Quasi-Periodic Schrödinger Cocycles with Positive Lyapunov Exponent are not Open in the Smooth Topology

Yiqian Wang Jiangong You

Abstract

One knows that the set of quasi-periodic Schrödinger cocycles with positive Lyapunov exponent is open and dense in the analytic topology. In this paper, we construct cocycles with positive Lyapunov exponent which can be approximated by ones with zero Lyapunov exponent in the space of $\mathcal{C}^l$ $(1 \leq l \leq \infty)$ smooth quasi-periodic cocycles. As a consequence, the set of quasi-periodic Schrödinger cocycles with positive Lyapunov exponent is not $\mathcal{C}^l$ open.

Keywords. Lyapunov exponent; Smooth quasi-periodic cocycles; Schrödinger operators.

1 Introduction and Results

Let X be a $\mathcal{C}^r$ compact manifold, $T : X \to X$ be ergodic with a normalized invariant measure μ and $A(x)$ be a $SL(2, \mathbb{R})$-valued function on X. The dynamical system: $(x, w) \to (T(x), A(x)w)$ in $X \times \mathbb{R}^2$ is called a $SL(2, \mathbb{R})$ cocycle (or cocycle for simplicity) over the base dynamics (X, T). We will simply denoted it as (T, A). If the base system is a rotation on torus, i.e., $X = \mathbb{T}^m = \mathbb{R}^m \backslash \mathbb{Z}^m$, $T = T_\omega : x \to x + \omega$ with rational independent ω, we call (T_ω, A) a quasi-periodic cocycle, which is simply denoted by (ω, A). If furthermore $A(x) = S_v(x)$ is of the form $S_v(x) = \begin{pmatrix} v(x) & -1 \\ 1 & 0 \end{pmatrix}$ with $v(x+1) = v(x)$, we call $(\omega, S_v(x))$ a quasi-periodic Schrödinger cocycle.

Y. Wang: Department of Mathematics, Nanjing University, Nanjing 210093, China; e-mail: yiqianw@nju.edu.cn

J. You: Department of Mathematics, Nanjing University, Nanjing 210093, China; e-mail: jyou@nju.edu.cn

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For any $n \in \mathbb{N}$ and $x \in X$, we denote

$$A^n(x) = A(T^{n-1}x) \cdots A(Tx)A(x)$$

and

$$A^{-n}(x) = A^{-1}(T^{-n}x) \cdots A^{-1}(T^{-1}x) = (A^n(T^{-n}x))^{-1}.$$

If the base dynamics (X, T, μ) is fixed, the (maximum) Lyapunov exponent of (T, A) is defined as

$$L(A) = \lim_{n\to\infty} \frac{1}{n} \int \log \|A^n(x)\| d\mu := \lim_{n\to\infty} \int L_n(A(x)) d\mu \in [0, \infty).$$

$L(A)$ measures the average growth rate of $\|A^n(x)\|$.

The regularity and positivity of the Lyapunov exponent (LE) are the central subjects in dynamical systems. One is also interested in the problem whether or not cocycles with positive LE are open and dense. The problems turn out to be very subtle, which depend on the base dynamics (X, T) and the smoothness of the matrix A.

Firstly, classical Furstenberg theory [26] showed that for certain special linear cocycles over Bernoulli shifts, the largest LE is positive under very general conditions. Furstenberg and Kifer [27] and Hennion [29] proved the continuity of the largest LE of i.i.d random matrices under a condition of almost irreducibility. Kotani [41] showed that the LE of Schrödinger cocycles S_{E-v} is positive for almost every energy E if the potential v is non-deterministic. Viana[49] proved that for any $s > 0$, the set of C^s linear cocycles over any hyperbolic ergodic transformation contains an open and dense subset of cocycles with nonzero LE; and the LE is continuous for $SL(2, \mathbb{R})$-cocycles over Markov shifts [44]. For other related results, see [7], [11] and [50].

When the base dynamics is uniquely ergodic (e.g., irrational rotation or skew shift on the torus), the positivity and continuity of the LE seem to be more sensitive to the smoothness of the matrix-valued function $A(x)$. The LE was proved to be discontinuous at any non-uniform cocycles in the C^0 topology by Furman [25] (Continuity at uniform hyperbolic cocyces and cocycles with zero LE is trivial). Motivated by Ma$\tilde{n}\acute{e}$ [42, 43], Bochi [12] further proved a stronger result that any non-uniformly hyperbolic $SL(2, \mathbb{R})$-cocycle over a fixed ergodic system on a compact space can be approximated by cocycles with zero LE in the C^0 topology, which shows that any non-uniform cocycle can not be an inner point of cocycles with positive LE in the C^0 topology. For further related results, we refer to [9], [13], [14], [29], [36], [37], [40], [48].

On the other hand, there are tremendously many positive results in the analytic topology. Herman [30] introduced the subharmonicity method and showed that the LE of $S_{E-2\lambda\cos x}$ is positive for $|\lambda| > 1$ and all E. Herman also proved the positivity of the LE for trigonometric polynomials if the coupling is large enough. The generalization to arbitrary one-frequency non-constant real analytic potentials was obtained by Sorets and Spencer [47]. Same results for *Diophantine* multi-frequency were established by Bourgain and Goldstein [18] and Goldstein and Schlag [28]. Zhang [55] gave a different proof of the results in [47] and applied it to a certain class of analytic Szegő cocycle. For more references, one can see [17], [23], [39].

For the continuity of the LE, Large Deviation Theorems (LDT) is an important tool, which was first established by Bourgain and Goldstein in [18] for real analytic potentials with *Diophantine* frequencies. In [28], Goldstein and Schlag proved that, by some sharp version of LDT and generalized Avalanche Principle(AP), $L(S_{E-v})$ is Hölder continuous in E if ω is a Diophantine, $v(x)$ is analytic and $L(S_{E-v}) > 0$. Jitomirskaya, Koslover and Schulteis [32] proved the continuity of the LE for a class of analytic one-frequency quasi-periodic $M(2,\mathbb{C})$-cocycles with singularities. We will briefly mention more results along this line at the end of this section. The continuity of the LE implies that the cocycles with positive LE are open in analytic topology. Together with the denseness result by Avila [1], one knows that the set of quasi-periodic cocycles with positive LE is open and dense in the analytic topology.

We have seen that the behavior of the LE in the C^0 topology is totally different from its behavior in the analytic topology. The smooth case is more subtle. Avila[1] proved, among other results, that the LE is positive for a dense subset of smooth quasi-periodic cocycles. Recently, with Benedicks-Carleson-Young's method[8, 54], the authors [51] constructed quasi-periodic cocycles (T_ω, A) where T_ω is an irrational rotation $x \to x+\omega$ on $\mathbb{S}^1$ with ω of bounded type and $A \in \mathcal{C}^l(\mathbb{S}^1, SL(2,\mathbb{R}))$, $0 \le l \le \infty$, such that the LE is not continuous at A in the $\mathcal{C}^l$ topology. Such an example in the Schrödinger class is also constructed in [51]. For $\mathcal{C}^2$ cosine-like potentials, Anderson Localization and the positivity of LE has been established by Sinai [46] and Fröhlich-Spencer-Wittwer [24], also see Bjerklöv [10]. For the model in [46], Wang and Zhang [52] showed the continuity of the LE, which implies that non-uniform quasi-periodic cocycles can be inner points of smooth quasi-periodic cocycles with positive exponents. An interesting problem is whether or not quasi-periodic cocycles with positive exponent are open and dense in the smooth topology as in the analytic topology. As we mentioned before, the denseness follows from the result of Avila [1]. In this paper, we will prove that, different from the analytic case, the set of smooth quasi-periodic cocycles with positive exponent are not open in smooth topology.

The LE of quasi-periodic Schrödinger cocycles have attracted so much attention not only because of its importance in dynamical systems, but also due to its close relation with quasi-periodic Schrödinger operators. The latter has strong background in physics. The LE of Schrödingier cocycles coming from the eigenvalue equations of quasi-periodic Schödinger operators encodes enormous information on the spectrum. It is known from Kotani theory that positive LE implies singular spectrum, and typically Anderson localization, see [31, 38, 45]; while zero Lyapunov spectrum usually implies continuous, typically absolutely continuous spectrum. The positivity of the LE is also a starting point for many other problems in dynamical systems and spectral theory, such as hölder continuity of LE, continuity and topological structure of spectrum set. The recent developed methods, such as Green's function estimates and Avalanche Principle, etc.(see [16]), depend crucially on the positivity of the LE.

Another related interesting question is the robustness of Anderson localization. i.e., wether or not the perturbations of a Schrödinger operator exhibiting Anderson localization still have Anderson localization? The answer is affirmative in the analytic category since the LE is continuous and thus the positivity of the LE is kept under perturbations.[3] We are interested in the question in smooth case, which is closely related to the problem whether or not the positivity of the LE is kept under perturbations in the smooth category, equivalently whether or not there exist smooth non-uniformly hyperbolic Schrödinger cocycles which can be accumulated by ones with zero LE in $\mathcal{C}^l$ topology ($l = 1, 2, \cdots, \infty$). If it is the case, the nature of the spectrum of Schrödinger operators might exhibit dramatically changes under small perturbations of the potential in smooth topology.

The following is the main result of this paper.

Theorem 1. *Consider quasi-periodic Schrödinger cocycles over $\mathbb{S}^1$ with ω being a fixed irrational number of bounded-type.*[4] *For any $0 \leq l \leq \infty$, there exists a Schrödinger cocycle S_v with arbitrarily large Lyapunov exponent and a sequence of Schrödinger cocycles S_{v_n} with zero Lyapunov exponent such that $v_n(x) \to v(x)$ in the C^l topology. As a consequence, the set of quasi-periodic Schrödinger cocycles with positive Lyapunov exponent is not C^l open.*

Theorem 1 can be obtained from Theorem 2 in the same way as in [51] to derive examples in Schrödinger cocycles from examples in $SL(2,\mathbb{R})$ cocycles. Thus we only need to prove Theorem 2.

Theorem 2. *Consider quasi-periodic $SL(2,\mathbb{R})$ cocycles over $\mathbb{S}^1$ with ω being a fixed irrational number of bounded-type. For any $0 \leq l \leq \infty$, there exists a cocycle $D_l \in \mathcal{C}^l(\mathbb{S}^1, SL(2,\mathbb{R}))$ with arbitrarily large Lyapunov exponent and a sequence of cocycles $C_k \in \mathcal{C}^l(\mathbb{S}^1, SL(2,\mathbb{R}))$ with zero Lyapunov exponent such that $C_k \to D_l$ in the C^l topology. As a consequence, the set of $SL(2,\mathbb{R})$-cocycles with positive Lyapunov exponent is not C^l open.*

Remark 1.1. *Completely different from the result in Theorem 1, Bonatti, Gómez-Mont and Viana [15] proved that there exist Hölder continuous cocycles over Bernoulli shift*

[3] More precisely, it is true for all almost all frequencies.

[4] Bounded type means $\frac{p_k}{q_k}$, the best approximation of ω, satisfies $q_{k+1} \leq M q_k$ for some $M > 0$.

with positive LE which can be approximated by continuous cocycles with zero LE, but not by Hölder ones, which shows that the base dynamics plays an important role in the regularity problem of the LE.

Remark 1.2. *Avila and Krikorian [6] showed that the LE is smooth in the space of smooth monotonic quasi-periodic cocycles. Our result shows that the monotonicity assumption in [6] is necessary, and behavior of the LE in smooth quasi-periodic Schrödinger cocycles homotopic to the identity are completely different from its behavior in the class of monotone cocycles.*

The proof of Theorem 2 is constructive. Recall in [51], we have constructed a smooth cocycles D_l with positive LE and a smooth cocycle A_1 in $\frac{1}{2k}$-neighborhood of D_l in the C^l topology for any given $k > 0$ such that the finite LE of A_1, defined by $L_{n_1}(A_1) = \frac{1}{n_1}\int_{\mathbb{S}^1}\log\|A_1^{n_1}(x)\|dx$, is smaller that $(1-\delta_2)L(D_l)$ for a fixed number $\delta_2 > 0$. As a consequence of subadditivity of finite LE, $L(A_1) < (1-\delta_2)L(A)$. It follows that the LE is discontinuous at D_l. However, the construction in [51] did not tell us how small $L(A_1)$ can be. In this paper we will define a new A_1 somehow different from the one in [51] but satisfies the same property stated as above. Then we further locally modify A_1 such that the modified cocycle, say A_2, satisfies $\|A_2-A_1\|_{C^l} < \frac{1}{4k}$ and $L_{n_2}(A_2) < (1-\delta_2)L_{n_1}(A_1)$. It follows that A_2 is in the δ-neighborhood of A and $L(A_2) < (1-\delta_2)^2L(A)$. Inductively, we locally modify A_i such that the modified cocycle, say A_{i+1}, satisfies $\|A_{i+1}-A_i\|_{C^l} < \frac{1}{2^ik}$ and $L_{n_{i+1}}(A_{i+1}) < (1-\delta_2)L_{n_i}(A_i)$, where $n_i \to \infty$ will be specified later. It follows that all A_i are in the $\frac{1}{k}$-neighborhood of D_l and $L(A_{i+1}) < (1-\delta_2)^iL(D_l)$. It is easy to see that A_i has a limit, say C_k, with $L(C_k) = 0$. Moreover, $\|C_k - D_l\|_{C^l} < \frac{1}{k}$. Theorem 2 is thus proved since k is arbitrary.

We remark that D_l and C_k we constructed are of the forms $\Lambda R_{\phi(x)}$ and $\Lambda R_{\phi_k(x)}$ where $\Lambda = diag\{\lambda, -\lambda\}, \lambda \gg 1$ with $L(D_l) \sim \ln\lambda$ and $L(C_k) = 0$. Moreover, $\phi_k(x)$ is an arbitrarily small modification of $\phi(x)$ in an arbitrarily small neighborhood of two special points (called critical points). So a small change makes a big difference! For Schrödinger cocycles, we actually construct, for arbitrarily large but fixed λ, smooth $v(x)$ and $\bar{v}(x)$ which are arbitrarily close to each other and slightly different only at the neighborhood of two critical points such that $L(S_{\lambda v(x)})$ is very big while $L(S_{\lambda\bar{v}(x)}) = 0$. The result is surprising as we have even not seen any example of smooth Schrödinger cocycles of the form $S_{\lambda\bar{v}(x)}$ with $\lambda \gg 1$ such that $L(S_{\lambda\bar{v}(x)}) = 0$.

