On dividends in the Phase–Type dual risk model

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Abstract

The dual risk model assumes that the surplus of a company decreases at a constant rate over time, and grows by means of upward jumps which occur at random times and at random sizes. In the present work, we study the dual risk renewal model when the waiting times are Phase–Type distributed. Using the roots of the fundamental and the generalized Lundberg's equations, we get expressions for the ruin probability and the Laplace transform of the time of ruin for an arbitrary single gain distribution. Furthermore, we calculate the expected discounted dividends when the individual common gains follow a Phase–Type distribution.

Keywords: Dual risk model; Phase–Type distribution; generalized Lundberg's equation; ruin probability; time of ruin; expected discounted dividends.

1 Introduction

We consider the dual risk model where the surplus or equity of the company is commonly described by the equation

$$U(t) = u - ct + S(t),\ t \geq 0,\ u \geq 0, \quad \text{where} \quad S(t) = \sum_{i=0}^{N(t)} X_i$$

is the aggregate gain process.

Here, $u \geq 0$ is the initial surplus, c is the constant rate at which the cost are paid, $\{X_i\}_{i=0}^{\infty}$ denote the sequence of random gains and $N(t)$ is the random number of gains occurring before time t. The model is called *dual* as opposed to the well known Cramér–Lundberg risk model, which consists of constant premiums instead of constant costs, and a sequence of claims rather than a sequence of gains.

We denote by W_i the waiting time between gains X_{i-1} and X_i. We assume that the sequences $\{X_i\}_{i=0}^{\infty}$ and $\{W_i\}_{i=0}^{\infty}$ are i.i.d. and independent from one another. Let $P(x)$ denote the cumulative distribution function of the sequence of gains $\{X_i\}_{i=0}^{\infty}$, $p(x)$ the density and $\hat{p}(s)$ its Laplace transform. We assume the existence of $\mu_1 = E[X_1]$, and the *net profit condition*, i.e. $cE(W_1) < E(X_1) = \mu_1$. These condition means that on average gains are greater than expenses, per unit time.

The dual risk model has an increasing interest in ruin theory since recent times. There are many possible interpretations for the model. We can look at the surplus as the amount of capital of a business engaged in research and development, where gains are random, at random instants, and costs are certain. More precisely, the company pays expenses which occur continuously along time for the research activity, gets occasional revenues according to an Erlang(n) distribution and of size driven by distribution $P(\cdot)$. Revenues can be interpreted as values of future gains from an invention or discovery, the decrease of surplus can represent costs of production, payments to employees, maintenance of equipment, etc.

Among pioneer works on the subject we can cite Cramér (1955), Takács (1967), Seal (1969), Bühlman (1970) and Gerber (1979). Recent works include those by Avanzi *et al.* (2007), Albrecher *et al.* (2008), Avanzi and Gerber (2008), Bayraktar and Egami (2008), Cheung and Drekic (2008), Gerber and Smith (2008), Song and Zhang (2008), Yang and Zhu (2008), Avanzi (2009), Ng (2009), Ng (2010), Cheung (2012), Afonso *et al.* (2013), Rodríguez-Martínez *et al.* (2013) and Sendova (2014).

Many published works, particularly those concerning the dual model, deal with the compound Poisson, or Erlang(1), dual model and the computation of discounted dividends. We particularly reference the work by Avanzi *et al.* (2007) that explains well where applications of the dual model are said to be appropriate. On this matter Bayraktar and Egami (2008) used it to model capital investments. On dividend and optimal strategies we underline the works by Avanzi *et al.* (2007), Avanzi and Gerber (2008) and Avanzi (2009). The latter is an excellent review paper, see also references therein. Among other works considering more general distributions, we can mention Rodríguez-Martínez *et al.* (2013) and Sendova (2014), who studied ruin probabilities and dividend problems for a dual risk model with Erlang and generalized Erlang distributed waiting times, respectively. We also underline the work by Afonso *et al.* (2013) who, among other problems, give a different view of the dividend problem calculation, by taking advantage of the relationship between the Cramér–Lundberg and the dual models.

