues are discrete, this is a contradiction. □

Proposition 3.7. *We have*

$$\lim_{x\uparrow b} w_\lambda(x) = 0 \qquad \textit{for all } \lambda > 0$$

if and only if $\lim_{x\uparrow b} p(x)r(x) = \infty$.

Proof. After transforming ℓ into the standard form (Remark 2.4), the result follows easily from [11, Lemma 3.7]. □

4 Product formula

The goal of this section is to prove the main result of the paper, which is stated as follows:

Theorem 4.1 (Product formula for w_λ)**.** *For each $x, y \in [a,b)$ there exists a measure $\boldsymbol{\nu}_{x,y} \in \mathcal{P}[a,b)$ such that the product $w_\lambda(x)\, w_\lambda(y)$ admits the integral representation*

$$w_\lambda(x)\, w_\lambda(y) = \int_{[a,b)} w_\lambda(\xi)\, \boldsymbol{\nu}_{x,y}(d\xi), \qquad x, y \in [a,b),\ \lambda \in \mathbb{C}.$$

4.1 The associated hyperbolic Cauchy problem

The proof of Theorem 4.1 relies crucially on the basic properties of the hyperbolic Cauchy problem associated with ℓ, i.e., of the boundary value problem defined by

$$(\ell_x f)(x,y) = (\ell_y f)(x,y) \quad (x, y \in (a,b)), \qquad f(x,a) = h(x), \qquad (\partial_y^{[1]} f)(x,a) = 0 \tag{4.1}$$

where h is a given function, $\partial^{[1]} u = pu'$ and the subscripts indicate the variable in which the operators act.

The assumptions on the coefficients of ℓ introduced in the previous sections allow for the higher order coefficient of ℓ to vanish at the endpoint a, in which case the hyperbolic Cauchy problem (4.1) is parabolically degenerate at the initial line. In general, such hyperbolic problems cannot be dealt with using the classical theory of hyperbolic equations in two variables. But, as we will show, the existence, uniqueness and positivity properties for the Cauchy problem (4.1) can be deduced by making use of the eigenfunction expansion of the Sturm-Liouville operator ℓ. In this subsection we only summarize the main results; the proofs are delayed to to Appendix A.

Theorem 4.2 (Existence and uniqueness of solution)**.** *If $h \in \mathcal{D}_{\mathcal{L}}^{(2)}$ and $\ell(h) \in \mathcal{D}_{\mathcal{L}}^{(2)}$, then there exists a unique solution $f \in \mathrm{C}^2\bigl((a,b)^2\bigr)$ of the Cauchy problem (4.1) satisfying the conditions*

- ***(i)*** *$f(\cdot, y) \in \mathcal{D}_{\mathcal{L}}^{(2)}$ for all $a < y < b$;*
- ***(ii)*** *There exists a zero $\boldsymbol{\rho}_{\mathcal{L}}$-measure set $\Lambda_0 \subset [\sigma^2, \infty)$ such that for each $\lambda \in [\sigma^2, \infty) \setminus \Lambda_0$ we have*

$$\mathcal{F}[\ell_y f(\cdot, y)](\lambda) = \ell_y[\mathcal{F} f(\cdot, y)](\lambda) \quad \textit{for all } a < y < b, \tag{4.2}$$

$$\lim_{y \downarrow a} [\mathcal{F} f(\cdot, y)](\lambda) = (\mathcal{F} h)(\lambda), \qquad \lim_{y \downarrow a} \partial_y^{[1]} \mathcal{F}[f(\cdot, y)](\lambda) = 0. \tag{4.3}$$

This unique solution is given by

$$f(x,y) = \int_{[\sigma^2, \infty)} w_\lambda(x)\, w_\lambda(y)\, (\mathcal{F} h)(\lambda)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda). \tag{4.4}$$

Proof. See Appendix A.1. □

Proposition 4.3 (Pointwise approximation by solutions of regularized problems)**.** *Let $\{a_m\}_{m \in \mathbb{N}}$ be a sequence $b > a_1 > a_2 > \ldots$ with $\lim a_m = a$. If $h \in \mathcal{D}_{\mathcal{L}}^{(2)}$ and $\ell(h) \in \mathcal{D}_{\mathcal{L}}^{(2)}$, then for each $m \in \mathbb{N}$ the function*

$$f_m(x,y) = \int_{[\sigma^2, \infty)} w_\lambda(x)\, w_{\lambda,m}(y)\, (\mathcal{F} h)(\lambda)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) \qquad \bigl(x \in (a,b),\ y \in (a_m, b)\bigr) \tag{4.5}$$

is a solution of the Cauchy problem

$$(\ell_x f_m)(x,y) = (\ell_y f_m)(x,y), \qquad f_m(x, a_m) = h(x), \qquad (\partial_y^{[1]} f_m)(x, a_m) = 0. \tag{4.6}$$

Moreover, we have

$$\lim_{m \to \infty} f_m(x,y) = f(x,y) \qquad \textit{pointwise for each } x, y \in (a,b). \tag{4.7}$$

where $f(x,y)$ is the solution (4.4) of the Cauchy problem (4.1).

Proof. See Appendix A.2. □

Proposition 4.4 (Positivity of solution for the regularized problem (4.6)). *Let $\{a_m\}_{m\in\mathbb{N}}$ as in the previous proposition and let $h \in \mathcal{D}_{\mathcal{L}}^{(2)}$ with $\ell(h) \in \mathcal{D}_{\mathcal{L}}^{(2)}$. If $h \geq 0$, then the function f_m given by (4.5) is such that*

$$f_m(x,y) \geq 0 \qquad \text{for } x \geq y > a_m. \tag{4.8}$$

If, in addition, $h \leq C$ (where C is a constant), then $f_m(x,y) \leq C$ for $x \geq y > a_m$.

Proof. See Appendix A.4. □

Corollary 4.5 (Positivity of solution for the Cauchy problem (4.1)). *Let $h \in \mathcal{D}_{\mathcal{L}}^{(2)}$ with $\ell(h) \in \mathcal{D}_{\mathcal{L}}^{(2)}$. If $h \geq 0$, then the function f given by (4.4) is such that*

$$f(x,y) \geq 0 \qquad \text{for } x, y \in (a,b).$$

If, in addition, $h \leq C$, then $f(x,y) \leq C$ for $x, y \in (a,b)$.

Proof. This is an immediate consequence of the pointwise convergence property (4.7) (note that the conclusion holds for all $x, y \in (a,b)$ because the function $f(x,y)$ is symmetric). □

We observe that the above existence, uniqueness and positivity results are valid, in particular, if the initial condition h belongs to the space

$$\mathrm{C}^4_{\mathrm{c},0} := \Bigl\{ u \in \mathrm{C}^4_{\mathrm{c}}[a,b) \Bigm| \ell(u), \ell^2(u) \in \mathrm{C}_{\mathrm{c}}[a,b), \; \lim_{x\downarrow a} u^{[1]}(x) = \lim_{x\downarrow a} [\ell(u)]^{[1]}(x) = 0 \Bigr\}$$

(clearly, if $h \in \mathrm{C}^4_{\mathrm{c},0}$ then $h, \ell(h) \in \mathcal{D}_{\mathcal{L}}^{(2)}$).

4.2 The time-shifted product formula

Before proving the product formula for the kernels $\{w_\lambda(\cdot)\}$ themselves, we will establish a product formula of the form (1.2) for the functions $\{e^{-t\lambda} w_\lambda(\cdot)\}$. This auxiliary result will be called the *time-shifted product formula* because $e^{-t\lambda} w_\lambda(x)$ is the $\mathcal{L}$-transform of the transition kernel $p(t,x,y)$ of the Feller semigroup generated by the Sturm-Liouville operator ℓ, cf. Proposition 2.7.

By the inversion formula (2.9) for the $\mathcal{L}$-transform, a natural candidate for the measure of the product formula for $\{w_\lambda(\cdot)\}$ is

$$\boldsymbol{\nu}_{x,y}(d\xi) = \int_{[\sigma^2,\infty)} w_\lambda(x) w_\lambda(y) w_\lambda(\xi) \boldsymbol{\rho}_{\mathcal{L}}(d\lambda)\, r(\xi) d\xi.$$

This is only a formal solution, because in general the integral does not converge. However, the uniform convergence of this integral always holds (under the present assumptions on ℓ) if the exponential term $e^{-t\lambda}$ is included in the integrand:

Lemma 4.6. *Let $t_0 > 0$ and K_1, K_2 compact subsets of (a,b). The integral*

$$\int_{[\sigma^2,\infty)} e^{-t\lambda}\, w_\lambda(x)\, w_\lambda(y)\, w_\lambda(\xi)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda)$$

converges absolutely and uniformly on $(t,x,y,\xi) \in [t_0,\infty) \times K_1 \times K_2 \times [a,b)$.

Proof. This follows from Lemma 2.3 and the uniform convergence property of the integral representation of the transition kernel of the Feller semigroup generated by $-\ell$ (Proposition 2.7). □

In what follows we write

$$q_t(x,y,\xi) := \int_{[\sigma^2,\infty)} e^{-t\lambda}\, w_\lambda(x)\, w_\lambda(y)\, w_\lambda(\xi)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda).$$

This function, which is (at least formally) the density of the measure of the time-shifted product formula, is for fixed t, x, y the density (with respect to $r(\xi)d\xi$) of a subprobability measure:

Lemma 4.7. *The function $q_t(x,y,\xi)$ is nonnegative and such that $\int_a^b q_t(x,y,\xi)\,r(\xi)d\xi \leq 1$ for all $(t,x,y) \in (0,\infty) \times (a,b) \times (a,b)$.*

Proof. Since $q_t(x,y,\cdot) \in \mathrm{C_b}[a,b]$, it suffices to show that for all $g \in \mathrm{C}_{\mathrm{c},0}^4$ with $0 \leq g \leq 1$ we have

$$0 \leq \mathcal{Q}_{t,g}(x,y) \leq 1 \qquad \bigl(t > 0,\; x,y \in (a,b)\bigr)$$

where $\mathcal{Q}_{t,g}(x,y) := \int_a^b g(\xi)\, q_t(x,y,\xi)\, r(\xi)d\xi$.

Fix $t > 0$ and $g \in \mathrm{C}_{\mathrm{c},0}^4$ with $0 \leq g \leq 1$. By changing the order of integration, we have

$$\mathcal{Q}_{t,g}(x,y) = \int_{[\sigma^2,\infty)} e^{-t\lambda}\, w_\lambda(x)\, w_\lambda(y)\, (\mathcal{F}g)(\lambda)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda).$$

Differentiating under the integral sign we easily check (by dominated convergence and using Lemma 2.6(b)) that $\ell_x \mathcal{Q}_{t,g} = \ell_y \mathcal{Q}_{t,g}$, $(\partial_y^{[1]} \mathcal{Q}_{t,g})(x,a) = 0$ and

$$\mathcal{Q}_{t,g}(x,a) = \int_{[\sigma^2,\infty)} e^{-t\lambda}\, w_\lambda(x)\, (\mathcal{F}g)(\lambda)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) = (\mathcal{P}_t g)(x)$$

where the last equality follows from (2.15) (here $\{\mathcal{P}_t\}$ is the Feller semigroup generated by $(\mathcal{L}^{(0)}, \mathcal{D}_{\mathcal{L}}^{(0)})$). The fact that $0 \leq g \leq 1$ clearly implies that $0 \leq (\mathcal{P}_t g)(x) \leq 1$ for $x \in (a,b)$. One can verify, again via Lemma 2.6(b), that the function $h(x) = (\mathcal{P}_t g)(x)$ is such that $h \in \mathcal{D}_{\mathcal{L}}^{(2)}$ and $\ell(h) \in \mathcal{D}_{\mathcal{L}}^{(2)}$. It then follows from the positivity property of the hyperbolic Cauchy problem (Corollary 4.5) that $0 \leq \mathcal{Q}_{t,g}(x,y) \leq 1$ for all $x,y \in (a,b)$, as claimed. ☐

Proposition 4.8 (Time-shifted product formula)**.** *The product $e^{-t\lambda}\, w_\lambda(x)\, w_\lambda(y)$ admits the integral representation*

$$e^{-t\lambda}\, w_\lambda(x)\, w_\lambda(y) = \int_a^b w_\lambda(\xi)\, q_t(x,y,\xi)\, r(\xi)d\xi, \qquad t > 0,\; x,y \in (a,b),\; \lambda \geq 0 \tag{4.9}$$

where the integral in the right hand side is absolutely convergent.