From our construction, one can see how and where to modify a cocycle so as to control the LE. This might be useful for other problems.

More results on the continuity of the LE in the analytic topology. When the base dynamics is a shift or skew-shift of a higher dimensional torus, the log-continuity of the LE was proved in [19] by Bourgain, Goldstein and Schlag. Recently, the result of [32] was generalized by Jitomirskaya and Marx [33] for all non-trivial singular analytic quasiperiodic cocycles with one-frequency with application to the extended Harper's model [34].

An arithmetic version of large deviations and inductive scheme were developed by Bourgain and Jitomirskaya in [20] allowing to obtain joint continuity of the LE for $\mathrm{SL}(2,\mathbb{C})$ cocycles, in frequency and cocycle map, at any irrational frequencies. This result has been crucial in many further important achievements, such as the proof of the Ten Martini problem [4], Avila's global theory of one-frequency cocycles [2, 3]. It was extended to multi-frequency case by Bourgain [17] and to general $\mathrm{M}(2,\mathbb{C})$ case by Jitomirskaya and Marx [34]. More recently, a completely different method without LDT or AP was developed by Avila, Jitomirskaya and Sadel [5] and was applied to prove the continuity of the LE in $\mathrm{M}(d,\mathbb{C})$, $d\geq 2$. For further works, see [21], [22], [35], [39], [53].

2 The construction of D_l

We consider the case $m=1$. We say a $SL(2,\mathbb{R})$-matrix A is hyperbolic if $\|A\|>1$. A quasi-periodic cocycle $(\omega, A(x))$ of degree d is defined by a matrix function $A(x)=R_{\psi(x)}\cdot\Lambda(x)\cdot R_{\phi(x)}$ on $\mathbb{R}$, with $\Lambda(x+1)=\Lambda(x)=diag\{\|A\|,\frac{1}{\|A\|}\}$, $\psi(x+1)=2\pi d+\psi(x), \phi(x+1)=2\pi d+\phi(x)$ where $R_\theta=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$. It is easy to see that $(\phi(x)+\psi(x-\omega))$ is uniquely determined by $A(x)$ up to $2\pi\mathbb{Z}$ and $L(A)=L(\Lambda(x)\cdot R_{\phi(x)+\psi(x-\omega)})$ as A is conjugated to $\Lambda(x)\cdot R_{\phi(x)+\psi(x-\omega)}$.

Let $\Lambda=diag\{\lambda,\frac{1}{\lambda}\}$ with $\lambda\gg 1$. In this section, we will construct a sequence of smooth cocycles B_k of the form $\Lambda\cdot R_{\xi_k(x)}$, converging in C^l such that $L(\lim B_k)>0$. Moreover $\xi_k(x)$ will be specially designed so that, in the next section, we can further constructed cocycles C_k with zero Lyapunov exponent in any small neighborhood of B_k. When λ is big, we will see that the Lyapunov exponent of B_k crucially depends on the local behavior, more precisely the degeneracy, of $\xi_k(x)$ at the critical points $\{c:\ \xi_k(c)=\frac{\pi}{2}\pmod{\pi}\}$ due to the cancelation. The construction in this section is in principle along the line of the construction in [51], the difference is in this paper, we use the decomposition of a matrix instead of the most expended and contracted direction of a matrix which makes the proof more transparent.

Let ω be a fixed irrational number and $\frac{p_k}{q_k}$ be its best approximation. Throughout the paper, we assume that ω is of the bounded type, i.e., $q_{k+1}\leq Mq_k$; $\epsilon>0$ is small. l is a fixed positive integer reflecting the smoothness of cocycles. Let λ and N are large enough so that

$$10l\sum_{k=N}^{\infty}\frac{\log q_{k+1}}{q_k}\leq\epsilon,\quad \lambda^{-1}\ll q_N^{-2}. \tag{2.1}$$

We define the decaying sequence $\{\lambda_k\}$ inductively by $\log\lambda_k = \log\lambda_{k-1} - \frac{10l\log q_k}{q_{k-1}}$ where $\lambda_N = \lambda \gg 1$. It is easy to see that λ_k converges to λ_∞ with $\lambda_\infty > \lambda^{1-\epsilon}$.

For $k \geq N$, let $\mathcal{C}_0 = \{0, \frac{1}{2}\}$, $I_{k,1} = [-\frac{1}{q_k^2}, \frac{1}{q_k^2}]$, $I_{k,2} = [\frac{1}{2} - \frac{1}{q_k^2}, \frac{1}{2} + \frac{1}{q_k^2}]$ and $I_k = I_{k,1}\bigcup I_{k,2}$. For $C \geq 1$, we denote by $\frac{I_{k,1}}{C} = [-\frac{1}{Cq_k^2}, \frac{1}{Cq_k^2}]$, $\frac{I_{k,2}}{C} = [\frac{1}{2} - \frac{1}{Cq_k^2}, \frac{1}{2} + \frac{1}{Cq_k^2}]$, and by $\frac{I_k}{C}$ the set $\frac{I_{k,1}}{C} \cup \frac{I_{k,2}}{C}$. Denote Lebesgue measure of I_k by $|I_k|$. For each $x \in I_k$, let $r_k^+(x)$ (respectively $r_k^-(x)$) be the smallest positive integer such that $T^{r_k^+(x)}(x) \in I_k$ (respectively $T^{-r_k^-(x)}(x) \in I_k$). Let $r_k^\pm = \min_{x\in I_k} r_k^\pm(x)$ and $r_k = \min\{r_k^+, r_k^-\}$. Obviously, $r_k \geq q_k$. Moreover, from the symmetry between $I_{k,1}$ and $I_{k,2}$, we have $r_k = r_k^+ = r_k^-$.

We define ξ_0 on $I = I_1 \bigcup I_2 = \{x : |x| \leq \frac{1}{2q_N^2}\} \bigcup \{x : |x - \frac{1}{2}| \leq \frac{1}{2q_N^2}\}$ by

$$\xi_0(x) = \begin{cases} \xi_{01}(x), & |x| \leq \frac{1}{2q_N^2}; \\ -\xi_{02}(x) \text{ (or } \xi_{02}(x)), & |x - \frac{1}{2}| \leq \frac{1}{2q_N^2} \end{cases} \tag{2.2}$$

where

$$\xi_{01}(x) = \mathrm{sgn}(x)|x|^{l+1}, \quad \xi_{02}(x) = \mathrm{sgn}(x - \frac{1}{2})|x - \frac{1}{2}|^{l+1}. \tag{2.3}$$

$\xi(x)$ is a lift of a 1-periodic C^l function satisfying

$$\xi(x) = \begin{cases} \xi_{01}(x), & |x| \leq \frac{1}{2q_N^2}; \\ -\xi_{02}(x) \text{ (or } \pi + \xi_{02}(x)), & |x - \frac{1}{2}| \leq \frac{1}{2q_N^2}, \end{cases} \tag{2.4}$$

and $|\xi(x)(\mathrm{mod}\ \pi)| > \frac{1}{2q_N^2}$ for any $x(\mathrm{mod}\ 1) \notin I$. See Figures 1 and 2 for the picture of $\xi(x)$.

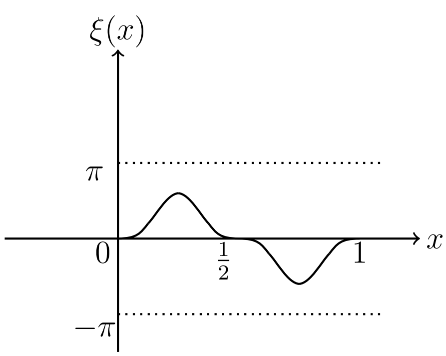

Figure 1: homotopic to identity

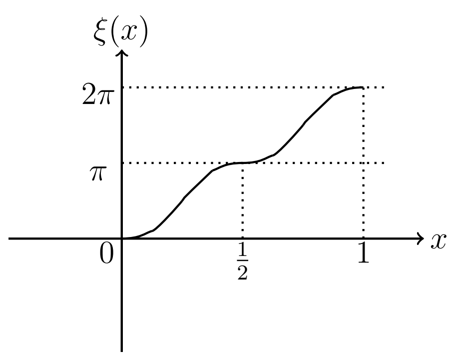

Figure 2: nonhomotopic to identity

In the following, we will use $c,\ C,\ C(l)$, etc, to denote universal positive constants independent of iterative steps. For any cocycle $A(x)$, $n \in Z^+$ and $x \in I$, we decompose $A^n(x)$ as $R_{\psi_{A,n}(x)} \cdot \Lambda_{A,n}(x) \cdot R_{\phi_{A,n}(x)}$ when $A^n(x)$ is hyperbolic in I and decompose $A^n(T^{-n}x)$ as $R_{\psi_{A,-n}(x)} \cdot \Lambda_{A,-n}(x) \cdot R_{\phi_{A,-n}(x)}$ when $A^n(T^{-n}x)$ is hyperbolic in I.

Let $\xi_N(x) = \xi(x)$ defined above. Define $B_N(x) = \Lambda R_{\frac{\pi}{2} - \xi_N(x)}$.

Proposition 2.1. *There are $\mathcal{C}^l$ functions $\xi_k(x)$ ($k = N+1, N+2, \cdots$) constructed inductively such that*

1. $|\xi_k(x) - \xi_{k-1}(x)|_{C^l} \leq C(l) \cdot \lambda_k^{-2r_k} \cdot |I_k|^{-l^2}$. (2.5)

2. Let $B_k(x) = \Lambda R_{\frac{\pi}{2} - \xi_k(x)}$. *For each* $x \in I_k$, *we have*

$$\|B_k^{r_k^\pm(x)}(x)\| \geq \lambda_k^{r_k^\pm(x)}. \tag{2.6}$$

3. For $x \in I_k$, *we have*

$$(1)_{\mathrm{k}} \quad \psi_{B_k, -r_k^-}(x) + \phi_{B_k, r_k^+}(x) - \tfrac{\pi}{2} = \xi_0(x) \quad \text{on } \tfrac{I_k}{10};$$

$$(2)_{\mathrm{k}} \quad |\psi_{B_k, -r_k^-}(x) + \phi_{B_k, r_k^+}(x) - \tfrac{\pi}{2}| \geq \tfrac{1}{(20q_k^2)^{l+1}}, \quad x \in I_k \backslash \tfrac{I_k}{10},$$

where $\xi_0(x)$ *is defined in (2.2) and (2.3).*

Remark 2.1. *It is easy to see from (2.5) that* B_k *converges to a limit* D_l *in* $\mathcal{C}^l$ *-topology. Moreover, from (2.5) and (2.6) as well as Theorem 3 in [51], we have* $L(D_l) \geq (1-\epsilon)\ln\lambda$.

To prove Proposition 2.1, we first give the following Lemma 2.1.

Lemma 2.1. *For any function* $\sigma(x)$ *defined on* S^1, *let* $d_k(\sigma) = \min_{x \notin I_k}\{|\sigma(x)|\}$. *Assume that for any* $x \in I_k$,

$$\log \|A^{r_k}(x)\| \gg -\log d_{k+1}, \tag{2.7}$$

where $d_{k+1} = d_{k+1}(\phi_{A, r_k^+}(x) + \psi_{A, -r_k^-}(x) - \frac{\pi}{2})$. *Furthermore assume that, for* $i \leq l$ *and* $m^\pm = r_k^\pm(x)$,

$$\begin{cases} \left|\frac{d^i}{dx^i}\phi_{A,m^+}(x)\right|, \quad \left|\frac{d^i}{dx^i}\psi_{A,-m^-}(x)\right| \leq C(i) \cdot d_{k+1}^{-i} & (1)_k \\ \left|\frac{d^i \|A^{\pm m}(x)\|}{dx^i}\right| \cdot \|A^{\pm m}(x)\|^{-1} \leq C(i) \cdot d_{k+1}^{-i}. & (2)_k \end{cases}$$

Then for $i \leq l$, $x \in I_{k+1}$ *and* $\hat{m}^\pm = r_{k+1}^\pm(x)$ *it holds that*

$$\begin{cases} \left|\frac{d^i}{dx^i}\phi_{A,\hat{m}^+}(x)\right|, \left|\frac{d^i}{dx^i}\psi_{A,-\hat{m}^-}(x)\right| \leq C(i) \cdot d_{k+1}^{-i}, & (1)_{k+1} \\ \left|\frac{d^i \|A^{\pm\hat{m}}(x)\|}{dx^i}\right| \cdot \|A^{\pm\hat{m}}(x)\|^{-1} \leq C(i) \cdot d_{k+1}^{-i}. & (2)_{k+1} \end{cases}$$

Moreover, for any $i \geq 0, x \in I_{k+1}$, *it holds that*

$$\begin{aligned} &\left|\frac{d^i}{dx^i}(\phi_{A,r_{k+1}^+}(x) - \phi_{A,r_k^+}(x))\right| \leq C(i) \cdot \|A^{r_k^+}\|^{-2} \cdot d_k^{-i}, \\ &\left|\frac{d^i}{dx^i}(\psi_{A,-r_{k+1}^-}(x) - \psi_{A,-r_k^-}(x))\right| \leq C(i) \cdot \|A^{r_k^-}\|^{-2} \cdot d_k^{-i}. \end{aligned} \tag{2.8}$$

The proof of Lemma 2.1 will be given in the Appendix.

