HYPERBOLIC BILLIARDS ON POLYTOPES WITH CONTRACTING REFLECTION LAWS

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Abstract. We study billiards on polytopes in $\mathbb{R}^d$ with contracting reflection laws, i.e. non-standard reflection laws that contract the reflection angle towards the normal. We prove that billiards on generic polytopes are uniformly hyperbolic provided there exists a positive integer k such that for any k consecutive collisions, the corresponding normals of the faces of the polytope where the collisions took place generate $\mathbb{R}^d$. As an application of our main result we prove that billiards on generic polytopes are uniformly hyperbolic if either the contracting reflection law is sufficiently close to the specular or the polytope is obtuse. Finally, we study in detail the billiard on a family of 3-dimensional simplexes.

Contents

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1. Introduction

Given a d-dimensional polytope P, a billiard trajectory inside P is a polygonal path described by a point particle moving with uniform motion in the interior of P. When the particle hits the interior of the faces of P, it bounces back according to a reflection law. Therefore, a billiard trajectory is determined by a sequence of reflections on the faces of P. Any reflection can be represented by a pair $x = (p, v)$ where p is a point belonging to a face of P and v is a unit velocity vector pointing inside P. The map $\Phi : x \mapsto x'$ that takes a reflection x to the next reflection x' is called the *billiard map*. The dynamics of billiards on polytopes has been mostly studied considering the specular reflection law. More recently, in the case of polygonal billiards, a new class of reflection laws has been introduced that contract the reflection angle towards the normal of the faces of the polygon [1, 6, 2, 3]. These are called *contracting reflection laws*. A billiard map with a contracting reflection law is called a *contracting billiard map*. It is known that strongly contracting billiard maps on generic convex polygons are uniformly hyperbolic and have finite number of ergodic SRB measures [4]. Recently, it has been proved that the same conclusion hods for contracting billiard maps on polygons with no parallel sides facing each other (even for contracting reflection laws close to the specular and for non-convex polygons) [5].

In this paper we extend some of the previous results to contracting billiard maps on polytopes.

In order to state our results we need to introduce the concept of escaping time. A precise definition can be found in section 2. Given a polytope P in $\mathbb{R}^d$ and a contracting reflection law, the *escaping time* of a reflection x is the number $T = T(x) \in \mathbb{N} \cup \{\infty\}$ which is the least positive integer $k \in \mathbb{N}$ such that for any sequence of k consecutive reflections, the corresponding normals of the faces of P where the k reflections took place generate $\mathbb{R}^d$.

Theorem 1.1. *If the contracting billiard map Φ of a generic polytope has an ergodic invariant probability measure μ such that T is integrable with respect to μ, then Φ is hyperbolic.*

Theorem 1.2. *If the contracting billiard map Φ of a generic polytope has an invariant set Λ such that T is bounded on Λ, then $\Phi|_\Lambda$ is uniformly hyperbolic.*

In section 4 we show that contracting billiards on polytopes have finite escaping time if either the contracting law is close to the specular or the polytope is obtuse. This together with Theorem 1.2 prove the following corollaries.

Corollary 1.3. *The contractive billiard map of a generic polytope with a contracting reflection law sufficiently close to the specular one is uniformly hyperbolic.*

Corollary 1.4. *The contracting billiard map of a generic obtuse polytope is uniformly hyperbolic.*

The paper is organized as follows. In section 2 we introduce some notation and define the contracting billiard on a polytope. We also derive several properties of contracting billiards maps and rigorously state our main result. In section 3 we show that polytopes on general position are generic and in section 4 we study the escaping time on polyhedral cones. Our main result is proved in section 5. Finally, in section 6 we study in detail the contracting billiard of a family of 3-dimensional simplexes.

2. Definitions and Statements

A half-space in $\mathbb{R}^d$ is any set of the form $\{\, x \in \mathbb{R}^d : \langle x, v\rangle \leq c \,\}$, for some non-zero vector $v \in \mathbb{R}^d$ and some real number $c \in \mathbb{R}$. A polyhedron is any finite intersection of half-spaces in $\mathbb{R}^d$. A polytope is a compact polyhedron. We call dimension of a polyhedron to the dimension of the affine subspace that it spans. Let $P \subset \mathbb{R}^d$ be a d-dimensional polytope.

The billiard on P is a dynamical system describing the linear motion of a point particle inside P. When the particle hits the boundary of P, it gets reflected according to a reflection law, usually the specular reflection law. In the following we rigorously define the billiard map Φ_P with the specular reflection law. But first, let us introduce some notation.

2.1. Basic Linear Algebra.

Given vectors $v_1, \ldots, v_n \in \mathbb{R}^d$, the linear subspace spanned by the vectors $v_1, \ldots, v_n$ is denoted by $\langle\!\langle v_1, \ldots, v_n \rangle\!\rangle$. Let $\mathbb{S}$ denote the unit sphere in $\mathbb{R}^d$, i.e. $\mathbb{S} = \{\, v \in \mathbb{R}^d : \|v\| = 1 \,\}$. Let $v, \eta \in \mathbb{S}$ be unit vectors and $u \in \mathbb{R}^d$. We denote by $\mathbb{S}^+_\eta$ the *hemisphere associated with* η,

$$\mathbb{S}^+_\eta = \{\, v \in \mathbb{S} : \langle v, \eta\rangle > 0 \,\} .$$

Let $\eta^\perp$ denote the orthogonal hyperplane to η. The *orthogonal projection of* u *onto the hyperplane* $\eta^\perp$ is,

$$P_{\eta^\perp}(u) = u - \langle u, \eta\rangle\, \eta = u - P_\eta(u) \; ,$$

where $P_\eta(u) = \langle u, \eta\rangle\, \eta$, is the *orthogonal projection of* u *onto the line spanned by* η. The *reflection of* u *about the hyperplane* $\eta^\perp$ is defined by,

$$R_\eta(u) = u - 2\,\langle u, \eta\rangle\, \eta.$$

Finally, the *parallel projection of u along v onto the hyperplane $\eta^\perp$* is

$$P_{v,\eta^\perp}(u) = u - \frac{\langle u, \eta\rangle}{\langle v, \eta\rangle}\, v \ .$$

Denote by $\angle(v,w)$ the angle between two non-zero vectors in $\mathbb{R}^d$, defined as

$$\angle(v,w) := \arccos\left(\frac{\langle v, w\rangle}{\|v\|\,\|w\|}\right) \ .$$

The angle between a non-zero vector $v \in \mathbb{R}^d$ and a linear subspace $E \subseteq \mathbb{R}^d$ is defined to be

$$\angle(v,E) := \min_{u\in E\setminus\{0\}} \angle(v,u) \ .$$

The angle between two linear subspaces E and E' of $\mathbb{R}^d$ of the same dimension is defined as

$$\angle(E,E') := \max\{ \max_{u\in E\setminus\{0\}} \angle(u,E'), \max_{u'\in E'\setminus\{0\}} \angle(u',E) \} \ .$$

Denoting by $\pi_{E,E'} : E \to E'$ and $\pi_{E',E} : E' \to E$ the orthogonal projections from each of these subspaces onto the other we have

(1) $\|\pi_{E,E'}\| = \max_{u\in E\setminus\{0\}} \frac{d(u,E')}{\|u\|} = \sin\left(\max_{u\in E\setminus\{0\}} \angle(u,E')\right)$,
(2) $\|\pi_{E',E}\| = \max_{u'\in E'\setminus\{0\}} \frac{d(u,E)}{\|u\|} = \sin\left(\max_{u'\in E'\setminus\{0\}} \angle(u,E)\right)$.

For all $v \in E$ and $v' \in E'$,

$$\langle \pi_{E,E'}(v), v'\rangle = \langle v, v'\rangle = \langle v, \pi_{E',E}(v')\rangle \ .$$

This shows that each of the projections $\pi_{E,E'}$ and $\pi_{E',E}$ is adjoint of the other. Therefore the maxima in the definition of $\angle(E,E')$ coincide with $\|\pi_{E,E'}\| = \|\pi_{E',E}\|$.

Lemma 2.1. *Let E, E' and H be linear subspaces of $\mathbb{R}^d$ such that*

(1) $\dim(E) = \dim(E')$,
(2) $\angle(h,E) \geq \varepsilon$, *for all* $h \in H \setminus \{0\}$.

Then

$$\sin\left(\angle(E+H, E'+H)\right) \leq \frac{\sin\left(\angle(E,E')\right)}{\sin\varepsilon}$$

Proof. First notice that

$$\angle(E+H, E'+H) = \angle((E+H)\cap H^\perp, (E'+H)\cap H^\perp) \ .$$

Given $u \in (E+H) \cap H^\perp$ we can write $u = v - h$ with $v \in E$ and $h \in H$. Hence, since $u \in H^\perp$,

$$\begin{aligned}\frac{d(u,(E'+H)\cap H^\perp)}{\|u\|} &= \frac{d(u,E'+H)}{\|u\|} = \frac{d(v,E'+H)}{\|u\|}\\ &\leq \frac{d(v,E')}{\|u\|} = \frac{\|v\|}{\|u\|}\frac{d(v,E')}{\|v\|} \leq \frac{\|v\|}{\|u\|}\sin\left(\angle(E,E')\right)\\ &= \frac{\sin\left(\angle(E,E')\right)}{\sin\left(\angle(v,h)\right)} \leq \frac{\sin\left(\angle(E,E')\right)}{\sin\varepsilon}\ .\end{aligned}$$

On the last equality we use that $v = h + u$ is an orthogonal decomposition with $h \in H$ and $u \in H^\perp$. Thus taking the sup in $u \in (E+H) \cap H^\perp \setminus \{0\}$ we get

$$\sin\left(\angle((E+H)\cap H^\perp,(E'+H)\cap H^\perp)\right) \leq \frac{\sin\left(\angle(E,E')\right)}{\sin\varepsilon}\ .$$

□

2.2. **Billiard map.** Suppose that P has N faces (of dimension $d-1$) which we denote by $F_1,\ldots,F_N$. For each $i = 1,\ldots,N$, denote by η_i the interior unit normal vector to the face F_i. Also denote by Π_i the hyperplane that supports the face F_i. We write the interior of F_i as F_i°, and its $(d-2)$-dimensional boundary as ∂F_i. Define $\partial P = \bigcup_{i=1}^N F_i$, and the $(d-2)$-skeleton $\Sigma P = \bigcup_{i=1}^N \partial F_i$. Finally define

$$M := \bigcup_{i=1}^N F_i^\circ \times \mathbb{S}^+_{\eta_i}\ .$$

The domain of the billiard map Φ_P is the set of points $(p,v) \in M$ such that the half-line $\{\,p + t\,v\ :\ t \geq 0\}$ does not intersect the skeleton ΣP. We denote this set by M'. Clearly, M' is the complement of a co-dimension two subset of M.