In particular, $\int_a^b q_t(x,y,\xi)\, r(\xi)d\xi = 1$ for all $t > 0$, $x,y \in (a,b)$.

Proof. The absolute convergence of the integral in the right hand side is immediate from Lemmas 2.3 and 4.7.

By Proposition 2.5, the equality in (4.9) holds $\boldsymbol{\rho}_{\mathcal{L}}$-almost everywhere. Since $\mathrm{supp}(\boldsymbol{\rho}_{\mathcal{L}}) = [\sigma^2,\infty)$ (Lemma 3.6), the fact that both sides of (4.9) are continuous functions of $\lambda \geq 0$ allows us to extend by continuity the equality (4.9) to all $\lambda \geq \sigma^2$. If $\sigma = 0$, we are done.

Suppose that $\sigma > 0$. By (3.2) and Lemma 4.7, together with standard results on the analyticity of parameter-dependent integrals, the function $\tau \mapsto \int_a^b w_{\tau^2+\sigma^2}(\xi)\, q_t(x,y,\xi)\, r(\xi)d\xi$ is an analytic function of τ in the strip $|\mathrm{Im}(\tau)| < \sigma$. It is also clear that $\tau \mapsto e^{-t(\tau^2+\sigma^2)}\, w_{\tau^2+\sigma^2}(x)\, w_{\tau^2+\sigma^2}(y)$ is an entire function. By analytic continuation we see that these two functions are equal for all τ in the strip $|\mathrm{Im}(\tau)| < \sigma$; consequently, (4.9) holds.

The last statement is obtained by setting $\lambda = 0$. ☐

4.3 The product formula for w_λ as the limit case

As one would expect, the product formula (4.10) will be deduced by taking (in a suitable way) the limit as $t \downarrow 0$ in the time-shifted product formula (4.9). First we present a lemma which will be needed to handle the case where the functions $w_\lambda(x)$ do not vanish at the limit $x \uparrow b$ (cf. Proposition 3.7).

Lemma 4.9. *For $-\infty < \kappa \leq 0$, consider the modified differential expression*

$$\ell^{\langle\kappa\rangle} = -\frac{1}{r^{\langle\kappa\rangle}} \frac{d}{dx}\Bigl(p^{\langle\kappa\rangle} \frac{d}{dx}\Bigr), \qquad x \in (a,b)$$

where $p^{\langle\kappa\rangle} = w_\kappa^2 \cdot p$ and $r^{\langle\kappa\rangle} = w_\kappa^2 \cdot r$. Then Assumption MP also holds for $\ell^{\langle\kappa\rangle}$, and the function

$$w_\lambda^{\langle\kappa\rangle}(x) := \frac{w_{\kappa+\lambda}(x)}{w_\kappa(x)}$$

is, for each $\lambda \in \mathbb{C}$, the unique solution of $\ell^{\langle\kappa\rangle}(w) = \lambda w$, $w(a) = 1$ and $(p^{\langle\kappa\rangle} w')(a) = 0$. Moreover, the spectral measure associated with $\ell^{\langle\kappa\rangle}$ is given by

$$\boldsymbol{\rho}_{\mathcal{L}}^{\langle\kappa\rangle}(\lambda_1, \lambda_2] = \boldsymbol{\rho}_{\mathcal{L}}(\lambda_1 + \kappa, \lambda_2 + \kappa] \qquad (-\infty < \lambda_1 \leq \lambda_2 < \infty).$$

Proof. Fix $\kappa < 0$. The functions A and $A^{\langle\kappa\rangle}$ associated to the operators ℓ and $\ell^{\langle\kappa\rangle}$ respectively (cf. (2.6)) are connected by $A^{\langle\kappa\rangle} = \widetilde{w}_\kappa^2 \cdot A$, where $\widetilde{w}_\kappa(\xi) = w_\kappa(\gamma^{-1}(\xi))$. Since w_κ is increasing and unbounded (this follows from classical results on the solutions of Sturm-Liouville type equations, e.g. [15, Sections 5.13–5.14]), $A^{\langle\kappa\rangle}$ is increasing. Letting $\eta^{\langle\kappa\rangle} := \eta + \frac{\widetilde{w}'_\kappa}{\widetilde{w}_\kappa}$ with η satisfying the conditions of Assumption MP, it is easily seen that $\eta^{\langle\kappa\rangle}$ satisfies the conditions corresponding to $A^{\langle\kappa\rangle}$, hence Assumption MP holds for $\ell^{\langle\kappa\rangle}$. A simple computation gives

$$\begin{aligned} -\frac{1}{r^{\langle\kappa\rangle}}\Big[p^{\langle\kappa\rangle}\Big(\frac{w_{\kappa+\lambda}}{w_\kappa}\Big)'\Big]' &= -\frac{1}{w_\kappa^2 \cdot r}\big[p\, w'_{\kappa+\lambda} w_\kappa - p\, w_{\kappa+\lambda} w'_\kappa\big]' \\ &= -\frac{1}{w_\kappa^2}\big[\ell(w_{\kappa+\lambda})\, w_\kappa - w_{\kappa+\lambda}\, \ell(w_\kappa)\big] = \lambda \frac{w_{\kappa+\lambda}(x)}{w_\kappa(x)} \end{aligned}$$

so that $\ell^{\langle\kappa\rangle}(w_\lambda^{\langle\kappa\rangle}) = \lambda w_\lambda^{\langle\kappa\rangle}$. The boundary conditions at a are also straightforwardly checked. To prove the last assertion, notice that the Fourier transforms associated with ℓ and $\ell^{\langle\kappa\rangle}$ are related through the identity

$$\Big(\mathcal{F}^{\langle\kappa\rangle}\frac{h}{w_\kappa}\Big)(\lambda) = (\mathcal{F}h)(\kappa + \lambda), \qquad h \in L_2(r)$$

and therefore

$$\|(\mathcal{F}h)\|_{L_2(\mathbb{R},\boldsymbol{\rho}_{\mathcal{L}})} = \|h\|_{L_2(r)} = \Big\|\frac{h}{w_\kappa}\Big\|_{L_2(r^{\langle\kappa\rangle})} = \big\|(\mathcal{F}h)(\kappa + \cdot)\big\|_{L_2(\mathbb{R},\boldsymbol{\rho}_{\mathcal{L}}^{\langle\kappa\rangle})}.$$

Recalling the uniqueness of the spectral measure for which the isometric property in Proposition 2.5 holds, we deduce that $\boldsymbol{\rho}_{\mathcal{L}}^{\langle\kappa\rangle}(\lambda_1, \lambda_2] = \boldsymbol{\rho}_{\mathcal{L}}(\lambda_1 + \kappa, \lambda_2 + \kappa]$. ☐

We are finally ready to prove the product formula for the $\mathcal{L}$-transform kernels $\{w_\lambda(\cdot)\}$. Recall that, by definition [2, §30], the complex measures μ_n converge weakly (respectively, vaguely) to $\mu \in \mathcal{M}_{\mathbb{C}}[a,b)$ if $\lim_n \int_{[a,b)} g(\xi)\mu_n(d\xi) = \int_{[a,b)} g(\xi)\mu(d\xi)$ for all $g \in \mathrm{C_b}[a,b)$ (respectively, for all $g \in \mathrm{C_0}[a,b)$). We use the notations $\xrightarrow{w}$ and $\xrightarrow{v}$ to denote weak and vague convergence, respectively.

Theorem 4.10 (Product formula for w_λ)**.** *For $x, y \in (a,b)$ and $t > 0$, let $\boldsymbol{\nu}_{t,x,y} \in \mathcal{P}[a,b)$ be the measure defined by $\boldsymbol{\nu}_{t,x,y}(d\xi) = q_t(x,y,\xi)\, r(\xi) d\xi$. Then for each $x, y \in (a,b)$ there exists a measure $\boldsymbol{\nu}_{x,y} \in \mathcal{P}[a,b)$ such that $\boldsymbol{\nu}_{t,x,y} \xrightarrow{w} \boldsymbol{\nu}_{x,y}$ as $t \downarrow 0$. Moreover, the product $w_\lambda(x)\, w_\lambda(y)$ admits the integral representation*

$$w_\lambda(x)\, w_\lambda(y) = \int_{[a,b)} w_\lambda(\xi)\, \boldsymbol{\nu}_{x,y}(d\xi), \qquad x, y \in (a,b),\ \lambda \in \mathbb{C}. \tag{4.10}$$

In particular, Theorem 4.1 holds.

Proof. Let $\{t_n\}_{n\in\mathbb{N}}$ be an arbitrary decreasing sequence with $t_n \downarrow 0$. Since any sequence of probability measures contains a vaguely convergent subsequence [2, p. 213], there exists a subsequence $\{t_{n_k}\}$ and a measure $\boldsymbol{\nu}_{x,y} \in \mathcal{M}_{\mathrm{b},+}[a,b)$ such that $\boldsymbol{\nu}_{t_{n_k},x,y} \xrightarrow{v} \boldsymbol{\nu}_{x,y}$ as $k \to \infty$. Let us show that all such subsequences $\{\boldsymbol{\nu}_{t_{n_k},x,y}\}$ have the same vague limit. Suppose that t_k^1, t_k^2 are two different sequences with $t_k^j \downarrow 0$ and that $\boldsymbol{\nu}_{t_k^j,x,y} \xrightarrow{v} \boldsymbol{\nu}_{x,y}^j$ as $k \to \infty$ ($j = 1,2$). For $g \in \mathrm{C}_{\mathrm{c},0}^4$ we have

$$\int_{[a,b)} g(\xi)\, \boldsymbol{\nu}_{x,y}^j(d\xi) = \lim_{k\to\infty} \int_{[a,b)} g(\xi)\, \boldsymbol{\nu}_{t_k^j,x,y}(d\xi)$$

$$= \lim_{k\to\infty} \int_{[\sigma^2,\infty)} e^{-t_k^j \lambda}\, w_\lambda(x)\, w_\lambda(y)\, (\mathcal{F}g)(\lambda)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda)$$
$$= \int_{[\sigma^2,\infty)} w_\lambda(x)\, w_\lambda(y)\, (\mathcal{F}g)(\lambda)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda)$$

(the second equality was justified in the proof of Lemma 4.7, and dominated convergence yields the last equality). In particular, $\int_{[a,b)} g(\xi)\, \boldsymbol{\nu}^1_{x,y}(\xi) = \int_{[a,b)} g(\xi)\, \boldsymbol{\nu}^2_{x,y}(\xi)$ for all $g \in \mathrm{C}^4_{\mathrm{c},0}$, and this implies that $\boldsymbol{\nu}^1_{x,y} = \boldsymbol{\nu}^2_{x,y}$. Since all subsequences have the same vague limit, we conclude that $\boldsymbol{\nu}_{t,x,y} \xrightarrow{v} \boldsymbol{\nu}_{x,y}$ as $t \downarrow 0$.

Suppose first that $\lim_{x\uparrow b} p(x)r(x) = \infty$. Then, by Proposition 3.7, we have $w_\lambda \in \mathrm{C}_0[a,b)$ for $\lambda > 0$. Accordingly, by taking the limit as $t \downarrow 0$ of both sides of (4.9) we deduce that the product formula (4.10) holds for all $\lambda > 0$.

