Measuring the impact of a *bonus-malus* system in finite and continuous time ruin probabilities for large portfolios in motor insurance

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Abstract

Motor insurance is a very competitive business where insurers operate with quite large portfolios, often decisions must be taken under short horizons and therefore ruin probabilities should be calculated in finite time. The probability of ruin, in continuous and finite time, is numerically evaluated under the classical Cramér-Lundberg risk process framework for a large motor insurance portfolio, where we allow for *a posteriori* premium adjustments, according to the claim record of each individual policyholder.

Focusing on the classical model for *bonus-malus* systems we propose that the probability of ruin can be interpreted as a measure to decide between different *bonus-malus* scales or even between different *bonus-malus* rules. In our work the required initial surplus can also be evaluated.

We consider an application of a *bonus-malus* system for motor insurance to study the impact of experience rating in ruin probabilities. For that we used a real commercial scale of an insurer operating in the portuguese market, and we also work various well known *optimal bonus-malus* scales estimated with real data from that insurer. Results involving these scales are discussed.

Keywords: Ruin probability; finite time ruin probability; bonus-malus; Markov chain; experience rating.

1 Introduction

Ruin probability, either finite or infinite, is often computed using the classical Cramér-Lundberg model, where the premium is paid continuously at a constant rate. See for instance Asmussen and Albrecher (2010) for comprehensive references. Extensions to that model where premia are variable and adjusted according to the past experience are known as well as extensions with particular *bonus-malus* applications for motor insurance [see recent paper by Li *et al.* (2015) and, in particular, Chapter VIII of Asmussen and Albrecher (2010)]. As far as models with varying premia and ruin probabilities are concerned, most known works consider ultimate ruin probabilities. Relevant of these works are addressed, and listed, below at the end of this section. Motor insurance is a very competitive business where insurers operate with large portfolios, often decisions must be taken under short horizons, so ruin probabilities should be calculated for shorter term periods, i.e., in finite time. With these features we particularly highlight the model developed by Afonso *et al.* (2009). This work is going to be the basis to our work, where a generic varying premium is considered, it is applicable to large portfolios and allows to compute finite-time ruin probabilities. There, a mix of calculation and simulation procedure is used. The same authors adapted the same model for the calculation of ruin probabilities where premia are updated using credibility estimation [see Afonso *et al.* (2010)]. In this manuscript, we adapt it for portfolios with *bonus-malus* updated premia, it allows to get fast results for a finite horizon and continuous time framework.

For application into motor insurance the model by Afonso *et al.* (2009) needs some adaptation. In Afonso *et al.* (2009) each year premium is depending on the surplus at the end of the previous year, calculated so that the ultimate ruin probability is given by a pre-determined value. Obviously, insurers benefit from a good global, portfolio behavior, or are penalized otherwise. In Afonso *et al.* (2010) it assumed a credibility theory approach, where premia can be different, and so individualized, according to each risk characteristic, assigned by a risk parameter, so that the portfolio is not completely homogeneous and we have a collection of different risk types. We can consider that the parameter is known or unknown. In the latter case we have to estimate each annual risk premium, based on past experience. It is quite different when we introduce a classical *bonus-malus* system (shortly BMS) into a motor portfolio. Like in the previous models, premia are variable on an annual basis, charged at the beginning, however, the next year premium for each risk is commonly depending on the current *bonus-malus* class, where the class in determined by the number of claims within the year. In this manuscript, we adopted classical or standard models only consider the class determination depending on the claim counts and not on the claim amounts. Models that consider claim amounts (e.g. see Ni *et al.* (2014) and references therein) must have a different build and are out of scope in this work. Technically, in the classical models the dynamics are estimated through Markov chain procedures. This procedure brings a higher variation in the premia during the lifetime of the portfolio [see e.g. Lemaire (1995)], when compared to a standard procedure. The model in Afonso *et al.* (2009), that we retrieve, is built under the assumption of an homogeneous classical compound Poisson process, with Afonso *et al.* (2010) we bring some sort of heterogeneity to the portfolio. Under a BMS we often work under a mixed Poisson framework and premium varying on claim counts only.

To be more precise, past annual claim counts allow for determining incoming year rating class of each individual policyholder and calculate the appropriate risk premium. In a portfolio perspective, the claim number distribution is essential to estimate the future allocation of policyholders among *bonus-malus* levels and, therefore, the aggregate risk premium of the portfolio. That together with aggregate claims, is necessary to compute ruin probabilities. The claim severity does not matter for the premium allocation in the BMS perspective, but it does for computing ruin probabilities.

To ease the reading we shorten common vocabulary in BMS (*bonus-malus* system or systems), so that a *bonus* scale means a particular BMS with a particular collection of premia, *bonus* level or class means a particular *bonus-malus* level in the *bonus* scale.

At the end of this paper, we will be contributing to answering questions like:

- How big is the impact of an evolving premium, along the years, in the portfolio ruin probability for a particular BMS, when compared with a fixed premium?

- For a given set of *bonus* scales, which scale provides a lower ruin probability for a given time span?

- For a given set of transition rules and claim frequency distribution, which combination of initial surplus, *bonus* scale and ruin probability obtained is acceptable for the manager?

Regarding the *a posteriori* ratemaking, we limit our study to the classical model for BMS [see Lemaire (1995), p. 6]. In order to evaluate the impact of a BMS in the ruin probabilities, we will compare the performance of several well known optimal *bonus* scales, estimated with real data from an automobile third-party liability portfolio of an insurer operating in the portuguese market, as well as his actual commercial scale. Those optimal scales are the ones proposed by Norberg (1976), Borgan *et al.* (1981), Gilde and Sundt (1989) and Andrade e Silva and Centeno (2005). This analysis may be used as a measure to help the insurer on the decision about which *bonus* scale should be implemented for the portfolio.