Now the billiard map $\Phi_P : M' \to M$ is defined as follows. Given $x = (p,v) \in M'$, let $\tau = \tau(p,v) > 0$ be minimum $t > 0$ such that $p + t\,v \in F_j'$ for some $j = 1,\ldots,N$. The real number τ is called the *flight time* of (p,v). Then the billiard map is defined by

$$\Phi_P(x) = (p + \tau\,v, R_{\eta_j}(v)).$$

Note that the billiard map Φ_P is a piecewise smooth map and it has finitely many domains of continuity. The number of domains of continuity is at most $N(N-1)$, which is the number of 2-permutations of N faces. If P is convex, then all permutations define a branch map.

Let $(p',v') = \Phi_P(p,v)$ for $(p,v) \in M'$. It is easy to obtain a formula for the branch maps and its derivatives.

Proposition 2.2. *Suppose that $(p_i',v_i') = \Phi_P(p_i,v_i)$ for some $p_i \in F_i^\circ$ such that $p_i' \in F_j'$ with $i \neq j$. For every $x = (p,v) \in F_i^\circ \times \mathbb{S}^+_{\eta_i}$ such that*

$p' \in F'_j$ we have

$$\Phi_P(x) = \Big(p_j + P_{v,\eta_j^\perp}(p - p_j), R_{\eta_j}(v)\Big) \ .$$

Moreover

$$D\Phi_P(x)(u,w) = \Big(\, P_{v,\eta_j^\perp}(u + \gamma(x)\, w), R_{\eta_j}(w)\Big) \ ,$$

where

$$\gamma(x) = \frac{\langle p - p_j, \eta_j \rangle}{\langle v, \eta_j \rangle} \, .$$

Proof. Recall that $p' = p + \tau(p,v)v$ where $\tau(p,v)$ is the length of the vector $p' - p$. Taking the inner product with η_j in both sides of the equation and noting that $\langle p' - p_j, \eta_j \rangle = 0$, we get

$$\tau(p,v) = \frac{\langle p' - p, \eta_j \rangle}{\langle v, \eta_j \rangle} = \frac{\langle p_j - p, \eta_j \rangle}{\langle v, \eta_j \rangle} \, .$$

So

$$p' = p_j + \left((p - p_j) - \frac{\langle p - p_j, \eta_j \rangle}{\langle v, \eta_j \rangle} v \right) = p_j + P_{v,\eta_j^\perp}(p - p_j) \, .$$

To prove the formula for the derivative, define the map $\Psi_\eta : (p,v) \mapsto P_{v,\eta^\perp}(p)$ for any given $\eta \in \mathbb{S}$. The claim follows from the formula

$$D\Psi_\eta(x)(u,w) = P_{v,\eta^\perp}(u) + \frac{\langle p, \eta \rangle}{\langle v, \eta \rangle} P_{v,\eta^\perp}(w).$$

□

2.3. **Contracting reflection laws.** A *contracting law* is any family $\{\, C_\eta : \mathbb{S}^+_\eta \to \mathbb{S}^+_\eta \,\}_{\eta \in \mathbb{S}}$ of class C^2 mappings that satisfies for every $\eta \in \mathbb{S}$,

(a) $C_\eta(\eta) = \eta$,

(b) there are non-negative C^2 functions $a_\eta, b_\eta : \mathbb{S}^+_\eta \to [0, +\infty)$ such that,

$$C_\eta(v) = a_\eta(v) P_\eta(v) + b_\eta(v) P_{\eta^\perp}(v), \quad \forall\, v \in \mathbb{S}^+_\eta.$$

(c) $0 < \sup\{\, \|DC_\eta(x)\| \,:\, x \in \mathbb{S}^+_\eta \,\} < 1$,

(d) $O \circ C_\eta = C_{O(\eta)} \circ O$, for every rotation $O \in \mathcal{O}(n, \mathbb{R})$.

A contracting law can be uniquely characterized by a single C^2 map of the interval $\big[0, \frac{\pi}{2}\big)$ as the following proposition shows.

Proposition 2.3. *Given a contracting law $\{\, C_\eta : \mathbb{S}^+_\eta \to \mathbb{S}^+_\eta \,\}_{\eta \in \mathbb{S}}$, there is a class C^2 mapping $f : \big[0, \frac{\pi}{2}\big) \to \big[0, \frac{\pi}{2}\big)$ such that*

(a) *$f(0) = 0$,*

(b) *$0 < \sup\{\, |f'(\theta)| \,:\, 0 \le \theta < \frac{\pi}{2} \,\} < 1$,*

(c) *for every* $\eta \in \mathbb{S}$, *and* $v \in \mathbb{S}_\eta^+$,

$$C_\eta(v) = \frac{\cos f(\theta)}{\cos\theta} P_\eta(v) + \frac{\sin f(\theta)}{\sin\theta} P_{\eta^\perp}(v) \ ,$$

where $\theta = \arccos\langle v, \eta\rangle$ *is the angle between* η *and* v,

(d) *for every* $\eta \in \mathbb{S}$,

$$\sup_{x\in\mathbb{S}_\eta^+} \|DC_\eta(x)\| = \sup_{0\le\theta<\pi/2} |f'(\theta)| \ .$$

Proof. Let $\eta \in \mathbb{S}$ and $v \in \mathbb{S}_\eta^+$. By item (b) of the definition of a contracting law we can write

$$C_\eta(v) = a_\eta(v)P_\eta(v) + b_\eta(v)P_{\eta^\perp}(v)$$

where a_η and b_η are non-negative C^2 functions. Taking the inner product with η on both sides of the previous equation we get,

$$a_\eta(v) = \frac{\langle C_\eta v, \eta\rangle}{\cos\theta},$$

where $\theta = \arccos\langle v, \eta\rangle \in [0, \frac{\pi}{2})$ is the angle formed by the vectors v and η. By item (d) we conclude that $\langle C_\eta(v), \eta\rangle = \langle C_{O(\eta)}(O(v)), O(\eta)\rangle$, thus its value depends only on the angle θ. So, there is a C^2 function $f : [0, \frac{\pi}{2}) \to [0, \frac{\pi}{2})$ such that $\langle C_\eta(v), \eta\rangle = \cos f(\theta)$. Similarly, we conclude that

$$b_\eta(v) = \frac{\sin f(\theta)}{\sin\theta}.$$

This shows (c). The remaining properties follow immediately. $\square$

A C^2 mapping $f : \left[0, \frac{\pi}{2}\right) \to \left[0, \frac{\pi}{2}\right)$ satisfying (a)-(d) above is called a *contracting reflection law*. We also define

$$\lambda(f) := \sup_{0\le\theta<\pi/2} |f'(\theta)|.$$

2.4. **Contracting billiard map.** Given a contracting law $\{C_\eta\}$ with contracting reflection law f, define the map $\chi_f : M \to M$ by $\chi_f(p, v) = (p, C_{\eta(p)}(v))$ where $\eta(p)$ denotes the interior unit normal of the face of the polytope where p lies. The *contracting billiard map* $\Phi_{f,P} : M' \to M$ is

$$\Phi_{f,P} = \chi_f \circ \Phi_P.$$

There is a system of coordinates which is convenient to represent the derivative of the contracting billiard map. For each $x = (p, v) \in M$ define $\Psi_x : T_xM \to v^\perp \times v^\perp$ by

$$\Psi_x(u, w) = (P_{v^\perp}(u), w) \ .$$

The previous linear isomorphism will be referred as *Jacobi coordinates* on the tangent space T_xM. We shall use the notation (J, J') to denote an element in $v^\perp \times v^\perp$. The following proposition gives a formula for the derivative of the contracting billiard map in terms of Jacobi coordinates.

Proposition 2.4. *Let $x = (p, v) \in M'$ and suppose that $x' = (p', v') = \Phi_{f,P}(x)$ with $p' \in F'_j$. Then $\Psi_{x'} \circ D\Phi_{f,P}(x) \circ \Psi_x^{-1}$ is given by*

$$(J, J') \mapsto \Big(P_{v'^\perp} \circ P_{v,\eta_j^\perp}(J + \tau(p, v)\, J'),\, (DC_{\eta_j})_{R_{\eta_j}(v)} R_{\eta_j}(J') \Big) \ .$$

Moreover, if $\theta = \arccos |\langle v, \eta_j\rangle|$, then

$$\left| \frac{\langle v', \eta_j\rangle}{\langle v, \eta_j\rangle} \right| = \frac{\cos f(\theta)}{\cos\theta} > 1 \ .$$

Proof. Immediate from Propositions 2.2 and 2.3. □

2.5. **Orbits, invariant sets and hyperbolicity.** Denote by M^+ the subset of points in M that can be iterated forward, i.e.

$$M^+ = \{x \in M : \Phi^n_{f,P}(x) \in M' \ \forall\, n \geq 0\}.$$

A *billiard orbit* is a sequence $\{x_n\}_{n\geq 0}$ in M' such that $x_{n+1} = \Phi_{f,P}(x_n)$ for every $n \geq 0$. A *billiard path or trajectory* is the polygonal path formed by segments of consecutive points of a billiard orbit.

Define

$$D := \bigcap_{n\geq 0} \Phi^n_{f,P}(M^+).$$

It is easy to see that D is an invariant set and $\Phi_{f,P}$ and its inverse are defined on D. Following Pesin we call the closure of D the *attractor* of $\Phi_{f,P}$. We say that $\Lambda \subset M$ is an *invariant set* if $\Lambda \subset D$ and $\Phi^{-1}_{f,P}(\Lambda) = \Lambda$.

Definition 2.1. Given an invariant set Λ, we say that $\Phi_{f,P}$ is *uniformly partially hyperbolic on* Λ if there exists a continuous splitting $T_\Lambda M = E^s \oplus E^u$ and constants $\lambda < 1$, $\sigma > \lambda$ and $C > 0$ such that for every $n \geq 1$ we have

$$\|D\Phi^n_{f,P}|_{E^s}\| \leq C\lambda^n \quad \text{and} \quad \|D\Phi^{-n}_{f,P}|_{E^u}\| \leq C\sigma^{-n}.$$

If $\sigma > 1$, then $\Phi_{f,P}$ is called *uniformly hyperbolic on* Λ. If $\Lambda = D$, then we simply say that $\Phi_{f,P}$ is *uniformly (partially) hyperbolic.*

We also say that a $\Phi_{f,P}$-invariant Borel probability measure μ is called *hyperbolic* if

$$\lim_{n\to\infty} \frac{1}{n} \int \log \|D\Phi^{-n}_{f,P}|_{E^u}\|\, d\mu < 0.$$

The proof of the following result is an adaptation of [4, Proposition 3.1].