As we said, the works particularly focusing the dual model and the discounted dividends problem mostly assume that the waiting times follow an exponential, Erlang or generalized Erlang distribution. In this paper, we study the dual risk model when the waiting times W_i are Phase–Type distributed, generalizing the work of Rodríguez-Martínez *et al.* (2013) and extending the results presented in Bergel and Egídio dos Reis (2014), the latter of which was based on the Cramér–Lundberg risk model.

In the next Section 2 we briefly introduce the Phase – Type distribution and the notation we use further in the paper. In Section 3 we study the fundamental and the generalized Lundberg's equations and the role of its solutions. In Section 4 we get expressions for the ruin probability and the Laplace transform of the time of ruin for an arbitrary individual gain distribution. We give some illustration examples. Finally, in section 5 we work on the problem of calculating the expected discounted dividends when the individual common gains X_i follow a Phase–Type distribution. We present some numerical illustrations.

2 The Phase – Type distribution

Phase–type distributions are the computational vehicle of much of modern applied probability. Typically, if a problem can be solved explicitly when the relevant distributions are exponentials, then the problem may admit an algorithmic solution involving a reasonable degree of computational effort, if one allows for the more general assumption of phase–type structure, and not in other cases. A proper knowledge of phase–type distributions seems therefore a must for anyone working in an applied probability area like risk theory.

We say that a distribution K on $(0, \infty)$ is Phase–Type(n) if K is the distribution of the lifetime of a terminating continuous time Markov process $\{J(t)\}_{t\geq 0}$ with finitely many states and time homogeneous transition rates. More precisely, we define a terminating Markov process $\{J(t)\}_{t\geq 0}$ with state space $E = \{1, 2, \ldots, n\}$ and intensity matrix $\mathbf{B}$ $(n \times n)$ as the restriction to E of a Markov process $\{\bar{J}(t)\}_{0\leq t<\infty}$ on $E_0 = E \cup \{0\}$ where 0 is some extra state which is absorbing, that is, $Pr(\bar{J}(t) = 0|\bar{J}(0) = i) = 1$ for all $i \in E$ and where all states $i \in E$ are transient. This implies in particular that the intensity matrix for $\{\bar{J}(t)\}$ can be written in block–partitioned form as

$$\left(\begin{array}{c|c} \mathbf{B} & \mathbf{b}^{\mathsf{T}} \\ \hline \mathbf{0} & 0 \end{array}\right) \tag{2.1}$$

The $1 \times n$ vector $\mathbf{b} = (b_1, \ldots, b_n)$ is the exit rate vector, i. e., the i–th component b_i gives the intensity in state i for leaving E and going to the absorbing state 0.

Note that since (2.1) is the intensity matrix of a non–terminating Markov process, the rows sums to zero which in matrix notation can be written as $\mathbf{b}^{\mathsf{T}} + \mathbf{B}\mathbf{1}^{\mathsf{T}} = \mathbf{0}$ where $\mathbf{1} = (1, 1, \ldots, 1)$ is the column vector with all components equal to one. In particular we have

$$\mathbf{b}^{\mathsf{T}} = -\mathbf{B}\mathbf{1}^{\mathsf{T}}$$

The intensity matrix $\mathbf{B}$ is denoted by $\mathbf{B} = (b_{i,j})_{i,j=1}^{n}$. This matrix satisfies the conditions: $b_{i,i} < 0$, $b_{i,j} \geq 0$ for $i \neq j$, and $\sum_{j=1}^{n} b_{i,j} \leq 0$ for $i = 1, \ldots, n$.

The vector of entry probabilities is given by $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i \geq 0$ for $i = 1, \ldots, n$, and $\sum_{i=1}^{n} \alpha_i = 1$, so $Pr(\bar{J}(0) = i) = \alpha_i$.