Proof of Proposition 2.1. For each $k \geq N$ and $x \in I_k$, since

$$\hat{f}_k(x) := (\psi_{B_{k-1},-r_k^-}(x) + \phi_{B_{k-1},r_k^+}(x)) - (\psi_{B_{k-1},-r_{k-1}^-}(x) + \phi_{B_{k-1},r_{k-1}^+}(x))$$

usually does not vanish on I_{k-1} and thus $\psi_{B_k,-r_k^-}(x) + \phi_{B_k,r_k^+}(x) - \frac{\pi}{2} \neq \xi_0(x)$ on I_k. To guarantee $(1)_\mathrm{k}$ in Proposition 2.1, we modify $\xi_{k-1}(x)$ on I_k as $\xi_k(x) = \xi_{k-1}(x) + f_k(x)$, where C^l periodic function $f_k(x)$ is defined as follows

$$f_k(x) = \begin{cases} \hat{f}_k(x) & x \in \frac{I_k}{10} \\ h_k^{\pm}(x), & x \in I_k \backslash \frac{I_k}{10} \\ 0, & x \in \mathbb{S}^1 \backslash I_k \end{cases}$$

where $h_k^{\pm}(x)$ is a polynomial of degree $2l+1$ restricted in each interval of $I_k \backslash \frac{I_k}{10}$ satisfying

$$\frac{d^j h_k^{\pm}}{dx^j}(\pm\frac{1}{10q_k^2}) = \frac{d^j \hat{f}_k}{dx^j}(\pm\frac{1}{10q_k^2})$$

$$\frac{d^j h_k^{\pm}}{dx^j}(\pm\frac{1}{q_k^2}) = 0, \quad i = 1, 2, \quad 0 \leq j \leq l.$$

From (2.8) in Lemma 2.1, we have

$$|(\psi_{B_{k-1},-r_k^-}(x)+\phi_{B_{k-1},r_k^+}(x))-(\psi_{B_{k-1},-r_{k-1}^-}(x)+\phi_{B_{k-1},r_{k-1}^+}(x))|_{C^l} \leq C(l)\cdot\lambda_k^{-2r_k}\cdot|I_k|^{-l^2}, \tag{2.9}$$

where (2.7) is fulfilled by conclusion 2 and 3 of the induction assumption for the case $k-1$.

In view of the definition of $f_k(x)$ we obtain

$$|f_k|_{C^l} \leq C(l) \cdot \lambda_k^{-2r_k} \cdot |I_k|^{-l^2}. \tag{2.10}$$

Let $B_k(x) = \Lambda \cdot R_{\frac{\pi}{2}-\xi_k(x)}$, then we have

Lemma 2.2. *For $x \in I_k$, it holds that*

$$B_k^{r_k^+(x)}(x) = B_{k-1}^{r_k^+(x)}(x) \cdot R_{-f_k(x)}$$

and

$$B_k^{r_k^-(x)}(T^{-r_k^-(x)}x) = B_{k-1}^{r_k^-(x)}(T^{-r_k^-(x)}x).$$

Proof. Obviously $T^i x \in \mathbb{S}^1 \backslash I_k$ for $x \in I_k$ and $1 \leq i \leq r_k^+(x)-1$. Since $B_k(x) = B_{k-1}(x)$ for $x \in \mathbb{S}^1 \backslash I_k$, we have that

$$B_k^{r_k^+(x)}(x) = B_{k-1}^{r_k^+(x)}(x) \cdot (B_{k-1}^{-1}(x) B_k(x)), \quad x \in I_k.$$

From the definition, we have $B_k(x) = B_{k-1}(x)\cdot R_{\xi_{k-1}(x)-\xi_k(x)}$, which implies $B_{k-1}^{-1}(x)B_k(x) = R_{\xi_{k-1}(x)-\xi_k(x)}$. Thus we obtain the first equation in Lemma 2.2. Similarly we can prove the second one. □

Lemma 2.3. *It holds that*

$$f_k(x) = (\psi_{B_{k-1},-r_k^-}(x) + \phi_{B_{k-1},r_k^+}(x)) - (\psi_{B_k,-r_k^-}(x) + \phi_{B_k,r_k^+}(x)), \quad x \in I_k.$$

Proof. Since a rotation does not change the norm of a vector, for a hyperbolic matrix A and a rotation matrix R_θ, we have

$$\phi_{A\cdot R_\theta} = \phi_A + \theta. \tag{2.11}$$

From Lemma 2.2, we have

$$\phi_{B_k,r_k^+}(x) = \phi_{B_{k-1},r_k^+}(x) - f_k(x), \quad \psi_{B_k,-r_k^-}(x) = \psi_{B_{k-1},-r_k^-}(x).$$

Thus

$$f_k(x) = (\psi_{B_{k-1},-r_k^-}(x) + \phi_{B_{k-1},r_k^+}(x)) - (\psi_{B_k,-r_k^-}(x) + \phi_{B_k,r_k^+}(x)), \quad x \in I_k,$$

which concludes the proof. □

Proof of $(1)_k$ *and* $(2)_k$ From the definition of $f_k(x)$, we have $f_k(x) = (\psi_{B_{k-1},-r_k^-}(x) + \phi_{B_{k-1},r_k^+}(x)) - (\psi_{B_{k-1},-r_{k-1}^-}(x) + \phi_{B_{k-1},r_{k-1}^+}(x))$ on $\frac{I_k}{10}$, which together with Lemma 2.3 implies that for each $x \in \frac{I_k}{10}$,

$$\psi_{B_k,-r_k^-}(x)+\phi_{B_k,r_k^+}(x) = (\psi_{B_{k-1},-r_k^-}(x)+\phi_{B_{k-1},r_k^+}(x))-f_k(x) = \psi_{B_{k-1},-r_{k-1}^-}(x)+\phi_{B_{k-1},r_{k-1}^+}(x).$$

Since $\psi_{B_{k-1},-r_{k-1}^-}(x) + \phi_{B_{k-1},r_{k-1}^+}(x) = \xi_0(x)$ on $\frac{I_{k-1}}{10}$ by induction assumption $(1)_{k-1}$, we obtain $(1)_k$ in proposition 2.1.

Obviously $\lambda_k^{q_{k-1}} \gg q_k^{2l}$. Hence $(2)_\mathrm{k}$ in Proposition 2.1 can be obtained from the induction assumption $(2)_{k-1}$ and (2.10).

Proof of conclusion 1 of Proposition 2.1. Conclusion 1 can be obtained from (2.9).

Proof of conclusion 2 of Proposition 2.1. For $x \in I_k$, let $i_1(x) < i_2(x) < \cdots < i_{j(x)}(x) \le r_k$ be the returning times of I_{k-1} less than r_k. Since $|I_k| \le \frac{1}{4}|I_{k-1}|$ (we can make a slight modification of the definition of I_k if necessary), from the symmetry between $I_{k,1}$ and $I_{k,2}$, we have that for any $x \in I_k$, we have $T^{r_k}x \in I_{k-1}$. Then we have that $i_{j(x)}(x) = r_k$. Since $T^{i_s(x)}x \notin I_k$ for $s < j(x)$, $|\theta_s - \frac{\pi}{2}| \ge \frac{1}{q_k^{2l}}$, where $\theta_s = \phi_{B_k,i_{s+1}(x)-i_s(x)}(T^{i_s(x)}x) + \psi_{B_k,i_s(x)-i_{s-1}(x)}(T^{i_{s-1}(x)}x)$. Together with the conclusion 3 of the induction assumption for $(k-1)$-th step we have that $|\tilde{\theta}_s - \frac{\pi}{2}| \ge \frac{1}{2q_k^{2l}}$, where $\tilde{\theta}_s = \phi_{B_k,i_{s+1}(x)-i_s(x)}(T^{i_s(x)}x) + \psi_{B_k,i_s(x)}(x)$. Thus from the definition of λ_k, we obtain the conclusion 2 for k-th step by repeated applications of Lemma A.1.

Remark 2.2. *In spirit, the proof of conclusion 2 of Proposition 2.1 coincides with the one of LDT.*

3 The construction of $C_k(x)$

Now we start to construct a C_k in any small $\mathcal{C}^l$-neighborhood of B_k such that $L(C_k) = 0$. It is obvious that $C_k \to D_l$ in $\mathcal{C}^l$ topology. C_k will be constructed as limit of a sequence of converging cocycles, say $A_{k,i}$, in any small neighborhood of B_k such that $L(A_{k,i}) \to 0$ as $i \to \infty$. By the construction, we can show that $L(C_k) = \lim_{i\to\infty} L(A_{k,i}) = 0$, see Corollary 3.1. In the following, we shall simply denote $A_{k,i}$ by A_i.

The following lemma is of key importance for the construction:

Iterative Lemma: *Let $A_0(x) = \Lambda \cdot R_{\frac{\pi}{2}-\theta_0(x)}$ satisfy that $\|A_0^{r_{n_0}(x)}(x)\| \geq \mu^{r_{n_0}(x)}$ with $\lambda \geq \mu \gg 1$, $n_0 \geq N$ and $\psi_{A_0,-r_{n_0}}(x) + \phi_{A_0,r_{n_0}}(x) - \frac{\pi}{2} = \xi_0(x)$, $x \in I_{n_0}$. Then we can find two small positive numbers $\delta_1 > \delta_2$ such that for any $i \geq 0$, there exist $A_i(x) = \Lambda \cdot R_{\frac{\pi}{2}-\theta_i(x)}$ and n_i, such that the following hold*

$$
(\mathrm{P_i}) : \begin{cases} (1). \quad \|A_i^{r_{n_i}(x)}(x)\| \geq \mu^{(1-\delta_1)^i \cdot r_{n_i}(x)} \text{ on } I_{n_i} \text{ and } \mu^{(1-\delta_1)^i \cdot q_{n_i}} \gg \frac{1}{|I_{n_i}|}; \\ (2). \quad \|A_i^{r_{n_j}(x)}(x)\| \leq \lambda^{(1-\delta_2)^j \cdot r_{n_j}(x)} \text{ for } x \in I_{n_j} \text{ and } j \leq i; \\ (3). \quad \bar{\mu}_{n_i} \leq \underline{\mu}_{n_i}^2; \\ (4). \quad \psi_{A_i,-r_{n_i}}(x) + \phi_{A_i,r_{n_i}}(x) - \frac{\pi}{2} = \xi_0(x) \text{ on } I_{n_i}; \\ (5). \quad |\theta_{i+1} - \theta_i|_{\mathcal{C}^l} \leq q_{n_i}^{4Ml^2} \cdot \mu^{-\frac{1}{2}(1-\delta_1)^i \cdot q_{n_i}} + q_{n_i}^{-2}. \end{cases}
$$

In the above, $\bar{\mu}_{n_i} = \max_{x\in I_{n_i}} \|A_i^{r_{n_i}(x)}(x)\|^{\frac{1}{r_{n_i}(x)}}$ and $\underline{\mu}_{n_i} = \min_{x\in I_{n_i}} \|A_i^{r_{n_i}(x)}(x)\|^{\frac{1}{r_{n_i}(x)}}$. Therefore, $\underline{\mu}_{n_i} \geq \mu^{(1-\delta_1)^i}$ and $\bar{\mu}_{n_i} \leq \lambda^{(1-\delta_2)^i}$.

The main result Theorem 2 is an easy consequence of the following corollary.

Corollary 3.1. *There exists a $SL(2,\mathbb{R})$-sequence $\{C_k\}_{k=N}^{\infty}$ with $L(C_k) = 0$ such that C_k tend to D_l in the $\mathcal{C}^l$ topology.*

Proof. For any $k \in \mathbb{N}$, we apply Iterative Lemma by setting $A_0 = B_k$, $n_0 = q_k$ and $\mu = \lambda^{1-\epsilon}$ where B_k is defined in Proposition 2.1. Hence for each i we obtain A_i such that $(\mathrm{P_i})$ holds true. By (5) and the inequality $\mu^{(1-\delta_1)^i \cdot q_{n_i}} \gg \frac{1}{|I_{n_i}|}$ in (1) of $(\mathrm{P_i})$, A_i has a limit, say C_k, in the $\mathcal{C}^l$ topology. From (2) of $(\mathrm{P_i})$, we obtain $\|C_k^{r_{n_j}(x)}(x)\|^{\frac{1}{r_{n_j}(x)}} \leq \lambda^{(1-\delta_2)^j}$ for any $j \leq i$ and $x \in I_{n_j}$, which by the subadditivity of Lyapunov exponent, implies $L(C_k) \leq (1-\delta_2)^j \log\lambda$ for any j. Let $j \to \infty$, we obtain $L(C_k) = 0$. Moreover, from (5) of $(\mathrm{P_i})$ it holds that

$$\begin{aligned}\|C_k - D_l\|_{\mathcal{C}^l} &\leq \|C_k - B_k\|_{\mathcal{C}^l} + \|B_k - D_l\|_{\mathcal{C}^l} \\ &= \|C_k - A_0\|_{\mathcal{C}^l} + \|B_k - D_l\|_{\mathcal{C}^l} \\ &\leq 2(q_k^{4M^2 l} \cdot \lambda^{-(1-\epsilon)(1-\delta_1)\cdot q_k} + q_k^{-2}) + \|B_k - D_l\|_{\mathcal{C}^l}.\end{aligned}$$

In the last inequality, we use (5) and the inequality $\mu^{(1-\delta_1)^i \cdot q_{n_i}} \gg \frac{1}{|I_{n_i}|}$ in (1) of $(\mathrm{P_i})$. It implies $C_k \to D_l$ in the $\mathcal{C}^l$ topology as $k \to \infty$. Moreover, $L(D_l) \geq (1-\epsilon)\ln\lambda$ by Remark 2.1. □

Proof of Iterative Lemma. Let $0 < \delta_0 \ll \max\{\frac{1}{l^2}, M^{-k_1}\}$ be a fixed number and $\delta_1 = 8\delta_0 l$, $\delta_2 = M^{-k_1}\cdot\delta_0 l$, where k_1 is defined in Proposition 3.1.

When $i = 1$, $(\mathrm{P_i})$ obviously holds true for A_1 with $\lambda \gg 1$. Assuming that $A_1, \cdots, A_{i-1}$ have been constructed with $(\mathrm{P_1}), \cdots, (\mathrm{P_{i-1}})$, we will construct A_i such that $(\mathrm{P_i})$ holds. From (3) of (P_{i-1}), we have $\|A_{i-1}^{r_{n_{i-1}}}(x)\| \leq \|A_{i-1}^{r_{n_{i-1}}}(y)\|^2$ for $x,\ y \in I_{n_{i-1}}$.