To prove that (4.10) is valid in the general case, let $\kappa < 0$ be arbitrary. We know that the operator $\ell_{\langle\kappa\rangle}$ from Lemma 4.9 satisfies Assumption MP and $\lim_{x\uparrow b} p^{\langle\kappa\rangle}(x) r^{\langle\kappa\rangle}(x) = \infty$. From the previous part of the proof,

$$w_\lambda^{\langle\kappa\rangle}(x)\, w_\lambda^{\langle\kappa\rangle}(y) = \int_a^b w_\lambda^{\langle\kappa\rangle}(\xi)\, \boldsymbol{\nu}^{\langle\kappa\rangle}_{x,y}(d\xi), \qquad x, y \in (a,b),\ \lambda > 0 \tag{4.11}$$

with $\boldsymbol{\nu}^{\langle\kappa\rangle}_{x,y}$ constructed as before. We easily verify that $q_t^{\langle\kappa\rangle}(x,y,\xi) r^{\langle\kappa\rangle}(\xi) = \frac{e^{t\kappa} w_\kappa(\xi)}{w_\kappa(x) w_\kappa(y)} q_t(x,y,\xi) r(\xi)$ and, consequently, $\boldsymbol{\nu}^{\langle\kappa\rangle}_{x,y}(d\xi) = \frac{w_\kappa(\xi)}{w_\kappa(x) w_\kappa(y)} \boldsymbol{\nu}_{x,y}(d\xi)$. It thus follows from (4.11) that

$$w_{\kappa+\lambda}(x)\, w_{\kappa+\lambda}(y) = \int_a^b w_{\kappa+\lambda}(\xi)\, \boldsymbol{\nu}_{x,y}(d\xi), \qquad x, y \in (a,b),\ \lambda > 0,$$

where $\kappa < 0$ is arbitrary; hence (4.10) holds for all $\lambda \in \mathbb{R}$. If we then set $\lambda = \tau^2 + \sigma^2$ in (4.10), we straightforwardly verify that both sides are entire functions of τ (for the right hand side, this follows from the Laplace-type representation (3.1) and the fact that the integral converges for all $\lambda < 0$), so by analytic continuation the product formula holds for all $\lambda \in \mathbb{C}$.

Given that $w_0(x) \equiv 1$, setting $\lambda = 0$ in (4.10) shows that $\boldsymbol{\nu}_{x,y} \in \mathcal{P}[a,b)$; consequently, the measures $\boldsymbol{\nu}_{t,x,y}$ converge to $\boldsymbol{\nu}_{x,y}$ in the weak topology (cf. [2, Theorem 30.8]). Clearly, the product formula (4.10) can be extended to $x, y \in [a,b)$ by setting $\boldsymbol{\nu}_{x,a} := \delta_x$ and $\boldsymbol{\nu}_{a,y} := \delta_y$, hence Theorem 4.1 holds. □

It is worth commenting that the reasoning used in this proof also allows us to justify that the time-shifted product formula (4.9) is valid for all $\lambda \in \mathbb{C}$.

As shown in the proof above, the measure $\boldsymbol{\nu}_{x,y}$ of the product formula (4.10) is characterized by the identity

$$\int_{[a,b)} h(\xi)\, \boldsymbol{\nu}_{x,y}(d\xi) = \int_{[\sigma^2,\infty)} w_\lambda(x)\, w_\lambda(y)\, (\mathcal{F}h)(\lambda)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda), \qquad h \in \mathrm{C}^4_{\mathrm{c},0}. \tag{4.12}$$

Furthermore, the relation between this measure and the measure $\boldsymbol{\nu}_{t,x,y}(d\xi) = q_t(x,y,\xi)\, r(\xi) d\xi$ of the time-shifted product formula (4.9) can be written explicitly:

Corollary 4.11. *The measure $\boldsymbol{\nu}_{t,x,y}$ can be written in terms of the measure $\boldsymbol{\nu}_{x,y}$ and the transition kernel $p(t,x,y)$ of the Feller semigroup generated by the Sturm-Liouville operator ℓ as*

$$\boldsymbol{\nu}_{t,x,y}(d\xi) = \int_a^b \boldsymbol{\nu}_{z,y}(d\xi)\, p(t,x,z)\, r(z) dz \qquad \big(t > 0,\ x, y \in (a,b)\big).$$

Proof. Recalling (2.15) and the proof of the previous proposition, we find that for $g \in \mathrm{C}^4_{\mathrm{c},0}$ we have

$$\begin{aligned}
\int_a^b \int_{[a,b)} g(\xi) \boldsymbol{\nu}_{z,y}(d\xi)\, p(t,x,z)\, r(z) dz &= \int_a^b \int_{[\sigma^2,\infty)} w_\lambda(z)\, w_\lambda(y)\, (\mathcal{F}g)(\lambda)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda)\, p(t,x,z)\, r(z) dz \\
&= \int_{[\sigma^2,\infty)} e^{-t\lambda}\, w_\lambda(x)\, w_\lambda(y)\, (\mathcal{F}g)(\lambda)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) \\
&= \int_a^b g(\xi)\, q_t(x,y,\xi)\, r(\xi) d\xi,
\end{aligned}$$

hence the measures $\boldsymbol{\nu}_{t,x,y}(d\xi)$ and $\int_a^b \boldsymbol{\nu}_{z,y}(d\xi)\, p(t,x,z)\, r(z) dz$ are the same. □

5 Generalized convolutions and hypergroups

5.1 The convolution measure algebra

As usual in the theory of generalized convolutions, we define the convolution $* : \mathcal{M}_{\mathbb{C}}[a,b) \times \mathcal{M}_{\mathbb{C}}[a,b) \longrightarrow \mathcal{M}_{\mathbb{C}}[a,b)$ as the natural extension of the mapping $(x,y) \mapsto \delta_x * \delta_y := \boldsymbol{\nu}_{x,y}$, where $\boldsymbol{\nu}_{x,y}$ is the measure of the product formula (4.10):

Definition 5.1. Let $\mu, \nu \in \mathcal{M}_{\mathbb{C}}[a,b)$. The complex measure

$$(\mu * \nu)(d\xi) = \int_{[a,b)} \int_{[a,b)} \boldsymbol{\nu}_{x,y}(d\xi)\, \mu(dx)\, \nu(dy)$$

is called the $\mathcal{L}$-*convolution* of the measures μ and ν.

The key tool for studying the properties of the $\mathcal{L}$-convolution is the extension of the $\mathcal{L}$-transform (2.8) to complex measures, defined by

$$\widehat{\mu}(\lambda) := \int_{[a,b)} w_\lambda(x)\, \mu(dx), \qquad \lambda \geq 0.$$

It is immediate from Lemmas 2.1 and 2.3 that $|\widehat{\mu}(\lambda)| \leq \widehat{\mu}(0) = \|\mu\|$ for all $\mu \in \mathcal{M}_+[a,b)$. In addition, this transformation has various properties which resemble those of the Fourier transform of complex measures:

Proposition 5.2. *The $\mathcal{L}$-transform $\widehat{\mu}$ of $\mu \in \mathcal{M}_{\mathbb{C}}[a,b)$ has the following properties:*

(i) *$\widehat{\mu}$ is continuous on $[0,\infty)$. Moreover, if a family of measures $\{\mu_j\} \subset \mathcal{M}_{\mathbb{C}}[a,b)$ is tight and uniformly bounded, then $\{\widehat{\mu_j}\}$ is equicontinuous on $[0,\infty)$.*

(ii) *Each measure $\mu \in \mathcal{M}_{\mathbb{C}}[a,b)$ is uniquely determined by $\widehat{\mu}|_{[\sigma^2,\infty)}$.*

(iii) *If $\{\mu_n\}$ is a sequence of measures belonging to $\mathcal{M}_+[a,b)$, $\mu \in \mathcal{M}_+[a,b)$, and $\mu_n \xrightarrow{w} \mu$, then*

$$\widehat{\mu_n} \xrightarrow[n\to\infty]{} \widehat{\mu} \qquad \textit{uniformly for } \lambda \textit{ in compact sets.}$$

(iv) *Suppose that $\lim_{x\uparrow b} w_\lambda(x) = 0$ for all $\lambda > 0$. If $\{\mu_n\}$ is a sequence of measures belonging to $\mathcal{M}_+[a,b)$ whose $\mathcal{L}$-transforms are such that*

$$\widehat{\mu_n}(\lambda) \xrightarrow[n\to\infty]{} f(\lambda) \qquad \textit{pointwise in } \lambda \geq 0 \tag{5.1}$$

for some real-valued function f which is continuous at a neighborhood of zero, then $\mu_n \xrightarrow{w} \mu$ for some measure $\mu \in \mathcal{M}_+[a,b)$ such that $\widehat{\mu} \equiv f$.

Proof. ***(i)*** Let us prove the second statement, which implies the first. Set $C = \sup_j \|\mu_j\|$. Fix $\lambda_0 \geq 0$ and $\varepsilon > 0$. By the tightness assumption, we can choose $\beta \in (a,b)$ such that $|\mu_j|(\beta,b) < \varepsilon$ for all j. Since the family $\{w_{(\cdot)}(x)\}_{x\in(a,\beta]}$ is equicontinuous on $[0,\infty)$ (this follows easily from the power series representation of $w_{(\cdot)}(x)$, cf. proof of Lemma 2.1), we can choose $\delta > 0$ such that

$$|\lambda - \lambda_0| < \delta \quad \Longrightarrow \quad |w_\lambda(x) - w_{\lambda_0}(x)| < \varepsilon \;\text{ for all } a < x \leq \beta.$$

Consequently,

$$\begin{aligned}\bigl|\widehat{\mu_j}(\lambda) - \widehat{\mu_j}(\lambda_0)\bigr| &= \biggl|\int_{(a,b)} \bigl(w_\lambda(x) - w_{\lambda_0}(x)\bigr)\mu_j(dx)\biggr| \\ &\leq \int_{(\beta,b)} \bigl|w_\lambda(x) - w_{\lambda_0}(x)\bigr||\mu_j|(dx) + \int_{(a,\beta]} \bigl|w_\lambda(x) - w_{\lambda_0}(x)\bigr||\mu_j|(dx)\end{aligned}$$

$$\leq 2\varepsilon + C\varepsilon = (C+2)\varepsilon$$

for all j, provided that $|\lambda - \lambda_0| < \delta$, which means that $\{\widehat{\mu_j}\}$ is equicontinuous at λ_0.

(ii) Let $\mu \in \mathcal{M}_{\mathbb{C}}[a,b)$ be such that $\widehat{\mu}(\lambda) = 0$ for all $\lambda \geq 0$. We need to show that μ is the zero measure. For each $h \in \mathrm{C}^4_{\mathrm{c},0}$, by (4.12) we have

$$\int_{[a,b)} h \, d(\delta_x * \mu) = \int_{[\sigma^2,\infty)} (\mathcal{F}h)(\lambda) \, w_\lambda(x) \, \widehat{\mu}(\lambda) \, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) \; = \; 0.$$

Since $h \in \mathrm{C}^4_{\mathrm{c},0}$, Theorem 4.2 assures that $\lim_{x\downarrow a} \int_{[a,b)} h \, d\boldsymbol{\nu}_{x,y} = h(y)$ for $y \geq 0$; therefore, by dominated convergence,

$$0 = \lim_{x\downarrow a} \int_{[a,b)} h \, d(\delta_x * \mu) = \lim_{x\downarrow a} \int_{[a,b)} \biggl(\int_{[a,b)} h \, d\boldsymbol{\nu}_{x,y} \biggr) \mu(dy) = \int_{[a,b)} h(y) \, \mu(dy)$$

This shows that $\int_{[a,b)} h(y) \, \mu(dy) = 0$ for all $h \in \mathrm{C}^4_{\mathrm{c},0}$ and, consequently, μ is the zero measure.