Proposition 2.5. $\Phi_{f,P}$ *is uniformly partially hyperbolic.*

Proof. To simplify the notation let us write $\Phi = \Phi_{f,P}$ and $\lambda = \lambda(f)$. Given $x = (p, v), x' = (p', v') \in M$ such that $x' = \Phi(x)$ we denote by $L(x, x')$ the map from $v^\perp \times v^\perp$ to $v'^\perp \times v'^\perp$ that represents the derivative

$D\Phi_x$ in the Jacobi coordinates (see proposition 2.4). This linear map is represented by a block upper triangular matrix of the form

$$L(x,x') = \begin{pmatrix} A(x,x') & B(x,x') \\ 0 & C(x,x') \end{pmatrix}$$

where $\|A(x,x')^{-1}\| \leq 1$ and $\|C(x,x')\| \leq \lambda < 1$, whose inverse is

$$L(x,x')^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{pmatrix}$$

where $A = A(x,x')$, etc. Given a linear map $H' : v'^{\perp} \to v'^{\perp}$ the pre-image of its graph by $L(x,x')$ is the graph of another linear function $H : v^{\perp} \to v^{\perp}$ called the *backward graph transform* of H' and denoted by $H =: \Gamma(x,x')H'$. The operator $\Gamma(x,x')$ is hence defined by the relation

$$L(x,x')^{-1}\mathrm{Graph}(H') = \mathrm{Graph}\left(\Gamma(x,x')H'\right).$$

A simple computation shows that

$$\Gamma(x,x')H' = A(x,x')^{-1}B(x,x') - A(x,x')^{-1}H'C(x,x').$$

We claim that writing $x_n = (p_n, v_n) = \Phi^n x$ and denoting by Z_n the zero endomorphism on $v_n^{\perp}$, the following limit exists

$$H^s(x) := \lim_{n\to+\infty} \Gamma(x,\Phi x)\ldots\Gamma(\Phi^{n-1}x, \Phi^n x)Z_n.$$

A recursive computation allows to explicit the right hand side composition $\Gamma(x,\Phi x)\ldots\Gamma(\Phi^{n-1}x,\Phi^n x)Z_n$, which is a partial sum of the following series

$$H^s(x) = \sum_{j=0}^{\infty} (-1)^j A_0^{-1}\cdots A_j^{-1}B_jC_{j-1}\cdots C_0$$

where $A_j = A(\Phi^j x, \Phi^{j+1}x)$, etc. This series converges because $\left\|A_j^{-1}\right\| \leq 1$ and $\|C_j\| \leq \lambda < 1$ for all $j \geq 0$.

By construction, the subspaces $E^s(x) := \Psi_x^{-1}\mathrm{Graph}(H^s(x))$ determine a $D\Phi$-invariant sub-bundle of TM satisfying $\left\|D\Phi_x|_{E^s(x)}\right\| \leq \lambda$ for all $x \in D$. Given $x = (p,v) \in D$, define $E^u(x) := \Psi_x^{-1}\{(J,0)\colon J \in v^{\perp}\}$. Clearly, E^u is invariant. Moreover, $\left\|D\Phi^{-1}|_{E^u(x)}\right\| \leq 1$ for all $x \in D$.

Finally, since $T_xM = E^s(x) \oplus E^u(x)$ the previous facts show that Φ is partially hyperbolic. $\square$

2.6. **Main results.**

Definition 2.2. Given $k \in \mathbb{N}$, we say that $x \in M^+$ is *k-generating* if the face normals along any orbit segment of length k of the orbit of x generate the Euclidean space $\mathbb{R}^d$.

Definition 2.3. Given $\varepsilon > 0$, the polytope P is called *ε-spanning* if for any d distinct faces $F_1, \ldots, F_d$ of P with interior normals $\eta_1, \ldots, \eta_d$, the angle between η_1 and $E := \langle\langle \eta_2, \ldots, \eta_d \rangle\rangle$ is at least ε, i.e. $\angle(\eta_1, E) \geq \varepsilon$.

We also say that P is a *spanning polytope* if it is ε-spanning for some $\varepsilon > 0$.

The following theorem is the main result of this paper. It shows that the contracting billiard map uniformly expands the unstable direction along the orbit of any k-generating point. Moreover, the expanding rate only depends on the polytope and contracting reflection law.

Theorem 2.6. *Suppose P is a spanning polytope. There exists $\sigma > 1$ depending only on the polytope P and contracting reflection law f such that for every k-generating $x \in D$,*

$$\|D\Phi_{f,P}^{-2k}|_{E^u(x)}\| \leq 1/\sigma$$

We prove this theorem and the following results in section 6.

Definition 2.4. Given $x \in M^+$, the *escaping time of* x, denoted by $T(x)$, is the least positive integer $k \in \mathbb{N}$ such that x is k-generating. If x is not k-generating for any $k \in \mathbb{N}$, then we set $T(x) = \infty$. We also call the function $T : M^+ \to \mathbb{N} \cup \{\infty\}$ the *escaping time of P with respect to f*.

Theorem 2.7. *Suppose P is a spanning polytope and μ is an ergodic $\Phi_{f,P}$-invariant Borel probability measure. If T is μ-integrable, then μ is hyperbolic.*

Theorem 2.8. *Suppose P is a spanning polytope and Λ an invariant set of $\Phi_{f,P}$. If T is bounded on Λ, then $\Phi_{f,P}$ is uniformly hyperbolic on Λ.*

The concept of polytope in general position, mentioned in the following corollaries, is defined below (see definition 3.1).

Corollary 2.9. *Suppose P is a polytope in general position. There exists $\lambda_0 = \lambda_0(P) > 0$ such that for every contracting reflection law f satisfying $\lambda(f) > \lambda_0$ the billiard map $\Phi_{f,P}$ is uniformly hyperbolic.*

A polytope P in general position is called *obtuse* if the barycentric angle at every vertex of P is greater than $\pi/4$ (see section 4 for a precise definition).

Corollary 2.10. *Suppose P is a polytope in general position and f any contracting reflection law. If P is obtuse, the $\Phi_{f,P}$ is uniformly hyperbolic.*

3. Generic Polytopes

Definition 3.1. A d-dimensional polytope P is said to be in general position if

(1) for any set of d faces of P, $(d-1)$-dimensional faces, their normals are linearly independent,

(2) the normals to the $(d-1)$-faces of P incident with any given vertex are linearly independent.

Proposition 3.1. *Given some d-dimensional polytope $P \subset \mathbb{R}^d$ in general position, each vertex has exactly d faces and d edges incident with it.*

Proof. Follows from condition (2). □

Consider the class $\mathcal{P}_N$ of d-dimensional polyhedra $P \subset \mathbb{R}^d$ that contain the origin, i.e., $0 \in \mathrm{int}(P)$, with exactly N faces. Given N points $(p_1, \ldots, p_N) \in (\mathbb{R}^d \setminus \{0\})^N$, define the polytope $Q(p_1, \ldots, p_N) \subset \mathbb{R}^d$,

$$Q(p_1, \ldots, p_N) := \cap_{j=1}^N \{\, x \in \mathbb{R}^b : \langle x, p_j \rangle \le \langle p_j, p_j \rangle \,\} \; .$$

The set

$$\mathcal{U} := \{\, (p_1, \ldots, p_N) \in (\mathbb{R}^d \backslash \{0\})^N : \; Q(p_1, \ldots, p_N) \;\text{ has exactly }\; N\text{-faces} \,\}$$

is open in $(\mathbb{R}^d \setminus \{0\})^N$, and the range of $Q : \mathcal{U} \to \cdot$ coincides with $\mathcal{P}_N$. Locally the map $Q : \mathcal{U} \to \mathcal{P}_N$ is one-to-one, and determines an atlas for a smooth structure on $\mathcal{P}_N$. We will consider on this manifold the Lebesgue measure obtained as push-forward of the Lebesgue measure on $(\mathbb{R}^d \setminus \{0\})^N$ by the map Q.

Let $\mathcal{P}_N$ denote the subset of polytopes in $\mathcal{P}_N$.

Proposition 3.2. *The subset of polytopes in general position is is open and dense, and has full Lebesgue measure in $\mathcal{P}_N$.*

Proof. Consider the subsets $\mathcal{N}_1 \subset \mathcal{P}_N$, resp. $\mathcal{N}_2 \subset \mathcal{P}_N$, of polytopes where condition (1), resp. (2), of definition 3.1 is violated. It is enough to observe that the sets $\mathcal{N}_1$ and $\mathcal{N}_2$ are finite unions of algebraic varieties of co-dimension one.

For any vector $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$, let $\hat{v} := (v_1, \ldots, v_d, \langle v, v \rangle) \in \mathbb{R}^{d+1}$. Then $\mathcal{N}_2$ is covered by the union over all $1 \le i_1 < i_2 < \ldots < i_{d+1} \le N$ of the hypersurfaces defined by the algebraic equation

$$\det[\, \hat{p}_{i_1}, \hat{p}_{i_2}, \ldots, \hat{p}_{i_{d+1}}] = 0 \; . \tag{3.1}$$

In fact, if there is a point $x_0 \in \mathbb{R}^d$ in the intersection of $d+1$ distinct hyperplanes

$$\langle p_{i_k}, x \rangle = \langle p_{i_k}, p_{i_k} \rangle \quad k = 1, \ldots, d+1$$

then the matrix with rows $\hat{p}_{i_1}, \hat{p}_{i_2}, \ldots, \hat{p}_{i_{d+1}}$ contains the vector $(x_0, -1) \in \mathbb{R}^{d+1}$ in its kernel, which implies (3.1).

Analogously, $\mathcal{N}_1$ is contained in the union over all $1 \le i_1 < i_2 < \ldots < i_d \le N$ of the hypersurfaces defined by the algebraic equation

$$\det[\, p_{i_1}, p_{i_2}, \ldots, p_{i_d}] = 0 \; .$$

□

4. Escaping Times

In this section we study the escaping times of billiards on polyhedral cones with contracting reflection laws.

Let $\Pi_1, \ldots, \Pi_s$ be s hyperplanes in $\mathbb{R}^d$ passing through the origin. For each hyperplane Π_i we take a unit normal vector η_i and we suppose that the set of hyperplanes are in general position, i.e. the normal vectors $\eta_1, \ldots, \eta_s$ are linearly independent. A set of s hyperplanes in general position define a *convex polyhedral cone*

$$Q = \{x \in \mathbb{R}^d \colon \langle x, \eta_i \rangle \geq 0\,, \quad i = 1, \ldots, s\}\,.$$

For polyhedral billiard with the specular reflection law, Sinai proved that there exists a constant $K > 0$, depending only on Q, such that every billiard trajectory in Q has at most K reflections [7]. In this case we say that Q has *finite escaping time*.

By projecting the billiard dynamics to the orthogonal complement of $\bigcap_{i=1}^s \Pi_i$, we may assume that the normal vectors $\eta_1, \ldots, \eta_s$ defining the polyhedral cone Q span $\mathbb{R}^d$. Thus, from now on we set $s = d$. Associated with a convex polyhedral cone Q there is a constant measuring the aperture of Q. It is defined as follows.