Different authors have addressed models with a varying premium based on claim counts. In Dubey (1977) the premium is also a function of the number of claims, but it does not consider a BMS. Dufresne (1988) also studies the ruin probability using simulation techniques, but in a stationary distribution environment. Wagner (2001) and then Wu *et al.* (2008) derive recursion formulae for the ruin probability in a two state Markov model but in an infinite time approach. Recently, Li *et al.* (2015) considered computing ruin probabilities where the Poisson parameter is a continuous random variable and use credibility theory arguments to adjust the premium rate *a posteriori*. Even more recently, Constantinescu *et al.* (2016) discusses ruin probabilities in a model with dependence on the number of claims that can be viewed as an application to a "no claims discount" system (where only *bonuses* are allowed, not malus). Again, in both cases, only ruin probabilities in an infinite time horizon are considered. With the method proposed in this paper we can compute ruin probabilities in a portfolio with a BMS at any time (year) moment.

The work in this paper evolves as follows. Section 2 introduces the basic framework, the model, reviews the classical BMS in the framework of an homogeneous Markov chain, and summarises the simulation and calculation procedure. Section 3 presents an illustration with data of a motor portfolio supplied by a Portuguese insurer, the model results on the effects of a BMS on the probability of ruin followed by a discussion. Some concluding remarks are written in the last section.

2 Basic framework

2.1 Modelling the ruin probability

We introduce our base model, main definitions and notation, retrieved from Afonso *et al.* (2009). We may define and introduce locally some other definitions and notation. Consider a risk process over an n-year period. We denote by $S(t)$ the aggregate claims up to time t, so that $S(0) = 0$ and by Y_i the aggregate claims in year i, so that $Y_i = S(i) - S(i-1)$. $\{Y_i\}_{i=1}^n$ is a sequence of independent and identically distributed (shortly *i.i.d.*) random variables with common compound Poisson distribution, whose first three moments exist. Poisson parameter is denoted as λ, i.e., λ (> 0) is the mean of the annual number of claims. Let us also set $f(\cdot, s)$ as the probability density function (*p.d.f.*) of $S(s)$ for $0 < s \leq 1$.

Let P_i denote the total amount of premium charged in the portfolio in year i, which depends on the distribution of policies through the *bonus* levels. Let $U(t)$ denote the insurer's surplus at time t, $0 \leq t \leq n$. It is assumed that premia are received continuously at a constant rate throughout each year. The initial surplus, u ($= U(0)$), and the initial premium, P_1, are known. For each year i, $i \geq 2$, the premium P_i and surplus level $U(i)$ are random variables since they both depend on the claims experience in previous years. We note that, as usual, whenever we wish to refer to a particular realization of these variables, we will use the lower case letters p_i and $u(i)$, respectively.

The evolution of the surplus of an insurance company or portfolio, $U(t)$, for any time t, $0 \leq t \leq n$, as defined in Afonso *et al.* (2009), is driven by equation:

$$U(t) = u + \sum_{j=1}^{i-1} P_j + (t - i + 1)P_i - S(t) \quad , \tag{2.1}$$

where i is the positive integer such that $t \in [i-1, i)$ and $\sum_{j=1}^{0} P_j = 0$, by convention. For a better perception of the following results, let us state basic assumptions for the portfolio:

- The portfolio is homogeneous with respect to claim severities;
- The portfolio is heterogeneous with respect to claim frequencies, following a mixed Poisson distribution;
- We consider a homogeneous claim frequency in each *bonus* level or class, in class j the number of claims in one year follows a Poisson distribution with parameter λ_j;
- The portfolio is closed for ingoing and outgoing policyholders.

Let $\psi(u, n)$ denote the probability of ruin in continuous time within a period of n years and $\psi(u(i-1), 1, u(i))$ the approximation to the probability of ruin within year i, given the surplus $u(i-1)$ at the start of the year, $u(i) \geq 0$ the surplus at the end of the year and a rate of premium income p_i during the year. For details see Afonso *et al.* (2009).

Let $H(s) + \kappa\, s$ be a random variable with a translated Gamma distribution whose first three moments match those of $S(s)$. We denote the parameters of the translated Gamma as α, β and κ, respectively the shape, scale and translation parameters, $\alpha, \beta > 0$ and $\kappa \in \mathbb{R}$. Denoting $F_G(\cdot, s)$ the cumulative distribution function and $f_G(\cdot, s)$ the *p.d.f.* of $H(s)$, Afonso *et al.* (2009) show how the approximation to the ruin probability in year i is defined and how to calculate the parameters α, β and κ. They obtained:

$$\psi(u(i-1),1,u(i)) = \frac{\int_{s=0}^{1-u(i)/p_i} f_G(u(i-1)+(p_i-\kappa)s,s)\frac{u(i)}{(1-s)}f_G((p_i-\kappa)(1-s)-u(i),1-s)ds}{f_G(u(i-1)+p_i-\kappa-u(i),1)} + \frac{f_G(u(i-1)+(p_i-\kappa)(1-\frac{u(i)}{p_i}),1-\frac{u(i)}{p_i})F_G(-\kappa u(i)/p_i,u(i)/p_i)}{f_G(u(i-1)+p_i-\kappa-u(i),1)}. \tag{2.2}$$

The estimated probability of ruin for a finite time, say n, will be obtained using this formula inserted in a simulation procedure that is described in Subsection 2.3.

2.2 BMS for homogeneous Markov chains

In this subsection we introduce main definitions and quantities from the classical BMS as known from the literature. Following Lemaire (1995), see also Denuit *et al.* (2007), for a BMS with transition rules based only on claim frequency we consider that the level/class, for each policyholder, in a given annual period is determined uniquely by the class of the preceding year and by the number of claims reported during that time period. The classical model for BMS is defined by the triplet $\Delta = (\mathbf{T}, \mathbf{b}, i_0)$, where $\mathbf{b} = (b_1, \ldots, b_L)'$ is the *bonus* scale, i_0 identifies the initial class and $\mathbf{T}$ denotes the $(L \times L)$ transition rules matrix for a BMS with L classes.

The probability of moving from class l to class j in one year, for a policyholder with annual claim frequency mean λ, denoted as $p_{T,\lambda}(l,j)$, is given by

$$p_{T,\lambda}(l,j) = \sum_{k=0}^{\infty} p_k(\lambda)\, t_{lj}(k)\ ,\ l,j = 1,\ldots,L\,, \tag{2.3}$$

where $p_k(\lambda)$ is the probability that an insured, with mean claim frequency λ, reports k claims in one year, $t_{lj}(k) = 1$ if k claims reported lead a policy to move from class l to class j and $t_{lj}(k) = 0$ otherwise.