(iii) Since $w_\lambda(\cdot)$ is continuous and bounded, the pointwise convergence $\widehat{\mu_n}(\lambda) \to \widehat{\mu}(\lambda)$ follows from the definition of weak convergence of measures. By Prokhorov's theorem [5, Theorem 8.6.2], $\{\mu_n\}$ is tight and uniformly bounded, thus (by part (i)) $\{\widehat{\mu_n}\}$ is equicontinuous on $[0,\infty)$. Invoking [17, Lemma 15.22], we conclude that the convergence $\widehat{\mu_n} \to \widehat{\mu}$ is uniform for λ in compact sets.

(iv) We only need to show that the sequence $\{\mu_n\}$ is tight and uniformly bounded. (Recall that a family $\{\mu_j\} \subset \mathcal{M}_{\mathbb{C}}[a,b)$ is said to be uniformly bounded if $\sup_j \|\mu_j\| < \infty$, and $\{\mu_j\}$ is said to be tight if for each $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset [a,b)$ such that $\sup_j |\mu_j|([a,b) \setminus K_\varepsilon) < \varepsilon$; these definitions are taken from [5].) Indeed, if $\{\mu_n\}$ is tight and uniformly bounded, then Prokhorov's theorem yields that for any subsequence $\{\mu_{n_k}\}$ there exists a further subsequence $\{\mu_{n_{k_j}}\}$ and a measure $\mu \in \mathcal{M}_+[a,b)$ such that $\mu_{n_{k_j}} \overset{w}{\longrightarrow} \mu$. Then, due to part (iii) and to (5.1), we have $\widehat{\mu}(\lambda) = f(\lambda)$ for all $\lambda \geq 0$, which implies (by part (ii)) that all such subsequences have the same weak limit; consequently, the sequence μ_n itself converges weakly to μ.

The uniform boundedness of $\{\mu_n\}$ follows immediately from the fact that $\widehat{\mu_n}(0) = \mu_n[a,b)$ converges. To prove the tightness, take $\varepsilon > 0$. Since f is continuous at a neighborhood of zero, we have $\frac{1}{\delta}\int_0^{2\delta} \bigl(f(0) - f(\lambda)\bigr) d\lambda \longrightarrow 0$ as $\delta \downarrow 0$; therefore, we can choose $\delta > 0$ such that

$$\biggl| \frac{1}{\delta} \int_0^{2\delta} \bigl(f(0) - f(\lambda)\bigr) d\lambda \biggr| < \varepsilon.$$

Next we observe that, due to the assumption that $\lim_{x\uparrow b} w_\lambda(x) = 0$ for all $\lambda > 0$, we have $\int_0^{2\delta} \bigl(1 - w_\lambda(x)\bigr) d\lambda \longrightarrow 2\delta$ as $x \uparrow b$, meaning that we can pick $\beta \in (a,b)$ such that

$$\int_0^{2\delta} \bigl(1 - w_\lambda(x)\bigr) d\lambda \geq \delta \qquad \text{for all } \beta < x < b.$$

By our choice of β and Fubini's theorem,

$$\begin{aligned}
\mu_n\bigl[\beta, b) &= \frac{1}{\delta} \int_{[\beta,b)} \delta \, \mu_n(dx) \\
&\leq \frac{1}{\delta} \int_{[\beta,b)} \int_0^{2\delta} \bigl(1 - w_\lambda(x)\bigr) d\lambda \, \mu_n(dx) \\
&\leq \frac{1}{\delta} \int_{[a,b)} \int_0^{2\delta} \bigl(1 - w_\lambda(x)\bigr) d\lambda \, \mu_n(dx) \\
&= \frac{1}{\delta} \int_0^{2\delta} \bigl(\widehat{\mu_n}(0) - \widehat{\mu_n}(\lambda)\bigr) d\lambda.
\end{aligned}$$

Hence, using the dominated convergence theorem,

$$\limsup_{n\to\infty} \mu_n[\beta, b) \leq \frac{1}{\delta} \limsup_{n\to\infty} \int_0^{2\delta} \bigl(\widehat{\mu_n}(0) - \widehat{\mu_n}(\lambda)\bigr) d\lambda$$

$$= \frac{1}{\delta} \int_0^{2\delta} \lim_{n\to\infty} \big(\widehat{\mu_n}(0) - \widehat{\mu_n}(\lambda)\big) d\lambda = \frac{1}{\delta} \int_0^{2\delta} \big(f(0) - f(\lambda)\big) d\lambda < \varepsilon$$

due to the choice of δ. Since ε is arbitrary, we conclude that $\{\mu_n\}$ is tight, as desired. ☐

An unsurprising consequence of the construction of the $\mathcal{L}$-convolution is that it is trivialized by the $\mathcal{L}$-transform of measures. Indeed:

Proposition 5.3. *Let $\mu, \nu, \pi \in \mathcal{M}_{\mathbb{C}}[a, b)$. We have $\pi = \mu * \nu$ if and only if*

$$\widehat{\pi}(\lambda) = \widehat{\mu}(\lambda)\,\widehat{\nu}(\lambda) \qquad \textit{for all } \lambda \geq 0.$$

Proof. Using the product formula (4.10), we compute

$$\begin{aligned} \widehat{\mu * \nu}(\lambda) &= \int_{[a,b)} w_\lambda(x)\,(\mu * \nu)(dx) \\ &= \int_{[a,b)} \int_{[a,b)} \int_{[a,b)} w_\lambda(\xi)\,\boldsymbol{\nu}_{x,y}(d\xi)\,\mu(dx)\nu(dy) \\ &= \int_{[a,b)} \int_{[a,b)} w_\lambda(x)\,w_\lambda(y)\,\mu(dx)\nu(dy) \;=\; \widehat{\mu}(\lambda)\,\widehat{\nu}(\lambda), \qquad \lambda \geq 0. \end{aligned}$$

This proves the "only if" part, and the converse follows from the uniqueness property in Proposition 5.2(ii). ☐

The next result summarizes the properties of the measure algebra determined by the $\mathcal{L}$-convolution:

Proposition 5.4. *The space $(\mathcal{M}_{\mathbb{C}}[a, b), *)$, equipped with the total variation norm, is a commutative Banach algebra over $\mathbb{C}$ whose identity element is the Dirac measure δ_a. The subset $\mathcal{P}[a, b)$ is closed under the $\mathcal{L}$-convolution. Moreover, the map $(\mu, \nu) \mapsto \mu * \nu$ is continuous (in the weak topology) from $\mathcal{M}_{\mathbb{C}}[a, b) \times \mathcal{M}_{\mathbb{C}}[a, b)$ to $\mathcal{M}_{\mathbb{C}}[a, b)$.*

Proof. Since $\widehat{\mu * \nu} = \widehat{\mu} \cdot \widehat{\nu}$ (Proposition 5.3), the commutativity, associativity and bilinearity of the $\mathcal{L}$-convolution follow at once from the uniqueness property of the $\mathcal{L}$-transform (Proposition 5.2(ii)). One can verify directly from the definition of the $\mathcal{L}$-convolution that the submultiplicativity property $\|\mu * \nu\| \leq \|\mu\| \cdot \|\nu\|$ holds, and that equality holds whenever $\mu, \nu \in \mathcal{M}_+[a, b)$; it is also clear that the convolution of positive measures is a positive measure. We conclude that the Banach algebra property holds and that $\mathcal{P}[a, b)$ is closed under convolution.

If $\lim_{x\uparrow b} w_\lambda(x) = 0$ for all $\lambda > 0$, the identity $\widehat{\boldsymbol{\nu}_{x,y}}(\lambda) = w_\lambda(x) w_\lambda(y)$ implies (by Proposition 5.2(iv)) that $(x, y) \mapsto \boldsymbol{\nu}_{x,y}$ is continuous in the weak topology. If the functions $w_\lambda(x)$ do not vanish at the limit $x \uparrow b$, let $\kappa < 0$ be arbitrary and let $h \in \mathrm{C_b}[a, b)$. Since w_κ is increasing and unbounded, $\frac{h}{w_\kappa} \in \mathrm{C_0}[a, b)$. By Lemma 4.9, the map $(x, y) \mapsto \boldsymbol{\nu}_{x,y}^{\langle\kappa\rangle}$ (where $\boldsymbol{\nu}_{x,y}^{\langle\kappa\rangle}$ is the measure defined in the proof of Theorem 4.10) is continuous, hence

$$(x, y) \;\longmapsto\; \int_{[a,b)} \frac{h(\xi)}{w_\kappa(\xi)} \boldsymbol{\nu}_{x,y}^{\langle\kappa\rangle}(d\xi) = \frac{1}{w_\kappa(x) w_\kappa(y)} \int_{[a,b)} h(\xi)\,\boldsymbol{\nu}_{x,y}(d\xi)$$

is continuous. This shows that $(x, y) \mapsto \int_{[a,b)} h(\xi)\,\boldsymbol{\nu}_{x,y}(d\xi)$ is continuous for all $h \in \mathrm{C_b}[a, b)$ and therefore $(x, y) \mapsto \boldsymbol{\nu}_{x,y}$ is continuous in the weak topology. Finally, for $h \in \mathrm{C_b}[a, b)$ and $\mu_n, \nu_n \in \mathcal{M}_{\mathbb{C}}[a, b)$ with $\mu_n \xrightarrow{w} \mu$ and $\nu_n \xrightarrow{w} \nu$ we have

$$\begin{aligned} \lim_n \int_{[a,b)} h(\xi)(\mu_n * \nu_n)(d\xi) &= \lim_n \int_{[a,b)} \int_{[a,b)} \biggl(\int_{[a,b)} h\; d\boldsymbol{\nu}_{x,y}\biggr) \mu_n(dx)\nu_n(dy) \\ &= \int_{[a,b)} \int_{[a,b)} \biggl(\int_{[a,b)} h\; d\boldsymbol{\nu}_{x,y}\biggr) \mu(dx)\nu(dy) \\ &= \int_{[a,b)} h(\xi)(\mu * \nu)(d\xi) \end{aligned}$$

due to the continuity of the function in parenthesis; this proves that $(\mu, \nu) \mapsto \mu * \nu$ is continuous. ☐

5.2 The nondegenerate case: Sturm-Liouville hypergroups

The goal of this section is to determine sufficient conditions in order that the $\mathcal{L}$-convolution defines a hypergroup structure on the interval $[a,b)$.

Let us recall the usual definition of a hypergroup, which was introduced in [16] (see also [4]). Let K be a locally compact space and $*$ a bilinear operator on $\mathcal{M}_{\mathbb{C}}(K)$. The pair $(K,*)$ is said to be a *hypergroup* if the following axioms are satisfied:

H1. If $\mu,\nu \in \mathcal{P}(K)$, then $\mu * \nu \in \mathcal{P}(K)$;

H2. $\mu * (\nu * \pi) = (\mu * \nu) * \pi$ for all $\mu,\nu,\pi \in \mathcal{M}_{\mathbb{C}}(K)$;

H3. The map $(\mu,\nu) \mapsto \mu * \nu$ is continuous (in the weak topology) from $\mathcal{M}_{\mathbb{C}}(K) \times \mathcal{M}_{\mathbb{C}}(K)$ to $\mathcal{M}_{\mathbb{C}}(K)$;

H4. There exists an element $\mathrm{e} \in K$ such that $\delta_{\mathrm{e}} * \mu = \mu * \delta_{\mathrm{e}} = \mu$ for all $\mu \in \mathcal{M}_{\mathbb{C}}(K)$;

H5. There exists a homeomorphism (called *involution*) $x \mapsto \check{x}$ of K onto itself such that $(\check{x})\check{} = x$ and $\mathrm{e} \in \operatorname{supp}(\delta_x * \delta_y)$ if and only if $y = \check{x}$;

H6. $(\mu * \nu)\check{} = \check{\nu} * \check{\mu}$, where $\check{\mu}$ is defined via $\int f(x)\check{\mu}(dx) = \int f(\check{x})\mu(dx)$;

H7. $(x,y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ is continuous from $K \times K$ into the space of compact subsets of K (endowed with the Michael topology, see [16]).