Definition 4.1. The normal vectors $\eta_1, \ldots, \eta_d$ determine a hyperplane H and a unit normal vector e such that

$$\langle \eta_i, e \rangle = \ell\,, \quad i = 1, \ldots, d\,,$$

where ℓ is the distance of H to the origin. The *barycentric angle* ϕ of Q is defined by $\sin\phi = \ell$ (see Figure 1). Note that $0 < \phi < \pi/2$. We say that a convex polyhedral cone Q is *obtuse* if $\phi > \pi/4$.

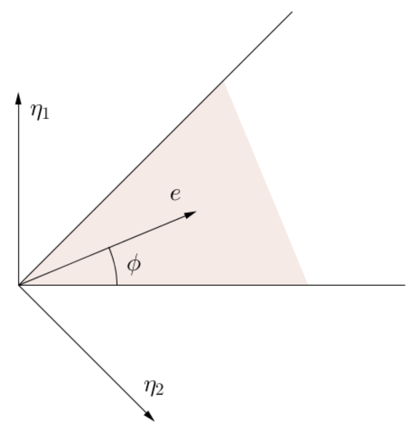


Figure 1. Barycentric angle ϕ.

4.1. **Zigzag reflections.** According to Proposition 2.3, given any billiard orbit $\{(p_k, v_k)\}_{k\geq 0}$, the sequence of *reflection velocities* satisfies

$$v_{k+1} = \frac{\cos f(\theta_k)}{\cos \theta_k} P_{\eta_{i_k}}(u_k) + \frac{\sin f(\theta_k)}{\sin \theta_k} P_{\eta_{i_k}^{\perp}}(u_k)\,, \quad k \geq 0, \tag{4.1}$$

where $u_k = R_{\eta_{i_k}}(v_k)$, $\theta_k = \arccos \langle u_k, \eta_{i_k}\rangle$ and η_{i_k} is the inward normal of P where the $k+1$-th collision took place.

Lemma 4.1. $\|v_{k+1} - v_k\| = 2\cos\left(\frac{f(\theta_k)+\theta_k}{2}\right)$ *for every* $k \geq 0$.

Proof. Simple computation using (4.1). □

Given a sequence of consecutive reflection velocities $v_0, \ldots, v_n$ we denote by L the length of the zigzag path formed by the reflections, i.e.

$$L(v_0, \ldots, v_n) = \sum_{k=0}^{n-1} \|v_{k+1} - v_k\|\,.$$

We say that Q has *bounded zigzag reflections* if there exists a constant $C > 0$ such that $L(v_0, \ldots, v_n) \leq C$ for every sequence of consecutive reflection velocities $v_0, \ldots, v_n$ and any $n \geq 0$.

Lemma 4.2. *A convex polyhedral cone has finite escaping time if and only if it has bounded zigzag reflections.*

Proof. If Q has finite escaping time, then there exists an integer $K > 0$ such that every billiard trajectory has at most K reflections. Since the zigzag length $L : \prod_{i=1}^{K} \mathbb{S}^m \to \mathbb{R}$ is a continuous function with compact domain, it has a maximum. Thus, Q has bounded zigzag reflections.

Now suppose that Q has not finite escaping time. This means that for every $K > 0$ there exists a billiard trajectory in Q that has at least K reflections with the faces of Q. By Lemma 4.1 we have $\|v_{k+1} - v_k\| \geq \delta > 0$ where $\delta := 2\cos\left(\frac{f(\pi/2)+\pi/2}{2}\right) > 0$. This means that for every $K > 0$ there exists a sequence of consecutive reflection velocities $v_0, \ldots, v_n$ such that $L(v_0, \ldots, v_n) \geq \delta K$. So Q cannot have bounded zigzag reflections. □

Next we provide a sufficient condition on the contracting reflection law that guarantees boundedness of zigzag reflections. Thus finite escaping time.

Lemma 4.3. *For every sequence of consecutive reflection velocities* $v_0, \ldots, v_n$ *we have*

$$\langle v_{k+1} - v_k, e\rangle = \|v_{k+1} - v_k\|\gamma_k\,, \quad k = 0, \ldots, n$$

where

$$\gamma_k = \cos\left(\frac{f(\theta_k) - \theta_k}{2}\right)\sin\phi + \sin\left(\frac{f(\theta_k) - \theta_k}{2}\right) h_k$$

and $h_k = \left\langle P_{\eta_{i_k}^\perp}(u_k)/\sin\theta_k, e\right\rangle$.

Proof. Follows from (4.1) that

$$v_{k+1} - v_k = \frac{\cos f(\theta_k) + \cos\theta_k}{\cos\theta_k} P_{\eta_{i_k}}(u_k) + \frac{\sin f(\theta_k) - \sin\theta_k}{\sin\theta_k} P_{\eta_{i_k}^\perp}(u_k)\,.$$

Taking into account that $P_{\eta_{i_k}}(u_k)/\cos\theta_k = \eta_{i_k}$ and $\langle \eta_{i_k}, e\rangle = \sin\phi$ we get

$$\langle v_{k+1} - v_k, e\rangle = (\cos f(\theta_k) + \cos\theta_k)\sin\phi + (\sin f(\theta_k) - \sin\theta_k)\, h_k\,,$$

where $h_k = \left\langle P_{\eta_{i_k}^\perp}(u_k)/\sin\theta_k, e\right\rangle$. Using classical trigonometric identities we can write

$$\langle v_{k+1} - v_k, e\rangle = 2\cos\left(\frac{f(\theta_k)+\theta_k}{2}\right)\gamma_k,$$

where

$$\gamma_k = \cos\left(\frac{f(\theta_k)-\theta_k}{2}\right)\sin\phi + \sin\left(\frac{f(\theta_k)-\theta_k}{2}\right) h_k.$$

To conclude the proof apply Lemma 4.1. □

Theorem 4.4. *If* $2\phi > \pi/2 - f(\pi/2)$ *then* Q *has finite escaping time.*

Proof. Let $v_0, \ldots, v_n$ be any sequence of consecutive reflection velocities. By Lemma 4.3,

$$2 \geq \langle v_n - v_0, e\rangle = \sum_{k=0}^{n-1} \|v_{k+1} - v_k\|\gamma_k\,, \tag{4.2}$$

where

$$\gamma_k = \cos\left(\frac{f(\theta_k)-\theta_k}{2}\right)\sin\phi + \sin\left(\frac{f(\theta_k)-\theta_k}{2}\right) h_k$$

and $h_k = \left\langle P_{\eta_{i_k}^\perp}(u_k)/\sin\theta_k, e\right\rangle$. To estimate γ_k from below note that $h_k \leq \cos\phi$. Thus

$$\gamma_k \geq \sin\left(\phi + \frac{f(\theta_k)-\theta_k}{2}\right) \geq \sin\left(\phi + \frac{f(\pi/2)-\pi/2}{2}\right).$$

By assumption $\mu := \phi + \frac{f(\pi/2)-\pi/2}{2} > 0$. Then, it follows from (4.2) that

$$L(v_0, \ldots, v_n) < \frac{2}{\sin\mu}\,,$$

for every sequence of consecutive reflection velocities $v_0, \ldots, v_n$. This proves that Q has bounded zigzag reflections. Thus, by Lemma 4.2, Q has finite escaping time. □

This theorem yields the following corollaries.

Corollary 4.5. *Any polyhedral cone* Q *with contracting reflection law* f *sufficiently close to the specular one has finite escaping time.*

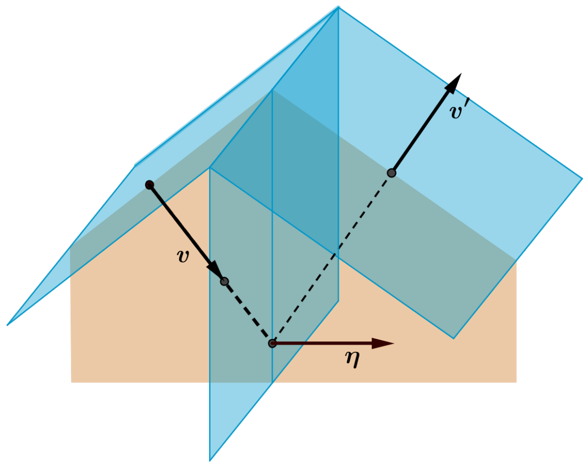


FIGURE 2. Composition of the projections $P_{v'^\perp} \circ P_{v,\eta^\perp}$

Proof. It is clear that $2\phi > \pi/2 - f(\pi/2)$ for every contraction f sufficiently close to the identity. Thus, Q has finite escaping time, by Theorem 4.4. □

Recall that a convex polyhedral cone Q is obtuse if $\phi > \pi/4$.

Corollary 4.6. *Any obtuse polyhedral cone Q has finite escaping time for every contracting reflection law f.*

Proof. If the polyhedral cone is obtuse then $\phi > \pi/4$. Thus, $2\phi > \pi/2 > \pi/2 - f(\pi/2)$ for every contraction f. Thus, Q has finite escaping time, by Theorem 4.4. □

5. Uniform Expansion

By Proposition 2.4, the first component of the derivative $D\Phi_{f,P}(p,v)$ of the billiard map is represented in Jacobi coordinates by the map

$$L_{v,\eta} = P_{v'^\perp} \circ P_{v,\eta^\perp} : \mathbb{R}^d \to \mathbb{R}^d$$

where $v', v, \eta \in \mathbb{R}^d$ are three coplanar unit vectors, $v' = C_\eta(R_\eta(v))$.

In this section we give conditions that ensure the uniform expansion of compositions of such maps. Since the second component of the billiard map is contractive (see Proposition 2.5), these conditions will imply the uniform hyperbolicity of the billiard map.

5.1. **Expansivity lemmas.** The first lemma says that $L_{v,\eta}$ has two singular values: $\lambda = 1$ with multiplicity $d-1$, and $\lambda = |\langle v', \eta\rangle / \langle v, \eta\rangle|$ multiplicity 1. See Figure 2.

Lemma 5.1. *Given coplanar unit vectors $v', v, \eta \in \mathbb{R}^d$, the composition $P_{v'^\perp} \circ P_{v,\eta^\perp} : \mathbb{R}^d \to \mathbb{R}^d$ satisfies:*

(a) $P_{v'^\perp} \circ P_{v,\eta^\perp}(v) = 0$,

(b) $P_{v'^\perp} \circ P_{v,\eta^\perp}(x) = x$, *for every $x \in \eta^\perp \cap v^\perp$,*

(c) $P_{v'^{\perp}} \circ P_{v,\eta^{\perp}}$ *maps the line* $v^{\perp} \cap W$ *onto the line* $v'^{\perp} \cap W$, *where* $W = \langle\!\langle v, \eta \rangle\!\rangle$, *multiplying the vector's norms by the factor* $|\langle v', \eta\rangle / \langle v, \eta\rangle|$.