Considering a given λ, a BMS can be modeled by a finite homogeneous Markov chain with state space $E = \{1, 2, \ldots, L\}$ and one step transition probability matrix

$$\mathbf{P}_{T,\lambda} = \left[p_{T,\lambda}(l,j)\right]_{L\times L} = \sum_{k=0}^{\infty} p_k(\lambda)\mathbf{T}_k\,.$$

with $\mathbf{T}_k = [t_{lj}(k)]\ ,\ l,j = 1,\ldots,L\ ,\ k \in \mathbf{N}_0$.

For the BMS defined above by Δ, let $\pi_{\Delta,\lambda}^{(i)}(j)$ be the conditional probability of an insured, for a given λ, belonging to class j after i steps. This probability is easily obtained from the i-step transition matrix and initial distribution. Assume that $\mathbf{T}_k$ is a set of transition rules that define an irreducible and aperiodic Markov chain in a classical BMS portfolio. It is known, see Parzen (1965), that the stationary distribution, denoting the probability of a policyholder belonging to class j in the long run, is given by the limiting distribution of the Markov chain:

$$\boldsymbol{\pi}_{T,\lambda} = \left[\pi_{T,\lambda}(j)\right]_{1\times L} = \left[\lim_{i\to+\infty} \pi_{\Delta,\lambda}^{(i)}(j)\right]_{1\times L}.$$

To express the heterogeneity of the portfolio with respect to the claim frequency, it is common to consider λ as an outcome of a positive random variable, say Λ, with distribution function denoted as $V_\Lambda(\cdot)$. As widely set in the BMS literature, the unconditional probability of an insured belonging to class j, after i steps, and the long run distribution, for a policyholder chosen at random from the portfolio, is expressed as the expectation with respect to Λ, respectively

$$\pi_\Delta^{(i)}(j) = \int_0^\infty \pi_{\Delta,\lambda}^{(i)}(j)\, dV(\lambda) \quad , \; j = 1, \ldots, L \tag{2.4}$$

and

$$\pi_T(j) = \int_0^\infty \pi_{T,\lambda}(j)\, dV(\lambda) \quad , \; j = 1, \ldots, L\,. \tag{2.5}$$

The total amount of premia to be charged annually for the set of policyholders in the portfolio, is not constant over time since it depends on the distribution of policyholders among the *bonus* levels and is the sum of total premia in each class. For a given year i and known involved quantities, total premium in the presence of a BMS can be computed using the formula,

$$P_i = (1+\theta) NPol \sum_{j=1}^{L} E\left[S(1)\right] \pi_\Delta^{(i)}(j)\, b_j \quad , \; i = 1, \ldots, n\,, \tag{2.6}$$

where $\theta > 0$ is the safety loading parameter, $NPol$ the total number of policies in the portfolio. Note that we consider $E\left[S(1)\right]$ to be dependent on j. For BMS based only on claim frequency, there is an implicit assumption that average claim size is constant across BMS classes. We consider that assumption in our developments. For a more detailed view over BMS please consider Lemaire (1995) or Denuit *et al.* (2007).

In our application quantities $E\left[S(1)\right]$ and $\pi_\Delta^{(i)}(j)$ have to be estimated with historical data, annual number of claims in class j will be considered Poisson distributed with mean λ_j, $j = 1, 2, \ldots, L$ (λ_j is going to be estimated as well). As said above mean claim size is constant across BMS classes. Also, starting premium, in year $i = 1$, is fixed and is given.

2.3 Simulation and calculation procedure

In this section we summarize the steps of the simulation (and calculation) procedure. Again, this is taken, with adaptations, from Afonso *et al.* (2009). This model is targeted for large portfolios, we need to calculate annual aggregate claims, the authors suggest the approximation by a translated gamma with parameters α, β and κ as introduced at the end of Subsection 2.1.

1. Estimation of λ_j , $j = 1, \ldots, L$.

   From historical data, estimate the mean claim frequency of bonus class j , $j = 1, \ldots, L$. Recall that, for class j, the number of claims is Poisson distributed with parameter λ_j.

2. Estimation of the expected number of claims for the portfolio, in year i , $i = 1, \ldots, n$.

   The expected number of claims in the portfolio, for year i, is given by

$$E[N_i] = NPol \sum_{j=1}^{L} \lambda_j\, \pi_\Delta^{(i)}(j)\,,\; i = 1, \ldots, n \tag{2.7}$$

estimated according to the Markov chain underlying the BMS, see (2.4).

3. For a given value of initial surplus, u, simulation of the aggregate claim amount for each year i, $\{Y_i\}_{i=1}^n$.

   Let Y_i be the aggregated claim amount in a given year i, assumed to have (approximately) a translated Gamma distribution. Calculate the parameters of the translated Gamma distribution, α_i, β_i, κ_i, for each year $i\,,: i = \ldots, n$, that match the first three moments of Y_i, considering the results obtained in step 2 and historical data for claim amounts, see Afonso *et al.* (2009).

4. Estimation of the premium collected in each year i, $P_i\,,\ i = 1, \ldots, n$.

   Estimate the total amount of premium collected in year i, using (2.6), for a given *bonus* scale $\mathbf{b} = (b_1, \ldots, b_L)'$.

5. Estimation of the ruin probability in year n, $\psi(u, n)$.

   This step is performed as follows:

   (a) From the simulated values of $\{Y_i\}_{i=1}^n$, say $\{y_i\}_{i=1}^n$, calculate successively the surplus at the end of each year: $u(1)$ $(= u + p_1 - y_1)$, and $u(i)$ $(= u(i-1) + p_i - y_i)$ for $i = 2, \ldots, n$.

   (b) Denote as $\psi_m(u, n)$ the ruin probability in simulation, or run, number m. In run m, if $u(i) < 0$ for any $i, i = 1, 2, \ldots, n$, we set $\psi_m(u, n) = 1$ and start simulation $m + 1$, $m = 1, \ldots, M - 1$, where M is the number of runs for each path set.