We saw in Proposition 5.4 that $\mathcal{L}$-convolution satisfies the axioms H1, H2, H3, H4 and H6 (with $K = [a,b)$ and $\mathrm{e} = a$ as the identity element; H6 holds for the identity involution $\check{x} = x$). In order to verify conditions H5 and H7, one needs to determine the support of $\boldsymbol{\nu}_{x,y} = \delta_x * \delta_y$. A crucial tool for determining $\operatorname{supp}(\boldsymbol{\nu}_{x,y})$ is the integral identity which we now state:

Lemma 5.5. *Let $\boldsymbol{\ell}^B$ be the differential expression $\boldsymbol{\ell}^B v := -v'' - \boldsymbol{\phi}_\eta v' + \boldsymbol{\psi}_\eta v$. For $\gamma(a) < c \leq y \leq x$, consider the triangle $\Delta_{c,x,y} := \{(\xi,\zeta) \in \mathbb{R}^2 \mid \zeta \geq c,\, \xi + \zeta \leq x+y,\, \xi - \zeta \geq x - y\}$, and let $v \in \mathrm{C}^2(\Delta_{c,x,y})$. Write $B(x) := \exp(\frac{1}{2}\int_\beta^x \eta(\xi)d\xi)$ (with $\beta > \gamma(a)$ arbitrary) and $A_B(x) = \frac{A(x)}{B(x)^2}$. Then the following integral equation holds:*

$$A_B(x)A_B(y)\, v(x,y) = H + I_0 + I_1 + I_2 + I_3 - I_4 \tag{5.2}$$

where

$$H := \tfrac{1}{2}A_B(c)\bigl[A_B(x-y+c)\, v(x-y+c,c) + A_B(x+y-c)\, v(x+y-c,c)\bigr] \tag{5.3}$$

$$I_0 := \tfrac{1}{2}A_B(c)\int_{x-y+c}^{x+y-c} A_B(s)(\partial_y v)(s,c)\, ds \tag{5.4}$$

$$I_1 := \tfrac{1}{2}\int_c^y A_B(s)A_B(x-y+s)\bigl[\boldsymbol{\phi}_\eta(s) + \boldsymbol{\phi}_\eta(x-y+s)\bigr]v(x-y+s,s)\, ds \tag{5.5}$$

$$I_2 := \tfrac{1}{2}\int_c^y A_B(s)A_B(x+y-s)\bigl[\boldsymbol{\phi}_\eta(s) - \boldsymbol{\phi}_\eta(x+y-s)\bigr]v(x+y-s,s)\, ds \tag{5.6}$$

$$I_3 := \tfrac{1}{2}\int_{\Delta_{c,x,y}} A_B(\xi)A_B(\zeta)\bigl[\boldsymbol{\psi}_\eta(\zeta) - \boldsymbol{\psi}_\eta(\xi)\bigr]v(\xi,\zeta)\, d\xi d\zeta \tag{5.7}$$

$$I_4 := \tfrac{1}{2}\int_{\Delta_{c,x,y}} A_B(\xi)A_B(\zeta)\,(\boldsymbol{\ell}_\zeta^B v - \boldsymbol{\ell}_\xi^B v)(\xi,\zeta)\, d\xi d\zeta. \tag{5.8}$$

Proof. See Appendix A.3. ∎

We note that the integral identity (5.2) is valid for any Sturm-Liouville operator ℓ satisfying Assumption MP. The proof of the positivity of the Cauchy problem (Proposition 4.4 and Corollary 4.5) also relies on this identity, see Appendix A.4.

A detailed study of $\operatorname{supp}(\boldsymbol{\nu}_{x,y})$ was carried out by Zeuner in [31]. The next proposition shows that the results of Zeuner can be applied to the $\mathcal{L}$-convolution, provided that the differential operator (2.1) has coefficients $p = r = A$ defined on $(0,\infty)$, and there exists $\eta \in \mathrm{C}^1[0,\infty)$ satisfying the conditions given in Assumption MP.

Proposition 5.6. *Suppose that*

$$\ell = -\frac{1}{A}\frac{d}{dx}\Big(A\frac{d}{dx}\Big), \qquad x \in (0,\infty)$$

where the function $p = r = A$ is increasing, and that there exists $\eta \in \mathrm{C}^1[0,\infty)$ such that $\eta \geq 0$, $\boldsymbol{\phi}_\eta \geq 0$ and the functions $\boldsymbol{\phi}_\eta$, $\boldsymbol{\psi}_\eta$ are both decreasing on $(0,\infty)$; assume also (without further loss of generality) that $\boldsymbol{\phi}_\eta(x) \searrow 0$ as $x \to \infty$. Let $x_0 = \sup\{x \geq 0 \mid \boldsymbol{\psi}_\eta(x) = \boldsymbol{\psi}_\eta(0)\}$ and $x_1 = \inf\{x > 0 \mid \boldsymbol{\phi}_\eta(x) = 0\}$. Then:

(a) *If $x_0 = \infty$, $x_1 = 0$ and $\eta(0) = 0$ then $\mathrm{supp}(\delta_x * \delta_y) = \{|x-y|, x+y\}$ for all $x, y \geq 0$.*

(b) *If $0 < x_0 < \infty$, $x_1 = 0$ and $\eta(0) = 0$ then*

$$\mathrm{supp}(\delta_x * \delta_y) = \begin{cases} \{|x-y|, x+y\}, & x+y \leq x_0 \\ \{|x-y|\} \cup [2x_0 - x - y, x+y], & x, y < x_0 < x+y \\ [|x-y|, x+y], & \max\{x,y\} \geq x_0. \end{cases}$$

(c) *If $x_0 = \infty$, $0 < x_1 < \infty$ and $\eta(0) = 0$ then*

$$\mathrm{supp}(\delta_x * \delta_y) = \begin{cases} [|x-y|, x+y], & \min\{x,y\} \leq 2x_1, \\ [|x-y|, 2x_1 + |x-y|] \cup [x+y-2x_1, x+y], & \min\{x,y\} > 2x_1. \end{cases}$$

(d) *If $0 < 3x_1 < x_0 < \infty$ and $\eta(0) = 0$ then*

$$\mathrm{supp}(\delta_x * \delta_y) = \begin{cases} [|x-y|, x+y], & \min\{x,y\} \leq 2x_1 \ \textit{or} \ \max\{x,y\} \geq x_0 - x_1, \\ [|x-y|, 2x_1 + |x-y|] \cup \\ \qquad \cup\, [x+y-2x_1, x+y], & \min\{x,y\} > 2x_1 \ \textit{and} \ \max\{x,y\} < x_0 - x_1. \end{cases}$$

(e) *If $x_0 \leq 3x_1$ or $\eta(0) > 0$ then $\mathrm{supp}(\delta_x * \delta_y) = [|x-y|, x+y]$ for all $x, y \geq 0$.*

Proof. Fix $z \geq 0$, and let $\{h_\varepsilon\} \subset \mathrm{C}^4_{\mathrm{c},0}$ be a family of functions such that

$$h_\varepsilon(\xi) > 0 \text{ for } z - \varepsilon < \xi < z + \varepsilon, \qquad h_\varepsilon(\xi) = 0 \text{ for } \xi \leq z - \varepsilon \text{ and } \xi \geq z + \varepsilon. \tag{5.9}$$

Observe that $z \in \mathrm{supp}(\delta_x * \delta_y)$ if and only if $\int_{[0,\infty)} h_\varepsilon \, d(\delta_x * \delta_y) > 0$ for all $\varepsilon > 0$. Now, we know from Theorem 4.2 and Corollary 4.5 that the function

$$f_{h_\varepsilon}(x,y) = \int_{[0,\infty)} h_\varepsilon \, d(\delta_x * \delta_y) = \int_{[\sigma^2,\infty)} w_\lambda(x)\, w_\lambda(y)\, (\mathcal{F} h_\varepsilon)(\lambda)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda)$$

(the second equality is due to (4.12)) is a nonnegative solution of the Cauchy problem (4.1) with $h \equiv h_\varepsilon$; writing $B(x) := \exp(\frac{1}{2}\int_0^x \eta(\xi) d\xi)$, it follows that $v_{h_\varepsilon}(x,y) = B(x)B(y) f_{h_\varepsilon}(x,y)$ is a solution of $\boldsymbol{\ell}_x^B v - \boldsymbol{\ell}_y^B v = 0$, $\boldsymbol{\ell}_x^B$ being the differential operator defined in Lemma 5.5. If we apply this lemma with $c > 0$ and then let $c \downarrow 0$, we deduce that the following integral equation holds:

$$A_B(x) A_B(y)\, v_{h_\varepsilon}(x,y) = H + I_0 + I_1 + I_2 + I_3 \tag{5.10}$$

where $H = \frac{1}{2}A(0)\big[\frac{A(x-y)}{B(x-y)}\, h_\varepsilon(x-y) + \frac{A(x+y)}{B(x+y)}\, h_\varepsilon(x+y)\big]$, $I_0 = \frac{\eta(0)}{4}\int_{x-y}^{x+y} \frac{A(s)}{B(s)} h_\varepsilon(s)\, ds$ and I_1, I_2, I_3 are given by (5.5)–(5.7) with $c = 0$ and $v = v_{h_\varepsilon}$. Since h_ε and f_{h_ε} are nonnegative, all the terms in the right-hand side of (5.10) are nonnegative; consequently, we have $z \in \mathrm{supp}(\delta_x * \delta_y)$ if and only if at least one of the terms in the right-hand side of (5.10) is strictly positive for all $\varepsilon > 0$. In order to ascertain whether this holds or not, one needs to perform a thorough analysis of the integrals I_0, I_1, I_2 and I_3. This has been done by Zeuner in [31, Proposition 3.9]; his results lead to the conclusion stated in the proposition. ∎

Theorem 5.7. *Let ℓ be a differential expression of the form (2.1). Suppose that $\gamma(a) > -\infty$ and that there exists $\eta \in \mathrm{C}^1[\gamma(a),\infty)$ satisfying the conditions given in Assumption MP. Then $\bigl([a,b),*\bigr)$ is a hypergroup.*

Proof. It was seen in Proposition 5.4 that the axioms H1, H2, H3, H4 and H6 hold for the $\mathcal{L}$-convolution (with $\check{x} = x$); we need to check that axioms H5 and H7 are also satisfied.

Assume first that ℓ satisfies the assumptions of Proposition 5.6. Then the explicit expressions for $\mathrm{supp}(\delta_x * \delta_y)$ show that (in each of the cases *(a)–(e)*) $\mathrm{supp}(\delta_x * \delta_y)$ depends continuously on (x,y) and contains $\mathrm{e} = 0$ if and only if $x = y$, hence axioms H5 and H7 hold. (Verifying the continuity is easy after noting that the topology in the space of compact subsets can be metrized by the Hausdorff metric, cf. [19, Subsection 4.1].)