Proof. Straightforward computation. □

The second lemma is abstract. Let V, V', V'' be Euclidean spaces of the same dimension, and $L : V \to V'$, $L' : V' \to V''$ be linear isomorphisms.

Given $\sigma \geq 1$ and a subspace $E \subset V$, we say that L is a *σ-expansion on E* if and only if $\|Lv\| \geq \sigma \|v\|$ for all $v \in E$. Given another linear subspace $H \subseteq V$ such that $E \subseteq H$ we say that L is a *relative σ-expansion on H w.r.t. E* if and only if the quotient map $\overline{L} : V/E \to V'/L(E)$ is a σ-expansion on H/E. Note that the quotient space V/E is an Euclidean space which can naturally be identified with $E^{\perp}$. Finally, we say that L is a *σ-expansion* to mean that L is a σ-expansion on its domain V.

If we do not need to specify the minimal rate of expansion we shall simply say that L is a uniform expansion on E, or that L is a relative uniform expansion on H w.r.t. E.

Lemma 5.2. *Given a linear subspace $H \subseteq V$, if*

(1) *L is a σ-expansion on H, and*
(2) *L is a relative σ-expansion on V w.r.t. H*

then L is a σ-expansion on V.

Proof. Follows immediately from the definition of σ-expansion and relative σ-expansion. □

5.2. **Trajectories.** Let P be a d-dimensional polytope in $\mathbb{R}^d$, and $\mathcal{N}_P$ be the set of its unit inward normals. Denote by $\mathbb{N}_0$ the set of natural numbers $\mathbb{N}$ including 0.

Definition 5.1. A sequence $\{(v_j, \eta_j)\}_{j \geq 0} \in (\mathbb{S} \times \mathcal{N}_P)^{\mathbb{N}_0}$ is called a *trajectory* if for all $j \in \mathbb{N}$

(1) $\langle v_{j-1}, \eta_j \rangle \leq 0$,
(2) $v_j = C_{\eta_j} \circ R_{\eta_j}(v_{j-1})$,

where R_η is the reflection introduced in section 2, and C_η is the contracting reflection law defined in subsection 2.3.

Trajectories relate with the billiard map orbits in the following way. Given a billiard orbit $\{(p_j, v_j)\}_{j \geq 0}$ of the contracting billiard map $\Phi_{f,P}$, denoting by η_j the inward unit normal of P at p_j, the sequence $\{(v_j, \eta_j)\}_{j \geq 0}$ is a trajectory.

Lemma 5.3. *Given any trajectory $\{(v_j, \eta_j)\}_{j \geq 0}$ there exist scalars $\alpha_j, \beta_j \in \mathbb{R}$ such that for any $j \geq 1$,*

$$v_j = \alpha_j \eta_j + \beta_j v_{j-1}$$

where
$$\cos\Big(\frac{\pi}{2}\lambda(f)\Big) < \alpha_j < 2 \quad \textit{and} \quad 0 \le \beta_j < 1.$$
Moreover,
$$\left|\frac{\langle v_j, \eta_j\rangle}{\langle v_{j-1}, \eta_j\rangle}\right| = \frac{\cos f(\theta_j)}{\cos\theta_j} \quad \textit{where} \quad \theta_j = \arccos|\langle v_{j-1}, \eta_j\rangle|.$$

Proof. According to Proposition 2.3,
$$v_j = (a_j + b_j)\cos\theta_j\,\eta_j + b_j v_{j-1}$$
where
$$a_j = \frac{\cos f(\theta_j)}{\cos\theta_j}, \quad b_j = \frac{\sin f(\theta_j)}{\sin\theta_j} \quad \text{and} \quad \theta_j = \arccos|\langle v_{j-1}, \eta_j\rangle|.$$
Since $\lambda(f) < 1$, we have $1 \le a_j + b_j < 2$ and $0 \le b_j < 1$. Moreover, $\cos\theta_j > \cos(\frac{\pi}{2}\lambda(f))$. The last claim is a simple computation. ☐

The following result says that the *trajectory space* $\mathcal{T} = \mathcal{T}_{f,P}$ of all trajectories is a compact space.

Proposition 5.4. *The space $\mathcal{T}$ is a closed subspace of the product space $(\mathbb{S} \times \mathcal{N}_P)^{\mathbb{N}_0}$. In particular, with the induced topology $\mathcal{T}$ is a compact space.*

Proof. The trajectory space $\mathcal{T}$ is closed in the product space because conditions (1) and (2) in Definition 5.1 are closed conditions. By Thychonoff's theorem $(\mathbb{S} \times \mathcal{N}_P)^{\mathbb{N}_0}$ is compact, and hence $\mathcal{T}$ is compact too. ☐

Given $i < j$ in $\mathbb{N}_0$, we denote by $[i, j] := \{i, i+1, \ldots, j\} \subseteq \mathbb{N}_0$ the time interval between the instants i and j. Given a trajectory $\{(v_j, \eta_j)\}_{j\ge 0}$ and a time interval $[i, j]$, the linear span $V_{[i,j]} := \langle\!\langle v_i, v_{i+1}, \ldots, v_j\rangle\!\rangle$ is called the *velocity front* of the trajectory along the time interval $[i, j]$. The linear span $N_{[i,j]} := \langle\!\langle \eta_i, \eta_{i+1}, \ldots, \eta_j\rangle\!\rangle$ is called the *normal front* of the trajectory along the time interval $[i, j]$. Given $i \in \mathbb{N}$, let $L_i : v_{i-1}^\perp \to v_i^\perp$ be the linear map defined by
$$L_i = P_{v_i^\perp} \circ P_{v_{i-1}, \eta_i^\perp}.$$
Finally we define the *velocity tangent flow* along $[i, j]$ to be the linear map $L_{[i,j]} : v_i^\perp \to v_j^\perp$ defined by
$$L_{[i,j]} = L_j \circ \ldots \circ L_{i+1}.$$
When the trajectory is associated to a billiard orbit $\{(p_l, v_l)\}_{l\ge 0}$ of $\Phi_{f,P}$, the linear map $L_{[i,j]}$ represents, in Jacobi coordinates, the first component of the derivative $D\Phi_{f,P}^{j-i}$ at (p_i, v_i). By definition, given $i < j < k$,
$$L_{[i,k]} = L_{[j,k]} \circ L_{[i,j]}\ .$$

Lemma 5.5. *Given a trajectory $\{(v_l, \eta_l)\}_{l\ge 0}$, for all intervals $[i, j]$,*

(1) $V_{[i,j]} = \langle\!\langle v_i \rangle\!\rangle + N_{[i+1,j]}$ *and* $V_{[i,j]}{}^{\perp} \subseteq v_i^{\perp} \cap v_j^{\perp}$.
(2) $L_{[i,j]} : v_i^{\perp} \to v_j^{\perp}$ *is the identity on* $V_{[i,j]}{}^{\perp}$.
(3) $L_{[i,j]}$ *is a* 1*-expansion on* $V_{[i,j]}$.

Proof. Straightforward computation. □

Definition 5.2. We say that the trajectory $\{(v_l, \eta_l)\}_{l\geq 0}$ is *generating on* $[i,j]$ if $N_{[i,j]} = \mathbb{R}^d$. Given $k \in \mathbb{N}$, we say that the trajectory is *k-generating* if it is generating on any interval $[i,j]$ with $j - i \geq k$.

We can now state this section's main result.

Theorem 5.6. *Given $\varepsilon > 0$, d-dimensional polytope P and contracting reflection law f, there exists a constant $\sigma = \sigma(\varepsilon, d, f) > 1$ such that for any trajectory $\{(v_j, \eta_j)\}_{j\geq 0}$ in $\mathcal{T}_{f,P}$ the following holds. If*
(1) *P is ε-spanning,*
(2) *$\{(v_j, \eta_j)\}_{j\geq 0}$ is k-generating, with $k \in \mathbb{N}$,*
then the velocity tangent flow $L_{[0,2k]} : v_0^{\perp} \to v_{2k}^{\perp}$ is a σ-expansion.

The proof of this theorem is done at the end of the section.

Remark 5.7. From the previous theorem's conclusion, for any $n \geq 0$,

$$\left\| L_{[0,n]}(v) \right\| \geq \sigma^{\frac{n}{2k}-1} \, \|v\| \qquad \text{all} \ \ v \in v_0^{\perp} \ .$$

This means, minimum growth expansion rate $\geq \sigma^{\frac{1}{2k}} > 1$.

5.3. **Collinearities.** Throughout the rest of this section, we assume that $\varepsilon > 0$ is fixed and that P is ε-spanning.

Consider a trajectory $\{(v_l, \eta_l)\}_{l\geq 0}$ in $\mathcal{T}$.

Definition 5.3. A time interval $[i,j]$ is called a *collinearity* of the trajectory $\{(v_l, \eta_l)\}_{l\geq 0}$ if its velocity and the normal fronts along the time interval $[i,j]$ coincide, i.e. $V_{[i,j]} = N_{[i,j]}$. The number $j - i$ will be referred as the length of the collinearity $[i,j]$.

Definition 5.4. A collinearity is called *minimal* if it contains no smaller subinterval which is itself a collinearity.

For instance, if $v_i \in \langle\!\langle \eta_i \rangle\!\rangle$ then $\{i\}$ is a minimal collinearity of length 0.

Proposition 5.8. *Given a trajectory $\{(v_l, \eta_l)\}_{l\geq 0}$, assume $v_i \in N_{[i,j]}$ with $i \leq j$. Then there is some $i' \in [i,j]$ such that the time interval $[i',j]$ is a collinearity.*

Proof. The proof goes by induction on the length $r = j - i$. If the length is 0 then $i = j$ and we have necessarily $v_i \in \langle\!\langle \eta_i \rangle\!\rangle$, in which case it is obvious that $[i,i] = \{i\}$ is collinearity. Assume now that the statement holds for all time intervals of length less than r, and let $v_i = \lambda_i \eta_i + \cdots + \lambda_j \eta_j$ with $j - i = r$. We consider two cases:

First suppose that $\lambda_i \neq 0$. By item (1) of Lemma 5.5,

$$V_{[i,j]} = \langle\!\langle v_i \rangle\!\rangle + N_{[i+1,j]} \subseteq N_{[i,j]}.$$

Conversely, because $\lambda_i \neq 0$ we have $\eta_i \in \langle\!\langle v_i \rangle\!\rangle + N_{[i+1,j]}$ which proves that

$$N_{[i,j]} \subseteq \langle\!\langle v_i \rangle\!\rangle + N_{[i+1,j]} = V_{[i,j]},$$

where in the last equality we have used again item (1) of Lemma 5.5. Therefore, $[i,j]$ is a collinearity in this case.