   (c) If $u(i) \geq 0$ for all $i, i = 1, 2, \ldots, n$, we calculate the approximation for run m $\psi_m(u(i-1), 1, u(i))$ using (2.2).

   We calculate the finite time ruin probability in run m, $\psi_m(u, n)$, as follows:

$$\psi_m(u, n) \quad = \quad 1 - \prod_{i=1}^{n} \left[1 - \psi_m(u(i-1), 1, u(i))\right] .$$

   (d) The estimate for the continuous and finite time n ruin probability, $\hat{\psi}(u, n)$, is set by the mean of the estimates obtained from each simulation, $\{\hat{\psi}_m(u, n)\}_{m=1}^M$.

   We note that this procedure also allow us to calculate the standard error of the obtained estimate.

In our model we introduce the simulation procedure as flexible as possible to be applied for different sorts of data.

3 Ruin probabilities in a portfolio with a BMS

In this section we discuss the effect of a BMS in the probability of ruin of a motor portfolio. We illustrate our model using a numerical example based on data from the automobile third-party liability portfolio of an insurer operating in Portugal, who wishes to remain anonymous. The portfolio and the BMS is specified in Section 3.1. The numerical results are discussed in Section 3.2.

3.1 Data and distribution fitting

The insurer's commercial scale has 18 premium entries, see column labeled "C" in Table 3.3 and the (18×18) transition rules matrix in Table 3.1. Here, entry (l, j) represents the number of claims reported by a policyholder, that origins a transition from class l to class j, $l, j = 1, \ldots, 18$.

In Table 3.2 we summarise data for the number of annual claims reported in the insurer portfolio, corresponding to a stable year of operation, we mean it can be considered to be under in stationarity. From that data we estimated a mixed Poisson distribution, where the parameter follows an inverse Gaussian distribution, whose maximum likelihood estimates for the mean and the shape parameter of this distribution, say μ and η, are $\hat{\mu} = 0.082401$ and $\hat{\eta} = 0.130271$, respectively. We performed an appropriate goodness-of-fit test. See Lemaire (1995) for details. In the table we considered all full year contracts, the insurer did not provide detailed information. We fit the data into the model explained in Subsection 2.1 by considering that in class j $(j = 1, \ldots, 18)$ number of annual claims follow a Poisson with parameter λ_j), parameter to be estimated from the data.

In Table 3.3 we show the seven *bonus* scales considered in Item 2 of Subsection 2.3, the actual number of policies in each level of the portfolio under study and the estimates for the claim frequency of each class, $\hat{\lambda}_j$, $j = 1, \ldots, 18$. Following the methods proposed by those scales we estimated the optimal value levels considering our commercial scale.

From the history of the insurer's portfolio claim amounts, we estimated a mean value of $1,766.31$, a variance of $71,097,953.5$ and a third central moment of $21,068,298,856,615$. They were then used to get estimates for the parameters α, β and κ of the translated Gamma approximation.

In the simulation procedure, summarised in Subsection 2.3, for the calculation of the ruin probability $\psi(u, n)$, we used $M = 50,000$ runs.

$$
\boldsymbol{T} = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18 \end{array}
\begin{array}{cccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
\end{array}
$$

$$
\left(\begin{array}{cccccccccccccccccc}
\{0\} & - & - & \{1\} & - & - & \{2\} & - & - & \{3\} & - & - & \{4\} & - & - & \{5\} & - & \{6,7,\ldots\} \\
\{0\} & - & - & - & \{1\} & - & - & \{2\} & - & - & \{3\} & - & - & \{4\} & - & - & \{5\} & \{6,7,\ldots\} \\
- & \{0\} & - & - & - & \{1\} & - & - & \{2\} & - & - & \{3\} & - & - & \{4\} & - & - & \{5,6,\ldots\} \\
- & - & \{0\} & - & - & - & \{1\} & - & - & \{2\} & - & - & \{3\} & - & - & \{4\} & - & \{5,6,\ldots\} \\
- & - & - & \{0\} & - & - & - & \{1\} & - & - & \{2\} & - & - & \{3\} & - & - & \{4\} & \{5,6,\ldots\} \\
- & - & - & - & \{0\} & - & - & - & \{1\} & - & - & \{2\} & - & - & \{3\} & - & - & \{4,5,\ldots\} \\
- & - & - & - & - & \{0\} & - & - & - & \{1\} & - & - & \{2\} & - & - & \{3\} & - & \{4,5,\ldots\} \\
- & - & - & - & - & - & \{0\} & - & - & - & \{1\} & - & - & \{2\} & - & - & \{3\} & \{4,5,\ldots\} \\
- & - & - & - & - & - & - & \{0\} & - & - & - & \{1\} & - & - & \{2\} & - & - & \{3,4,\ldots\} \\
- & - & - & - & - & - & - & - & \{0\} & - & - & - & \{1\} & - & - & \{2\} & - & \{3,4,\ldots\} \\
- & - & - & - & - & - & - & - & - & \{0\} & - & - & - & \{1\} & - & - & \{2\} & \{3,4,\ldots\} \\
- & - & - & - & - & - & - & - & - & - & \{0\} & - & - & - & \{1\} & - & - & \{2,3,\ldots\} \\
- & - & - & - & - & - & - & - & - & - & - & \{0\} & - & - & - & \{1\} & - & \{2,3,\ldots\} \\
- & - & - & - & - & - & - & - & - & - & - & - & \{0\} & - & - & - & \{1\} & \{2,3,\ldots\} \\
- & - & - & - & - & - & - & - & - & - & - & - & - & \{0\} & - & - & - & \{1,2,\ldots\} \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & \{0\} & - & - & \{1,2,\ldots\} \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & \{0\} & - & \{1,2,\ldots\} \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & \{0\} & \{1,2,\ldots\}
\end{array}\right).
$$

