Multinomial method for option pricing under Variance Gamma

Nicola Cantarutti* , João Guerra

CEMAPRE - Centre for Applied Mathematics and Economics
ISEG - University of Lisbon

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Abstract

This paper presents a multinomial method for option pricing when the underlying asset follows an exponential Variance Gamma process. The continuous time Variance Gamma process is approximated by a discrete time Markov chain with the same firsts four cumulants. This approach is particularly convenient for pricing American and Bermudan options, which can be exercised at any time up to expiration date. Numerical computations of European and American options are presented, and compared with results obtained with finite differences methods and with the Black Scholes model.

Keywords: American option, Lévy processes, Moment Matching, Multinomial tree, Variance Gamma.

1 Introduction

Since the early nineties, a lot of research has been done on the topic of pure jump Lévy processes to describe the dynamics of the asset returns. The main contributions are due to Madan and Seneta (1990), Eberlein and Keller (1995), Geman et al. (1998), Barndorff-Nielsen (1998).

Lévy processes are stochastic processes with independent and stationary increments that have nice analytical properties and reproduce quite well the statistical features of the financial data. For example, in Figure 1 we show four histograms of the daily log-returns of four indices: the S&P 500 Stock Index, the KOSPI (Korea Composite Stock Price Index), XAO (All Ordinaries Australian Index) and TAIEX (Taiwan Capitalization weighted Stock Index). The histograms are plotted together with the fitted Normal and Variance Gamma (VG) densities. It is straightforward to check that the VG density reproduces much better the high peaks near the origin, and the heavy tails of the empirical data.

*Corresponding author. Email: nicolacantarutti@gmail.com

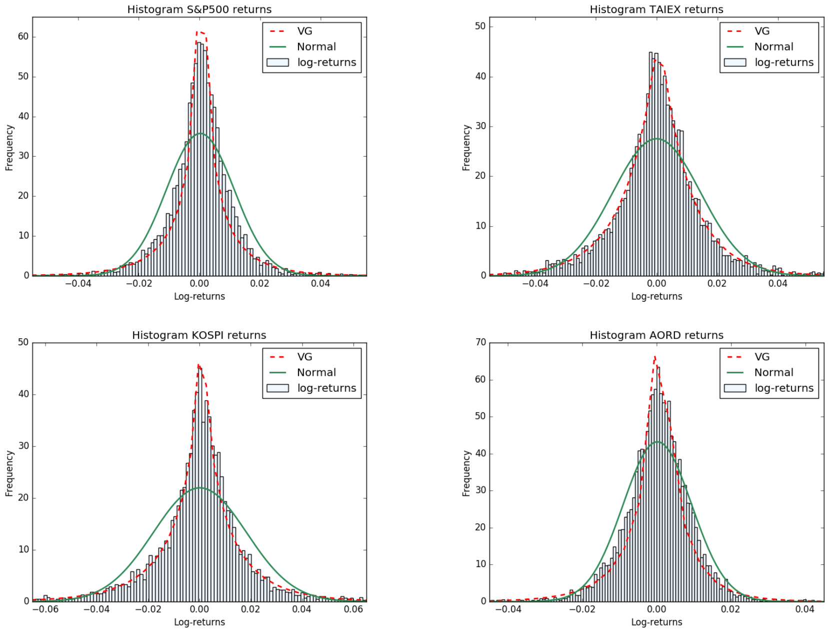

Figure 1: Histogram of daily log-returns for S&P500, KOSPI, XAO and TAIEX. The dashed line corresponds to the VG density. The continuous line is the normal density. The VG parameters are obtained with the method of moments, as explained in Seneta (2004).

The *Variance Gamma* process is a pure jump Lévy process with infinite activity. This means that when the magnitude of the jumps becomes infinitesimally small, the arrival rate of jumps tends to infinity. The first complete presentation of the symmetric VG model is due to Madan and Seneta (1990) where, with respect to the Normal case, only an additional parameter is introduced to control the kurtosis, while the skewness is still not considered. The authors model the log-returns as a driftless Brownian motion with a random Gamma distributed variance. This is the origin of the name "Variance Gamma".

There are two representations of the VG process. In the first, the VG process is obtained by time changing a Brownian motion with drift: the Brownian motion is evaluated at random times that are Gamma distributed. A possible interpretation is that the economical relevant times are random. The non-symmetric VG process is described by Madan et al. (1998), where the authors also presented an explicit form of the return density function and closed form formula for the price of a vanilla European option. The authors consider a Brownian motion with drift, and this gives the possibility to control the skewness as well.

As a pure jump process, the VG process does not have a continuous martingale component. It resembles the Brownian motion because it has an infinite number of jumps in any time interval, but unlike Brownian motion it has finite variation, so the sum of the absolute value of the jumps in any time interval converges. This property can be derived easily by the second representation of the VG process as the difference of two (finite variation) Gamma processes. The proof can be found in Madan et al. (1998), where the authors show that the two representations are equivalent, and also derive the VG characteristic function as the product of two Gamma characteristic functions. This representation has another interesting economical interpretation as the difference of gains and losses. The Gamma processes are always increasing, therefore this representation is coherent with independent gains and losses.

The VG process was first presented in the context of option pricing in Madan and Milne (1991), where it was used in the pricing of European options. The problem for European options can be easily solved by the analytical formula of Madan et al. (1998) or numerically by different tecniques. Monte Carlo methods for VG are presented in Fu (2000). A finite difference scheme for the VG Partial Integro-Differential Equation (PIDE) is described in Cont and Voltchkova (2005). In Carr and Madan (1998), the authors show how to solve the option pricing problem using the Fourier transform method. The problem for American options is considered in Hirsa and Madan (2001) and Almendral and Oosterlee (2007), where finite difference schemes are applied to solve the American options PIDE for VG. However, the option pricing problem for an asset following a risk neutral VG process, has never (to our knowledge) been faced in the literature using a *tree method*. The tree method was first introduced by Cox et al. (1979) for a market where the log-price can change only in two different ways: an upward jump, or a downward jump. For this reason this discrete model is called *binomial model*. The authors prove that when the number of time steps increses, the discrete random walk of the log-price converges to the Brownian motion and the option price converges to that of Black and Scholes (1973). Multinomial models are a generalization of the binomial model and at each time step it considers more than just two possible future states. In this work we consider multinomial methods as developed by Yamada and Primbs (2001), Yamada and Primbs (2003) and Yamada and Primbs (2004).