Assume next that $\lambda_i = 0$. By Lemma 5.3, there are scalars α_{i+1} and β_{i+1} such that $v_{i+1} = \alpha_{i+1}\eta_{i+1} + \beta_{i+1}v_i$. We may assume that $\beta_{i+1} \neq 0$. Otherwise $v_{i+1} \in \langle\!\langle \eta_{i+1} \rangle\!\rangle$ and $[i+1,j]$ is a collinearity. Thus

$$\lambda_{i+1}\eta_{i+1} + \ldots + \lambda_j\eta_j = v_i = \frac{1}{\beta_{i+1}}\left(v_{i+1} - \alpha_{i+1}\eta_{i+1}\right) .$$

In this case

$$v_{i+1} = \beta_{i+1}\left[\left(\lambda_{i+1} - \frac{\alpha_{i+1}}{\beta_{i+1}}\right)\eta_{i+1} + \lambda_{i+2}\eta_{i+2} + \ldots + \lambda_j\eta_j\right] .$$

and the conclusion follows by the induction hypothesis applied to the time interval $[i+1,j]$ of length $p-1$. ☐

Proposition 5.9. *Given a trajectory $\{(v_l,\eta_l)\}_{l\geq 0}$ and $i < j \leq j'$ the following holds.*

1. *If $[i,j]$ is a collinearity then $[i,j']$ is also a collinearity.*
2. *If $v_j \in V_{[i,j-1]}$ and $\eta_j \notin N_{[i,j-1]}$, then there is some $i < i' \leq j$ such that $[i',j]$ is a collinearity.*

Proof. Let $i < j \leq j'$.

(1) Assume $V_{[i,j]} = N_{[i,j]}$. Then by Lemma 5.5,

$$V_{[i,j']} = \langle\!\langle v_i \rangle\!\rangle + N_{[i+1,j]} + N_{[j+1,j']} = N_{[i,j]} + N_{[j+1,j']} = N_{[i,j']}.$$

(2) Assume now $v_j \in V_{[i,j-1]}$. By Lemma 5.3,

$$\eta_j = \frac{1}{\alpha_j}\left(v_j - \beta_j v_{j-1}\right),$$

where $\alpha_j \neq 0$. Thus $\eta_j \in V_{[i,j-1]}$. By Lemma 5.5 we can write $\eta_j = \lambda_i v_i + u$ for some $u \in N_{[i+1,j-1]}$. By assumption, $\lambda_i \neq 0$. Thus $v_i \in N_{[i+1,j]}$. Again by Lemma 5.3, we conclude that $v_{i+1} \in N_{[i+1,j]}$. Now the claim follows by Proposition 5.8.

☐

Corollary 5.10. *Given a trajectory $\{(v_l,\eta_l)\}_{l\geq 0}$, let $[i,k]$ be a time segment that contains no subinterval which is a collinearity. Then for every $j \in [i,k]$ either*

1. *$\eta_j \in \{\eta_{i+1},\ldots,\eta_{j-1}\}$, or else*
2. *$v_j \notin V_{[i,j-1]}$.*

Proof. This corollary is a reformulation of item (2) of Proposition 5.9. ☐

5.4. **Quantifying collinearities.** We are now going to prove quantified versions of Propositions 5.8, 5.9 and Corollary 5.10. The following abstract continuity lemma will be useful.

Lemma 5.11. *Let $\mathcal{X}$ be a compact topological space and $f, g : \mathcal{X} \to \mathbb{R}$ be continuous functions such that $g(x) = 0$ for all $x \in \mathcal{X}$ with $f(x) = 0$. Given $\delta > 0$ there is $\delta' > 0$ such that for all $x \in \mathcal{X}$, if $f(x) < \delta'$ then $g(x) < \delta$.*

Proof. Assume, to get a contradiction, that the claimed statement does not hold. Then there is $\delta > 0$ such that for all $n \in \mathbb{N}$ there is a point $x_n \in \mathcal{X}$ with $f(x_n) < \frac{1}{n}$ and $g(x_n) \geq \delta$. Since $\mathcal{X}$ is compact, by taking a subsequence we can assume $x_n \to x$ in $\mathcal{X}$. By continuity of f and g, $f(x) = 0$ and $g(x) \geq \delta$, which contradicts the lemma hypothesis. □

Definition 5.5. Given $\delta > 0$, we call *δ-collinearity* of a trajectory $\{(v_l, \eta_l)\}_l$ to any time interval $[i, j]$ such that $\dim V_{[i,j]} = \dim N_{[i,j]}$ and

$$\angle \left(V_{[i,j]}, N_{[i,j]}\right) < \delta \, .$$

Proposition 5.12. *Given $\delta > 0$ there exists $\delta' > 0$ such that for any trajectory $\{(v_l, \eta_l)\}_l$ the following holds. If*

$$\angle \left(v_i, N_{[i,j]}\right) < \delta'$$

for some $0 \leq i \leq j$, then there exists $i' \in [i, j]$ for which the time interval $[i', j]$ is a δ-collinearity of the given trajectory.

Proof. Notice that, because the space of trajectories $\mathcal{T}$ is shift invariant, there is no loss of generality in assuming that $[i, j] = [0, p]$. Define the functions $f_k, g_k : \mathcal{T} \to \mathbb{R}$ by

$$\begin{aligned} f_k \left(\{(v_l, \eta_l)\}_l\right) &= \angle \left(v_0, N_{[0,k]}\right) , \\ g_k \left(\{(v_l, \eta_l)\}_l\right) &= \min_{0 \leq i \leq k} \angle \left(V_{[i,k]}, N_{[i,k]}\right) . \end{aligned}$$

These functions are clearly continuous.

Proposition 5.8 shows that for all $x \in \mathcal{T}$ and $0 \leq k \leq p$, $f_k(x) = 0$ implies $g_k(x) = 0$. Thus, given $\delta > 0$, by Lemma 5.11, there exists $\delta' > 0$ such that for any $0 \leq k \leq p$ and $x \in \mathcal{T}$,

$$f_k(x) < \delta' \quad \Rightarrow \quad g_k(x) < \delta \, .$$

□

Proposition 5.13. *Given any trajectory $\{(v_l, \eta_l)\}_l$, $i < j \leq j'$ and $\delta > 0$ the following holds.*

1. *If $[i, j]$ is a δ-collinearity, then $[i, j']$ is a δ'-collinearity, where $\delta' = \arcsin(\frac{\sin \delta}{\sin \varepsilon})$.*
2. *There exists $\delta' > 0$ such that, if*

$$\angle(v_j, V_{[i,j-1]}) < \delta'$$

and $\eta_j \notin N_{[i,j-1]}$, then there is some $i < i' \leq j$ such that $[i', j]$ is a δ-collinearity.

Proof. (1) Denote by H the linear space spanned by the 'new' normals η_l in the range $j < l \leq j'$, i.e., normals which are not in $\{\eta_i, \ldots, \eta_j\}$. By definition of H we have,

$$V_{[i,j']} = V_{[i,j]} + H \ ,$$
$$N_{[i,j']} = N_{[i,j]} + H \ .$$

Hence by Lemma 2.1, if $[i, j]$ is a δ-collinearity,

$$\sin \angle \left(V_{[i,j']}, N_{[i,j']}\right) \leq \frac{1}{\sin \varepsilon} \sin \angle \left(V_{[i,j]}, N_{[i,j]}\right) \leq \frac{\sin \delta}{\sin \varepsilon} = \sin \delta',$$

which proves that $[i, j']$ is a δ'-collinearity.

(2) As in the proof of Proposition 5.12, there is no loss of generality in assuming that $[i, j] = [0, p]$. Define the functions $f_k, g_k : \mathcal{T} \to \mathbb{R}$ by

$$f_k \left(\{(v_l, \eta_l)\}_l\right) = \angle \left(v_k, V_{[0,k-1]}\right) ,$$
$$g_k \left(\{(v_l, \eta_l)\}_l\right) = \min_{1 \leq i \leq k} \angle \left(V_{[i,k]}, N_{[i,k]}\right) .$$

These functions are clearly continuous.

Item (2) of Proposition 5.9 shows that for every $x = \{(v_l, \eta_l)_l\} \in \mathcal{T}$ and for every $1 \leq k \leq p$ for which $\eta_k \notin N_{[0,k-1]}$, $f_k(x) = 0$ implies $g_k(x) = 0$. Thus, given $\delta > 0$, by Lemma 5.11, there exists $\delta' > 0$ such that for every $x = \{(v_l, \eta_l)_l\} \in \mathcal{T}$ and for every $1 \leq k \leq p$ for which $\eta_k \notin N_{[0,k-1]}$,

$$f_k(x) < \delta' \quad \Rightarrow \quad g_k(x) < \delta \ .$$

□

Corollary 5.14. *Given $\delta > 0$ there is $\delta' > 0$ such that the following dichotomy holds. Let $[i + 1, j]$ be a time segment of a trajectory that contains no subinterval which is a δ-collinearity of that trajectory. Then for every $l \in [i + 1, j]$ either*

(1) $\eta_l \in \{\eta_{i+1}, \ldots, \eta_{l-1}\}$, *or else*
(2) $\angle(v_l, V_{[i,l-1]}) \geq \delta'$.

Proof. This corollary is a reformulation of Proposition 5.13 (2). □

5.5. **The expansivity argument.** In this subsection we relate collinearities with expansion of the velocity tangent flow, and then prove Theorem 5.6.

Recall that we are assuming that P is ε-spanning.

Proposition 5.15. *There exists $\sigma > 1$, depending only on d, f and ε, such that given a collinearity $[i, j_0]$ of some trajectory, for all $j > j_0$, the velocity flow $L_{[i,j]}$ is a relative σ-expansion on $v_i^{\perp} \cap V_{[i,j]}$ w.r.t. $v_i^{\perp} \cap V_{[i,j_0]}$.*

Proof. Assume $\{(v_l, \eta_l)\}_l$ is a trajectory with collinearity $[i, j_0]$. Because P is ε-spanning, for all $j > j_0$ such that $\eta_j \notin N_{[i,j-1]}$ we have $\angle(\eta_j, N_{[i,j-1]}) \geq \varepsilon$.

Notice that $V_{[i,j-1]} = N_{[i,j-1]}$, for all $j > j_0$, and by Lemma 5.3, we have $v_j = \alpha_j \eta_j + \beta_j v_{j-1}$ with $\alpha_j \geq \cos(\frac{\pi}{2}\lambda(f)) > 0$. Hence, by Lemma 2.1 there is some $0 < \varepsilon' < \varepsilon$ depending on ε and on $\lambda(f)$, such that for all $j > j_0$ with $\eta_j \notin N_{[i,j-1]}$,

$$\angle(v_j, V_{[i,j-1]}) \geq \varepsilon' \ .$$

By Lemma 5.1, the linear map $L_{v_{j-1},\eta_j} : v_{j-1}^\perp \to v_j^\perp$ adds expansion in some new direction. Two features are important to retain here:

(1) By Lemmas 5.1 and 5.3, the expansion rate in this new direction is given by
$$\left| \frac{\langle v_j, \eta_j \rangle}{\langle v_{j-1}, \eta_j \rangle} \right| = \frac{\cos f(\theta)}{\cos \theta}$$
where the angle $\theta = \angle(v_{j-1}, \eta_j)$ is at least ε. This shows that this expansion rate is bounded away from 1.