We list the most important properties of K.

$$\begin{array}{rrcl} \text{Density} & k(t) & = & \boldsymbol{\alpha} e^{\mathbf{B}t}\mathbf{b}^{\mathsf{T}}, \quad t \geq 0, \\ \text{C.D.F} & K(t) & = & 1 - \boldsymbol{\alpha} e^{\mathbf{B}t}\mathbf{1}^{\mathsf{T}}, \quad t \geq 0, \\ \text{Laplace T.} & \hat{k}(s) & = & \boldsymbol{\alpha}(s\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}^{\mathsf{T}}, \\ \text{Mean} & E[W_1] & = & -\boldsymbol{\alpha}\mathbf{B}^{-1}\mathbf{1}^{\mathsf{T}}, \\ & k^{(j)}(0) & = & \boldsymbol{\alpha}\mathbf{B}^{j}\mathbf{b}^{\mathsf{T}}, \quad j \geq 0, \end{array} \tag{2.2}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix.

From this point we denote by $K(t)$ the Phase–Type(n) distribution of the waiting times W_i, and we call our model the Phase–Type(n) dual risk model. Before we continue to the next section, it is important to notice that can write the corresponding *net profit condition* $cE(W_1) < E(X_1) = \mu_1$ in the following way

$$-c\boldsymbol{\alpha}\mathbf{B}^{-1}\mathbf{1}^{\mathsf{T}} < \mu_1. \tag{2.3}$$

3 The Lundberg's equations

In this section we study the Lundberg's equations

$$E\left[e^{-s(X_1-cW_1)}\right] = 1, \qquad E\left[e^{-\delta W_1}e^{-s(X_1-cW_1)}\right] = 1,\ s \in \mathbb{C},\ \delta > 0, \tag{3.1}$$

(see e.g. Landriault and Willmot (2008) or Rodríguez-Martínez *et al.* (2013).)
As we can see from the works of Gerber and Shiu (2005) and Ren (2007), these equations can be expressed in the form

$$\hat{k}(-cs)\hat{p}(s) = 1, \qquad \hat{k}(\delta - cs)\hat{p}(s) = 1, \quad \text{respectively,} \tag{3.2}$$

A very important result we will use in the rest of our paper is the following

Remark 3.1. *For a Phase–Type(n) dual risk model the Lundberg's equations have exactly n roots with positive real parts, see Albrecher and Boxma (2005). Denote them by $\rho_1, \cdots, \rho_n$.*

The roots of the Lundberg's equations play an important role in the calculation of many quantities that are fundamental in risk and ruin theory. Namely, the ultimate and finite time ruin probabilities, the Laplace transform of the ruin time, the expected discounted future dividends, among others. All those calculations depend on the nature of the roots of the Lundberg's equation, particularly its roots with positive real parts. A study on the multiplicities of these roots can be found in Bergel and Egídio dos Reis (2014).

Notice that in order to solve equations (3.2) numerically we need to determine a rational expression for the Laplace transform $\hat{k}(\delta - cs)$. Since

$$\hat{k}(\delta - cs) = \boldsymbol{\alpha}((\delta - cs)\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}^{\mathsf{T}},$$

the main difficulty is to compute the inverse matrix $((\delta - cs)\mathbf{I} - \mathbf{B})^{-1}$. Before we go further we give some definitions from linear algebra.

Definition 3.1. *Let $\mathbf{A} = (a_{i,j})_{i,j=1}^n$ be a $n \times n$ matrix.*
Define, for $1 \le i_1 < i_2 < \ldots < i_k \le n$

$$\mathbf{M}_{i_1,i_2\ldots i_k} = \begin{pmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \cdots & a_{i_1,i_k} \\ a_{i_2,i_1} & a_{i_2,i_2} & \cdots & a_{i_2,i_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_k,i_1} & a_{i_k,i_2} & \cdots & a_{i_k,i_k} \end{pmatrix},\ 1 \le k \le n,$$

then

$$tr_k(\mathbf{A}) = \sum_{1\le i_1<i_2<\ldots<i_k\le n} det(\mathbf{M}_{i_1,i_2\ldots i_k}).$$

We call $tr_k(\mathbf{A})$ the k-generalized trace of the matrix A. In particular $tr_1(\mathbf{A}) = trace(\mathbf{A})$ and $tr_n(\mathbf{A}) = det(\mathbf{A})$.