Step 1. Definition of n_i *and* I_{n_i}. Choose $n_i \gg n_{i-1}$ such that $\mu^{(1-\delta_1)^i\cdot q_{n_i}} \gg q_{n_i}^2 \gg \lambda^{2\delta_0 \cdot r_{n_{i-1}}}$ and I_{n_i} is defined as before. The Diophantine condition implies that $r_{n_i} \geq q_{n_i}$.

Step 2. Modification of A_{i-1}. For our purpose, we first make a local modification for A_{i-1} on $I_{n_{i-1}}$ such that there is a low platform in the image of $\phi_{\tilde{A}_{i-1}, r_{n_{i-1}}}(x) + \psi_{\tilde{A}_{i-1}, -r_{n_{i-1}}}(x) - \frac{\pi}{2}$ for the new cocycle $\tilde{A}_{i-1}$, see Figure 3.

Consider the sub-interval $[0, \frac{1}{q_{n_{i-1}}^2}]$ of $I_{n_{i-1}}$. Define $0 < c < \tilde{c} < d < \frac{1}{q_{n_{i-1}}^2}$ such that $|c| = \underline{\mu}_{n_{i-1}}^{-2\delta_0\cdot r_{n_{i-1}}}$, $|\tilde{c}| = M^2 \cdot |c|$, $d = \frac{1}{2q_{n_{i-1}}^2}$, see Figure 3. From the definition of n_i, we have $|I_{n_i}| < |c|$.

Define

$$e_i^0(x) = \begin{cases} 2|c|^{l+1} - (\phi_{A_{i-1}, r_{n_{i-1}}}(x) + \psi_{A_{i-1}, -r_{n_{i-1}}}(x) - \frac{\pi}{2}), & x \in [\tilde{c}, d]; \\ 0, & x \notin [c, \frac{1}{q_{n_{i-1}}^2}] \\ \tilde{h}_i(x) & x \in [c, \tilde{c}] \cup\ [d, \frac{1}{q_{n_{i-1}}^2}], \end{cases}$$

where $\tilde{h}_i(x)$ are polynomials of degree $2l+1$ restricted on each interval and for $0 \leq j \leq l$ satisfies

$$\frac{d^j \tilde{h}_i}{dx^j}(\frac{1}{q_{n_{i-1}}^2}) = 0, \quad \frac{d^j \tilde{h}_i}{dx^j}(d) = \frac{d^j(2|c|^{l+1} - (\phi_{A_{i-1}, r_{n_{i-1}}} + \psi_{A_{i-1}, -r_{n_{i-1}}} - \frac{\pi}{2}))}{dx^j}(d),$$

$$\frac{d^j \tilde{h}_i}{dx^j}(c) = 0, \qquad \frac{d^j \tilde{h}_i}{dx^j}(\tilde{c}) = \frac{d^j(2|c|^{l+1} - (\phi_{A_{i-1}, r_{n_{i-1}}} + \psi_{A_{i-1}, -r_{n_{i-1}}} - \frac{\pi}{2}))}{dx^j}(\tilde{c}).$$

$e_i^1(x)$ on the subinterval $[\frac{1}{2}, \frac{1}{2} + \frac{1}{q_{n_{i-1}}^2}]$ of $I_{n_{i-1}}$ is defined similarly. Let $e_i(x) = e_i^0(x) + e_i^1(x)$. We have the following estimates for $e_i(x)$.

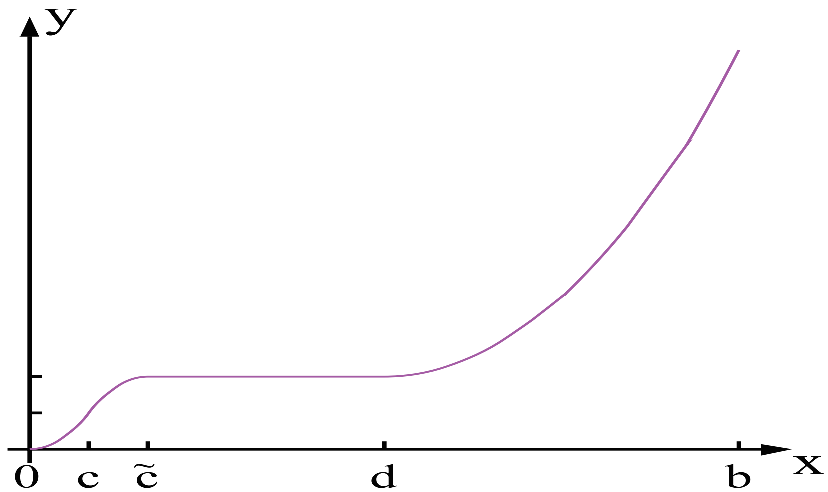

Figure 3, $b = \frac{1}{q_{n_{i-1}}^2}$

Lemma 3.1. *It holds that* $|e_i(x)|_{\mathcal{C}^l} \leq C \cdot q_{n_{i-1}}^{-2}$.

Proof. From (1) in $(\mathrm{P_{i-1}})$ we have $\mu_{n_{i-1}}^{r_{n_{i-1}}} \geq \mu^{(1-\delta_1)^i \cdot q_{n_{i-1}}} \gg \frac{1}{|I_{n_{i-1}}|}$. Then From (4) in $(\mathrm{P_{i-1}})$ and the definition of c, it holds for $0 \leq j \leq l$ that

$$|(2|c|^{l+1} - (\phi_{A_{i-1}, r_{n_{i-1}}} + \psi_{A_{i-1}, -r_{n_{i-1}}} - \frac{\pi}{2}))(x)|_{\mathcal{C}^j} \leq C \cdot q_{n_{i-1}}^{-2(l+1-j)}.$$

Hence from Cramer's rule we have that $|\tilde{h}_i(x)|_{\mathcal{C}^l} \leq C \cdot q_{n_{i-1}}^{-2}$. Consequently, $|e_i(x)|_{\mathcal{C}^l} \leq C \cdot q_{n_{i-1}}^{-2}$. □

Let $\tilde{\theta}_i = \theta_{i-1} + e_i(x)$ and $\tilde{A}_{i-1} = \Lambda \cdot R_{\frac{\pi}{2} - \tilde{\theta}_i}$, we have $\psi_{\tilde{A}_{i-1}, -r_{n_{i-1}}}(x) + \phi_{\tilde{A}_{i-1}, r_{n_{i-1}}}(x) - \frac{\pi}{2}$ on the part $[0, \frac{1}{q_{n_{i-1}}^2}]$ of $I_{n_{i-1}}$ is of the shape in Figure 3.

Step 3. The estimate on the lower bound.

Lemma 3.2. *Let* $A_{i,0} = \Lambda \cdot R_{\tilde{\theta}_i} := \Lambda \cdot R_{\theta_{i,0}}$ *satisfy* $\|A_{i,0}^{r_{n_{i-1}}(x)}(x)\| \geq \nu_0^{r_{n_{i-1}}(x)}$ *for* $x \in I_{n_i}$ *with* $\nu_0 = \mu^{(1-\delta_1)^{i-1}}$. *Then for any* $n_i - n_{i-1} \geq j \geq 1$, *there exist* $\theta_{i,j}$ *and* $A_{i,j} = \Lambda \cdot R_{\frac{\pi}{2} - \theta_{i,j}}$ *such that the following properties hold true :*

$$(\tilde{\mathrm{P}}_{\mathrm{i,j}}) : \begin{cases} (\tilde{1}). \quad \|A_{i,j}^{r_{n_{i-1}+j}(x)}(x)\| \geq \nu_j^{r_{n_{i-1}+j}(x)} \text{on } I_{n_i}; \\ (\tilde{2}). \quad \phi_{A_{i,j}, r_{n_{i-1}+j}}(x) + \psi_{A_{i,j}, -r_{n_{i-1}+j}}(x) - \frac{\pi}{2} = \theta_{i,0}(x) \text{ on } I_{n_i}; \\ (\tilde{3}). \quad |\theta_{i,j} - \theta_{i,j-1}|_{\mathcal{C}^l} \leq r_{i,j} \cdot \nu_{j-1}^{-q_{n_{i-1}+j-1}}, \ r_{i,j} \approx \max\{\nu_{j-1}^{2l^2 \delta_0 q_{n_{i-1}}}, q_{n_{i-1}+j}^{2l^2}\}, \end{cases}$$

where ν_j *are iteratively defined by*

$$\nu_j = \nu_0 \cdot \nu_0^{-\delta_0(\sqrt{2}^{-(j-1)} + \cdots + \sqrt{2}^{-1} + 1) \cdot 2(l+1)} \geq \nu_0^{(1 - 8\delta_0 \cdot (l+1))} = \mu^{(1-\delta_1)^i}.$$

Let $\underline{\mu}_{i,j} = \min_{x\in I_{n_i}} \|(A_{i,j}^{r_{n_{i-1}+j}}(x))\|^{\frac{1}{r_{n_{i-1}+j}}}$ *for any* $j \le n_i - n_{i-1}$. *We have* $\underline{\mu}_{i,j} \ge \nu_j$ *and* $\underline{\mu}_{n_i} = \underline{\mu}_{i,n_i-n_{i-1}} \ge \nu_{n_i-n_{i-1}}$.

Proof. For $j=1$, from (1) of (P_{i-1}) and the definitions of $\tilde{\theta}_i$ and μ_0, we have

$$\frac{1}{r_{n_{i-1}+1}(x)} \log \|A_{i,1}^{r_{n_{i-1}+1}(x)}(x)\| \ge \log \underline{\mu}_{i,n_{i-1}} + \frac{1}{q_{n_{i-1}+1}} \log \underline{\mu}_{i,n_{i-1}}^{-2(l+1)\delta_0 q_{n_{i-1}}\cdot\frac{q_{n_{i-1}+1}}{q_{n_{i-1}}}}$$

$$= (1-2(l+1)\delta_0)\log \underline{\mu}_{i,n_{i-1}} \ge \nu_1.$$

Thus we obtain $(\tilde{1})$. Moreover $(\tilde{2})$ and $(\tilde{3})$ can be proved by Proposition 2.1 and Lemma 2.1 with $d_{n_{i-1}+1} \ge \frac{1}{q_{n_{i-1}+1}^{2(l+1)}}$.

Assume $(\tilde{P}_{i,j})$ hold true. We will prove $(\tilde{P}_{i,j+1})$. Define $\theta_{i,j+1}(x)$ by modifying $\theta_{i,j}(x)$ in the same way as we define ξ_{k+1} by modifying ξ_k in Proposition 2.1. Applying Lemma 2.1 with $d_{n_{i-1}+j} \ge \min\{\nu_{j-1}^{-2(l+1)\delta_0 q_{n_{i-1}}}, q_{n_{i-1}+j}^{-2(l+1)}\}$, we get $(\tilde{2})$ and $(\tilde{3})$.

Now we prove $(\tilde{1})$. In case that $|\tilde{c}| < |I_{n_{i-1}+j}|$, we have $q_{m+1} \ge \sqrt{2}\cdot q_m$ for each m, and thus

$$\frac{1}{r_{n_{i-1}+j}(x)} \log \|A_{i,j+1}^{r_{n_{i-1}+j}(x)}(x)\| \ge \log \nu_j + \frac{1}{q_{n_{i-1}+j+1}} \cdot \log \underline{\mu}_{i,n_{i-1}}^{-\delta_0\cdot q_{n_{i-1}}\cdot\frac{q_{n_{i-1}+j+1}}{q_{n_{i-1}+j}}\cdot 2(l+1)}$$

$$\ge (1-\delta_0(\sqrt{2}^{-(j-1)}+\cdots+\sqrt{2}^{-1}+1)\cdot 2(l+1))\log \nu_0 - \delta_0\cdot\sqrt{2}^{-j}\cdot 2(l+1)\log \nu_0$$

$$= \log \nu_{j+1}.$$

Now we consider the case $|\tilde{c}| \ge |I_{n_{i-1}+j}|$. Let j^* be the smallest integer such that $|I_{q_{n_{i-1}+j^*}}| \le |\tilde{c}|$ (Obviously, j^* depends on n_{i-1} and we can choose n_i large enough such that $j^* \ll n_i$). Since for any s, it holds that $M^2\cdot|I_{s+1}| \ge |I_s|$. Thus from the definition of $|c|$ and $|\tilde{c}|$, we have $|I_{q_{n_{i-1}+j^*}}| \ge |c|$ since $|I_{q_{n_{i-1}+j^*-1}}| \ge |\tilde{c}|$. Notice that $\nu_{j^*}^{q_{n_{i-1}+j^*}} \gg q_{n_{i-1}+j^*+1}^{2l}$. We construct ψ_{i,j^*+m} and $A_{i,j^*+m} = \Lambda\cdot R_{\psi_{i,j^*+m}}$ as in Proposition 2.1,such that

$$\underline{\mu}_{i,n_{i-1}+j^*+m} \ge \nu_{j^*+m}, \quad \text{for } m \ge 1,$$

which thus implies $(\tilde{1})$. □

Define $\theta_i(x) = \theta_{i,j^*+m^*}(x)$ and $A_i(x) = \Lambda\cdot R_{\frac{\pi}{2}-\theta_i(x)}$, where $m^* = n_i - n_{i-1} - j^*$. Then (1) of $(\mathrm{P_i})$ can be proved by $(\tilde{1})$ in $(\tilde{P}_{i,j})$. From the inequality $0 < \delta_0 \ll \frac{1}{l^2}$, (5) of $(\mathrm{P_i})$ can be proved by $(\tilde{3})$ in $(\tilde{P}_{i,j})$ and Lemma 3.1. (4) of $(\mathrm{P_i})$ is obvious from the construction of $A_{i,j}$.

Step 4. The estimate on the upper bound.

Now we prove (2) of $(\mathrm{P_i})$, i.e., an upper bound estimate for the Lyapunov exponent. We need the following proposition in [51]:

Proposition 3.1. *Let I_1 be a small interval in S^1, $I_2 = I_1 + 1/2$, $I = I_1 \bigcup I_2$. Let*

$$\min r(x) = \min_{x \in I} \min\{i > 0 | T^i x \ (mod\ 2\pi) \in I\},$$

$$\max r(x) = \max_{x \in \frac{1}{10} I_1} \min\{i > 0 | T^i x \ (mod\ 2\pi) \in \frac{1}{10} I_1\}.$$

Then there exists $k_1 \in \mathbb{N}$ such that $M^{-k_1} \leq \frac{\min r(x)}{\max r(x)} \leq 1$.