Now, in the general case of an operator ℓ of the form (2.1), note that $\gamma(a) > -\infty$ means that $\sqrt{\frac{r(y)}{p(y)}}$ is integrable near a, so that we may assume that $\gamma(a) = 0$ (otherwise, replace the interior point c by the endpoint a in the definition of the function γ). By hypothesis, the transformed operator $\widetilde{\ell} = -\frac{1}{A}\frac{d}{d\xi}(A\frac{d}{d\xi})$ defined via (2.6) satisfies the assumptions of Proposition 5.6; by the above, the associated convolution, which we denote by $\widetilde{*}$, satisfies axioms H5 and H7. From the product formulas for the solutions $w_\lambda(x)$ and $\widetilde{w}_\lambda(\xi) = w_\lambda(\gamma^{-1}(\xi))$ we deduce that

$$\int_{[a,b)} w_\lambda \, d(\delta_x * \delta_y) = w_\lambda(x) w_\lambda(y) = \widetilde{w}_\lambda(\gamma(x))\widetilde{w}_\lambda(\gamma(y)) = \int_{[0,\infty)} w_\lambda(\gamma^{-1}(z))\bigl(\delta_{\gamma(x)} \,\widetilde{*}\, \delta_{\gamma(y)}\bigr)(dz)$$

and, consequently, $\delta_x * \delta_y = \gamma^{-1}(\delta_{\gamma(x)} \,\widetilde{*}\, \delta_{\gamma(y)})$ (the right hand side denoting the pushforward of the measure $\delta_{\gamma(x)} \,\widetilde{*}\, \delta_{\gamma(y)}$ under the map $\xi \mapsto \gamma^{-1}(\xi)$). In particular, $\mathrm{supp}(\delta_x * \delta_y) = \gamma^{-1}\bigl(\mathrm{supp}(\delta_{\gamma(x)} \,\widetilde{*}\, \delta_{\gamma(y)})\bigr)$; since γ and γ^{-1} are continuous, we immediately conclude that the convolution $*$ also satisfies H5 and H7. ☐

A *hypergroup isomorphism* between $(K_1,*)$ and $(K_2,\diamond)$ is an isomorphism between the Banach algebras $(\mathcal{M}_{\mathrm{b},\mathbb{C}}(K_1),*)$ and $(\mathcal{M}_{\mathrm{b},\mathbb{C}}(K_2),\diamond)$ which preserves involution and point measures. The proof of Theorem 5.7 shows that the hypergroups $\bigl([a,b),*\bigr)$ and $\bigl([0,\infty),\widetilde{*}\bigr)$ associated with the differential operators ℓ and $\widetilde{\ell}$ are isomorphic, the isomorphism being the pushforward map $\mu \in \mathcal{M}_{\mathbb{C}}[a,b) \longmapsto \gamma^{-1}(\mu) \in \mathcal{M}_{\mathbb{C}}[0,\infty)$.

Let us write $\mathrm{C}^\infty_{\mathrm{c,even}} := \{h : [0,\infty) \to \mathbb{C} \mid h \text{ is the restriction of an even } \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R})\text{-function}\}$. The next definition was introduced by Zeuner [30]:

Definition 5.8. A hypergroup $([0,\infty),*)$ is said to be a *Sturm-Liouville hypergroup* if there exists a function A on $[0,\infty)$ satisfying the condition

SL0 $A \in \mathrm{C}[0,\infty) \cap \mathrm{C}^1(0,\infty)$ and $A(x) > 0$ for $x > 0$

such that, for every function $h \in \mathrm{C}^\infty_{\mathrm{c,even}}$, the convolution

$$v_h(x,y) = \int_{[0,\infty)} h(\xi)(\delta_x * \delta_y)(d\xi) \tag{5.11}$$

belongs to $\mathrm{C}^2\bigl([0,\infty)^2\bigr)$ and satisfies $(\ell_x v_h)(x,y) = (\ell_y v_h)(x,y)$, $(\partial_y v_h)(x,0) = 0$ $(x > 0)$, where $\ell_x = -\frac{1}{A}\frac{\partial}{\partial x}(A(x)\frac{\partial}{\partial x})$.

A fundamental existence theorem for Sturm-Liouville hypergroups, which was proved by Zeuner [31, Theorem 3.11], states: *Suppose that A satisfies SL0 and is such that*

SL1 One of the following assertions holds:

- **SL1.1** $A(0) = 0$ and $\frac{A'(x)}{A(x)} = \frac{\alpha_0}{x} + \alpha_1(x)$ for x in a neighbourhood of 0, where $\alpha_0 > 0$ and $\alpha_1 \in \mathrm{C}^\infty(\mathbb{R})$ is an odd function;
- **SL1.2** $A(0) > 0$ and $A \in \mathrm{C}^1[0,\infty)$.

SL2 There exists $\eta \in \mathrm{C}^1[0,\infty)$ such that $\eta \geq 0$, $\boldsymbol{\phi}_\eta \geq 0$ and the functions $\boldsymbol{\phi}_\eta$, $\boldsymbol{\psi}_\eta$ are both decreasing on $(0,\infty)$ ($\boldsymbol{\phi}_\eta$, $\boldsymbol{\psi}_\eta$ are defined as in Assumption MP).

Define the convolution $*$ via (5.11) *where, for $h \in \mathrm{C}^\infty_{\mathrm{c,even}}$, v_h denotes the unique solution of $\ell_x v_h = \ell_y v_h$, $v_h(x,0) = v_h(0,x) = h(x)$, $(\partial_y v_h)(x,0) = (\partial_x v_h)(0,y) = 0$. Then $\bigl([0,\infty),*\bigr)$ is a Sturm-Liouville hypergroup.*

To the best of our knowledge, this is the most general known result giving sufficient conditions for the existence of a Sturm-Liouville hypergroup on $[0,\infty)$ associated with a given function A. In fact, as far as the authors are aware, all the concrete examples of hypergroup structures on $[0,\infty)$ which were known prior to this work are particular cases of Sturm-Liouville hypergroups satisfying conditions SL0, SL1 and SL2 (see [4, 13]). However, we can prove as a corollary of Theorem 5.7 that an existence theorem very similar to that of Zeuner continues to hold if the condition SL1 is removed:

Corollary 5.9. *Suppose that A satisfies SL0 and SL2. For $h \in \mathrm{C}^4_{\mathrm{c},0}$, denote by v_h the unique solution of $\ell_x v_h = \ell_y v_h$, $v_h(x,0) = v_h(0,x) = h(x)$, $(\partial_y^{[1]} v_h)(x,0) = (\partial_x^{[1]} v_h)(0,y) = 0$ such that conditions (i)–(ii) in Theorem 4.2 hold for $f = v_h$. Define the convolution $*$ via (5.11). Then $\bigl([0,\infty),*\bigr)$ is a hypergroup.*

Proof. Just notice that, by (4.12) and Theorem 4.2, the definition of convolution given in the statement of the corollary is equivalent to Definition 5.1. ∎

This corollary shows that it is natural to modify the definition of Sturm-Liouville hypergroup (Definition 5.8) by replacing the space $\mathrm{C}^\infty_{\mathrm{c,even}}$ by $\mathrm{C}^4_{\mathrm{c},0}$ and replacing ∂_y by $\partial_y^{[1]}$ in the initial condition, because in this way we are able to extend the class of Sturm-Liouville hypergroups to all functions A satisfying conditions SL0 and SL2.

We emphasize that condition SL1 imposes a great restriction on the behavior of the Sturm-Liouville operator $\ell(u) = -u'' - \frac{A'}{A}u'$ near zero: in the singular case $A(0) = 0$, SL1 requires that $\frac{A'(x)}{A(x)} \sim \frac{\alpha_0}{x}$. Therefore, as shown in the next example, Corollary 5.9 leads, in particular, to a considerable extension of the class of singular operators for which an associated hypergroup exists:

Example 5.10. If A satisfies SL0 and the function $\frac{A'}{A}$ is nonnegative and decreasing, then SL2 is satisfied with $\eta := 0$. Therefore, Corollary 5.9 assures that there exists a hypergroup associated with the operator $\ell(u) = -u'' - \frac{A'}{A}u'$. Notice that this existence result holds without any restriction on the growth of $\frac{A'(x)}{A(x)}$ as $x \downarrow 0$.

5.3 The degenerate case: degenerate hypergroups of full support

The goal of this subsection is to prove that in the degenerate case $\gamma(a) = -\infty$ the pair $\bigl([a,b),*\bigr)$ is a degenerate hypergroup of full support, in the sense of the following definition:

Definition 5.11. Let K be a locally compact space and $*$ a bilinear operator on $\mathcal{M}_{\mathbb{C}}(K)$. The pair $(K,*)$ is said to be a *degenerate hypergroup of full support* if conditions H1–H4 and H6 hold, together with the following axiom:

DH. $\mathrm{supp}(\delta_x * \delta_y) = K$ for all $x, y \in K \setminus \{\mathrm{e}\}$.

As we saw in the proof of Proposition 5.6, in order to determine the support of $\delta_x * \delta_y$ one needs to know when the solution of the Cauchy problem (4.1) is strictly positive. Our first step is to use Lemma 5.5 in order to derive an integral inequality which will turn out to be useful for studying the strict positivity of solution.

Lemma 5.12. *Write $R(x) := \frac{A(x)}{B(x)}$, where $B(x) = \exp(\frac{1}{2}\int_\beta^x \eta(\xi)d\xi)$ (with $\beta > \gamma(a)$ arbitrary). Take $h \in \mathcal{D}_{\mathcal{L}}^{(2)}$ with $\ell(h) \in \mathcal{D}_{\mathcal{L}}^{(2)}$ and $h \geq 0$. Let $u(x,y) := f(\gamma^{-1}(x), \gamma^{-1}(y))$, where f is the solution (4.4) of the Cauchy problem. Then the following inequality holds:*

$$R(x)R(y)u(x,y) \geq \tfrac{1}{2}\int_{\gamma(a)}^y R(s)R(x-y+s)\bigl[\phi_\eta(s) + \phi_\eta(x-y+s)\bigr]u(x-y+s,s)\,ds$$

$$+\tfrac{1}{2}\int_{\gamma(a)}^{y} R(s)R(x+y-s)\big[\boldsymbol{\phi}_\eta(s)-\boldsymbol{\phi}_\eta(x+y-s)\big]u(x+y-s,s)\,ds$$
$$+\tfrac{1}{2}\int_{\Delta} R(\xi)R(\zeta)\big[\boldsymbol{\psi}_\eta(\zeta)-\boldsymbol{\psi}_\eta(\xi)\big]u(\xi,\zeta)\,d\xi d\zeta$$

where $\Delta \equiv \Delta_{\gamma(a),x,y} = \{(\xi,\zeta)\in\mathbb{R}^2 \mid \zeta\geq\gamma(a),\, \xi+\zeta\leq x+y,\, \xi-\zeta\geq x-y\}$.

Proof. Let $\{a_m\}_{m\in\mathbb{N}}$ be a sequence $b>a_1>a_2>\ldots$ with $\lim a_m = a$. For $m\in\mathbb{N}$, define $u_m(x,y) := f_m(\gamma^{-1}(x),\gamma^{-1}(y))$, where f_m is given by (4.5). The function $v_m(x,y) = B(x)B(y)u_m(x,y)$ is a solution of

$$\begin{aligned}
(\boldsymbol{\ell}_x^B v_m)(x,y) &= (\boldsymbol{\ell}_y^B v_m)(x,y), && x,y>\tilde{a}_m\\
v_m(x,\tilde{a}_m) &= B(x)B(\tilde{a}_m)h(\gamma^{-1}(x)), && x>\tilde{a}_m\\
(\partial_y v_m)(x,\tilde{a}_m) &= \tfrac{1}{2}\eta(\tilde{a}_m)B(x)B(\tilde{a}_m)h(\gamma^{-1}(x)), && x>\tilde{a}_m
\end{aligned}$$

where $\boldsymbol{\ell}^B v := -v'' - \boldsymbol{\phi}_\eta v' + \boldsymbol{\psi}_\eta v$. Clearly, $v_m(x,\tilde{a}_m)$, $(\partial_y v_m)(x,\tilde{a}_m)\geq 0$. By Lemma 5.5, the integral equation (5.2) holds with $v=v_m$ and $c=a_m$. It is clear that we have $H\geq 0$, $I_0\geq 0$ and $I_4=0$ in the right hand side of (5.2); moreover, it follows from Proposition 4.4 and Assumption MP that the integrands of I_1, I_2 and I_3 are nonnegative. Consequently, for $\alpha\in[\tilde{a}_m,y]$ we have

$$\begin{aligned}
R(x)R(y)u_m(x,y) \geq{}& \tfrac{1}{2}\int_{\alpha}^{y} R(s)R(x-y+s)\big[\boldsymbol{\phi}_\eta(s)+\boldsymbol{\phi}_\eta(x-y+s)\big]u_m(x-y+s,s)\,ds\\
&+\tfrac{1}{2}\int_{\alpha}^{y} R(s)R(x+y-s)\big[\boldsymbol{\phi}_\eta(s)-\boldsymbol{\phi}_\eta(x+y-s)\big]u_m(x+y-s,s)\,ds\\
&+\tfrac{1}{2}\int_{\Delta_{\alpha,x,y}} R(\xi)R(\zeta)\big[\boldsymbol{\psi}_\eta(\zeta)-\boldsymbol{\psi}_\eta(\xi)\big]u_m(\xi,\zeta)\,d\xi d\zeta
\end{aligned}\tag{5.12}$$

where $\Delta_{\alpha,x,y} = \{(\xi,\zeta)\in\mathbb{R}^2 \mid \zeta\geq\alpha,\, \xi+\zeta\leq x+y,\, \xi-\zeta\geq x-y\}$. Since by Proposition 4.3 $\lim_{m\to\infty} u_m(x,y) = u(x,y)$ pointwise for $x,y\in(\gamma(a),\infty)$, by taking the limit we deduce that for each fixed $\alpha\in(\gamma(a),y]$ the inequality (5.12) holds with u_m replaced by u. If we then take the limit $\alpha\downarrow\gamma(a)$, the desired integral inequality follows. ☐

The next lemma will be helpful for verifying the strict positivity of the integrands in the above integral inequality.