In Section 2 we present the basic features of Lévy processes, in particular finite variation processes. The VG process and exponential VG are introduced in the successive subsections. A short summary of some useful concepts such as Poisson integration, and the relation between the Lévy symbol with the cumulants are collected in the Appendices A and B. In Section 3 we review the construction of the multinomial tree that approximates the VG process following the method of moment matching proposed by Yamada and Primbs (2001). We prove that the multinomial tree converges to the continuous time jump process that we introduce to approximate the VG process. In Section 4, which is the most important of the paper, we describe the algorithm for pricing options with the multinomial method and show the numerical results for European and American options. In Section 5 we presents a topic that deserves further research. We show how to obtain the parameters of the discrete time Markov chain that approximate the VG process, by discretizing its infinitesimal generator. However, using this method, the transition probabilities are not always positive. These coincide with the probabilities obtained with the moment matching condition only for a particular choice of the parameters. This topic can be further investigated. In Section 6 there are the conclusions.

2 Lévy processes

Let X_t be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{t\geq 0}), \mathbb{P})$, X_t is said to be a *Lévy process* if it satisfies the three properties:

1. $X_0 = 0$.
2. X_t has independent and stationary increments.
3. X_t is stochastically continuous: $\forall \epsilon, t > 0 \;\; \lim_{h\to 0} \mathbb{P}\big(|X_{t+h} - X_t| > \epsilon\big) = 0$.

The characteristic function of every Lévy process X_t has the Lévy-Khintchine representation:

$$\begin{aligned}\phi_{X_t}(u) &= \mathbb{E}[e^{iuX_t}] \\ &= e^{t\eta(u)} \\ &= \exp\left[t\bigg(ibu - \frac{1}{2}\tilde{\sigma}^2u^2 + \int_{\mathbb{R}} \big(e^{iux} - 1 - iux\mathbb{1}_{(|x|<1)}(x)\big)\nu(dx)\bigg)\right],\end{aligned} \tag{1}$$

where $\eta(u)$ is called *Lévy symbol*, $b \in \mathbb{R}$ and $\tilde{\sigma} \geq 0$ are constants[1] and $\nu(dx)$ is the *Lévy measure* which satisfies:

$$\nu(\{0\}) = 0, \qquad \int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < \infty. \tag{2}$$

The *Lévy triplet* $(b, \tilde{\sigma}, \nu)$ completely characterizes a Lévy process. Every Lévy process can be written as the superposition of a Brownian motion with drift and a pure jump process. This is the so called *Lévy-Itō decomposition*:

$$dX_t = bdt + \tilde{\sigma}dW_t + \int_{|x|\geq 1} xN(dt, dx) + \int_{|x|<1} x\tilde{N}(dt, dx), \tag{3}$$

where W_t is a standard Brownian motion, $N(dt, dx)$ and $\tilde{N}(dt, dx)$ are the Poisson random measure and the compensated Poisson random measure (see Appendix A).

We are interested in particular in processes with finite variation and finite moments. We see that the Lévy measure contains all the information we need:

- A Lévy process with triplet $(b, \tilde{\sigma}, \nu)$ is of finite variation if and only if

$$\tilde{\sigma} = 0 \quad \text{and} \quad \int_{|x|<1} |x|\nu(dx) < \infty. \tag{4}$$

- A Lévy process has finite moment of order n, $\mathbb{E}[X_t^n] < \infty$, if and only if

$$\int_{|x|\geq 1} |x|^n \nu(dx) < \infty. \tag{5}$$

[1] The diffusion coefficients is usually referred as σ. Here we call it $\tilde{\sigma}$ because we will call σ a parameter of the VG process.

For a proof see Applebaum (2009), Theorem 2.4.25 and Theorem 2.5.2. As a conseguence of these two properties, the truncator term $\mathbb{1}_{(|x|<1)}$ can be absorbed in the parameter b. It is easy to verify that every finite variation Lévy process is a compound Poisson process:

$$X_t = b't + \int_{\mathbb{R}} xN(t,dx), \tag{6}$$

with $b' = b - \int_{|x|<1} x\nu(dx)$. The Lévy symbol is:

$$\eta(u) = ib'u + \int_{\mathbb{R}} (e^{iux} - 1)\nu(dx). \tag{7}$$

2.1 The Variance Gamma process

The VG process is obtained by time changing a Brownian motion with drift. The new time variable is a stochastic process T_t whose increments are Gamma distributed and $T_t \sim \Gamma(\mu t, \kappa)$ with density[2]:

$$f_{T_t}(x) = \frac{(\frac{\mu}{\kappa})^{\frac{\mu^2 t}{\kappa}}}{\Gamma(\frac{\mu^2 t}{\kappa})} x^{\frac{\mu^2 t}{\kappa}-1} e^{-\frac{\mu x}{\kappa}} \qquad x \geq 0. \tag{8}$$

The Gamma process T_t is a *subordinator*. A subordinator is a one dimensional Lévy process that is non-decreasing almost surely. Therefore it is consistent to represent a time variable. It is possible to prove that every subordinator is a finite variation process (see Applebaum (2009)).

Consider a Brownian motion with drift $X_t = \theta t + \sigma W_t$, with $W_t \sim \mathcal{N}(0,t)$, and replace the time variable by the Gamma subordinator $T_t \sim \Gamma(t, \kappa)$ (with $\mu = 1$). We obtain the *Variance Gamma* process:

$$X_t = \theta T_t + \sigma W_{T_t}. \tag{9}$$

It depends on three parameters:

- θ, the drift of the Brownian motion,
- σ, the volatility of the Brownian motion,
- κ, the variance of the Gamma process.

The characteristic function of the VG process can be computed easily by conditioning on the realization of the Gamma time (Proposition 1.3.27 of Applebaum (2009))

$$\begin{aligned}\phi_{X_t}(u) &= \Big(1 - i\kappa\big(u\theta + \frac{i}{2}\sigma^2 u^2\big)\Big)^{-\frac{t}{\kappa}} &(10)\\ &= \Big(1 - i\theta\kappa u + \frac{1}{2}\sigma^2 \kappa u^2\Big)^{-\frac{t}{\kappa}}, &(11)\end{aligned}$$

[2] Usually the Gamma distribution is paramentrized by a shape and scale positive parameters $X \sim \Gamma(\rho, \zeta)$. The Gamma process $X_t \sim \Gamma(\rho t, \zeta)$ has pdf $f_{X_t}(x) = \frac{\zeta^{-\rho t}}{\Gamma(\rho t)} x^{\rho t - 1} e^{-\frac{x}{\zeta}}$ and has moments $\mathbb{E}[X_t] = \rho\zeta t$ and $\mathrm{Var}[X_t] = \rho\zeta^2 t$. Here we use a parametrization as in Madan et al. (1998) such that $\mathbb{E}[X_t] = \mu t$ and $\mathrm{Var}[X_t] = \kappa t$, so $\zeta = \frac{\kappa}{\mu}$, $\rho = \frac{\mu^2}{\kappa}$.

and the Lévy symbol is thus

$$\eta(u) = -\frac{1}{\kappa}\log(1 - i\theta\kappa u + \frac{1}{2}\sigma^2\kappa u^2). \tag{12}$$