Obviously, we have

$$|\phi_{A_i, r_{n_{i-1}}}(x) + \psi_{A_i, -r_{n_{i-1}}}(x) - \frac{\pi}{2}| \leq 2|c|^l = 2\underline{\mu}_{n_{i-1}}^{-2l\delta_0 q_{n_{i-1}}} \tag{3.12}$$

for $x \in [0, \frac{1}{2q_{n_{i-1}}^2}]$. Apply Proposition 3.1 with $I_1 = [0, \frac{1}{2q_{n_{i-1}}^2}]$. Then from (3.12) and Lemma A.1 we have that

$$\bar{\mu}_{n_i} \leq \bar{\mu}_{n_{i-1}} \cdot \underline{\mu}_{n_{i-1}}^{-2lM^{-k_1} \cdot \delta_0}.$$

Subsequently (3) of (P_{i-1}) and the definition of δ_2 imply that

$$\bar{\mu}_{n_i} \leq \bar{\mu}_{n_{i-1}}^{1-\delta_2} \leq \lambda^{(1-\delta_2)^i}.$$

Step 5. The comparison between the lower and upper bounds. For (3) of (P_0), the upper bound for w_0 can be achieved by choosing $\lambda \gg N \gg 1$. For $i > 0$, we have the following argument. For any x, y with $|x - y| \leq \frac{1}{q_{n_i}^2}$ we have that $\|A_i(x) - A_i(y)\|_{C^l} \leq \frac{C}{q_{n_i}^2}$. From (1) of (P_{i-1}) it holds that $\mu^{(1-\delta_1)^i \cdot q_{n_{i-1}}} \gg \frac{1}{|I_{n_{i-1}}|}$. Once n_{i-1} is determined, for any $\tilde{\epsilon} > 0$, we can find $n_i \gg \tilde{n}_i \gg n_{i-1}$ such that $|I_{\tilde{n}_i}| \ll |c|$ and for any x, y with $|x - y| \leq \frac{1}{q_{\tilde{n}_i}^2}$, it holds that

$$(1 - \tilde{\epsilon}) \cdot L_{r_{\tilde{n}_i}}(A_i(y)) \leq L_{r_{\tilde{n}_i}}(A_i(x)) \leq (1 + \tilde{\epsilon}) \cdot L_{r_{\tilde{n}_i}}(A_i(y)),$$

$$\|\phi_{A_i, \pm r_{\tilde{n}_i}^{\pm}}(x) - \phi_{A_i, \pm r_{\tilde{n}_i}^{\pm}}(y)\|_{C^l} + \|\psi_{A_i, \pm r_{\tilde{n}_i}^{\pm}}(x) - \psi_{A_i, \pm r_{\tilde{n}_i}^{\pm}}(y)\|_{C^l} \leq \tilde{\epsilon}.$$

Apply the inductive process for A_i from step $\tilde{n}_i$ to n_i for $x \in I_{n_i}$. Thus similar to Remark 2.1, if $\tilde{n}_i$ is large enough such that $\mu^{(1-\delta_1)^i \cdot q_{\tilde{n}_i}} \gg \frac{1}{|I_{\tilde{n}_i}|}$, we have that for any $x \in I_{n_i}$,

$$\prod_j \|A_i^{r_{\tilde{n}_{i,j}}}(x_j)\| \geq \|A_i^{r_{n_i}}(x)\| \geq (\prod_j \|A_i^{r_{\tilde{n}_{i,j}}}(x_j)\|)^{1-\tilde{\epsilon}},$$

where x_j's are the points on orbits of x when returning to $I_{\tilde{n}_i}$ and $\tilde{n}_{i,j}$'s are the corresponding returning times. Hence for any $x, y \in I_{n_i}$ we have

$$(1 - 2\tilde{\epsilon}) L_{r_{n_i}}(A_i(x)) \leq L_{r_{n_i}}(A_i(y)) \leq (1 + 2\tilde{\epsilon}) L_{r_{n_i}}(A_i(x)).$$

Then it follows that (3) of (P_i) holds true. This ends the proof of Iterative Lemma. □

4 The proof for the $\mathcal{C}^\infty$ case

In this section, we will prove Theorem 1 and 2 for the $\mathcal{C}^\infty$ case. The basic idea is same as the one for the finitely smooth case. Essentially, we only need to modify cocycles in C^∞ category. We will focus on the difference between the two cases. First we follow the steps in section 3 to construct a sequence of C^∞ cocycle which is $\mathcal{C}^1$-convergent. Then we will prove that it actually converges in $\mathcal{C}^\infty$ topology.

Assume $\lambda \gg e^{q_N^{a+1}} \gg 1$ with $0 < a < \frac{1}{10}$. For $n > N$, let $\lambda_{n+1}^{q_{n+1}} = \lambda_n^{q_{n+1}} \cdot e^{-(10q_{n+1}^2)^a}$ with $\lambda_N = \lambda$. From the definition of λ_n, we have $\lambda_n^{q_n} \geq \lambda_{n-1}^{q_n} \cdot e^{-q_n^{2a}} \geq \lambda_{n-2}^{q_n} \cdot e^{-q_n \cdot q_{n-1}^{2a-1}} \geq \cdots \geq \lambda^{q_n} \cdot \lambda_N^{-C\cdot q_n^{2a}} \geq \lambda^{(1-\epsilon)q_n}$ for some small positive ϵ if $\lambda \gg 1$ and $N \gg 1$. It implies that λ_n decrease to $\lambda_\infty > \lambda^{1-\epsilon}$.

Construction of $B_N(x)$ Let

(a)

$$\xi_0(x) = \begin{cases} \xi_{01}(x) \quad \text{for } |x| \leq \delta, \\ \xi_{02}(x)(\text{or } -\xi_{02}(x)) \quad \text{for } |x-1/2| \leq \delta, \end{cases}$$

where $\xi_{01}(x) = \mathrm{sgn}(x)e^{-\frac{1}{|x|^a}}$ and $\xi_{02}(x) = \mathrm{sgn}(x-1/2)e^{-\frac{1}{|x-1/2|^a}}$, $\delta > 0$ is a small number. Let $\xi(x)$ be a lift of a $\mathcal{C}^\infty$ 1-periodic function satisfying

$$\xi(x) = \begin{cases} \xi_{01}(x), & |x| \leq \delta; \\ -\xi_{02}(x) \text{ (or } \pi + \xi_{02}(x)), & |x-\frac{1}{2}| \leq \delta. \end{cases} \tag{4.13}$$

(b) $\forall\ |x (\mathrm{mod}\ 1)| > \delta$ and $|(x-1/2)(\mathrm{mod}\ 1)| > \delta$, $|\xi(x)(\mathrm{mod}\ \pi)| > e^{-\frac{1}{\delta^a}}$.

Define $\xi_N(x) = \xi(x)$ and $B_N(x) = \Lambda \cdot R_{\frac{\pi}{2}-\xi_N(x)}$.

We restate Lemma 5.1 in [51] as follows:

Lemma 4.1. *For each $n \geq N$, there exist a $g_n(x) \in \mathcal{C}^\infty$ be a 1-periodic function such that*

$$g_n(x) : \begin{cases} = 1, & x \in \frac{I_n}{10}, \\ \in [0,1], & x \in I_n \backslash \frac{I_n}{10} \\ = 0, & x \in \mathbb{S}^1 \backslash I_n \end{cases}$$

and

$$\left| \frac{d^r g_n(x)}{dx^r} \right| \leq q_n^{3r}, \quad 0 \leq r \leq [q_n^{\frac{1}{10}}]. \tag{4.14}$$

Using the same argument as that in finite smooth case, we have that for any $x \in I_N$, $\|B_N^{r_N^+(x)}(x)\| \geq \lambda_N^{r_N^+(x)}$ and

$$|\phi_{B_N, r_N}(x) + \psi_{B_N, -r_N}(x) - \frac{\pi}{2} - \xi_0(x)|_{\mathcal{C}^1} \leq \lambda_N^{-1} \tag{4.15}$$

for $x \in I_N$.

Define a 1-periodic function $e_N(x) \in \mathcal{C}^\infty$ such that $e_N(x) = -(\phi_{B_N,r_N}(x)+\psi_{B_N,-r_N}(x)-\frac{\pi}{2}-\xi_0(x))$ for $x \in I_N$.

Let $\hat{e}_N(x) = e_N(x) \cdot g_N(x)$ and $\xi_{N+1}(x) = \xi_N(x) + \hat{e}_N(x)$ for $x \in \mathbb{S}^1$. Define $B_{N+1}(x) = \Lambda \cdot R_{\frac{\pi}{2}-\xi_{N+1}(x)}$. Obviously, $B_{N+1}(x) = B_N(x) \cdot R_{-\hat{e}_N(x)}$. Then for any $x \in I_N$, $\|B_{N+1}^{r_N^+(x)}(x)\| \geq \lambda_N^{r_N^+(x)}$ and $\phi_{B_{N+1},r_N}(x) + \psi_{B_{N+1},-r_N}(x) = \phi_{B_N,r_N}(x) + \psi_{B_N,-r_N}(x) - \hat{e}_N(x)$, which implies $\phi_{B_{N+1},r_N}(x) + \psi_{B_{N+1},-r_N}(x) - \frac{\pi}{2} = \xi_0(x)$ on $\frac{I_N}{10}$. (4.15) implies that $|\hat{e}_N(x)|_{\mathcal{C}^1} \leq \lambda_N^{-1}$ in I_N. Thus we have $|\phi_{B_{N+1},r_N}(x) + \psi_{B_{N+1},-r_N}(x) - \frac{\pi}{2}| \geq \frac{1}{2} \cdot e^{-(10 \cdot q_N^2)^a}$ on $I_N \backslash \frac{I_N}{10}$.

For any $n \geq N$, define a 1-periodic function $e_n(x) \in \mathcal{C}^\infty$ such that

$$e_n(x) = (\phi_{B_n,r_n}(x) + \psi_{B_n,-r_n}(x)) - (\phi_{B_n,r_{n+1}}(x) + \psi_{B_n,-r_{n+1}}(x)) \quad x \in I_n.$$

Define $\hat{e}_n(x) = e_n(x) \cdot g_n(x)$, $\xi_n(x) = \xi_{n-1}(x) + \hat{e}_n(x)$ and $B_n(x) = \Lambda \cdot R_{\frac{\pi}{2}-\xi_n(x)}$. Obviously, $B_n(x) = B_{n-1}(x) \cdot R_{-\hat{e}_n(x)}$. Then we obtain (2.5), (2.6) of Proposition 2.1 and

$$\begin{array}{ll} |\phi_{B_n,r_n}(x) + \psi_{B_n,-r_n}(x) - \frac{\pi}{2}| & = e^{-|x|^{-a}} (\text{ or } e^{-|x-1/2|^{-a}}), \quad x \in \frac{I_{n,i}}{10}, i = 1, 2 \\ |\phi_{B_n,r_n}(x) + \psi_{B_n,-r_n}(x) - \frac{\pi}{2}| & \geq \frac{1}{2} \cdot e^{-(10 \cdot q_n^2)^a}, \quad x \in I_n \backslash \frac{I_n}{10}. \end{array}$$

From (2.5), one easily sees that $B_N(x), B_{N+1}(x), \cdots$, is $\mathcal{C}^1$-convergent to some $D_\infty(x)$. Furthermore, from (2.6), the Lyapunov exponent of $D_\infty(x)$ has a lower bound $\log \lambda_\infty > (1-\epsilon) \log \lambda$.

In the following, we will prove that $B_N(x), B_{N+1}(x), \cdots$, is also convergent to $D_\infty(x)$ in $\mathcal{C}^\infty$-topology.

Lemma 4.2. $B_N(x), B_{N+1}(x), \cdots$, *is also convergent to* $D_\infty(x)$ *in* $\mathcal{C}^\infty$*-topology.*

Proof. It is equivalent to prove that $\xi_n(x),\ n = N, N+1, \cdots$ is $\mathcal{C}^\infty$-convergent. From the definition of $\xi_n(x)$, we have $\xi_n(x) - \xi_{n-1}(x) = \hat{e}_n(x)$. From the definition of $\hat{e}_n(x)$, it is sufficient to estimate $e_n(x)$ and $g_n(x)$. Since $e_n(x)$ is determined by $\phi_{B_n,r_n}(x) - \phi_{B_n,r_{n+1}}(x)$ and $\psi_{B_n,-r_n}(x) - \psi_{B_n,-r_{n+1}}(x)$, with the help of Lemma 2.1, we have

$$\left| \frac{d^r e_n(x)}{dx^r} \right| \leq C(r) \cdot \lambda_n^{-q_{n-1}}, \quad 0 \leq r \leq [q_{n-1}^{\frac{1}{10}}].$$

Note that $C(r)$ is independent of n. Thus for any fixed $R \in \mathbb{N}$, we can choose n large enough such that $C(r) \leq \lambda_n^{\frac{1}{2}q_{n-1}}$ for any $r \leq R$. This together with (4.14) ends the proof. □

Construction of $C_k(x)$ Next we will construct the sequence $C_k(x) (k = N, N+1, \cdots)$ with $L(C_k) = 0$ such that $\mathcal{C}^\infty$ converge to D_∞.

Consider the sub-interval $[0, \frac{1}{q_{n_{i-1}}^2}]$ of $I_{n_{i-1}}$. Define $0 < c < \tilde{c} < d < \frac{1}{q_{n_{i-1}}^2}$ such that $|c| = (2\delta_0 \cdot r_{n_{i-1}} \cdot \log \underline{\mu}_{n_{i-1}})^{-1/a}, |\tilde{c}| = M^2 \cdot |c|, d = \frac{1}{2q_{n_{i-1}}^2}$. Let n_i be sufficiently large such that $I_{n_i} \not\supseteq [0, c]$.