(2) The new expanding direction makes an angle bounded away from 0 (at least ε') with previous velocity front $V_{[i,j-1]}$.

Together these two facts show that $L_{v_{j-1},\eta_j} : v_{j-1}^\perp \to v_j^\perp$ is a relative uniform expansion of $v_i^\perp \cap V_{[i,j]}$ w.r.t. $v_i^\perp \cap V_{[i,j-1]}$. The general conclusion for the composite map $L_{[i,j]}$ follows easily by induction. □

Corollary 5.16. *Given the constant $\sigma > 1$ in Proposition 5.15, and $1 < \sigma' < \sigma$, there is $\delta > 0$ such that for every trajectory $\{(v_l, \eta_l)\}_l$, if $[i, j_0]$ is a δ-collinearity then for all $j > j_0$, the velocity flow $L_{[i,j]}$ is a relative σ'-expansion on $v_i^\perp \cap V_{[i,j]}$ w.r.t. $v_i^\perp \cap V_{[i,j_0]}$.*

Proof. This follows from Proposition 5.15 with a continuity argument like the one used in the proof of Proposition 5.12. □

Proposition 5.17. *Given $\delta > 0$ there exists $\sigma > 1$, depending on depending on d, f, ε and δ, such that if a time interval $[i+1, j]$ of some trajectory contains no subinterval which is a δ-collinearity then $L_{[i,j]}$ is a σ-expansion on $v_i^\perp \cap V_{[i,j]}$.*

Proof. Let $[i, j]$ be a time interval such that $[i+1, j]$ contains no subinterval which is itself a δ-collinearity. By Corollary 5.14, there is $\delta' > 0$ such that for every $l \in [i+1, j]$ either $\eta_l \in \{\eta_{i+1}, \ldots, \eta_{l-1}\}$, or else

$$\angle(v_l, V_{[i,l-1]}) \geq \delta' \ .$$

Therefore, using Lemma 5.1, we see by induction that $L_{[i,j]}$ is a uniform expansion on $v_i^\perp \cap V_{[i,j]}$. □

Proof of Theorem 5.6. Take the constant $\sigma > 1$ given in Proposition 5.15. Set $\sigma' = \frac{1}{2} + \frac{1}{2}\sigma \in]1, \sigma[$, and pick $\delta = \delta(\sigma') > 0$ as provided by Corollary 5.16. Fix the constant $\sigma'' = \sigma(\delta) > 1$ given by Proposition 5.17 and set $\sigma_0 = \min\{\sigma', \sigma''\}$.

Let $\{(v_j, \eta_j)\}_{j\in\mathbb{Z}}$ be a trajectory. We consider three cases:

- If $[0, k]$ contains no δ-collinearity, by Proposition 5.17 $L_{[0,k]}$ is a σ''-expansion on $v_0^\perp \cap V_{[0,k]}$. But since any trajectory is generating on $[0, k]$, we have $v_0^\perp = v_0^\perp \cap V_{[0,k]}$, which proves that $L_{[0,k]}$ is a σ''-expansion. Finally, because $L_{[k,2k]}$ is non contracting, $L_{[0,2k]} = L_{[k,2k]} \circ L_{[0,k]}$ is also a σ''-expansion.
- If $[0, k]$ contains a δ-collinearity $[i, j] \subseteq [0, k]$, we can assume it is minimal, in the sense that $[i, j]$ contains no proper subinterval which is itself a δ-collinearity. Consider first the case $j \geq i + 1$. By Proposition 5.17, $L_{[i,j]}$ is a σ''-expansion on $v_i^\perp \cap V_{[i,j]}$. Because $L_{[i,2k]} = L_{[j,2k]} \circ L_{[i,j]}$, and $L_{[j,2k]}$ is non contracting, the map $L_{[i,2k]}$ is also a σ''-expansion on $v_i^\perp \cap V_{[i,j]}$. Remark that since $i \leq k$, the trajectory is generating on $[i, 2k]$, and hence $v_i^\perp = v_i^\perp \cap V_{[i,2k]}$. Hence by Proposition 5.16, $L_{[i,2k]}$ is a relative σ'-expansion on $v_i^\perp$ w.r.t. $v_i^\perp \cap V_{[i,j]}$. Thus by Lemma 5.2, $L_{[i,2k]}$ is a σ_0-expansion, which implies so is $L_{[0,2k]}$.
- Finally we consider the case $[0, k]$ contains δ-collinearities, but the minimal ones have length zero, say $\{i\} \subset [0, k]$ is a δ-collinearity. In this case we have $\angle(v_i, \eta_i) < \delta$, and the proof is somehow simpler. By Lemma 2.1

$$\begin{aligned}\angle(V_{[i,j-1]}, N_{[i,j-1]}) &= \angle(\langle\!\langle v_i \rangle\!\rangle + N_{[i+1,j-1]}, \langle\!\langle \eta_i \rangle\!\rangle + N_{[i+1,j-1]}) \\ &\leq \arcsin\left(\frac{\sin\delta}{\sin\varepsilon}\right) =: \hat{\delta}\ .\end{aligned}$$

On the other hand, because $v_j = \alpha_j\,\eta_j + \beta_j\,v_{j-1}$ with $\alpha_j \geq c$ and $c = \cos(\frac{\pi}{2}\lambda(f))$, whenever $\eta_j \notin \{\eta_i, \ldots, \eta_{j-1}\}$ we have

$$\begin{aligned}\angle(v_j, V_{[i,j-1]}) &\geq \frac{c}{2}\angle(\eta_j, V_{[i,j-1]}) \\ &\geq \frac{c}{2}\,\angle(\eta_j, N_{[i,j-1]}) - \frac{c\,\hat{\delta}}{2} \\ &\geq \frac{c}{2}\,(\varepsilon - \hat{\delta}) \geq \frac{c\,\varepsilon}{4}\ ,\end{aligned}$$

provided δ is small enough. Thus, using Lemma 5.1 we get by induction that $L_{[i,i+k]}$ is a uniform expansion, and as before that $L_{[0,2k]}$ is also a uniform expansion.

Therefore, $L_{[0,2k]}$ is a σ_0-expansion in all cases. □

6. Proof of the Main Statements

Denote by $\Phi : D \to D$ the billiard map for the polytope P and the contracting reflection law f.

Proof of Theorem 2.6. Let $x = (p, v) \in D$ be any k-generating point. We can identify the tangent space $T_x M$ with $v^\perp \times v^\perp$ using the Jacobi coordinates. From the proof of Proposition 2.5 we know that $E^u(x) = \{(J, J') \in v^\perp \times v^\perp \colon J' = 0\}$. Moreover, by Theorem 5.6, there exists $\sigma > 1$ depending only on P and f such that

$$\|D\Phi^{2k}(x)(J,0)\| = \|L_{[0,2k]}(J)\| \geq \sigma \|J\|, \quad \forall\, J \in v^\perp .$$

This uniform minimum growth expansion on E^u proves the theorem. □

Proof of Theorem 2.7 . Assume that $\int T\, d\mu < +\infty$. Consider the partition $\{A_n = T^{-1}\{n\}\}_{n\in\mathbb{N}}$ of D, and define the measurable function $\tilde{T} : D \to \mathbb{N}$, $\tilde{T} = n$ on $A'_n := \Phi(A_n)$. This function satisfies

$$T\left(\Phi^{-\tilde{T}(x)}(x)\right) = \tilde{T}(x) \quad \text{for all } \ x \in D\, .$$

Moreover $\int \tilde{T}\, d\mu = \int T\, d\mu < +\infty$. From Theorem 2.6 we have

$$\left\| D\Phi_x^{-2\,\tilde{T}(x)}|_{E^u} \right\| \leq \sigma^{-1} \quad \text{for all } \ x \in D\, .$$

Define recursively the following sequence of backward iterates and stopping times

$$\begin{cases} x_0 &= x \\ t_0 &= 2\,\tilde{T}(x_0) \end{cases} \qquad \begin{cases} x_{j+1} &= \Phi^{-t_j}(x_j) \\ t_{j+1} &= 2\,\tilde{T}(x_{j+1}) \end{cases} .$$

Let us write $\tau_n = \sum_{j=0}^{n-1} t_j$. Since $t_j \geq 2\,d$ for all j, this sequence tends to $+\infty$, and we have

$$\begin{aligned} -\frac{1}{\tau_n} \log \left\| D\Phi_x^{-\tau_n}|_{E^u} \right\| &\geq -\frac{1}{\sum_{j=0}^{n-1} t_j} \sum_{j=0}^{n-1} \log \left\| D\Phi_{x_j}^{-t_j}|_{E^u} \right\| \\ &\geq -\frac{n}{\sum_{j=0}^{n-1} t_j} \log \sigma^{-1} = \frac{\log \sigma}{\frac{1}{n} \sum_{j=0}^{n-1} \tilde{T}(x_{j-1})} . \end{aligned}$$

Thus, by Birkhoff's ergodic theorem, for μ-almost every $x \in D$,

$$\limsup_{n\to+\infty} -\frac{1}{n} \log \left\| D\Phi_x^{-n}|_{E^u} \right\| \geq \frac{\log \sigma}{\int \tilde{T}\, d\mu} > 0\, .$$

By Kingman's ergodic theorem, the above lim sup is actually a limit, and applying Fatou's lemma,

$$\lim_{n\to+\infty} \int -\frac{1}{n} \log \left\| D\Phi_x^{-n}|_{E^u} \right\|\, d\mu \geq \frac{\log \sigma}{\int \tilde{T}\, d\mu} > 0\, .$$

This proves that μ is a hyperbolic measure. □

Proof of Theorem 2.8 . By Proposition 2.5, Φ is uniformly partially hyperbolic on Λ. Moreover, it follows from Theorem 2.6 that there exists a constant $C > 0$ depending only on P and f such that

$$\|D\Phi_{f,P}^{-n}|_{E^u(x)}\| \leq C\left(\frac{1}{\sigma}\right)^{\frac{n}{2k}}$$

for every $x \in \Lambda$ that is k-generating. Since the escaping time function T is bounded on Λ, every $x \in \Lambda$ is τ-generating where $\tau := \sup_{x\in\Lambda} T(x)$. So the expansion rate can be made uniform and equal to $\sigma^{1/\tau} > 1$. This shows that Φ is uniformly hyperbolic on Λ. □