Using the formula (66) in Appendix B for the cumulants we derive:

$$\begin{aligned}
c_1 &= t\theta \\
c_2 &= t(\sigma^2 + \theta^2\kappa) \\
c_3 &= t(2\theta^3\kappa^2 + 3\sigma^2\theta\kappa) \\
c_4 &= t(3\sigma^4\kappa + 12\sigma^2\theta^2\kappa^2 + 6\theta^4\kappa^3).
\end{aligned} \tag{13}$$

The VG Lévy measure is[3]:

$$\nu(dx) = \frac{e^{\frac{\theta x}{\sigma^2}}}{\kappa|x|}\exp\left(-\frac{\sqrt{\frac{2}{\kappa} + \frac{\theta^2}{\sigma^2}}}{\sigma}|x|\right)dx. \tag{14}$$

It satisfies conditions (4) and (5). The finite variation process can be represented as a compound Poisson process as in (6) and (7), with no additional drift $b' = 0$.

$$X_t = \int_{\mathbb{R}} xN(t, dx). \tag{15}$$

All the informations are contained in the Lévy measure (14), which completely describes the process. Even if the process has been created by Brownian subordination, it has no diffusion component. The Lévy triplet is $(0, 0, \nu)$. Using the formalism of Poisson integrals in Appendix A, the Lévy symbol (12) has the representation[4]:

$$\eta(u) = \int_{\mathbb{R}}(e^{iux} - 1)\nu(dx). \tag{16}$$

2.2 Exponential VG model

Under the risk neutral measure $\mathbb{Q}$, the dynamics of the stock price is described by an *exponential Lévy model*:

$$S_t = S_0 e^{rt + X_t}, \tag{17}$$

where r is the risk free interest rate, and X_t is a general Lévy process. Under $\mathbb{Q}$, the discounted price is a $\mathbb{Q}$-martingale:

$$\mathbb{E}^{\mathbb{Q}}\big[S_t e^{-rt}\big|S_0\big] = \mathbb{E}^{\mathbb{Q}}\big[S_0 e^{X_t}\big|S_0\big] = S_0, \tag{18}$$

and so $\mathbb{E}^{\mathbb{Q}}[e^{X_t}|X_0 = 0] = 1$. The condition for the existence of the exponential moment $\mathbb{E}[e^{X_t}] < \infty$ is equivalent to

$$\int_{|x|>1} e^x \nu(dx) < \infty, \tag{19}$$

[3] In Madan et al. (1998) the authors derive the expression for the Lévy measure using the VG representation as the difference of two Gamma processes and then change the parameters.

[4] See Example 8.10 in Sato (1999).

as proved in the Lemma 25.7 in Sato (1999). For the VG process it is easy to verify that it is satisfied. We need to add a correction term to X_t to satisfy the martingale condition[5]. The following process is a martingale:

$$S_t = S_0 e^{(r+\omega)t+X_t}. \tag{20}$$

where $w = \frac{1}{\kappa}\log(1-\theta\kappa - \frac{1}{2}\sigma^2\kappa)$. Passing to the log-prices $Y_t = \log(S_t)$, we get a Poisson process as in Eq. (6) with $b' = r+\omega$

$$Y_t = Y_0 + (r+\omega)t + \int_{\mathbb{R}} xN(t,dx). \tag{21}$$

Let $V(t,Y_t)$ be the value of an option at time t. By the *martingale pricing theory*, the discounted price of the option is a martingale. From this it is possible to derive the partial integro-differential equation (PIDE) for the price of the option

$$\mathbb{E}^{\mathbb{Q}}\Big[d\big(e^{-rt}V(t,Y_t)\big)\Big] = \frac{\partial V(t,y)}{\partial t} + \mathcal{L}^{Y_t}V(t,y) - rV(t,y) = 0, \tag{22}$$

where $\mathcal{L}^{Y_t}V(t,Y_t)$ is the infinitesimal generator of the log-price process (21). The resulting PIDE is

$$\frac{\partial V(t,y)}{\partial t} + (r+\omega)\frac{\partial V(t,y)}{\partial y} + \int_{\mathbb{R}}\big[V(t,y+x) - V(t,y)\big]\nu(dx) = rV(t,y). \tag{23}$$

3 The multinomial method

In this section we introduce the multinomial method proposed in Yamada and Primbs (2004). The stock price is considered as a Markov chain with L possible future states at each time. In this setting, the time $t \in [t_0, T]$ is discretized as $t_n = t_0 + n\Delta t$ for $n = 0, ..., N$ and $\Delta t = (T - t_0)/N$. We denote the stock price at time t_n as $S(t_n) = S_n$.

Consider the up/down factors $u > d > 0$ and write the discrete evolution of the stock price S_n as:

$$S_{n+1} = u^{L-l}d^{l-1}S_n \qquad l = 1, ..., L \tag{24}$$

where each future state has transition probability p_l, satisfying $\sum_{l=1}^{L} p_l = 1$. The value of the stock at time t_n can assume $j \in [1, n(L-1)+1]$ possible values:

$$S_n^{(j)} = u^{n(L-l)+1-j}d^{j-1}S_0. \tag{25}$$

The multinomial tree is recombining if for a constant $c > 1$, $u/d = c$. Regarding our work, we only consider five branches, $L = 5$. As explained in Yamada and Primbs (2004), this number of branches is enough to model the features of a stochastic process up to its fourth moment.

[5] To find the correction ω we have to find the exponential moment of X_t using its characteristic function:

$$\mathbb{E}[e^{X_t}] = \phi_{X_t}(-i) = e^{-\omega t}$$

.

3.1 Moment matching

To determine the parameters of the Markov chain we ask for the local moments to be equal to that of the continuous process. First, rewrite the continuous process (21) as the sum of a drift term and a martingale term:

$$\begin{aligned} Y_{t+\Delta t} - Y_t &= (r+\omega)\Delta t + \int_{\mathbb{R}} xN(\Delta t, dx) \\ &= (r+\omega+\theta)\Delta t + \int_{\mathbb{R}} x\tilde{N}(\Delta t, dx) \end{aligned} \tag{26}$$

where $\theta = \int_{\mathbb{R}} x\nu(dx)$ is the mean of the Poisson process (for $\Delta t = 1$), and the compensated Poisson integral term is a martingale (see Appendix A).

We can pass to log-prices $Y_n = \log(S_n)$ in the discrete Eq. (24), and write it as the sum of a drift component and a random variable with L possible outcomes:

$$\begin{aligned} \Delta Y = Y_{n+1} - Y_n &= (L-l)\log(u) + (l-1)\log(d) \\ &= b\,\Delta t + (L-2l+1)\alpha(\Delta t). \end{aligned} \tag{27}$$

The term $b\,\Delta t$ is the drift term, while $(L-2l+1)\alpha(\Delta t)$ is a random variable that satisfies the martingale condition

$$\mathbb{E}\big[(L-2l+1)\alpha(\Delta t)\big] = \alpha(\Delta t)\sum_{l=1}^{L} p_l(L-2l+1) = 0,$$

with $\alpha(\Delta t)$ a function of Δt.