Lemma 5.13. *If $\gamma(a)=-\infty$, then at least one of the functions $\boldsymbol{\phi}_\eta$, $\boldsymbol{\psi}_\eta$ defined in Assumption MP is non-constant on every neighbourhood of $-\infty$.*

Proof. Suppose by contradiction that $\gamma(a)=-\infty$ and $\boldsymbol{\phi}_\eta$, $\boldsymbol{\psi}_\eta$ are both constant on an interval $(-\infty,\kappa]\subset\mathbb{R}$. Recall from the proof of Proposition 3.6 that $\mathcal{L}$ is unitarily equivalent to a self-adjoint realization of $-\frac{d^2}{d\xi^2}+\mathfrak{q}$, where $\mathfrak{q}$ is given by (3.4). Clearly, $\mathfrak{q}(\xi)=\mathfrak{q}_\infty := \frac{1}{4}\boldsymbol{\phi}_\eta^2(\kappa)+\boldsymbol{\psi}_\eta(\kappa) < -\infty$ for all $\xi\in(-\infty,\kappa)$. It therefore follows from [29, Theorem 15.3] that the essential spectrum of any self-adjoint realization of ℓ restricted to an interval (a,c) (for $a<c<b$) contains $[\mathfrak{q}_\infty,\infty)$. However, it follows from the boundary condition (2.2) and [23, Theorem 3.1] that self-adjoint realizations of ℓ restricted to (a,c) have purely discrete spectrum. This contradiction proves the lemma. ☐

We are now ready to prove that in the case $\gamma(a)=-\infty$ the solution of the (nontrivial) Cauchy problem (4.1) always has full support on $(a,b)^2$, even when the initial condition is compactly supported:

Theorem 5.14 (Strict positivity of solution for the Cauchy problem (4.1))**.** *Suppose that $\gamma(a)=-\infty$. Take $h\in\mathcal{D}_{\mathcal{L}}^{(2)}$ with $\ell(h)\in\mathcal{D}_{\mathcal{L}}^{(2)}$. If $h\geq 0$ and $h(\tau_0)>0$ for some $\tau_0\in(a,b)$, then the function f given by (4.4) is such that*

$$f(x,y)>0 \qquad \textit{for } x,y\in(a,b).$$

Proof. Let $u(x,y) := f(\gamma^{-1}(x), \gamma^{-1}(y))$ and $\tilde{\tau}_0 = \gamma(\tau_0)$. Fix $x_0 \geq y_0 > -\infty$. Since $\lim_{y\to-\infty} u(\tilde{\tau}_0, y) = h(\tau_0) > 0$, there exists $\kappa \in (-\infty, \min\{y_0, \tau_0\})$ such that $u(\tilde{\tau}_0, y) > 0$ for all $y \leq \kappa$.

Suppose $\boldsymbol{\phi}_\eta$ is non-constant on every neighbourhood of $-\infty$. Choosing a smaller κ if necessary, we may assume that $\boldsymbol{\phi}_\eta(\kappa) > \boldsymbol{\phi}_\eta(\xi)$ for all $\xi > \kappa$. For each $x > \tilde{\tau}_0$ and $y \leq \kappa$ we have by Lemma 5.12

$$R(x)R(y)u(x,y) \geq \tfrac{1}{2} \int_{-\infty}^{y} R(s)R(x-y+s)\bigl[\boldsymbol{\phi}_\eta(s) + \boldsymbol{\phi}_\eta(x-y+s)\bigr]u(x-y+s,s)\, ds$$

and the integrand in the right hand side is continuous and strictly positive at $s = y - x + \tilde{\tau}_0$, so the integral is positive and therefore $u(x,y) > 0$ for all $x \geq \tilde{\tau}_0$ and $y \leq \kappa$. Again by Lemma 5.12,

$$R(x_0)R(y_0)u(x_0,y_0) \geq \tfrac{1}{2} \int_{-\infty}^{y_0} R(s)R(x_0+y_0-s)\bigl[\boldsymbol{\phi}_\eta(s) - \boldsymbol{\phi}_\eta(x_0+y_0-s)\bigr]u(x_0+y_0-s,s)\, ds$$

with the integrand being strictly positive for $s < \min\{\kappa, x_0 + y_0 - \tilde{\tau}_0\}$, thus $u(x_0,y_0) > 0$.

Suppose now that $\boldsymbol{\psi}_\eta$ is non-constant on every neighbourhood of $-\infty$ and that κ is chosen such that $\boldsymbol{\psi}_\eta(\kappa) > \boldsymbol{\psi}_\eta(\xi)$ for all $\xi > \kappa$. The integral inequality of Lemma 5.12 yields

$$R(x_0)R(y_0)u(x_0,y_0) \geq \tfrac{1}{2} \int_{\Delta} R(\xi)R(\zeta)\bigl[\boldsymbol{\psi}_\eta(\zeta) - \boldsymbol{\psi}_\eta(\xi)\bigr]u(\xi,\zeta)\, d\xi d\zeta.$$

where $\Delta = \{(\xi,\zeta) \in \mathbb{R}^2 \mid \xi + \zeta \leq x_0 + y_0,\, \xi - \zeta \geq x_0 - y_0\}$. Clearly, the integrand is continuous and > 0 on $\{(\tau_0, \zeta) \mid \zeta \leq \min(y_0 - |x_0 - \tau_0|, \kappa)\} \subset \Delta$, and it follows at once that $u(x_0,y_0) > 0$.

By Lemma 5.13 it follows that $u(x_0,y_0) > 0$. Since $x_0 \geq y_0 > -\infty$ are arbitrary we conclude that $f(x,y) > 0$ for $b > x \geq y > a$ and, by symmetry, for $x, y \in (a,b)$. ☐

Corollary 5.15 (Existence theorem for degenerate hypergroups of full support)**.** *Let ℓ be a differential expression of the form* (2.1) *and satisfying* (2.2)*. Suppose that $\gamma(a) = -\infty$. Then $\bigl([a,b), *\bigr)$ is a degenerate hypergroup of full support.*

Proof. By Proposition 5.4, the pair $\bigl([a,b), *\bigr)$ satisfies axioms H1–H4 and H6. As in the proof of Proposition 5.6, $z \in [a,b)$ belongs to $\mathrm{supp}(\delta_x * \delta_y)$ if and only if $\int_{[\sigma^2,\infty)} w_\lambda(x)\, w_\lambda(y)\, (\mathcal{F}h_\varepsilon)(\lambda)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) > 0$ for all $\varepsilon > 0$, where $\{h_\varepsilon\} \subset \mathrm{C}^4_{\mathrm{c},0}$ is a family of functions satisfying (5.9). But it follows from Theorem 5.14 that $f_{h_\varepsilon}(x,y) = \int_{[\sigma^2,\infty)} w_\lambda(x)\, w_\lambda(y)\, (\mathcal{F}h_\varepsilon)(\lambda)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) > 0$ for all $x, y \in (a,b)$. Hence each $z \in [a,b)$ belongs to all the sets $\mathrm{supp}(\delta_x * \delta_y)$, $x, y \in (a,b)$; therefore, $\bigl([a,b), *\bigr)$ satisfies axiom DH. ☐

As discussed in the Introduction, the notion of degenerate hypergroup of full support is motivated by the example of the so-called Whittaker convolution, associated with the normalized Whittaker differential operator $\ell = -x^2 \frac{d^2}{dx^2} - (1 + 2(1-\alpha)x)\frac{d}{dx}$ and studied by the authors in [25, 26]. Corollary 5.15 shows that many other Sturm-Liouville differential expressions yield convolution algebras with the full support property.

Example 5.16. Let $\boldsymbol{\zeta} \in \mathrm{C}^1(0,\infty)$ be a nonnegative decreasing function and let $\kappa > 0$. The differential expression

$$\ell = -x^2 \frac{d^2}{dx^2} - \bigl[\kappa + x\bigl(1 + \boldsymbol{\zeta}(x)\bigr)\bigr]\frac{d}{dx}, \qquad 0 < x < \infty$$

is a particular case of (2.1), obtained by considering $p(x) = x e^{-\kappa/x + I_{\boldsymbol{\zeta}}(x)}$ and $r(x) = \frac{1}{x} e^{-\kappa/x + I_{\boldsymbol{\zeta}}(x)}$, where $I_{\boldsymbol{\zeta}}(x) = \int_1^x \boldsymbol{\zeta}(y) \frac{dy}{y}$. (If $\kappa = 1$ and $\boldsymbol{\zeta}(x) = 1 - 2\alpha > 0$, we recover the normalized Whittaker operator.) The change of variable $z = \log x$ transforms ℓ into the standard form $\widetilde{\ell} = -\frac{d^2}{dz^2} - \frac{A'(z)}{A(z)}\frac{d}{dz}$, where $\frac{A'(z)}{A(z)} = \kappa e^{-\kappa z} + \boldsymbol{\zeta}(e^z)$. It is clear that $\gamma(a) = -\infty$ and that ℓ satisfies Assumption MP with $\eta = 0$, and it is not difficult to show that the boundary condition (2.2) holds. Consequently, the Sturm-Liouville operator ℓ gives rise to a convolution structure such that $\mathrm{supp}(\delta_x * \delta_y) = [0,\infty)$ for all $x, y > 0$.

Appendix A Proofs of existence, uniqueness and positivity of solution for the associated hyperbolic Cauchy problem

A.1 Proof of Theorem 4.2

We start by proving that there exists at most one solution of (4.1) satisfying the given conditions. Let $f_1, f_2 \in \mathrm{C}^2\bigl((a,b)^2\bigr)$ be two solutions of $\ell_x f = \ell_y f$ such that *(i)–(ii)* hold for $f \in \{f_1, f_2\}$. Fix $\lambda \in [0,\infty) \setminus \Lambda_0$ and let $\Psi_j(y,\lambda) := [\mathcal{F} f_j(\cdot,y)](\lambda)$. We have

$$\ell_y \Psi_j(y,\lambda) = \mathcal{F}[\ell_y f_j(\cdot,y)](\lambda) = \mathcal{F}[\ell_x f_j(\cdot,y)](\lambda) = \lambda \Psi_j(y,\lambda), \qquad a<y<b$$

where the first equality is due to (4.2) and the last step follows from (2.10). Moreover,

$$\lim_{y\downarrow a} \Psi_j(y,\lambda) = (\mathcal{F}h)(\lambda) \quad \text{ and } \quad \lim_{y\downarrow a} \partial_y^{[1]} \Psi_j(y,\lambda) = 0$$

by (4.3). It thus follows from Lemma 2.1 that

$$[\mathcal{F} f_j(\cdot,y)](\lambda) = \Psi_j(y,\lambda) = (\mathcal{F}h)(\lambda)\, w_\lambda(y), \qquad a<y<b.$$

This equality takes place for $\boldsymbol{\rho}_{\mathcal{L}}$-almost every λ, so the isometric property of $\mathcal{F}$ gives $f_1(\cdot,y) = f_2(\cdot,y)$ Lebesgue-almost everywhere; since the f_j are continuous, we conclude that $f_1(x,y) \equiv f_2(x,y)$ for all $x,y \in (a,b)$.