Define

$$\bar{e}_i(x) = \begin{cases} e^{-|c|^{-a}} - (\phi_{A_{i-1}, r_{n_{i-1}}}(x) + \psi_{A_{i-1}, -r_{n_{i-1}}}(x)), & x \in [\tilde{c}, d], \\ 0, & x \notin [c, \frac{1}{q_{n_{i-1}}^2}], \\ \bar{h}_i^{\pm}(x) & x \in [c, \tilde{c}] \cup [d, \frac{1}{q_{n_{i-1}}^2}], \end{cases}$$

where $\bar{h}_i^{\pm}(x)$ is of a $\mathcal{C}^\infty$ connection between the parts in $[0, c]$ and $[\tilde{c}, d]$ as well as between the part in $[\tilde{c}, d]$ and the end point $\frac{1}{q_{n_{i-1}}^2}$ of I_{n_i}. Then similar to Lemma 4.1, we have

$$\left| \frac{d^r \bar{h}_i(x)}{dx^r} \right| \leq C(r) \cdot q_{n_i}^{3r}, \quad 0 \leq r \leq [q_{n_i}^{\frac{1}{10}}].$$

Thus the $\mathcal{C}^\infty$-convergence of $C_k(x)$ is similar to the above argument. The remain part of the proof is same as Section 3.

A Product of hyperbolic matrices

Let A be a hyperbolic $SL(2, R)$-matrix, i.e., $\|A\| > 1$. It is know that A can be written uniquely as $A = R_\psi \cdot \Lambda_A \cdot R_\phi$ with $\Lambda_A = diag(\|A\|, \|A\|^{-1})$. It is known that $-\phi$ is the most expanded direction of A and ψ is the most contracted direction of A^{-1}.

For two hyperbolic matrices $A = R_{\psi_A} \cdot \Lambda_A \cdot R_{\phi_A}, B = R_{\psi_B} \cdot \Lambda_B \cdot R_{\phi_B}$ with big norms, let $BA = R_{\psi_{BA}} \cdot \Lambda_{BA} \cdot R_{\phi_{BA}}$. We firstly investigate how ϕ_{BA}, ψ_{BA} and $\|BA\|$ depend on A and B.

Lemma A.1. *Let A, B be hyperbolic $SL(2, \mathbb{R})$ cocycles and $\theta = \phi_B + \psi_A$. Then it holds that $\frac{1}{4} N(\|A\|, \|B\|, \theta) \leq \|BA\|^2 \leq N(\|A\|, \|B\|, \theta)$, where $N(\|A\|, \|B\|, \theta) = (\|A\|^2 \|B\|^2 + \|A\|^{-2} \|B\|^{-2}) \cdot \cos^2 \theta + (\|A\|^2 \|B\|^{-2} + \|A\|^{-2} \|B\|^2) \cdot \sin^2 \theta$.*

Proof. For any $SL(2, \mathbb{R})$ matrix $A = (a_{ij})_{2 \times 2}$, it is known that $\frac{1}{4} \sum_{i,j} a_{ij}^2 \leq \|A\|^2 \leq \sum_{i,j} a_{ij}^2$.

It is easy to see that

$$\begin{aligned} \|BA\| &= \left\| \begin{pmatrix} \|B\| & 0 \\ 0 & \|B\|^{-1} \end{pmatrix} \cdot \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \cdot \begin{pmatrix} \|A\| & 0 \\ 0 & \|A\|^{-1} \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} \|A\| \|B\| \cos \theta & -\|A\|^{-1} \|B\| \sin \theta \\ \|A\| \|B\|^{-1} \sin \theta & \|A\|^{-1} \|B\|^{-1} \cos \theta \end{pmatrix} \right\|. \end{aligned}$$

It thus implies the conclusion. $\square$

Lemma A.2. *Let* $\phi = \phi_A - \phi_{BA}$, $\psi = \psi_{BA} - \psi_B$. *Assume* $\theta \in [0, \pi)$. *Then*

$$\phi(\|A\|, \|B\|, \theta) = \begin{cases} 0, & \text{for } \theta = 0 \\ -\frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(a\cot\theta + b\tan\theta)\right), & \text{for } 0 < \theta < \frac{\pi}{2} \\ \frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(a\cot\theta + b\tan\theta)\right), & \text{for } \frac{\pi}{2} < \theta < \pi \text{ if } b \geq 0 \\ -\frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(a\cot\theta + b\tan\theta)\right), & \text{for } \frac{\pi}{2} < \theta < \pi \text{ if } b < 0 \\ 0, & \text{for } \theta = \frac{\pi}{2} \text{ if } b \geq 0 \\ -\frac{\pi}{2}, & \text{for } \theta = \frac{\pi}{2} \text{ if } b < 0, \end{cases} \tag{A.16}$$

where

$$a = \frac{\|A\|^2\|B\|^2 - \|A\|^{-2}\|B\|^{-2}}{2(\|B\|^2 - \|B\|^{-2})}, \quad b = \frac{\|A\|^2\|B\|^{-2} - \|A\|^{-2}\|B\|^2}{2(\|B\|^2 - \|B\|^{-2})}.$$

Similarly,

$$\psi(\|A\|, \|B\|, \theta)) = \begin{cases} 0, & \text{for } \theta = 0 \\ -\frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(a'\cot\theta - b'\tan\theta)\right), & \text{for } 0 < \theta < \frac{\pi}{2} \\ \frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(a'\cot\theta - b'\tan\theta)\right), & \text{for } \frac{\pi}{2} < \theta < \pi \text{ if } b' \geq 0 \\ -\frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(a'\cot\theta - b'\tan\theta)\right), & \text{for } \frac{\pi}{2} < \theta < \pi \text{ if } b' < 0 \\ 0, & \text{for } \theta = \frac{\pi}{2} \text{ if } b' \geq 0 \\ -\frac{\pi}{2}, & \text{for } \theta = \frac{\pi}{2} \text{ if } b' < 0, \end{cases} \tag{A.17}$$

where

$$a' = \frac{\|A\|^2\|B\|^2 - \|A\|^{-2}\|B\|^{-2}}{2(\|A\|^2 - \|A\|^{-2})}, \quad b' = \frac{\|A\|^2\|B\|^{-2} - \|A\|^{-2}\|B\|^2}{2(\|A\|^2 - \|A\|^{-2})}.$$

Proof. Let

$$\begin{aligned} V(s) &= \begin{pmatrix} \|B\| & 0 \\ 0 & \|B\|^{-1} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \|A\| & 0 \\ 0 & \|A\|^{-1} \end{pmatrix} \cdot \begin{pmatrix} \cos s \\ \sin s \end{pmatrix} \\ &= \begin{pmatrix} \|B\| & 0 \\ 0 & \|B\|^{-1} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \cdot \|A\| \cdot \cos s - \sin\theta \cdot \|A\|^{-1} \cdot \sin s \\ \sin\theta\|A\| \cdot \cos s + \cos\theta \cdot \|A\|^{-1} \cdot \sin s \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta \cdot \|A\|\|B\| \cdot \cos s - \sin\theta \cdot \|A\|^{-1} \cdot \|B\| \sin s \\ \sin\theta\|A\|\|B\|^{-1} \cdot \cos s + \cos\theta \cdot \|A\|^{-1}\|B\|^{-1} \cdot \sin s \end{pmatrix}. \end{aligned}$$

Thus

$$|V(s)|^2 = (\cos\theta\|A\|\|B\|)^2 + (\sin^2\theta\|A\|^{-2}\|B\|^2 - \cos^2\theta\|A\|^2\|B\|^2)\sin^2 s + \sin^2\theta\|A\|^2\|B\|^{-2}$$

$$+(\cos^2\theta\|A\|^{-2}\|B\|^{-2} - \sin^2\theta\|A\|^2\|B\|^{-2})\sin^2 s + 2(\|B\|^{-2} - \|B\|^2)\sin\theta\cos\theta\sin s\cos s.$$

Obviously $\frac{d}{ds}(|V(s)|^2) = 0$ at ϕ since $|V(s)|^2$ attains its extreme at ϕ, a simple computation leads to

$$\big((\|A\|^2\|B\|^2 - \|A\|^{-2}\|B\|^{-2})\cos^2\theta + (\|A\|^2\|B\|^{-2} - \|A\|^{-2}\|B\|^2)\sin^2\theta\big)\sin 2\phi$$
$$= -2(\|B\|^2 - \|B\|^{-2})\sin 2\theta\cos 2\phi.$$

Thus

$$-\cot 2\phi = \frac{\|A\|^2\|B\|^2 - \|A\|^{-2}\|B\|^{-2}}{2(\|B\|^2 - \|B\|^{-2})}\cot\theta + \frac{\|A\|^2\|B\|^{-2} - \|A\|^{-2}\|B\|^2}{2(\|B\|^2 - \|B\|^{-2})}\tan\theta.$$

With the help of the inequality $\frac{d^2}{ds^2}(|V(s)|^2) \leq 0$, we obtain the unique ϕ corresponding the maximum $\|BA\|^2$ of $|V(s)|^2$, which satisfies (A.16).

(A.17) is proved similarly. □

Later we will see that both $\|A\|$ and $\|B\|$ are very big. Thus

$$a = \frac{\|A\|^2\|B\|^2 - \|A\|^{-2}\|B\|^{-2}}{\|B\|^2 - \|B\|^{-2}} \sim \|A\|^2,$$

$$b = \frac{\|A\|^2\|B\|^{-2} - \|A\|^{-2}\|B\|^2}{\|B\|^2 - \|B\|^{-2}} \lesssim \max\{\|A\|^{-2}, \frac{\|A\|^2}{\|B\|^4}\}.$$

If A, B are hyperbolic, the functions $\phi(\|A\|, \|B\|, \theta), \psi(\|A\|, \|B\|, \theta)$ defined above are continuous in all variables. In the following, we estimate the derivatives of ϕ and ψ with respect to θ, $\|A\|$ and $\|B\|$.

Lemma A.3. *It holds that*

$$|\phi(mod\,\pi)| \leq C(0)\cdot\|A\|^{-2}\cdot|\theta - \frac{\pi}{2}|^{-1} \tag{A.18}$$

and

$$|\psi(mod\,\pi)| \leq C(0)\cdot\|B\|^{-2}\cdot|\theta - \frac{\pi}{2}|^{-1}. \tag{A.19}$$

Suppose $|\theta - \frac{\pi}{2}|^{-1} \ll \|A\|^2$*. Then, for* $i \geq 1$*, we have that*

$$|\frac{\partial^i\phi}{\partial\theta^i}| \leq C(i)\cdot\|A\|^{-2}\cdot|\theta - \frac{\pi}{2}|^{-i-1}, \tag{A.20}$$

$$|\frac{\partial^i\phi}{\partial\|A\|^i}| \leq C(i)\cdot\|A\|^{-2}\cdot\|A\|^{-i}\cdot|\theta - \frac{\pi}{2}|^{-1}, \tag{A.21}$$

and

$$|\frac{\partial^i\phi}{\partial\|B\|^i}| \leq C(i)\cdot\|A\|^{-2}\cdot\|B\|^{-i}\cdot|\theta - \frac{\pi}{2}|^{-1}. \tag{A.22}$$

More generally, for $i+j+k \geq 1$, we have

$$|\frac{\partial^{i+j+k}\phi}{\partial\theta^i\partial\|A\|^j\partial\|B\|^k}| \leq C(i,j,k)\cdot|\theta-\frac{\pi}{2}|^{-i-1}\|A\|^{-2-j}\cdot\|B\|^{-k}; \tag{A.23}$$

Similarly, suppose $|\theta-\frac{\pi}{2}|^{-1} \ll \|B\|^2$. Then we have

$$|\frac{\partial^{i+j+k}\psi}{\partial\theta^i\partial\|A\|^j\partial\|B\|^k}| \leq C(i,j,k)\cdot|\theta-\frac{\pi}{2}|^{-i-1}\|A\|^{-j}\cdot\|B\|^{-2-k}. \tag{A.24}$$

Proof. To prove (A.18), it is sufficient to consider the situation $\theta \approx \frac{\pi}{2}$. We only consider the case $0 \leq \theta \leq \frac{\pi}{2}$ since the proof for the other cases is similar. From the fact $\lim_{x\to\infty}\frac{\frac{\pi}{2}-\arctan x}{x^{-1}} = 1$ and the definition of a, we have $|\phi| \leq C(0)\cdot a^{-1}\cdot|\theta-\frac{\pi}{2}|^{-1} \leq C(0)\cdot\|A\|^{-2}\cdot|\theta-\frac{\pi}{2}|^{-1}$. Thus we obtain (A.18). We can obtain (A.19) similarly.

for $i \geq 1$, from the definition of ϕ, we have

$$\frac{\partial^i\phi}{\partial\theta^i} = -\frac{1}{2}\sum_{l_1+\cdots+l_k=i}\frac{d^{k-1}(\frac{1}{1+f^2})}{df^{k-1}}\cdot\frac{\partial^{l_1}f}{\partial\theta^{l_1}}\cdots\frac{\partial^{l_k}f}{\partial\theta^{l_k}},$$

where $f(\|A\|,\|B\|,\theta) = a\cot\theta + b\tan\theta$. To estimate $\frac{\partial^i\phi}{\partial\theta^i}$, we have that

$$|\frac{\partial^{l_s}f}{\partial\theta^{l_s}}| = |\frac{\partial^{l_s}}{\partial\theta^{l_s}}(a\cot\theta+b\tan\theta)| \leq |a|\cdot|\cot^{(l_s)}(\theta)| + |b|\cdot|\tan^{(l_s)}(\theta)|.$$

By a direct computation, we have

$$|\tan^{(l_s)}\theta| = |(\cos^{-2}\theta)^{(l_s-1)}| \leq |\sum_{\kappa_1+\cdots+\kappa_t=l_s-1}\cos^{-(2+t)}\theta\cdot\cos^{(\kappa_1)}\theta\cdots\cos^{(\kappa_t)}\theta|$$

and

$$|\cot^{(l_s)}\theta| = |(\sin^{-2}\theta)^{(l_s-1)}| \leq |\sum_{\kappa_1+\cdots+\kappa_t=l_s-1}\sin^{-(2+t)}\theta\cdot\sin^{(\kappa_1)}\theta\cdots\sin^{(\kappa_t)}\theta|.$$