The corresponding up/down factors have the following representation:

$$u = \exp\left(\frac{b}{L-1} + \alpha(\Delta t)\right) \qquad d = \exp\left(\frac{b}{L-1} - \alpha(\Delta t)\right), \tag{28}$$

and we can readly see that if u/d is constant, the tree recombines.

Given the mean $c_1 = \mathbb{E}[\Delta Y] = b\Delta t$, the k-central moment is

$$\mathbb{E}\big[(\Delta Y - c_1)^k\big] = \alpha(\Delta t)^k\,\mathbb{E}\big[(L-2l+1)^k\big]. \tag{29}$$

The moment matching condition requires that the central moments of the discrete process (27) are equal to the central moments of the continuous process (26):

$$\alpha(\Delta t)^k\,\mathbb{E}\big[(L-2l+1)^k\big] = \mu_k. \tag{30}$$

Using the relation between central moments and cumulants (Eq. (67) in Appendix B) we can solve the linear system of equations for the transition probabilities:

$$p_1 = \frac{1}{196\alpha(t)^4}\left[\frac{3}{2}c_2^2 - 2c_2\alpha(t)^2 + 2c_3\alpha(t) + \frac{1}{2}c_4\right] \tag{31}$$
$$p_2 = \frac{1}{196\alpha(t)^4}\left[-6c_2 + 32c_2\alpha(t)^2 - 4c_3\alpha(t) - 2c_4\right]$$
$$p_3 = 1 + \frac{1}{196\alpha(t)^4}\left[3c_4 + 9c_2^2 - 60c_2\alpha(t)^2\right]$$
$$p_4 = \frac{1}{196\alpha(t)^4}\left[-6c_2 + 32c_2\alpha(t)^2 + 4c_3\alpha(t) - 2c_4\right]$$
$$p_5 = \frac{1}{196\alpha(t)^4}\left[\frac{3}{2}c_2^2 - 2c_2\alpha(t)^2 - 2c_3\alpha(t) + \frac{1}{2}c_4\right].$$

The drift parameter (for $\Delta t = 1$) can be easily computed as $b = r + \omega + \theta$. The only missing parameter to determine is $\alpha(\Delta t)$. This is a function of the time increment Δt and can be be determined using the higher order in the moment matching condition together with the condition of positive probabilities.

Recall that in the well known binomial model for a diffusion process, it takes the value $\alpha(\Delta t) = \tilde{\sigma}\sqrt{\Delta t}$, and represents the volatility of the increments in Δt, see Cox et al. (1979). In the trinomial model, it takes the well known value $\alpha(\Delta t) = \frac{3}{4}\tilde{\sigma}\sqrt{\Delta t}$, see for instance Yamada and Primbs (2001). For the multinomial method a good representation for the parameter is

$$\alpha(\Delta t) = \sqrt{c_2}\sqrt{\frac{3+\bar{\kappa}}{12}}, \tag{32}$$

where $\bar{\kappa} = c_4/c_2^2$ is the excess of kurtosis[6]. We refer to the paper of Yamada and Primbs (2004) for the derivation. This choiche guarantees that the probabilities p_i for $i = 1...5$ are always positive and sum to one. We can use the formula (32), together with (31), to obtain the simpler form:

$$[p_1, p_2, p_3, p_4, p_5] = \left[\frac{3+\bar{\kappa}+s\sqrt{9+3\bar{\kappa}}}{4(3+\bar{\kappa})^2}, \frac{3+\bar{\kappa}-s\sqrt{9+3\bar{\kappa}}}{2(3+\bar{\kappa})^2}, \right. \tag{33}$$
$$\left. \frac{3+2\bar{\kappa}}{2(3+\bar{\kappa})}, \frac{3+\bar{\kappa}+s\sqrt{9+3\bar{\kappa}}}{2(3+\bar{\kappa})^2}, \frac{3+\bar{\kappa}-s\sqrt{9+3\bar{\kappa}}}{4(3+\bar{\kappa})^2}\right].$$

where $s = c_3/\sqrt{c_2^3}$ is the skewness.

Remark 1. *The standard deviation of a Lévy process with finite moments follows the square root rule. This means that the term $\alpha(\Delta t)$ has to be proportional to the square root of Δt. In the binomial and trinomial models, the proportionality constant is explicit, while for the pentanomial method it is implicit in the formula (32). Expanding the formula using the expression (13) for the cumulants, it is possible to check that the square root rule is satisfied at first order in $\sqrt{\Delta t}$.*

[6] I use the bar over κ, to distinguish the kurtosis from the variance of the gamma process κ.

3.2 Convergence

In this section we prove that the multinomial method converges to a compound Poisson process that is an approximation of the Variance Gamma process.

We call a generic compound Poisson process (6), with the same first four cumulants (13) of the original VG process (21), the *approximated VG process* X^A. The cumulant generating function of the increment ΔX^A has the following series representation (see Appendix (B)):

$$H_{\Delta X^A}(u) = ic_1 u - \frac{c_2 u^2}{2} - \frac{ic_3 u^3}{3!} + \frac{c_4 u^4}{4!} + \mathcal{O}(u^5). \tag{34}$$

We can check that this expression holds for the VG increments as well, simply by using a Taylor expansion up to the fourth order on the VG Lévy exponent (12), and adding the addidional drift term $c_1 = b\Delta t = (r + \omega + \theta)\Delta t$.

Theorem 3.1. *The increments of the discrete Markov chain (27) and the increments of the approximated VG process X^A have the same distribution.*

Proof. The idea of the proof is to show that the cumulant generating function of the discrete process (27) coincides with that of the approximated VG process (34). We prove it using the moment matching condition (30).

$$\begin{aligned} H_{\Delta Y}(u) &= \log\big(\phi_{\Delta Y}(u)\big) = \log\Bigg(\mathbb{E}\big[e^{iu\Delta Y}\big]\Bigg) \\ &= \log\Bigg(\mathbb{E}\Bigg[e^{iu\big(b\Delta t + (L-2l+1)\alpha(\Delta t)\big)}\Bigg]\Bigg) \\ &= iub\Delta t + \log\Bigg(\mathbb{E}\Bigg[e^{iu\big((L-2l+1)\alpha(\Delta t)\big)}\Bigg]\Bigg). \end{aligned} \tag{35}$$