Table 3.1: Transition matrix of the BMS

No. Claims	No. Policies
0	408,348
1	31,993
2	2,010
3	133
4	6
Total	442,490

Table 3.2: Number of Reported Claims in the Portfolio

				(%)						
j	C	N	LN	GN	B	LB	GB	No. Policies	$\hat{\lambda}_j$	j
1	45	33.4	32.4	41.0	48.8	46.0	45.7	174,173	0.034516	1
2	45	48.0	39.9	45.3	58.0	52.0	49.8	109,113	0.072883	2
3	50	49.5	47.4	50.0	60.1	58.0	54.4	42,736	0.076425	3
4	55	51.1	54.9	55.2	62.3	64.0	59.3	29,134	0.080265	4
5	60	66.5	62.4	61.0	63.7	70.0	64.7	23,730	0.126855	5
6	65	69.9	69.9	67.3	66.2	76.0	70.6	4,241	0.135954	6
7	70	74.6	77.5	74.3	68.1	82.0	77.0	2,759	0.148393	7
8	80	87.3	85.0	82.0	69.9	88.0	84.0	24,829	0.181802	8
9	90	92.9	92.5	90.6	72.3	94.0	91.7	11,747	0.195919	9
10	100	100.0	100.0	100.0	100.0	100.0	100.0	166	0.213730	10
11	110	109.8	107.5	110.4	105.6	106.0	109.1	2,882	0.237433	11
12	120	117.0	115.0	121.9	113.0	112.0	119.0	7,632	0.255984	12
13	130	125.3	122.5	134.6	124.3	118.0	129.9	250	0.277505	13
14	150	134.5	130.1	148.6	148.8	124.0	141.7	710	0.301956	14
15	180	143.4	137.6	164.0	162.6	130.0	154.6	2,256	0.327931	15
16	250	153.3	145.1	181.1	181.9	136.0	168.6	2,643	0.358676	16
17	325	164.1	152.6	199.9	209.1	142.0	184.0	1,304	0.395719	17
18	400	176.0	160.1	220.7	235.0	148.0	200.7	2,183	0.441571	18
								442,490		

Table 3.3: Number of policies, Poisson parameter and *bonus* scales by class

3.2 Results and comments

From the actual data, say "year 0", we used the figures in column 9 (No. Policies) of Table 3.3 to estimate the starting distribution of the policyholders among the *bonus* levels, column "0" of Table 3.4, so that we find the premium for the first period in our model. The premium indices (in percentage) for the different scales are shown in Table 3.3 as well as the estimated $\hat{\lambda}_j$'s for the starting simulations as referred in Item 1.

For comparing performances we consider seven different *bonus* scales as well as the risk premium, as follows.

$P_0 = E(Y_i)$ is denoted as the risk premium for each year for the portfolio when no SBM is considered. It is taken constant all along the years. For the premium calculation we used the expected value principle with a loading $\theta = 0.8$ (i.e., $(1+0.8)E(S(1))$ so that the calculated total premium without BMS for the portfolio with 442,490 policies is $P_0 = 115,838,792$.

C is denoted as the premium obtained with the insurer commercial scale, a real scale. We further estimate six different optimal *bonus* scales: the scale proposed by Norberg (1976) (denoted N), the one proposed by Borgan *et al.* (1981) (denoted B) as well as Linear Norberg (denoted LN), Geometric Norberg (denoted GN), Linear Borgan (denoted LB) and Geometric Borgan (denoted GB). Linear Norberg and Linear Borgan are the application of Gilde and Sundt (1989) to the optimal scale of Norberg and Borgan, respectively. Geometric Norberg and Geometric Borgan are the application of Andrade e Silva and Centeno (2005) to the same scales.

The total amount of premium for each year, for each *bonus* scale shown in Table 3.3, is calculated using (2.6) according to the expected number of polices in each class in each year.

Yet, a note related with the choice of an 80% loading coefficient. We were never told about the loading used by the insurer or the *capital requirements* for this portfolio, we chose a loading so that a ten year ruin probability for a fixed initial surplus would be around 1%, roughly, if no BMS system were considered. In our illustration we got an estimated ruin probability of 0.01246 for an initial surplus of 350,000 monetary units, see column for P_0 in Table 3.5. Then, with the application of a BMS we could compare ruin probability figures in two ways:

1. For each different BMS scale we could see the effect on ruin probabilities for a given initial surplus, when compared to the *no BMS* situation and between each other. And,

2. For a fixed finite time ruin probability of around 1%, see the initial surplus needed (we use round figures).

Table 3.4 shows the distribution of the policies through the $s = 18$ classes of the portfolio under study for years "0", 2, 5, 10 and in stationarity situation, i.e., estimates for $\pi_{\Delta}^{(0)}(j)$, $\pi_{\Delta}^{(2)}(j)$, $\pi_{\Delta}^{(5)}(j)$, $\pi_{\Delta}(j)$ and $\pi_T(j)$, $j = 1, 2, \ldots, 18$. In year 0, the time of the data collection, we see that around 64% of the policies belong to the two classes with higher discount. Ten years later we would expect about 78% of the policies in the same two classes.

In Figure 1 we represent the evolution of the premia according to the different *bonus* scales for the portfolio. The straight line corresponds to the estimate for the expected value of aggregate claims ($E(S(1))$). The premium $P_0 = 115,838,792$ is not shown for scale matter reasons. The other premia were obtained applying the *bonus* scales shown in Table 3.3 to premium P_0, as referred in (2.6). The dashed line, labeled S^*, is the claims estimated mean according to their class placement or evolution along time, calculated with estimated class claim frequency $\hat{\lambda}_j$ from Table 3.3, $j = 1, \ldots, 18$ (we stopped that calculation at year 10).