From the condition $|\theta-\frac{\pi}{2}|^{-1} \ll \|A\|^2$ and the fact that the signs of $\|B\|^2\cot\theta$ and $\|B\|^{-2}\tan\theta$ are the same, we have

$$|\frac{\partial^{l_s}f}{\partial\theta^{l_s}}| \leq C(l_s)\cdot(|a|\cdot|\theta-\frac{\pi}{2}|^{-(l_s-1)}+|b|\cdot|\theta-\frac{\pi}{2}|^{-(l_s+1)}) \leq C(l_s)\cdot|f|\cdot|\theta-\frac{\pi}{2}|^{-l_s}. \tag{A.25}$$

On the other hand, we have

$$\left|\frac{d^{k-1}(\frac{1}{1+f^2})}{df^{k-1}}\right| \lesssim |f|^{-k-1} \quad \text{if } k \geq 1.$$

Thus from $|f| \gtrsim \|A\|^2\cdot|\cot\theta|$ we obtain

$$|\frac{\partial^i\phi}{\partial\theta^i}| \leq C(i)|\theta-\frac{\pi}{2}|^{-i}\cdot\frac{1}{|f|} \leq C(i)\|A\|^{-2}|\theta-\frac{\pi}{2}|^{-i-1}.$$

Next we estimate

$$\begin{aligned} |\tfrac{\partial^i \phi}{\partial \|A\|^i}| &\leq \textstyle\sum_{l_1+\cdots+l_k=i} |\tfrac{d^{k-1}(\frac{1}{1+f^2})}{df^{k-1}}| \cdot |\tfrac{\partial^{l_1} f}{\partial \|A\|^{l_1}}| \cdots |\tfrac{\partial^{l_k} f}{\partial \|A\|^{l_k}}| \quad l_j \geq 1,\ 1 \leq j \leq k \\ &\leq \textstyle\sum_{l_1+\cdots+l_k=i} |f|^{-k-1} \cdot |\tfrac{\partial^{l_1} f}{\partial \|A\|^{l_1}}| \cdots |\tfrac{\partial^{l_k} f}{\partial \|A\|^{l_k}}|. \end{aligned} \tag{A.26}$$

It is easy to see that $|f| \sim |a| \cdot |\cot\theta|$ with the condition $|\theta - \frac{\pi}{2}|^{-1} \ll \|A\|^2$. We also have

$$|\frac{\partial^{l_s} f}{\partial \|A\|^{l_s}}| \leq |\cot\theta||\frac{\partial^{l_s} a}{\partial \|A\|^{l_s}}| + |\tan\theta||\frac{\partial^{l_s} b}{\partial \|A\|^{l_s}}|.$$

By a direct computation, we obtain

$$|\frac{\partial^{l_s} a}{\partial \|A\|^{l_s}}| = \left|\frac{\partial^{l_s}}{\partial \|A\|^{l_s}} \left(\frac{\|A\|^2\|B\|^2 - \|A\|^{-2}\|B\|^{-2}}{\|B\|^2 - \|B\|^{-2}}\right)\right| \leq C(l_s) \cdot |a| \cdot \|A\|^{-l_s}$$

and

$$|\tfrac{\partial^{l_s} b}{\partial \|A\|^{l_s}}| = \left|\tfrac{\partial^{l_s}}{\partial \|A\|^{l_s}} \left(\tfrac{\|A\|^2\|B\|^{-2} - \|A\|^{-2}\|B\|^{2}}{\|B\|^2 - \|B\|^{-2}}\right)\right| \leq C(l_s) \cdot (\|A\|^{-2} + \tfrac{\|A\|^2}{\|B\|^4}) \cdot \|A\|^{-l_s}.$$

Thus we have

$$\begin{aligned} |\tfrac{\partial^{l_s} f}{\partial \|A\|^{l_s}}| &\leq C(l_s) \cdot \left\{|\theta - \tfrac{\pi}{2}|\|A\|^{2-l_s} + |\theta - \tfrac{\pi}{2}|^{-1} \cdot \|A\|^{-l_s} \cdot (\|A\|^{-2} + \tfrac{\|A\|^2}{\|B\|^4})\right\} \\ &\leq C(l_s) \cdot |f| \cdot \|A\|^{-l_s}. \end{aligned} \tag{A.27}$$

With the fact that $|f| \gtrsim \|A\|^2 \cdot |\cot\theta|$, it follows that

$$\begin{aligned} |f|^{-2} \cdot |\tfrac{\partial^{l_s} f}{\partial \|A\|^{l_s}}| &\leq C(l_s)|f|^{-1} \cdot \|A\|^{-l_s} \\ &\leq C(l_s) \cdot \|A\|^{-2-l_s} \cdot |\theta - \tfrac{\pi}{2}|^{-1}. \end{aligned} \tag{A.28}$$

Combining (A.26), (A.27) with (A.28), we obtain (A.21). Similarly, we have (A.22) and (A.23). The estimates for ψ can be proved similarly. □

B Proof of Lemma 2.1

In this section, we first give estimates on most contracted and expanded directions of the product of hyperbolic blocks. Then we will give the proof of Lemma 2.1.

Let $A(x), B(x), \theta(x), \phi(x)$ and $\psi(x)$ be defined as in Lemma A.1 and A.2.

Lemma B.1. *Let $|\theta - \frac{\pi}{2}|^{-1} \ll \|A\|^2, \|B\|^2$. Suppose that, for any $i \geq 0$,*

$$\left|\frac{d^i\|A\|}{dx^i}\right| \leq C(i) \cdot \|A\| \cdot |\theta - \frac{\pi}{2}|^{-i-1}, \quad \left|\frac{d^i\|B\|}{dx^i}\right| \leq C(i) \cdot \|B\| \cdot |\theta - \frac{\pi}{2}|^{-i-1}, \ |\frac{d^i\theta}{dx^i}| \leq C(i) \cdot |\theta - \frac{\pi}{2}|^{-i-1}. \tag{B.29}$$

Then we have

$$|\tfrac{d^i\phi}{dx^i}| \le C(i)\cdot|\theta-\tfrac{\pi}{2}|^{-i-1}\cdot\|A\|^{-2},$$
$$|\tfrac{d^i\psi}{dx^i}| \le C(i)\cdot|\theta-\tfrac{\pi}{2}|^{-i-1}\cdot\|B\|^{-2}, \tag{B.30}$$
$$\left|\tfrac{d^i\|BA\|}{dx^i}\right| \le C(i)\cdot\|BA\|\cdot|\theta-\tfrac{\pi}{2}|^{-i-1}.$$

Proof. From Lemma A.3 and

$$\left|\tfrac{d^i\phi}{dx^i}\right| = \textstyle\sum_{t_1+\cdots+t_{i_1}+s_1+\cdots+s_{i_2}+j_1+\cdots+j_{i_3}=i}\left|\tfrac{\partial^{i_1+i_2+i_3}\phi}{\partial\|A\|^{i_1}\cdot\partial\|B\|^{i_2}\partial\theta^{i_3}}\cdot\tfrac{d^{t_1}\|A\|}{dx^{t_1}}\cdots\tfrac{d^{t_{i_1}}\|A\|}{dx^{t_{i_1}}}\right|$$
$$\cdot\left|\tfrac{d^{s_1}\|B\|}{dx^{s_1}}\cdots\tfrac{d^{s_{i_2}}\|B\|}{dx^{s_{i_2}}}\cdot\left(\tfrac{d^{j_1}\theta}{dx^{j_1}}\right)\cdots\left(\tfrac{dj_{i_3}\theta}{dx^{j_{i_3}}}\right)\right|, \tag{B.31}$$

we have

$$\left|\tfrac{\partial^{i_1+i_2+i_3}\phi}{\partial\|A\|^{i_1}\cdot\partial\|B\|^{i_2}\cdot\partial\theta^{i_3}}\right| \le C(i_1,i_2,i_3)\cdot|\theta-\tfrac{\pi}{2}|^{-(1+i_3)}\cdot\|A\|^{-i_1-2}\cdot\|B\|^{-i_2}. \tag{B.32}$$

Then from (B.29), (B.31), (B.32), we have

$$\left|\frac{d^i\phi}{dx^i}\right| \le C(i)\cdot\|A\|^{-2}\cdot|\theta-\frac{\pi}{2}|^{-(i+1)},$$

thus the first inequality of (B.30) is proved. In the above inequality, we used the fact

$$\left|\frac{\partial f}{\partial\|A\|}\cdot\frac{\partial^j\|A\|}{\partial x^j}\right| \le |\theta-\frac{\pi}{2}|^{-(j+1)}\cdot|f|.$$

The second inequality is proved similarly. Now we prove the third inequality. By a direct computation, we have

$$\frac{\partial^i\|BA\|}{\partial\phi^i} = \frac{\partial^i(g^{\frac12})}{\partial\phi^i} = \sum_{l_1+\cdots+l_k=i}(g^{\frac12})^{(k)}\cdot\frac{\partial^{l_1}g}{\partial\phi^{l_1}}\cdots\frac{\partial^{l_k}g}{\partial\phi^{l_k}}, \tag{B.33}$$

where

$$g=g_1^2+g_2^2,\quad g_1=\|A\|\cdot\|B\|\cdot\cos\theta\cos\phi-\|A\|^{-1}\cdot\|B\|^{-1}\sin\theta\sin\phi,$$
$$g_2=\|A\|\cdot\|B\|^{-1}\cdot\sin\theta\cdot\cos\phi+\|A\|^{-1}\cdot\|B\|^{-1}\cdot\cos\theta\cdot\sin\phi. \tag{B.34}$$

It is not difficult to see that $|(g^{\frac12})^{(k)}|\le C(k)\cdot g^{\frac12-k}$. From the definition of g, we have

$$\left|\frac{\partial^{l_s}g}{\partial\phi^{l_s}}\right| \le \left|\frac{\partial^{l_s}(g_1^2)}{\partial\phi^{l_s}}\right| + \left|\frac{\partial^{l_s}(g_2^2)}{\partial\phi^{l_s}}\right|.$$

From

$$\left|\tfrac{\partial^{l_s}(g_1^2)}{\partial\phi^{l_s}}\right| \le \textstyle\sum_{l_{s,1}+l_{s,2}=l_s}\left|\tfrac{\partial^{l_{s,1}}g_1}{\partial\phi^{l_{s,1}}}\right|\cdot\left|\tfrac{\partial^{l_{s,2}}g_1}{\partial\phi^{l_{s,2}}}\right|,$$

it follows that

$$\begin{aligned}\left|\tfrac{\partial^{l_{s,1}} g_1}{\partial \phi^{l_{s,1}}}\right| &\leq \|A\|\|B\||\cos\theta|\cdot|\cos(\phi+\tfrac{\pi}{2}\cdot l_{s,1})|+\|A\|^{-1}\|B\||\sin\theta|\cdot|\sin(\phi+\tfrac{\pi}{2}\cdot l_{s,1})| \\ &\leq \|A\|\cdot\|B\|\cdot|\cos\theta| \lesssim \|BA\|.\end{aligned}$$

Then we obtain

$$\left|\frac{\partial^{l_s}(g_1^2)}{\partial\phi^{l_s}}\right| \lesssim \|BA\|^2. \tag{B.35}$$

Similarly, we have

$$\left|\frac{\partial^{l_s}(g_2^2)}{\partial\phi^{l_s}}\right| \lesssim \|BA\|^2. \tag{B.36}$$

Combining (B.33) with (B.35), (B.36), we then have

$$\left|\frac{\partial^i \|BA\|}{\partial\phi^i}\right| \leq C(i)\cdot\max_{k\leq i}(\|BA\|^{2k}\cdot g^{\frac{1}{2}-k}) = C(i)\cdot\|BA\|. \tag{B.37}$$

Similarly, it holds that

$$\left|\frac{\partial^i \|BA\|}{\partial\theta^i}\right| \leq C(i)\cdot\max_{k\leq i}\left(g^{\frac{1}{2}-k}\cdot(\|A\|\|B\|)^{2k}|\cos\theta|^k\right) \leq C(i)\cdot|\theta-\frac{\pi}{2}|^{1-i} \tag{B.38}$$

and

$$\left|\frac{\partial^i \|BA\|}{\partial\|A\|^i}\right| \leq C(i)\cdot\|BA\|\cdot\|A\|^{-i}, \quad \left|\frac{\partial^i \|BA\|}{\partial\|B\|^i}\right| \leq C(i)\cdot\|BA\|\cdot\|B\|^{-i}. \tag{B.39}$$

Similar to (B.37)-(B.39), we have

$$\left|\tfrac{\partial^{i_1+\cdots+i_4}\|BA\|}{\partial\|A\|^{i_1}\cdot\partial\|B\|^{i_2}\cdot\partial^{i_3}\phi\cdot\partial^{i_4}\theta}\right| \leq C(i)\cdot\|BA\|\cdot\|A\|^{-i_1}\cdot\|B\|^{-i_2}\cdot|\theta-\tfrac{\pi}{2}|^{-i_4}.$$

Combining with (B.1), the first inequality in (B.30) and the fact

$$\frac{d^i\|BA\|}{dx^i} = \sum \frac{\partial^{i_1+\cdots+i_4}\|BA\|}{\partial\|A\|^{i_1}\cdot\partial\|B\|^{i_2}\cdot\partial^{i_3}\phi\cdot\partial^{i_4}\theta}\cdot\frac{\partial^{j_{1,1}}\|A\|}{\partial x^{j_{1,1}}}\cdots\frac{\partial^{j_{i_1,1}}\|A\|}{\partial x^{j_{i_1,1}}}\cdots\frac{\partial^{j_{1,4}}\theta}{\partial x^{j_{1,4}}}\cdots\frac{\partial^{j_{i_4,4}}\theta}{\partial x^{j_{i_4,4}}},$$

we prove the third inequality in (B.30). □

Proof of Lemma 2.1. For any $x \in I_{k+1}$, let $r_k(x) := r_{k,0}(x) < r_{k,1}(x) < \cdots < r_{k,s(x)}(x) := r_{k+1}(x)$ such that $T^{r_{k,j}(x)}x \in I_k$, $0 \leq j \leq s(x) \leq C(M)$. Consider $A^{r_{k,0}(x)+r_{k,1}(x)}(x) = A^{r_{k,1}(x)}(T^{r_{k,0}(x)}(x))\cdot A^{r_{k,0}(x)}(x)$.