We can expand in Taylor series up to the fourth order in u, and use the moment matching condition (30) to obtain:

$$\begin{aligned} H_{\Delta Y}(u) &= \log\Bigg(\sum_{k=0}^{4} \frac{(iu)^k}{k!} \big(\alpha(\Delta t)\big)^k \mathbb{E}\Big[\big(L - 2l + 1\big)^k\Big] + \mathcal{O}(u^5)\Bigg) \\ &= \log\Bigg(\sum_{k=0}^{4} \frac{(iu)^k}{k!} \mu_k + \mathcal{O}(u^5)\Bigg) \\ &= \sum_{k=0}^{4} \frac{(iu)^k}{k!} c_k + \mathcal{O}(u^5) \\ &= H_{X^A}(u), \end{aligned} \tag{36}$$

where $c_0 = 0$. $\square$

Remark 2. *The proof can be easly generalized for a Taylor expansion of order n. For $n \to \infty$, the approximated VG process converges to the original VG process. However the number of branches of the discrete tree goes to infinity as well. The five branches we consider are enough to describe the features of the underlying process and, at the same time, keep the numerical problem quite simple.*

Theorem 3.2. *The distribution of the multinomial tree at time N converges to the distribution of the approximated VG process at time N, when $\Delta t \to 0$.*

For the proof of this theorem we refer to the proof in Section 4.2 of Yamada and Primbs (2004). They prove that when the $\Delta t \to 0$ the characteristic funcion of the multinomial tree converges to the characteristic function of a Poisson process.

4 Numerical results

In this section we present the steps to implement the algorithm to price European and American options by the multinomial method. Then we compare the results with the ones obtained by the PIDE method and Black-Scholes model.

4.1 Algorithm

In order to implement the multinomial method, we suggest the following algorithm:

1. Estimate the four risk neutral parameters $(b, \theta, \sigma, \kappa)$ of the VG process with drift (26). This can be done as described in Seneta (2004).
2. Compute the cumulants of the VG process with parameters estimated in 1. using the relations in (66). Then compute the mean, variance, skewness and kurtosis.
3. Compute the transition probabilities vector (33).
4. Compute the up/down factors u and d (28) and then the vector of prices S_N at terminal time N as in Eq. (25).
5. Evaluate the payoff of the option $V^N(S_N)$ at terminal time N.
6. Compute the values of the option at the previous time level. The value is the conditional expectation given the current value of the price of the five future option values:

$$V^n = e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[V^{n+1}(S_{n+1})\middle| S_n^{(k)} = s_n^{(k)}\right]. \tag{37}$$

7. In computing the price of an American option, the value at the previous time level is the maximum between the conditional expectation and the intrinsic value of the option. For an American put we have:

$$V^n = \max\left\{e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[V^{n+1}(S_{n+1})\middle| S_n^{(k)} = s_n^{(k)}\right], K - S_n^{(k)}\right\}. \tag{38}$$

8. Iterate the algorithm until the initial time t_0.

The points 1. and 2. can be skipped if using a non-parametric model and the moments can be computed directly from the data. However, in this work we want to start from a parametric model. We assume the following values for the VG parameters:

r	θ	σ	κ
0.06	-0.1	0.2	0.2

Table 1: r is the risk free interest rate. θ, σ, κ are the VG parameters.

4.2 European options

We compare the numerical results obtained for European call and put options with the values obtained solving the VG PIDE, Eq. (23).

- *VG PIDE*: We solve the partial integro-differential equation following the method proposed by Cont and Voltchkova (2005). The Lévy measure is singular in the origin and this is a problem for the computation of the integral term. The authors propose to approximate the small jumps with infinite activity, with a Brownian motion. Therefore the original VG process becomes an approximated jump diffusion process. The associated PIDE is then solved with the implicit-explicit scheme proposed in the same paper.

- *Multinomial*: We follow the algorithm proposed in the previous section. The number of time steps for all the computations is $N = 2000$.

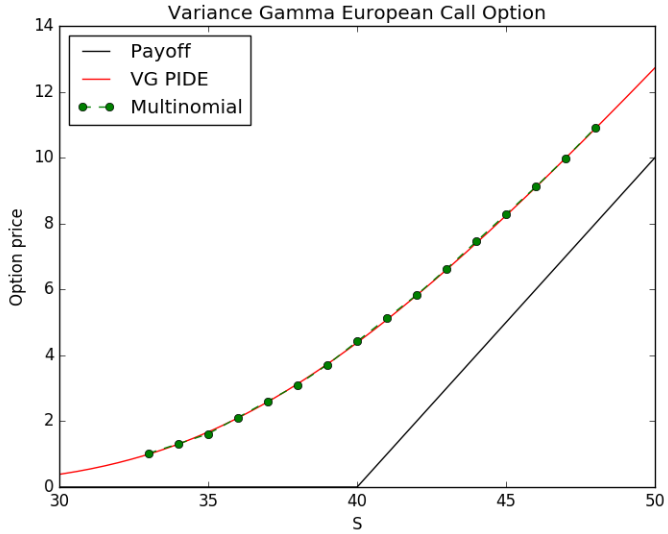

Figure 2: European call option with strike $K = 40$ and time to maturity 1 year.

Figures (2) and (3) show that the prices obtained by the multinomial method agree with the prices obtained by solving the VG PIDE.

There are many other methods to compute the price of an European call and put option, such as the closed formula developed by Madan et al. (1998), the FFT method of Carr and Madan (1998) and the Monte Carlo algorithms explained in Cont and Tankov (2003). The big advantage of the multinomial method is in the computation of the price of American options, where the other

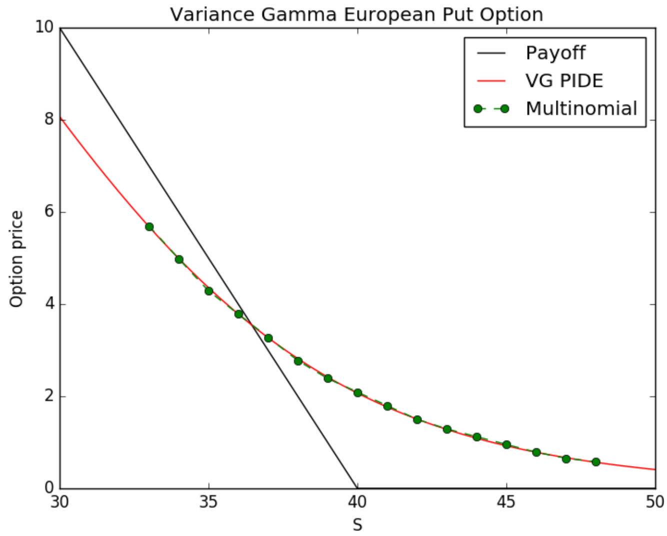

Figure 3: European put option with strike $K = 40$ and time to maturity 1 year.

algorithms (in particular PIDEs and Least Squares Monte Carlo) are difficult to implement and are much slower.

4.3 American options

In this section we present the numerical results for American put options and compare them with the prices obtained with the Black-Scholes model.