Analysing the figures we see that P_0 (the premium calculated if no BMS is applied) is extremely high when compared to the premium obtained by application of a *bonus* scale, any scale. We note that premia obtained with scales N, LN and GN are always below than the estimated expected value of one year aggregate claims, line $E(S)$. We emphasise that with these optimal scales, in order not to be ruined with high probability, either the initial surplus has to be very high or the loading in practice needs to be very high, as we will show later. At the beginning of our timeline the premia obtained with scales B, LB, GB are above the expected value of aggregate claims, but after some years all of them will be below. Indeed, we figure that in the long run the BMS set in practice will put most policyholders in the classes

$j\backslash$ Year	0	2	5	10	Stationarity
1	0.39362	0.63576	0.67436	0.71953	0.73121
2	0.24659	0.05681	0.04089	0.05678	0.04913
3	0.09658	0.06896	0.08045	0.04929	0.05394
4	0.06584	0.05938	0.06980	0.04927	0.05941
5	0.05363	0.01651	0.01678	0.01904	0.01871
6	0.00958	0.05774	0.01747	0.02528	0.01678
7	0.00623	0.03644	0.03326	0.01645	0.01411
8	0.05611	0.00444	0.01265	0.01037	0.00837
9	0.02655	0.00793	0.00659	0.00840	0.00725
10	0.00037	0.02375	0.00878	0.00977	0.00612
11	0.00651	0.00480	0.01277	0.00576	0.00495
12	0.01725	0.00188	0.00596	0.00451	0.00453
13	0.00056	0.00620	0.00541	0.00407	0.00419
14	0.00160	0.00818	0.00344	0.00435	0.00398
15	0.00510	0.00294	0.00403	0.00332	0.00398
16	0.00597	0.00468	0.00208	0.00326	0.00410
17	0.00295	0.00195	0.00210	0.00429	0.00438
18	0.00493	0.00164	0.00317	0.00629	0.00488

Table 3.4: Portfolio distribution over time and classes

with higher *bonuses*. It may be good for attracting customers for other lines of business of the same company but not so good for this particular portfolio.

We analise now the ruin probabilities for years 2, 5 and 10, $\psi(u,2)$, $\psi(u,5)$, $\psi(u,10)$ respectively, given a known initial surplus u. Based on the first choice for premium P_0, for each premium scale we chose u in order to obtain roughly $\hat{\psi}(u,10) = 1\%$. Figures are shown in Table 3.5 and we highlight in bold the ruin probability numbers in each scale for situations around 1%, roughly. We can see how different is the need for initial capital u for each premium scale in order to get an estimate $\hat{\psi}(u,10) \simeq 1\%$, in all cases it is much higher than the no BMS situation. In particular, scales N and LN show a need for very high initial surplus. In our calculations we experienced that if ruin is going to occur it will in the first 2 years. In most cases results for $t = 5$ and $t = 10$ are very stable and very close (in many cases equal) to the results for $t = 2$. Although we don't show, standard deviations of our estimates are all quite small, ranging from 3.3×10^{-40} to 3.92×10^{-6} (zero in the case where $\widehat{\psi}(u,t) = 1,\ t = 1, 2, \cdots$). The introduction of this BMS results in a significant increase in the ruin probabilities, the best premium scale is the Borgan *et al.* (1981). Note that this scale is the one that offers less discount for lower classes that contain a high proportion of policies, as shown in Tables 3.3 and 3.4.

Table 3.6 shows figures for the average of the within the year ruin probability for each *bonus* scale and a particular collection of initial surpluses (denoted $\bar{\psi}(\cdot)$ in the table). The surplus choice corresponds to those initial surpluses with "boldface figures" in Table 3.5. Analysing the table we can gain some insight into the results and look to what happens, on average, for the within the year ruin probabilities. With the exception of scales N and LN, if ruin occurs it does in the first year. This highlights the need for having a control on a short term basis.

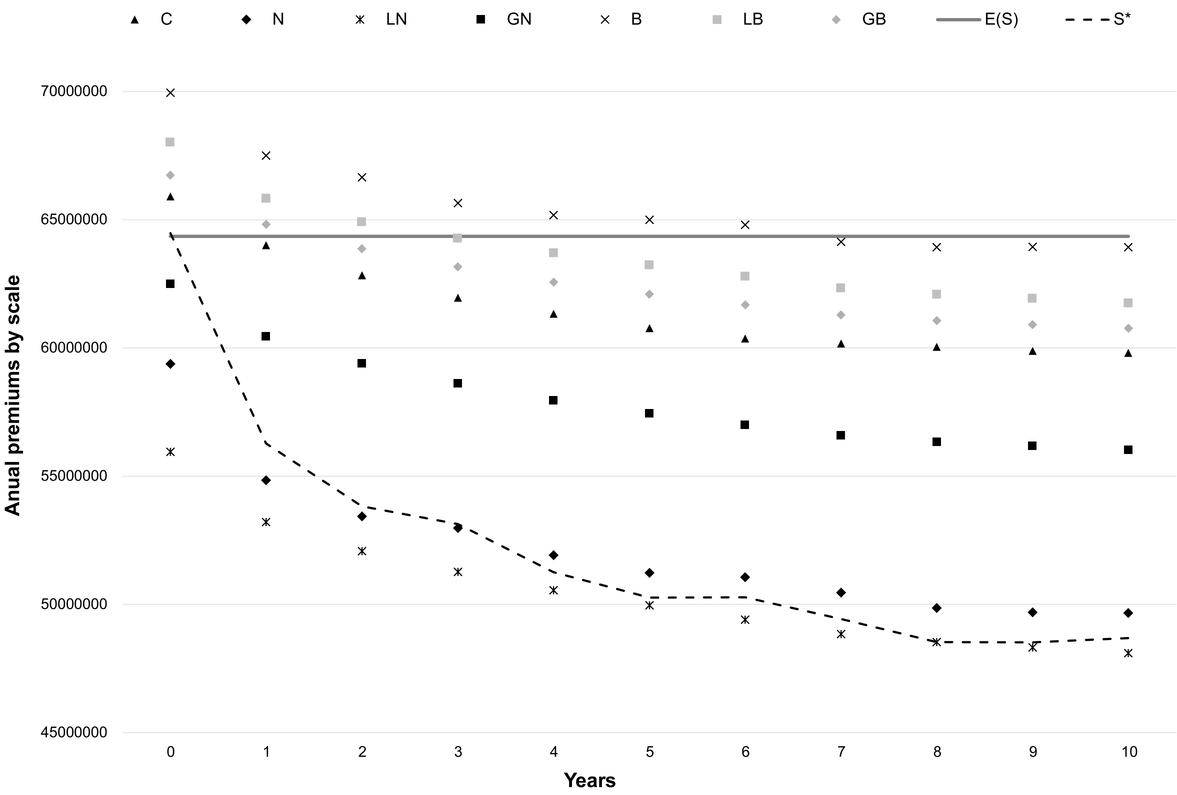

Figure 1: Evolution of the different premiums over time

In Figure 2 we present some graphs with paths of the surplus $U(t)$, for each *bonus* scale and respective initial capital u. The scale range in figures "Norberg" and "Linear Norberg" are very small when compared with the others. For this reason more paths appear more visible.