In order to prove that (4.4) is the (unique) solution, we need to justify that $\ell_x f$ can be computed via differentiation under the integral sign. It follows from (2.3) that $w_\lambda^{[1]}(x) = -\lambda \int_a^x w_\lambda(\xi)\, r(\xi) d\xi$ and therefore (by Lemma 2.3) $|w_\lambda^{[1]}(x)| \leq \lambda \int_a^x r(\xi) d\xi$. Hence

$$\int_{[\sigma^2,\infty)} \bigl|(\mathcal{F}h)(\lambda)\, w_\lambda^{[1]}(x)\, w_\lambda(y)\bigr| \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) \leq \int_a^x r(\xi) d\xi \cdot \int_{[\sigma^2,\infty)} \lambda \, \bigl|(\mathcal{F}h)(\lambda)\, w_\lambda(y)\bigr| \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) < \infty, \tag{A.1}$$

where the convergence (which is uniform on compacts) follows from (2.10) and Lemma 2.6(b). Due to the convergence of the differentiated integral, we have $\partial_x^{[1]} f(x,y) = \int_{[\sigma^2,\infty)} (\mathcal{F}h)(\lambda)\, w_\lambda^{[1]}(x)\, w_\lambda(y)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda)$. Since $(\ell w_\lambda)(x) = \lambda w_\lambda(x)$, in the same way we check that $\int_{[\sigma^2,\infty)} (\mathcal{F}h)(\lambda)\, (\ell w_\lambda)(x)\, w_\lambda(y)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda)$ converges absolutely and uniformly on compacts and is therefore equal to $(\ell_x f)(x,y)$. Consequently,

$$(\ell_x f)(x,y) = (\ell_y f)(x,y) = \int_{[\sigma^2,\infty)} \lambda\, (\mathcal{F}h)(\lambda)\, w_\lambda(x)\, w_\lambda(y)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda).$$

Concerning the boundary conditions, Lemma 2.6(b) together with the fact that $w_\lambda(a) = 1$ imply that $f(x,a) = h(x)$, and from (A.1) we easily see that $\lim_{y\downarrow a} \partial_y^{[1]} f(x,y) = 0$. This shows that f is a solution of the Cauchy problem (4.1). □

A.2 Proof of Proposition 4.3

Let us begin by justifying that $\partial_x^{[1]} f_m(x,y)$ and $(\ell_x f_m)(x,y)$ can be computed via differentiation under the integral sign. The differentiated integrals are given by

$$\int_{[\sigma^2,\infty)} w_\lambda^{[1]}(x)\, w_{\lambda,m}(y)\, (\mathcal{F}h)(\lambda)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) \tag{A.2}$$

$$\int_{[\sigma^2,\infty)} w_\lambda(x)\, w_{\lambda,m}(y)\, [\mathcal{F}(\ell(h))](\lambda)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) \tag{A.3}$$

(for the latter, we used the identities $(\ell w_\lambda)(x) = \lambda w_\lambda(x)$ and (2.10)), and their absolute and uniform convergence on compacts follows from the fact that $h, \ell(h) \in \mathcal{D}_{\mathcal{L}}^{(2)}$, together with Lemma 2.6(b) and the inequality $|w_{\lambda,m}(\cdot)| \leq 1$ (which follows from Lemma 2.3 if we replace a by a_m). This justifies that $\partial_x^{[1]} f_m(x,y)$ and $(\ell_x f_m)(x,y)$ are given by (A.2), (A.3) respectively.

We also need to ensure that $\partial_y^{[1]} f_m(x,y)$ and $(\ell_y f_m)(x,y)$ are given by the corresponding differentiated integrals, and to that end we must check that

$$\int_{[\sigma^2,\infty)} w_\lambda(x)\, w_{\lambda,m}^{[1]}(y)\, (\mathcal{F}h)(\lambda)\, \boldsymbol{\rho}_{\mathcal{L}}(d\lambda)$$

converges absolutely and uniformly. Indeed, it follows from (2.4) that for $y \geq a_m$ we have $w_{\lambda,m}^{[1]}(y) = \lambda \int_{a_m}^y w_{\lambda,m}(\xi)\, r(\xi) d\xi$ and consequently $|w_{\lambda,m}^{[1]}(y)| \leq \lambda \int_{a_m}^y r(\xi) d\xi$; hence

$$\int_{[\sigma^2,\infty)} \bigl|w_\lambda(x)\, w_{\lambda,m}^{[1]}(y)\, (\mathcal{F}h)(\lambda)\bigr| \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) \leq \int_{a_m}^y r(\xi) d\xi \cdot \int_{[\sigma^2,\infty)} \lambda \bigl|w_\lambda(x)(\mathcal{F}h)(\lambda)\bigr| \boldsymbol{\rho}_{\mathcal{L}}(d\lambda) \tag{A.4}$$

and the uniform convergence in compacts follows from (2.10) and Lemma 2.6(b).

The verification of the boundary conditions is straightforward: Lemma 2.6(b) together with the fact that $w_{\lambda,m}(a_m) = 1$ imply that $f_m(x, a_m) = h(x)$, and from (A.4) we easily see that $\partial_y^{[1]} f_m(x, a_m) = 0$. This shows that the function f_m defined by (4.5) is a solution of the Cauchy problem (4.6).

Since $w_{\lambda,m}(y) \to w_\lambda(y)$ as $m \to \infty$ (Lemma 2.2), the pointwise convergence $f_m(x,y) \to f(x,y)$ follows from the dominated convergence theorem (which is applicable due to Lemmas 2.3 and 2.6(b)). ∎

A.3 Proof of Lemma 5.5

Just compute

$$\begin{aligned}
I_4 - I_3 &= \tfrac{1}{2} \int_{\Delta_{c,x,y}} \biggl(\frac{\partial}{\partial \xi} \bigl[A_B(\xi) A_B(\zeta)\, (\partial_\xi v)(\xi,\zeta)\bigr] - \frac{\partial}{\partial \zeta} \bigl[A_B(\xi) A_B(\zeta)\, (\partial_\zeta v)(\xi,\zeta)\bigr] \biggr) d\xi d\zeta \\
&= I_0 - \tfrac{1}{2} \int_0^y A_B(s) A_B(x-y+s)\, (\partial_\zeta v + \partial_\xi v)(x-y+s, s)\, ds \\
&\quad - \tfrac{1}{2} \int_0^y A_B(s) A_B(x+y-s)\, (\partial_\zeta v - \partial_\xi v)(x+y-s, s)\, ds \\
&= I_0 + I_1 - \int_c^y \frac{d}{ds} \bigl[A_B(s) A_B(x-y+s)\, v(x-y+s, s)\bigr] ds \\
&\quad + I_2 - \int_c^y \frac{d}{ds} \bigl[A_B(s) A_B(x+y-s)\, v(x+y-s, s)\bigr] ds
\end{aligned}$$

where in the second equality we used Green's theorem, and the third equality follows easily from the fact that $(A_B)' = \boldsymbol{\phi}_\eta A_B$.

A.4 Proof of Proposition 4.4

The proof depends on the following maximum principle for the (standardized) hyperbolic equation:

Proposition A.1 (Weak maximum principle). *Suppose Assumption MP holds, and let $\gamma(a) < c \leq y_0 \leq x_0$. If $u \in \mathrm{C}^2(\Delta_{c,x_0,y_0})$ satisfies*

$$\begin{aligned}
(\widetilde{\ell}_x u - \widetilde{\ell}_y u)(x,y) &\leq 0, && (x,y) \in \Delta_{c,x_0,y_0} \\
u(x,c) &\geq 0, && x \in [x_0 - y_0 + c, x_0 + y_0 - c] \\
(\partial_y u)(x,c) + \tfrac{1}{2}\eta(c) u(x,c) &\geq 0, && x \in [x_0 - y_0 + c, x_0 + y_0 - c]
\end{aligned} \tag{A.5}$$

then $u \geq 0$ in Δ_{c,x_0,y_0}.

Proof. Pick a function $\omega \in \mathrm{C}^2[c,\infty)$ such that $\boldsymbol{\ell}^B \omega < 0$, $\omega(c) > 0$ and $\omega'(c) \geq 0$ (where $\boldsymbol{\ell}_x^B$ is the differential operator defined in Lemma 5.5). Clearly, it is enough to show that for all $\delta > 0$ we have $v(x,y) := B(x) B(y) u(x,y) + \delta \omega(y) > 0$ for $(x,y) \in \Delta_{c,x_0,y_0}$.

By Lemma 5.5, the integral equation (5.2) holds for the function v. Assume by contradiction that there exist $\delta > 0$, $(x,y) \in \Delta_{c,x_0,y_0}$ for which we have $v(x,y) = 0$ and $v(\xi,\zeta) \geq 0$ for all $(\xi,\zeta) \in \Delta_{c,x,y} \subset \Delta_{c,x_0,y_0}$.

It is clear from the choice of ω that $v(\cdot,c) > 0$, thus we have $H \geq 0$ in the right hand side of (5.2). Similarly, $(\partial_y v)(\cdot,c) = B(x)B(y)\bigl[(\partial_y u)(\cdot,c) + \frac{1}{2}\eta(c)u(\cdot,c)\bigr] + \delta\omega'(c) \geq 0$, hence $I_0 \geq 0$. Since $\boldsymbol{\phi}_\eta$ is positive and decreasing and $\boldsymbol{\psi}_\eta$ is decreasing (cf. Assumption MP) and we are assuming that $u \geq 0$ on $\Delta_{c,x,y}$, it follows that $I_1 \geq 0$, $I_2 \geq 0$ and $I_3 \geq 0$. In addition, $I_4 < 0$ because $(\boldsymbol{\ell}_\zeta^B v - \boldsymbol{\ell}_\xi^B v)(\xi,\zeta) = B(x)B(y)(\widetilde{\ell}_\zeta u - \widetilde{\ell}_\xi u)(\xi,\zeta) + (\boldsymbol{\ell}^B \omega)(\zeta) < 0$. Consequently, (5.2) yields $0 = A_B(x)A_B(y)v(x,y) \geq -I_4 > 0$. This contradiction shows that $v(x,y) > 0$ for all $(x,y) \in \Delta_{c,x_0,y_0}$. ☐

Proof of Proposition 4.4. It follows from Proposition 4.3 that the function $u_m(x,y) := f_m(\gamma^{-1}(x),\gamma^{-1}(y))$ is a solution of the Cauchy problem

$$(\widetilde{\ell}_x u_m)(x,y) = (\widetilde{\ell}_y u_m)(x,y), \qquad x,y > \tilde{a}_m \tag{A.6}$$

$$u_m(x,\tilde{a}_m) = h(\gamma^{-1}(x)), \qquad x > \tilde{a}_m \tag{A.7}$$

$$(\partial_y u_m)(x,\tilde{a}_m) = 0, \qquad x > \tilde{a}_m \tag{A.8}$$

where $\tilde{a}_m = \gamma(a_m)$. Clearly, u_m satisfies the inequalities (A.5) for arbitrary $x_0 \geq y_0 \geq \tilde{a}_m$ (here $c = \tilde{a}_m$). By Proposition A.1, $u_m(x_0,y_0) \geq 0$ for all $x_0 \geq y_0 > \tilde{a}_m$; consequently, (4.8) holds.

The proof that $h \leq C$ implies $f_m \leq C$ is straightforward: if we have $h \leq C$, then $\widetilde{u}_m(x,y) = C - u_m(x,y)$ is a solution of (A.6) with initial conditions $\widetilde{u}_m(x,\tilde{a}_m) = C - h(\gamma^{-1}(x)) \geq 0$ and (A.8), thus the reasoning of the previous paragraph yields that $C - u_m \geq 0$ for $x \geq y > \tilde{a}_m$. ☐

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