S_0	T	BS Eu. Put	VG Eu. Put	BS Am. Put	VG Am. Put
36	1	3.8443	3.7837	4.4867	4.3173
36	2	3.7632	3.7695	4.8483	4.8817
38	1	2.8521	2.7756	3.2573	3.2034
38	2	2.9901	3.0232	3.7512	3.8401
40	1	2.0660	2.0908	2.3194	2.3767
40	2	2.3553	2.4046	2.8897	2.9997
42	1	1.8413	1.5014	1.6214	1.6947
42	2	1.4648	1.9252	2.2167	2.3366
44	1	1.4296	1.1229	1.1132	1.2267
44	2	1.0171	1.5205	1.6936	1.8449

Table 2: Values for European and American put options using Black-Scholes and Variance Gamma model. Strike $K = 40$. BS values have $\sigma = 0.2$.

In Table (2), we present some results for European and American put options using the Black Scholes and the Variance gamma models. We choose the parameters and the values of strike and spot price in order to compare with other computations in the literature. The reader may compare our results with those obtained in Longstaff and Schwartz (2001).

Even if we compare results obtained with two different processes, the comparison makes sense as long as the processes have the same mean and variance. At first order approximation, we can ignore the term in θ^2 appearing in the formula for c_2. Therefore $c_2 = t(\sigma^2 + \theta^2\kappa) \approx t\sigma^2$.

The Black-Scholes prices are computed using a binomial algorithm. Of course the same values can be obtained with the multinomial algorithm in the limit of $\theta, \kappa \to 0$. Recall that under the Black-Scholes model, the log-returns follow a Brownian motion. Looking at the definition of the VG process (9), it is easy to see that when the drift θ and the variance of the Gamma subordinator κ are zero, the process turns out to be a Brownian motion:

$$X_t^{VG} \underset{\theta,\kappa\to 0}{\to} \sigma W_t$$

As a conseguence, the price process (20) converges to the Geometric Brownian Motion:

$$S_t = S_0 e^{(r+\omega)t + X_t} \underset{\theta,\kappa\to 0}{\to} S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}$$

where:

$$\begin{aligned}\lim_{\theta,\kappa\to 0} w &= \lim_{\theta,\kappa\to 0} \frac{1}{\kappa}\log(1-\theta\kappa-\frac{1}{2}\sigma^2\kappa) \\ &= -\frac{1}{2}\sigma^2\end{aligned}$$

5 Finite difference approximation

Consider the VG PIDE (23):

$$\frac{\partial V(t,x)}{\partial t} + (r+\omega)\frac{\partial V(t,x)}{\partial x} + \int_{\mathbb{R}} \big[V(t,x+y) - V(t,x)\big]\nu(dy) = rV(t,x). \quad (39)$$

We can expand $V(t,x+y)$ using the Taylor formula up to the fourth order:

$$\begin{aligned}V(t,x+y) = V(t,x) &+ \frac{\partial V(t,x)}{\partial x}y + \frac{1}{2}\frac{\partial^2 V(t,x)}{\partial x^2}y^2 \\ &+ \frac{1}{6}\frac{\partial^3 V(t,x)}{\partial x^3}y^3 + \frac{1}{24}\frac{\partial^4 V(t,x)}{\partial x^4}y^4\end{aligned} \quad (40)$$

and use the expression for the cumulants (see Appendix A). We call $\tilde{c}_n$ the cumulant evaluated at $t=1$:

$$\tilde{c}_n = \int_{\mathbb{R}} y^n \nu(dy). \quad (41)$$

The approximated equation is a fourth order PDE:

$$\begin{aligned}&\frac{\partial V(t,x)}{\partial t} + (r+\omega+\tilde{c}_1)\frac{\partial V(t,x)}{\partial x} + \frac{1}{2}\tilde{c}_2\frac{\partial^2 V(t,x)}{\partial x^2} \\ &+ \frac{1}{6}\tilde{c}_3\frac{\partial^3 V(t,x)}{\partial x^3} + \frac{1}{24}\tilde{c}_4\frac{\partial^4 V(t,x)}{\partial x^4} = rV(t,x)\end{aligned} \quad (42)$$

Consider the variable x in the interval $[x_{min}, x_{max}]$ and discretize time and space, such that $h = \Delta x = \frac{x_{max}-x_{min}}{N}$ and $\Delta t = \frac{T-t_0}{M}$ for $N, M \in \mathbb{N}$. Using the variables $x_i = x_{min} + ih$ for $i = 0, ..., N$ and $t_n = t_0 + n\Delta t$ for $n = 0, ..., M$, we use the short notation

$$V(t_n, x_i) = V_i^n.$$

We can use the following discretization for the time derivative, corresponding to an explicit method:

$$\frac{\partial V(t,x)}{\partial t} = \frac{V_i^{n+1} - V_i^n}{\Delta t}, \tag{43}$$

and the central difference for the spatial derivative:

$$\begin{aligned}
\frac{\partial V(t,x)}{\partial x} &= \frac{V_{i+h}^{n+1} - V_{i-h}^{n+1}}{2h}, \\
\frac{\partial^2 V(t,x)}{\partial x^2} &= \frac{V_{i+h}^{n+1} + V_{i-h}^{n+1} - 2V_i^{n+1}}{h^2}, \\
\frac{\partial^3 V(t,x)}{\partial x^3} &= \frac{V_{i+2h}^{n+1} - V_{i-2h}^{n+1} + 2V_{i-h}^{n+1} - 2V_{i+h}^{n+1}}{2h^3}, \\
\frac{\partial^4 V(t,x)}{\partial x^4} &= \frac{V_{i-2h}^{n+1} + V_{i+2h}^{n+1} - 4V_{i-h}^{n+1} - 4V_{i+h}^{n+1} + 6V_i^{n+1}}{h^4}.
\end{aligned} \tag{44}$$

The discretized equation is:

$$\begin{aligned}
&\left(\frac{V_i^{n+1} - V_i^n}{\Delta t}\right) + (r + \omega + \tilde{c}_1)\left(\frac{V_{i+h}^{n+1} - V_{i-h}^{n+1}}{2h}\right) \\
&+ \frac{1}{2}\tilde{c}_2\left(\frac{V_{i+h}^{n+1} + V_{i-h}^{n+1} - 2V_i^{n+1}}{h^2}\right) + \frac{1}{6}\tilde{c}_3\left(\frac{V_{i+2h}^{n+1} - V_{i-2h}^{n+1} + 2V_{i-h}^{n+1} - 2V_{i+h}^{n+1}}{2h^3}\right) \\
&+ \frac{1}{24}\tilde{c}_4\left(\frac{V_{i-2h}^{n+1} + V_{i+2h}^{n+1} - 4V_{i-h}^{n+1} - 4V_{i+h}^{n+1} + 6V_i^{n+1}}{h^4}\right) = rV_i^n
\end{aligned} \tag{45}$$