					$\widehat{\psi}(u,t)$ (%)				
u	t	P_0	C	N	LN	GN	B	LB	GB
350,000	2	**1.246**	60.502	99.999	100	96.129	25.665	38.028	50.392
	5	**1.246**	60.502	99.999	100	96.129	25.665	38.028	50.392
	10	**1.246**	60.502	99.999	100	96.129	25.665	38.028	50.392
1,500,000	2	0	14.807	99.954	100	75.749	**0.953**	3.407	8.374
	5	0	14.807	99.969	100	75.749	**0.953**	3.407	8.374
	10	0	14.807	99.969	100	75.749	**0.953**	3.407	8.374
2,000,000	2	0	7.569	99.854	100	63.277	0.233	**1.186**	3.716
	5	0	7.569	99.902	100	63.277	0.233	**1.186**	3.716
	10	0	7.569	99.902	100	63.277	0.233	**1.186**	3.716
2,550,000	2	0	3.495	99.601	100	48.724	0.052	0.369	**1.463**
	5	0	3.495	99.716	100	48.724	0.052	0.369	**1.463**
	10	0	3.495	99.717	100	48.724	0.052	0.369	**1.463**
3,250,000	2	0	**1.206**	98.790	100	31.551	0.009	0.083	0.426
	5	0	**1.206**	99.150	100	31.551	0.009	0.083	0.426
	10	0	**1.206**	99.154	100	31.551	0.009	0.083	0.426
6,400,000	2	0	0.008	74.639	99.923	**1.245**	0	0	0.003
	5	0	0.008	81.050	99.986	**1.245**	0	0	0.003
	10	0	0.008	81.163	99.989	**1.245**	0	0	0.003
15,000,000	2	0	0	**0.360**	31.246	0	0	0	0
	5	0	0	**1.580**	70.362	0	0	0	0
	10	0	0	**1.623**	74.868	0	0	0	0
25,000,000	2	0	0	0	**0.002**	0	0	0	0
	5	0	0	0	**0.879**	0	0	0	0
	10	0	0	0	**2.037**	0	0	0	0

Table 3.5: Estimates for the probability of ruin for different u's and $t = 2, 5, 10$ years for each BMS.

u	350,000	3,250,000	15,000,000	25,000,000	6,400,000	1,500,000	2,000,000	2,550,000
Scales	P_0	C	N	LN	GN	B	LB	GB
i \				$\bar{\psi}(u(i-1), 1, u(i))$				
1	0.012455245	0.011341473	2.028E-11	1.18E-132	0.00862	0.0095224	0.0117586	0.0142202
2	0	0.0054895	0.0003543	2.958E-09	0.0109755	0.0001391	0.0014271	0.0043680
3	0	7.113E-06	0.0035845	1.728E-05	0.0002886	3.707E-30	1.211E-14	1.447E-08
4	0	3.073E-47	0.0095027	0.0012845	9.317E-11	0	5.55E-105	9.214E-65
5	0	1.73E-170	0.0113343	0.0048442	8.457E-27	0	0	0
6	0	0	0.0080704	0.0080609	1.054E-68	0	0	0
7	0	0	0.0042740	0.0112567	0	0	0	0
8	0	0	0.0021026	0.0136769	0	0	0	0
9	0	0	0.0006633	0.0114318	0	0	0	0
10	0	0	0.0001815	0.0070086	0	0	0	0

Table 3.6: Average of the within the year ruin probabilities for each BMS and different initial surpluses