Using this definition enables us to express the characteristic polynomial of the matrix $\mathbf{B}$ as:

$$det(s\mathbf{I} - \mathbf{B}) = \sum_{i=0}^{n} (-1)^{n-i} tr_{n-i}(\mathbf{B}) s^i.$$

Moreover, the inverse matrix $(s\mathbf{I} - \mathbf{B})^{-1}$ can be obtained as follows:

Theorem 3.1. *The inverse matrix $(s\mathbf{I} - \mathbf{B})^{-1}$ has the expression*

$$(s\mathbf{I} - \mathbf{B})^{-1} = \frac{N(s, \mathbf{B})}{det(s\mathbf{I} - \mathbf{B})},$$

where the matrix $N(s, \mathbf{B})$ takes the form

$$N(s, \mathbf{B}) = \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1-i} (-1)^j tr_j(\mathbf{B}) \mathbf{B}^{n-1-i-j} \right) s^i.$$

Proof. See appendix. □

From the Theorem (3.1) we get the rational expression for the Lundberg's equations (3.2). The generalized Lundberg's equation for the Phase–Type(n) dual risk model becomes

$$\frac{1}{\hat{k}(\delta - cs)} = \frac{det((\delta - cs)\mathbf{I} - \mathbf{B})}{\boldsymbol{\alpha} N(\delta - cs, \mathbf{B}) \mathbf{b}^\top} = \hat{p}(s), \tag{3.3}$$

and we obtain the corresponding fundamental Lundberg's equation by setting $\delta = 0$ in equation (3.3)

$$\frac{1}{\hat{k}(-cs)} = \frac{det((-cs)\mathbf{I} - \mathbf{B})}{\boldsymbol{\alpha} N(-cs, \mathbf{B}) \mathbf{b}^\top} = \hat{p}(s). \tag{3.4}$$

Although the new expressions for the Lundberg's equations found in (3.3) and (3.4) are already in rational form, they are not adequate for our purposes. The reason for this will be clear in the following section when we will calculate ruin probabilities using integro–differential equations.
Therefore, we have rewritten the generalized Lundberg's equation in the form

$$B_\delta(-s) = q_\delta(-s)\hat{p}(s), \qquad s \in \mathbb{C}, \tag{3.5}$$

where B and q are polynomials in s given by

$$B_\delta(s) = \frac{det(\mathbf{B} - \delta\mathbf{I} - cs\mathbf{I})}{det(\mathbf{B})} = \sum_{i=0}^{n} B_i \left(s + \frac{\delta}{c} \right)^i,$$

and

$$q_\delta(s) = \sum_{j=0}^{n-1} \tilde{B}_j \left(s + \frac{\delta}{c}\right)^j.$$

The equivalent fundamental Lundberg's equation (for $\delta = 0$) is

$$B(-s) = q(-s)\hat{p}(s), \qquad s \in \mathbb{C}. \tag{3.6}$$

The coefficients B_i and $\tilde{B}_j$ of the polynomials B and q, respectively, are given by the following expressions

$$B_i = (-c)^i \frac{tr_{n-i}(\mathbf{B})}{det(\mathbf{B})}, \qquad \tilde{B}_j = \sum_{i=j+1}^{n} B_i \left(\frac{1}{c}\right)^{i-j} k^{(i-1-j)}(0).$$

Theorem 3.2. *The expressions* (3.3) *and* (3.5) *are equivalent forms of the generalized Lundberg's equation. Analogously, expressions* (3.4) *and* (3.6) *both represent the fundamental Lundberg's equation.*

Proof. The proof is simple and follows by rearranging and comparing the coefficients of the above mentioned versions of the Lundberg's equations. Namely, it is easy to prove that

$$\frac{det((\delta - cs)\mathbf{I} - \mathbf{B})}{\boldsymbol{\alpha} N(\delta - cs, \mathbf{B})\mathbf{b}^\mathsf{T}} = \frac{B_\delta(-s)}{q_\delta(-s)}.$$

□

4 The time of ruin and its Laplace transform

In this section we study the ruin probability and the Laplace transform of the time of ruin in the Phase–Type(n) dual risk model. Let

$$\tau_u = \begin{cases} \min\{t > 0 : U(t) = 0 \mid U(0) = u\} \\ \infty \;\text{ if }\; U(t) \geq 0 \quad \forall t \geq 0 \end{cases}$$

be the time to ruin, $\psi(u) = P(\tau_u < \infty)$ be the ultimate ruin probability and

$$\psi(u, \delta) = E[e^{-\delta \tau_u} I(\tau_u < \infty) \mid U(0) = u]$$

be the Laplace transform of the time to ruin, where $\delta > 0$ and $I(.)$ is the indicator function. This Laplace transform can be interpreted as the expected value of one monetary unit received at the time of ruin discounted at the constant force of interest δ.