Proof of Corollary 2.9. Use Theorem 2.8 and Corollary 4.5. □

Proof of Corollary 2.10. Use Theorem 2.8 and Corollary 4.6. □

7. Examples

In this section we provide some examples of uniformly hyperbolic polytopal billiards with contracting reflection laws. The examples are 3-dimensional simplexes in $\mathbb{R}^4$. We denote by Δ_h^3 the convex hull of the vertexes $e_1, e_2, e_3, e = (e_1 + e_2 + e_3)/3 + he_4$, where the vectors e_i stand for the canonical basis of $\mathbb{R}^4$. For any set of three faces of Δ_h^3, their normals are linearly independent. Therefore, Δ_h^3 is in general position (see Definition 3.1) and it is spanning (see Definition 2.3). First it will be shown that:

Theorem 7.1. *For every $h > 0$ and every contracting reflection law f satisfying $\lambda(f) > \lambda_0(h)$ the billiard map Φ_{f,Δ_h^3} is uniformly hyperbolic, where $\lambda_0(h)$ is given by*

$$\begin{cases} -4/\pi \arcsin\left(3h/\sqrt{12\sqrt{6h^2+1}+45h^2+12}\right)+1 & \text{for } h \leq 2/\sqrt{3}, \\ -4/\pi \arcsin\left(1/\sqrt{6h^2+1}\right)+1 & \text{for } h > 2/\sqrt{3}. \end{cases}$$

Proof. This theorem follows from Theorem 2.8 and Theorem 4.4. For that the barycentric angles for $\phi(\Delta_h^3)$ is needed to be compute. Note that there are two distinct barycentric angles: one associated to the polyhedral cone at the vertex e and other one associated to the polyhedral cone at the each of other vertexes.

Let n_i be the normal to the face with vertexes $\{e, e_1, e_2, e_3\} \setminus \{e_i\}$ for $i \in \{1, 2, 3\}$, and n be the normal to the base of the pyramid with vertexes $\{e_1, e_2, e_3\}$. Then, the normals are

$$n_i = \frac{1}{\sqrt{6h^2+1}}(-2he_i + h\sum_{j\neq i} e_j + e_4) \quad \text{and} \quad n = -e_4.$$

Define w_1 and w_2 to be respectively the orthogonal vector to the hyperplane $\prod_1$ determined by vectors n_i's and the orthogonal vector to the hyperplane $\prod_2$ determined by the vectors n and n_i's with $i \neq 1$.

Note that the hyperplanes are considered in the orthogonal complement subspace of the vector $(1,1,1,0)$. It can be seen that

$$w_1 = e_4 \quad \text{and} \quad w_2 = (-2,1,1,\frac{3h}{\sqrt{6h^2+1}+1}).$$

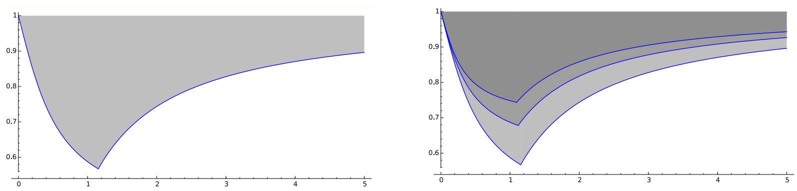
The barycentric angles (see Definition 4.1) are the arcsin of the distance of the hyperplanes $\prod_1$ and $\prod_2$ from the origin, i.e.,

$$\phi_1 = \arcsin(|\langle w_1, n_2\rangle|/\|w_1\|) = 1/\sqrt{6h^2+1},$$

$$\phi_2 = \arcsin(|\langle w_2, n_2\rangle|/\|w_2\|) = 3h/\sqrt{12\sqrt{6h^2+1}+45h^2+12}.$$

Now, by Theorem 4.4, if $2\phi_i > \pi/2 - f(\pi/2) > \pi/2 - \lambda\pi/2$ for $i = 1,2$ then the polyhedral cones have bounded escaping time. This is the case when $\lambda > 1 - 4\min\{\phi_1,\phi_2\}/\pi$. Thus, by Theorem 2.8, it is enough to take $\lambda_0 = 1 - 4\min\{\phi_1,\phi_2\}/\pi$. □

Remark 7.2. Figure 3(a) shows the region where the parameter λ can be chosen. We can do same calculation for polytopal billiard with dimension grater than three. As dimension increases the region for appropriate λ gets smaller and smaller, see Figure 3(b).



(a) (b)

FIGURE 3

Theorem 7.3. *For any $0 < h < \frac{\sqrt{3}}{3}$ there exists a compact set $\Lambda \subset M$ and $\lambda_0 = \lambda_0(h) > 0$ such that for every contracting reflection law f satisfying $\lambda(f) < \lambda_0(h)$,*

1. *$D \subset \Phi_{f,\Delta_h^3}(\Lambda \cap M') \subset \Lambda$,*
2. *The orbit of every point in M^+ eventually enters Λ.*
3. *the billiard map Φ_{f,Δ_h^3} is uniformly hyperbolic.*

Proof. For simplicity, we omit the subindex Δ_h^3 in Φ_{f,Δ_h^3}. First, let assume that $\lambda(f) = 0$, that is the bounce on each face will be along the normal direction. Since after the first iterate of the billiard map the angle is zero, we can take the billiard map as a map from Δ_h^3 to itself and simply denote it by Φ. We show that there is an invariant set on the base of the pyramid under two iterates of the billiard map, i.e., Φ^2. The base of the pyramid is the triangle $A_1A_2A_3$. Let C_0

denote the center of $A_1A_2A_3$, i.e., the point mapped by Φ to the top vertex of the pyramid. Then, the base triangle is partitioned in to three triangle with vertexes $A_i, A_{i+1}, C_0(\mathrm{mod}4)$. Since $\Phi(C_0)$ is at the meet of three faces, it has three distinct images by Φ. By simple calculation, it can be seen that when $h < \frac{\sqrt{3}}{3}$, then these images belong to the base of the pyramid and we denote them by C_1, C_2 and C_3. The image of the triangle $A_iA_{i+1}C_0$ under Φ^2 is the triangle $A_iA_{i+1}C_{i+2}(\mathrm{mod}4)$. Therefore Φ^2 maps the triangle $A_1A_2A_3$ to itself.

We claim that the polygon $\mathcal{H} = M_1M_2M_3M_4M_5M_6$ is a trapping region for Φ^2, see left triangle in Figure 4. This hexagon is constructed as follows. The point M_1 is the intersection of A_1C_2 with the perpendicular to A_1C_0 through C_1. Likewise, M_2 is the intersection of A_2C_1 with the perpendicular to A_2C_0 through C_2. The other M_j's are similarly defined. The hexagon $\mathcal{H}$ is the union of three pentagons, each is mapped by Φ to a different face of the pyramid, see the left triangle in Figure 4. Each of these pentagons is mapped by Φ^2 into the hexagon $\mathcal{H}$. On the right triangle of Figure 4 we can see the image $\Phi^2(\mathcal{P}) = C_0'C_1'M_1'M_2'C_2'$ of the pentagon $\mathcal{P} = C_0C_1M_1M_2C_2$.

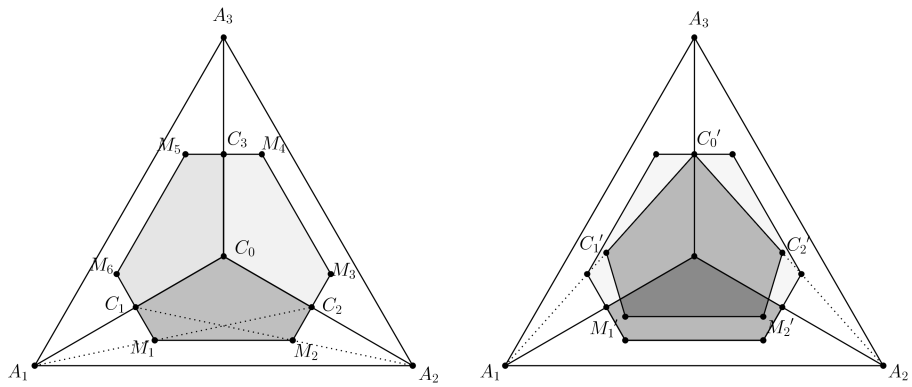


FIGURE 4

Moreover, the intersection of the image $\Phi^2(\mathcal{P})$ with the boundary of $\mathcal{H}$ is just the point $C_0' = C_3$. Hence, for some small enough neighborhood if $\mathcal{V}$ of $\mathcal{H}$ in the triangle $A_1A_2A_3$ we have $\Phi^2(\overline{\mathcal{V}}) \subset \mathcal{V}$. Therefore, if $\lambda(f) \le \lambda_0$ with $\lambda_0 = \lambda_0(h)$ small enough (λ_0 tends to zero as h gets close to $\sqrt{3}/3$), we also have $\Phi_f^2\left(\overline{\mathcal{V} \times \mathbb{S}^+_{\lambda_0}}\right) \subset \mathcal{V} \times \mathbb{S}^+_{\lambda_0}$, where $\mathbb{S}^+_{\lambda_0} = \{v \in \mathbb{S} : \langle v, e_4\rangle > \cos(\lambda_0\pi/2) \quad \text{and} \quad v \perp (1,1,1,0)\}$. Let

$$\Lambda = \overline{\mathcal{V} \times \mathbb{S}^+_{\lambda_0}} \cup \Phi_f\left(\overline{\mathcal{V} \times \mathbb{S}^+_{\lambda_0}}\right).$$

Obviously, Λ satisfies the statement (1). Remember that we always exclude the point that is not forward iterate defined. Moreover, Λ is a basin of attraction: every point in M^+ eventually enters Λ. This holds since $\overline{\mathcal{V}}$ is a basin of attraction for Φ^2. To see this fact, note that every point in triangle $\mathcal{T} = A_1A_2C_0$ eventually enters the pentagon $\mathcal{P}$ or when for the first time it is out of $\mathcal{T}$, it is either in $\mathcal{H}$ or in one of the triangles $A_1C_1M_6$ or $A_2C_2M_3$. In latter case, the point with a zigzag motion in the triangle $A_1M_1M_6$ or $A_2M_2M_3$ eventually enters $\mathcal{H}$. At the end, the escaping time $T(x)$ for $x \in \Lambda$ is uniformly bounded. The uniform boundedness of the escaping time is a result of this observation that for some n the set $\mathcal{P}_n = \Phi^2(\mathcal{P}_{n-1} \cap \mathcal{T})$ is empty, where $\mathcal{P}_0 = \mathcal{P}$. Then, the orbit segment $x, \Phi(x), \Phi^2(x), \ldots, \Phi^{2n}(x), \Phi^{2n+1}(x)$ contains in three different faces of Δ_h^3. Therefore, the statement (3) is obtained by Theorem 2.8.

□

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