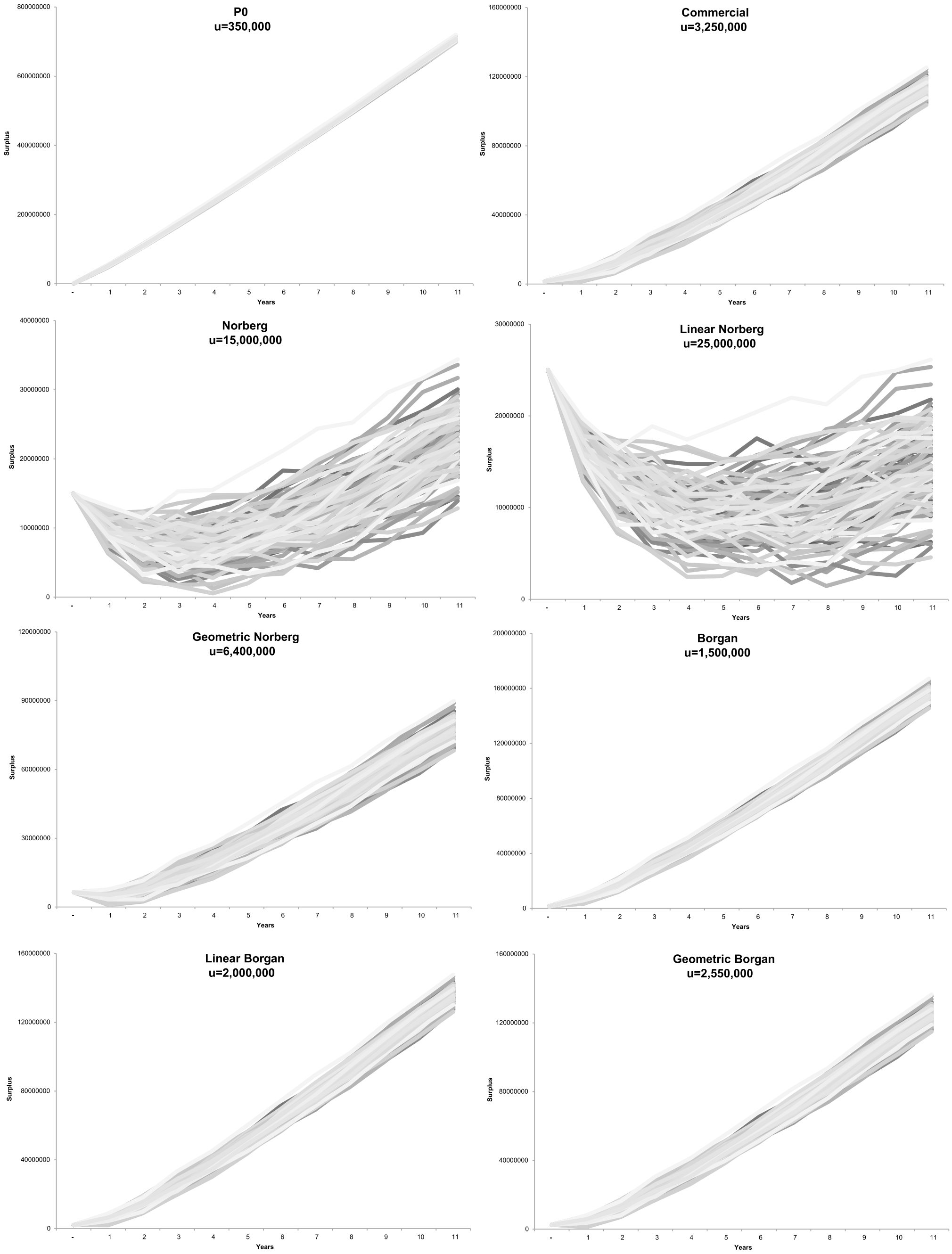

Figure 2: Some paths for the reserve over time for each *bonus* scale and respective u.

4 Final remarks

Throughout this paper we used the method proposed in Afonso *et al.* (2009) to evaluate, in a motor insurance portfolio, the impact in the ruin probability of the introduction of a BMS in the portfolio, when compared to the "em classical no BMS situation". The proposed model is applicable to large portfolios and the SBM does not necessarily assume a system at a stability stage as most literature do.

We could compute the within one year ruin probability. Here, analysing the average ruin probability, the insurer may foresee the time where the ruin probability is reaching an intolerable level and start prepare either a tariff revision or an increase in the capital requirement, or reserve amounts, or both. In a very competitive business it is importatnt to have a model prepared to adapt quickly to market changes, we mean, in shorter term situations. Like other existing BMS, this BMS is going to put a high proportion of policyholders in the classes with higher *bonuses* in the long run. This results into an increase in the ruin probabilities, and we can estimate the magnitude of that increase. Having high discounts can attract new (suposedly "good") customers however, that, together with high penalties, may lead to a *bonus hunger* situation. If the first may increase the premium receipts, the latter certainly lead to an artificial decrease in those receipts. This is not an easy issue and certainly has an effect on ruin probabilities.

The estimation of ruin probabilities in the presence of a BMS may also outcome as a means to decide among a set of optimal and/or commercial scales. We realized that small changes in the scales, even almost no perceptible, may lead to a big impact on initial surplus u required to meet a given level in the ruin probability.

This model provides a simple and effective methodology for assessing scales and *bonus malus* schemes. It is quite flexible, it can be applied to other BMS providing decision makers to choose a suitable BMS concerning the ruin probabilities. It can be applied also for the Solvency II purposes to obtain the estimates of ruin probabilities in one year period.

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References

Afonso, L. B., Egídio dos Reis, A. D., and Waters, H. R. (2009). Calculating continuous time ruin probabilities for a large portfolio with varying premiums. *ASTIN Bulletin*, 39(1):117–136.

Afonso, L. B., Egídio dos Reis, A. D., and Waters, H. R. (2009). Numerical evaluation of continuous time ruin probabilities for a portfolio with credibility updated premiums. *ASTIN Bulletin*, 40(1):399–414.

Andrade e Silva, J. and Centeno, M. L. (2005). A note on bonus scales. *Journal of Risk and Insurance*, 72(4):601–607.

Asmussen, S. and Albrecher, H. (2010). *Ruin Probabilities*, Advanced Series on Statistical Science & Applied Probability: Volume 14. World Scientific.

Borgan, Ø., Hoem, J., and Norberg, R. (1981). A non asymptotic criterion for the evaluation of automobile bonus system. *Scandinavian Actuarial Journal*, 1981: 165–178.

Constantinescu, C., Dai, S., Ni, W. and Palmowski, Z. (2016). Ruin probabilities with dependence on the number of claims within a fixed time window. *Risks*, 4(2), 17. doi:10.3390/risks4020017

Denuit, M.; Maréchal, X.; Pitrebois, S. and Walhin, J.-F. (2007). *Actuarial Modelling of Claim Counts*. Willey.

Dubey, A. (1977). Probabilité de ruine lorsque le paramètre de poisson est ajusté a posteriori. *Mitt. Verein. Schweiz. VersicherungsMath.*, 77:131–141.

Dufresne, F. (1988). Distributions stationnaires d'un système bonus-malus et probabilité de ruine. *ASTIN Bulletin*, 18(1):31–46.

Gilde, V. and Sundt, B. (1989). On bonus systems with credibility scales. *Scandinavian Actuarial Journal*, 1989:13–22.

Lemaire, J. (1995). *Bonus-malus systems in automobile insurance*. Springer.

Li, B.; Ni, W. and Constantinescu, C. (2015). Risk models with premiums adjusted to claims number, *Insurance: Mathematics and Economics*, 65(2015):94-102. (http://dx.doi.org/10.1016/j.insmatheco.2015.09.001).

Ni, W., Constantinescu, C. and Pantelous, A. A. (2014). Bonus-Malus systems with Weibull distributed claim severities. *Annals of Actuarial Science*, 8(2):217–233. (doi:10.1017/S1748499514000062)

Norberg, R. (1976). A credibility theory for automobile bonus system. *Scandianvian Actuarial Journal*, 1976:92–107.

Parzen, E. (1965). *Stochastic processes*. Holden-Day series in probability and statistics. Holden-Day, San Francisco, London, Amsterdam.

Wagner, C. (2001). A note on ruin in a two state markov model. *ASTIN Bulletin*, 32(2):349–358.

Wu, X., Chen, M., Guo, J. and Jin, C. (2015). On a discrete-time risk model with claim correlated premiums. *Annals of Actuarial Science* 9(2):322–342.

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