Rearranging the terms we obtain:

$$\begin{aligned}
(1 + r\Delta t)V_i^n =& V_{i+h}^{n+1}\left[\frac{(r + \omega + \tilde{c}_1)\Delta t}{2h} + \frac{\tilde{c}_2\Delta t}{2h^2} - \frac{\tilde{c}_3\Delta t}{6h^3} - \frac{\tilde{c}_4\Delta t}{6h^4}\right] \\
&+ V_{i-h}^{n+1}\left[\frac{-(r + \omega + \tilde{c}_1)\Delta t}{2h} + \frac{\tilde{c}_2\Delta t}{2h^2} + \frac{\tilde{c}_3\Delta t}{6h^3} - \frac{\tilde{c}_4\Delta t}{6h^4}\right] \\
&+ V_{i+2h}^{n+1}\left[\frac{\tilde{c}_3\Delta t}{12h^3} + \frac{\tilde{c}_4\Delta t}{24h^4}\right] \\
&+ V_{i-2h}^{n+1}\left[-\frac{\tilde{c}_3\Delta t}{12h^3} + \frac{\tilde{c}_4\Delta t}{24h^4}\right] \\
&+ V_i^{n+1}\left[1 - \frac{\tilde{c}_2\Delta t}{h^2} + \frac{\tilde{c}_4\Delta t}{4h^4}\right].
\end{aligned} \tag{46}$$

If we rename the coefficients, the equation is:

$$(1 + r\Delta t)V_i^n = V_{i+h}^{n+1}p_{+h} + V_{i-h}^{n+1}p_{-h} + V_{i+2h}^{n+1}p_{+2h} + V_{i-2h}^{n+1}p_{-2h} + V_i^{n+1}p_0. \tag{47}$$

The coefficients can be interpreted as the (risk neutral) transition probabilities for the Markov chain:

$$X(t_{n+1}) = \begin{cases} X(t_n) + 2h & \text{with } \mathbb{P}(x_i \to x_i + 2h) = p_{+2h} \\ X(t_n) + h & \text{with } \mathbb{P}(x_i \to x_i + h) = p_{+h} \\ X(t_n) & \text{with } \mathbb{P}(x_i \to x_i) = p_0 \\ X(t_n) - h & \text{with } \mathbb{P}(x_i \to x_i - h) = p_{-h} \\ X(t_n) + 2h & \text{with } \mathbb{P}(x_i \to x_i - 2h) = p_{-2h} \end{cases}$$

It is straightforward to verify that the probabilities sum to 1. The value of the option in the previous time step is thus the discounted expectation under the risk neutral probability measure $\mathbb{Q}$:

$$V_i^n = \frac{1}{1 + r\Delta t} \mathbb{E}^{\mathbb{Q}} \left[V^{n+1}(X(t_{n+1})) \middle| X(t_n) = x_i \right]. \tag{48}$$

Define the increments $\Delta X = X(t_{n+1}) - X(t_n)$. We check that the local properties for the moments of the Markov chain are satisfied:

$$\mu' = \mathbb{E}[\Delta X] = \big(r + \omega + \tilde{c}_1\big)\Delta t \tag{49}$$

$$\mu_2' = \mathbb{E}[\Delta X^2] = \tilde{c}_2 \Delta t \tag{50}$$

$$\mu_3' = \mathbb{E}[\Delta X^3] = \big((r + \omega + \tilde{c}_1)h^2 + \tilde{c}_3\big)\Delta t \tag{51}$$

$$\mu_4' = \mathbb{E}[\Delta X^4] = \big(\tilde{c}_2 h^2 + \tilde{c}_4\big)\Delta t. \tag{52}$$

At first order in Δt we can calculate the variance, skewness and kurtosis[7] :

$$\mathrm{Var}[\Delta X] \approx \tilde{c}_2 \Delta t \tag{53}$$

$$\mathrm{Skew}[\Delta X] \approx \frac{(r + \omega + \tilde{c}_1)}{(\tilde{c}_2)^{3/2}} \frac{h}{\sqrt{\Delta t}} + \frac{\tilde{c}_3}{(\tilde{c}_2)^{3/2}} \frac{1}{\sqrt{\Delta t}} \tag{54}$$

$$\mathrm{Kurt}[\Delta X] \approx \frac{h^2}{\tilde{c}_2} \frac{1}{\Delta t} + \frac{\tilde{c}_4}{(\tilde{c}_2)^2} \frac{1}{\Delta t}. \tag{55}$$

So, with a step size h proportional to the square root of Δt as in (32), we confirm that the local variance, skewness and kurtosis are consistent with their definition in terms of cumulants, up to a constant factor.

Using the a step size $h = 2\alpha(\Delta t)$, these probabilities can be approximated by the probabilities in (33). However, the probabilities obtained by the discretization of the PDE are not always positive. The two sets of probabilities are close only for a well determined set of parameters. This can be a topic of further research. We can plot, for example, the two probabilities obtained respectively by Moment Matching and PDE discretization:

$$p_3^{MM} = \frac{3 + 2\bar{\kappa}}{2(3 + \bar{\kappa})} \quad \text{and} \quad p_3^{PDE} = 1 - \frac{\tilde{c}_2 \Delta t}{h^2} + \frac{\tilde{c}_4 \Delta t}{4h^4}$$

[7]Remind that $\mathrm{Skew}[X] = \frac{\mu_3}{\mu_2^{3/2}}$ and $\mathrm{Kurt}[X] = \frac{\mu_4}{\mu_2^2}$, with μ_i the central i-th moment. Remind also that $\mu_3 = \mu_3' - 3\mu'\mu_2' + 2\mu'^3$ and $\mu_4 = \mu_4' - 4\mu'\mu_3' + 6\mu'^2\mu_2' - 3\mu'^4$

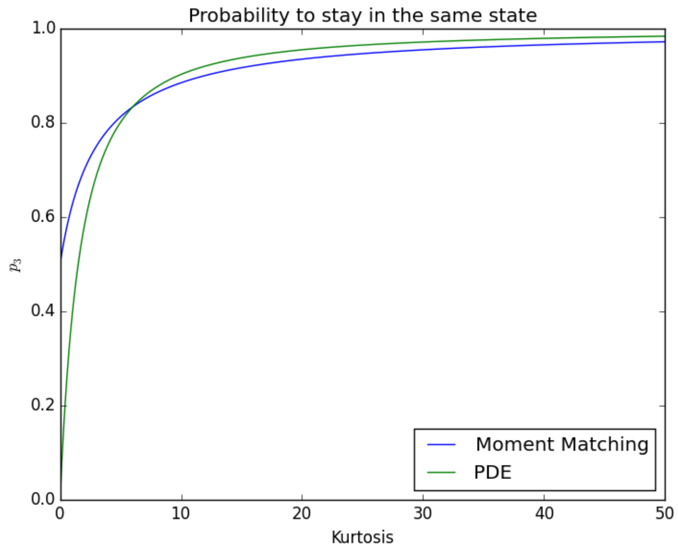

Figure 4: Probabilities p_3^{MM} and p_3^{PDE} as functions of the kurtosis.