Let $A^{r_{k,0}(x)}(x) := R_{\psi_k^-(x)}\cdot\Lambda_{k^-}(x)\cdot R_{\phi_k^-(x)}$ and $A^{r_{k,1}(x)}(T^{r_{k,0}(x)}x) := R_{\psi_k^+(x)}\cdot\Lambda_{k^+}(x)\cdot R_{\phi_k^+}(x)$. Then

$$A^{r_{k,0}(x)+r_{k,1}(x)}(x) = R_{\psi_k^+(x)}\cdot\Lambda_{k^+}(x)\cdot R_{\phi_k^++\psi_k^-}(x)\cdot\Lambda_{k^-}(x)\cdot R_{\phi_k^-(x)} := R_{\psi_{k+1,1}}(x)\cdot\Lambda_{k+1,1}(x)\cdot R_{\phi_{k+1,1}}(x).$$

Since $T^{r_{k,j}(x)}x \in I_k \backslash I_{k+1}$ for $j < s(x)$, it holds that $|\phi_k^+ + \psi_k^- - \frac{\pi}{2}| \geq d_{k+1}$. From (2.7) and Lemma B.1, we have

$$\left| \frac{d^i(\phi_{k+1,1} - \phi_k^-)}{dx^i} \right| \leq C(i) \cdot d_k^{-i} \cdot \|A^{r_{k,1}(x)}(x)\|^{-2}. \tag{B.40}$$

Similarly, it holds that

$$\left| \frac{d^i(\psi_{k+1,1} - \psi_k^+)}{dx^i} \right| \leq C(i) \cdot d_k^{-i} \cdot \|A^{r_{k,0}(x)}(x)\|^{-2}. \tag{B.41}$$

In the above, we regard $\phi_{k+1,1} - \phi_k^-$ and $\phi_k^+ + \psi_k^-$ as ϕ and θ in Lemma B.1, respectively. Moreover, for each $x \in I_k$, $|\theta(x) - \frac{\pi}{2}| = |\phi_k^+(x) + \psi_k^-(x) - \frac{\pi}{2}| \geq d_{k+1}$ from the definition of d_{k+1}. It implies that

$$\begin{aligned} &\left| \tfrac{d^i \phi_{k+1,1}}{dx^i} \right|, \quad \left| \tfrac{d^i \psi_{k+1,1}}{dx^i} \right| \leq C(i) \cdot d_k^{-i} + \left| \tfrac{d^i \phi_k^-}{dx^i} \right| + \left| \tfrac{d^i \psi_k^+}{dx^i} \right| \\ &\leq C(i) \cdot (d_k^{-i} + d_{k-1}^{-i}) \leq C(i) \cdot d_k^{-i}. \end{aligned}$$

The last inequality is obtained from $(1)_k$.

From Lemma B.1 we obtain $\left| \frac{d^i \|A^{r_{k,0}(x)+r_{k,1}(x)}(x)\|}{dx^i} \right| \leq C(i) \cdot \|A^{r_{k,0}(x)+r_{k,1}(x)}(x)\| \cdot d_k^{-i}$. Since $s(x) \leq C(M)$, it follows from no more than $C(M)$-applications of the above argument that $(1)_{k+1}$ and $(2)_{k+1}$ hold true.

Iterating (B.40) and (B.41) less than $C(M)$ times, we get

$$\left| \frac{d^i}{dx^i} (\phi_{A, r_{k+1}^+}(x) - \phi_{A, r_k^+}(x)) \right| \leq C(i) \cdot |\theta_k - \frac{\pi}{2}|^{-i} \cdot \|A^{r_k^+}\|^{-2}$$

and

$$\left| \frac{d^i}{dx^i} (\psi_{A, -r_{k+1}^-}(x) - \psi_{A, -r_k^-}(x)) \right| \leq C(i) \cdot |\theta_k - \frac{\pi}{2}|^{-i} \cdot \|A^{r_k^-}\|^{-2}.$$

Since $|\theta_k - \frac{\pi}{2}| \geq d_k$, we obtain (2.8).

□

Remark B.1. *In the proof of Lemma 2.1, it is not necessary that $s(x)$ is bounded by a constant. We make such an assumption only for the simplicity. And the condition that ω is of bound type is only used in constructing $C_k(x)$.*

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References

[1] A. Avila, Density of positive Lyapunov exponents for $SL(2,\mathbb{R})$ cocycles, *Journal of the American Mathematical Society* **24**(2011), 999-1014.

[2] A. Avila, Global theory of one-frequency Schrödinger operators I: Stratified analyticity of the Lyapunov exponent and the boundary of nonuniform hyperbolicity, arXiv:0905.3902v1(2009).

[3] A. Avila, Global theory of one-frequency Schrödinger operators II: Acriticality and finiteness of phase transitions for typical potentials, preprint (2013).

[4] A. Avila and S. Jitomirskaya, The Ten Martini Problem, *Annals of Mathematics* **170**(2009), 303-342.

[5] A. Avila, S. Jitomirskaya and C. Sadel, Complex one-frequency cocycles, *J. Eur. Math. Soc.* **16**(9)(2014), 1915-1935.

[6] A. Avila and R. Krikorian, Quasiperiodic $SL(2,\mathbb{R})$ cocycles which are not homotopic to the identity, arXiv:1310.0703v1(2010).

[7] A. Avila and M. Viana, Extremal Lyapunov exponents: an Invariance Principle and applications, *Inventiones Math.* **181** (2010), 115-189.

[8] M. Benedicks and L. Carleson, The dynamics of the Hénon map, *Annals of Mathematics* **133**(1991), 73-169.

[9] K. Bjerklöv, Explicit examples of arbitrarily large analytic ergodic potentials with zero Lyapunov exponent, *Geom. Funct. Anal.* **16**(6)(2006), 1183-1200.

[10] K. Bjerklöv, The dynamics of a class of quasi-periodic Schrödinger cocycles, *Ann. Henri Poincaré Online*, 2014.

[11] C. Bocher-Neto and M. Viana, Continuity of Lyapunov exponents for random $2D$ matrices, arXiv:1012.0872v1(2010).

[12] J. Bochi, Genericity of zero Lyapunov exponents, *Ergodic Theory and Dynamical Systems* **22**(6)(2002), 1667-1696.

[13] J. Bochi and B. Fayad, Dichotomies between uniform hyperbolicity and zero Lyapunov exponents for $SL(2,\mathbb{R})$ cocycles, *Bull. Braz. Math. Soc.*, New Series **37**(3)(2006), 307-349.

[14] J. Bochi and M. Viana, The Lyapunov exponents of generic volume perserving and symplectic maps, *Ann. of Math.* **161**(2005), 1-63.

[15] C. Bonatti, X. Gómez-Mont and M. Viana, Généricité d'exposants de Lyapunov non-nuls pour des produits déterministes de matrices, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **20**(2003), 579-624.

[16] J. Bourgain, Green's function estimates for lattice Schrödinger operators and applications, *Annals of Mathematics Studies*, 158. Princeton University Press, Princeton, NJ, 2005.

[17] J. Bourgain, Positivity and continuity of the Lyapunov exponent for shifts on $\mathbb{T}^d$ with arbitrary frequency vector and real analytic potential, *J. Anal. Math.* **96**(2005), 313-355.

[18] J. Bourgain and M. Goldstein, On nonperturbative localization with quasi-periodic potential, *Ann. of Math.* **152** (2000), 835-879.

[19] J. Bourgain, M. Goldstein and W. Schlag, Anderson localization for Schrödinger operators on $\mathbb{Z}$ with potentials given by skew-shift, *Commun. Math. Physics* **220**(3)(2001), 583-621.

[20] J. Bourgain and S. Jitomirskaya, Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential, *Journal of Statistical Physics* **108**(5-6)(2002), 1203-1218.

[21] J. Bourgain and W. Schlag, Anderson localization for Schrödinger operators on $\mathbb{Z}$ with strongly mixing potentials, *Commun. Math. Phys.* **215**(2000), 143-175.

[22] P. Duarte and S.Klein, Continuity of the Lyapunov exponents for quasiperiodic cocycles, *Comm. Math. Phys.* **332**(3)(2014), 1113-1166.

[23] L.H. Eliasson, Discrete one-dimensional quasi-periodic Schrödinger operators with pure point spectrum, *Acta Math.* **179**(2) (1997), 153-196.

[24] J. Fröhlich, T. Spencer and P. Wittwer, Localization for a class of one-dimensional quasi-periodic Schrödinger operators, *Comm. Math. Phys.* **132**(1990), 5-25.

[25] A. Furman, On the multiplicative ergodic theorem for the uniquely ergodic systems, *Ann. Inst. Henri Poincaré* **33**(1997), 797-815.

[26] H. Furstenberg, Noncommuting random products, *Trans. Amer. Math. Soc.* **108** (1963), 377-428.

[27] H. Furstenberg and Y. Kifer, Random matrix products and measures in projective spaces, *Israel J. Math* **10**(1983), 12-32.

[28] M. Goldstein and W. Schlag, Hölder continuity of the integrated density of states for quasi-periodic Schrödinger equations and averages of shifts of subharmonic functions, *Annals of Math.* **154**(2001), 155-203.

[29] H. Hennion, Loi des grands nombres et perturbations pour des produits réductibles de matrices aléatoires indépendantes, *Z. Wahrsch. Verw. Gebiete* **67**(1984), 265-278.

[30] M. Herman, Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2, *Comment. Math. Helv.* **58**(1983), 453-502.

[31] K. Ishii, Localization of eigenstates and transport phenomena in one-dimensional disordered systems, *Suppl. Prog. Theoret. Phys.* **53** (1973), 77-138.

[32] S. Jitomirskaya, D. Koslover and M. Schulteis, Continuity of the Lyapunov exponent for general analytic quasiperiodic cocycles, *Ergodic theory and dynamical systems* **29**(2009), 1881-1905.

[33] S. Jitomirskaya and C. Marx, Continuity of the Lyapunov exponent for analytic quasi-periodic cocycles with singularities, *Journal of Fixed Point Theory and Applications* **10**(2011), 129-146.

[34] S. Jitomirskaya and C. Marx, Analytic quasi-perodic cocycles with singularities and the Lyapunov exponent of extended Harper's model, *Comm. Math. Phys.* **316**(1)(2012), 237-267.

[35] S. Jitomirskaya and C. Marx, Analytic quasi-periodic Schrödinger operators and rational frequency approximants, *Geom. Funct. Anal.* **22**(5)(2012), 1407-1443.

[36] S. Jitomirskaya and R. Mavi, Continuity of the measure of the spectrum for quasiperiodic Schrödinger operators with rough potentials, *Commun. Math. Phys.* **325**(2) (2014), 585-601.

[37] S. Jitomirskaya and R. Mavi, Dynamical bounds for quasiperiodic Schrödinger operators with rough potentials, arXiv:1412.0309(2014).

[38] S. Kotani, Ljapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators, *Stochastic Analysis*(Katata/Kyoto, 1982), (North-Holland Math. Library 32, North-Holland, Amsterdam)(1984), 225-247.

[39] S. Klein, Anderson localization for the discrete one-dimensional quasi-periodic Schrödinger operator with potential defined by a $\mathcal{C}^{\infty}$-class function, *Journal of Functional Analysis* **218**(2)(2005), 255-292.

[40] O. Knill, The upper Lyapunov exponent of $SL(2,\mathbb{R})$ cocycles: Discontinuity and the problem of positivity, Lecture notes in Math. **1486**, Lyapunov exponents (Oberwolfach, 1990) 86-97, 1991.

[41] S. Kotani, Jacobi matrices with random potentials taking finitely many values, *Rev. Math. Phys.* **1**(1989), 129-133.

[42] R. Ma$\tilde{n}\acute{e}$, Oseledec's theorem from the generic viewpoint, In Proceedings of the ICM (Warsaw, 1983), 1269-1276, PWN, Warsaw, 1984.

[43] R. Ma$\tilde{n}\acute{e}$, The Lyapunov exponents of generic area preserving diffeomorphisms, In International Conference on Dynamical Systems (Montevideo, 1995), 110-119, Pitman Res. Notes Math. **362**, Longman, Harlow, 1996.

[44] E. C. Malheiro and M. Viana, Lyapunov exponents of linear cocycles over Markov shifts, preprint (2014).

[45] L. A. Pastur, Spectral properties of disordered systems in one-body approximation, *Comm. Math. Phys.* **75** (1980), 179-196.

[46] Ya. G. Sinai, Anderson localization for one-dimensional difference Schrödinger operator with quasiperiodic potential. *J. Statist. Phys.* **46**(1987), 861-909.

[47] E. Sorets and T. Spencer, Positive Lyapunov exponents for Schrödinger operators with quasi-periodic potentials, *Commun. Math. Phys.***142**(1991), 543-566.

[48] J. Thouvenot, An example of discontinuity in the computation of the Lyapunov exponents, *Proc. Stekolov Inst. Math.* **216**(1997), 366-369.

[49] M. Viana, Almost all cocycles over any hyperbolic system have nonvanishing Lyapunov exponents, *Annals of Mathematics* **167**(2008), 643-680.

[50] M. Viana and J. Yang, Physical measures and absolute continuity for one-dimensional center direction, *Annales Inst. H. Poincaré-Analyse Non-Linéaire* **30**(2013), 845-877.

[51] Y. Wang and J. You, Examples of discontinuity of Lyapunov exponent in smooth quasi-periodic cocycles, *Duke Math. J.* **162**(2013), 2363-2412.

[52] Y. Wang and Z. Zhang, Uniform Positivity and Continuity of Lyapunov Exponents for a class of C^2 Quasiperiodic Schrödinger Cocycles, preprint(2013).

[53] J. You and S. Zhang, Hölder continuity of the Lyapunov exponent for analytic quasiperiodic Schröinger cocycle with weak Liouville frequency. Ergodic Theory Dynam. Systems **34** (2014), 1395-1408.

[54] L. Young, Lyapunov exponents for some quasi-periodic cocycles, *Ergodic Theory and Dynamical Systems*, **17**(1997), 483-504.

[55] Z. Zhang, Positive Lyapunov exponents for quasiperiodic Szegö cocycles, *Nonlinearity* **25**(2012), 1771-1797.