6 Conclusions

In this paper we show how to price options using a multinomial method when the underlying price is modelled as a Variance Gamma process. The multinomial method is well known in the literature, see for example Cont and Tankov (2003), Yamada and Primbs (2001), Yamada and Primbs (2003) and Yamada and Primbs (2003), but in the literature the are no works (to our knowledge) that analyse the VG process and compare it with other results.

The VG process is approximated by a general jump process that has the same firsts four cumulants of the original VG process. We proved that the multinomial method converges to this approximated process. We obtained numerical results for European and American options, and compared them with PIDE methods and with results computed within the Black Scholes framework. It turns out that the multinomial method is faster than Finite Differences methods and easier to implement.

We proposed a topic of further research in Section 5. The probabilities obtained by discretizing the approximated PDE are not always positive. They are related with the probabilities obtained by moment matching for some particular choice of the parameters. This relation can be further investigated. Another possible topic of further research is the comparison of our results for the American options with other numerical methods such as the Least Square Monte Carlo (Longstaff and Schwartz (2001)) and finite differences (Almendral and Oosterlee (2007)).

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A Poisson integration

A convenient tool for analysing the jumps of a Lévy process is the random measure of jumps. With this formalism is possible to describe jump processes with infinite activity, like the VG. The jump process associated to the Lévy process X_t is defined, for each $0 \leq t \leq T$, by:

$$\Delta X_t = X_t - X_{t-} \tag{56}$$

where $X_{t-} = \lim_{s \uparrow t} X_s$. Consider a set $A \in \mathcal{B}(\mathbb{R} \backslash \{0\})$, the *random measure* of the jumps of the process X_t is defined by:

$$\begin{aligned} N(t, A)(\omega) &= \#\{\Delta X_s(\omega) \in A \ : \ 0 \leq s \leq t\} \\ &= \sum_{s \leq t} 1_A(\Delta X_s(\omega)). \end{aligned} \tag{57}$$

This measure counts the number of jumps of size in A, up to time t. Fix $A \in \mathcal{B}(\mathbb{R} \backslash \{0\})$. The process $N(t, A)(\omega)$ is a Poisson process with intensity

$$\nu(A) = \mathbb{E}[N(1, A)], \tag{58}$$

(see Applebaum (2009) theorem 2.3.5). The process $N(t, A)$ is called *Poisson random measure*. The Lévy measure corresponds to the intensity of the Poisson measure. The *Compensated Poisson random measure* is defined by

$$\tilde{N}(t, A) = N(t, A) - t\nu(A), \tag{59}$$

which is a martingale.

The next step is to define the integration with respect to a random measure. Following Applebaum (2009), let $f : \mathbb{R} \to \mathbb{R}$ be a Borel-measurable function. We define the *Poisson integral* of f as:

$$\int_A f(x) N(t, dx)(\omega) = \sum_{x \in A} f(x) N(t, \{x\})(\omega). \tag{60}$$

For the case of integration of the identity funcion, we see that every compound Poisson process can be represented by:

$$X_t = \sum_{s \in [0,t]} \Delta X_s = \int_0^t \int_{\mathbb{R}} x N(dt, dx). \tag{61}$$

We can also define in the same way the *compensated Poisson integral* with respect the compensated Poisson measure.

We present a last formula for computing the moments of a general compound Poisson process. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function and $X_t = \int_A f(x)N(t,dx)$, the characteristic function of X_t is:

$$\begin{aligned}\mathbb{E}[e^{iuX_t}] &= \mathbb{E}\left[\exp\left(iu\int_A f(x)N(t,dx))\right)\right] \qquad (62)\\ &= \exp\left(t\int_A [e^{iuf(x)}-1]\nu(dx)\right).\end{aligned}$$

Assuming that $\mathbb{E}[X_t^n] < \infty$, all the moments can be computed from (62) by differenciation using eq:

$$\mathbb{E}[X_t^n] = \frac{1}{i^n}\frac{\partial^n \phi_{X_t}(u)}{\partial u^n}\bigg|_{u=0}, \quad \forall n \in \mathbb{N}. \qquad (63)$$

For the case of f identity function and $A = \mathbb{R}$, we find the expression for the cumulants using (66).

$$c_n = t\int_{\mathbb{R}} x^n \nu(dx). \qquad (64)$$

The cumulants of X_t are thus the moments of its Lévy measure.

B Cumulants

The cumulant generating function $H_{X_t}(u)$ of X_t is defined as the natural logarithm of its characteristic function (see Cont and Tankov (2003)). Using the Lévy-Khintchine representation for the characteristic function (1), it is easy to find its relation with the Lévy simbol:

$$\begin{aligned}H_{X_t}(u) &= \log(\phi_{X_t}(u)) \qquad (65)\\ &= t\eta(u)\\ &= \sum_{n=1}^{\infty} c_n\frac{(iu)^n}{n!}\end{aligned}$$

The *cumulants* of a Lévy process are thus defined by:

$$c_n = \frac{t}{i^n}\frac{\partial^n \eta(u)}{\partial u^n}\bigg|_{u=0}. \qquad (66)$$

The cumulants are closely related to the central moments μ_n:

$$\mu_0 = 1, \quad \mu_1 = 0, \quad \mu_n = \sum_{k=1}^{n}\binom{n-1}{k-1}c_k\mu_{n-k} \quad \text{for } n > 1. \qquad (67)$$

For a Poisson process with finite firsts n moments, all the information about the cumulants is contained inside the Lévy measure. Expand in Taylor series the exponential

$$e^{iux} \approx 1 + iux - \frac{u^2x^2}{2} - \frac{iu^3x^3}{3!} + \frac{u^4x^4}{4!} + \dots$$

The Lévy symbol from the representation (7) becomes:

$$\begin{aligned} t\eta(u) &= ib'ut + t\int_{\mathbb{R}} (e^{iux}-1)\nu(dx) \qquad (68) \\ &= i\Big(b - \int_{|x|<1} x\nu(dx)\Big)ut + iut\int_{\mathbb{R}} x\nu(dx) - \frac{u^2}{2}t\int_{\mathbb{R}} x^2\nu(dx) \\ &\quad - \frac{iu^3}{3!}t\int_{\mathbb{R}} x^3\nu(dx) + \frac{u^4}{4!}t\int_{\mathbb{R}} x^4\nu(dx) + \dots \\ &= ic_1u - \frac{c_2u^2}{2} - \frac{ic_3u^3}{3!} + \frac{c_4u^4}{4!} + \dots \end{aligned}$$

with $c_1 = t\big(b + \int_{|x|\geq 1} x\nu(dx)\big)$.

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