4.1 The ruin probability

The ruin probability in the dual risk model with exponential waiting times $(k(t) = \lambda e^{-\lambda t})$, satisfies the following renewal equation

$$\psi(u) = e^{-\lambda t_0} + \int_0^{t_0} \lambda e^{-\lambda t} \int_0^{\infty} p(x)\psi(u - ct + x)dx\, dt,$$

where $t_0 = \frac{u}{c}$ is the time of ruin if no gain arrives. See e.g. Afonso *et al.*(2013).

Differentiation with respect to u gives an integro–differential equation for $\psi(u)$

$$\psi(u) + \left(\frac{c}{\lambda}\right) \frac{d}{du}\psi(u) = \int_0^\infty p(x)\psi(u+x)dx.$$

We can write this equation as

$$\left(\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)\psi(u) = \int_0^\infty p(x)\psi(u+x)dx,$$

where $\mathcal{I}$ is the identity operator and $\mathcal{D}$ is the differentiation operator, $\mathcal{D} = d/du$.

In the exponential case, Gerber (1979) found that $\psi(u) = e^{-\rho u}$, where ρ is the unique positive root of the Fundamental Lundberg's equation for $n = 1$.

For the Phase–Type(n) dual risk model the renewal equation becomes

$$\psi(u) = 1 - K\left(\frac{u}{c}\right) + \int_0^{\frac{u}{c}} k(t)\int_0^\infty p(x)\psi(u-ct+x)dx\,dt, \tag{4.1}$$

and the analogous integro–differential equation is given in the following theorem:

Theorem 4.1. *The ruin probability $\psi(u)$ satisfies the following integro–differential equation*

$$B(\mathcal{D})\psi(u) = q(\mathcal{D})W(u), \tag{4.2}$$

where $W(u) = \int_0^\infty p(x)\psi(u+x)dx$ and B, q are the same polynomials described before for the fundamental Lundberg's equation (3.6)*. The operator $\mathcal{D}$ is the differentiation with respect to u, as before.*

The boundary conditions of (4.2) *are given by*

$$\begin{aligned} \psi(0) &= 1, \\ \frac{d^j}{du^j}\psi(u)\bigg|_{u=0} &= -\frac{1}{c^j}k^{(j-1)}(0) + \sum_{i=0}^{j-1}\frac{1}{c^{i+1}}k^{(i)}(0)W^{(j-1-i)}(0), \\ & \quad j = 1, \ldots, n-1. \end{aligned} \tag{4.3}$$

Proof. See appendix. □

For the Phase–Type(n) dual risk model, we found that the ruin probability can be written as follows

Theorem 4.2. *The ultimate ruin probability $\psi(u)$ can be written in the general form*

$$\psi(u) = \sum_{i=1}^{L}\sum_{j=1}^{\beta_i} a_{ij}u^{j-1}e^{-\rho_i u},$$

where $\rho_1, \ldots, \rho_L$ are the only roots of the Fundamental Lundberg's equation which have positive real parts, and ρ_i has multiplicity β_i, with $\sum_{i=1}^{L}\beta_i = n$.

Proof. It is very simple to verify that if ρ is a single root of the fundamental Lundberg's equation $B(-s) = q(-s)\hat{p}(s)$ then the function $f(u) = e^{-\rho u}$ satisfies the integro–differential equation $B(\mathcal{D})f(u) = q(\mathcal{D})W_f(u)$, where $W_f(u) = \int_0^\infty p(x)f(u+x)dx$.

Moreover, it is also straightforward to show that if ρ is a root of the fundamental Lundberg's equation with multiplicity $j \geq 1$ then the function $f(u) = u^{j-1}e^{-\rho u}$ is solution of the same integro–differential equation.

Since the functions $u^{j-1}e^{-\rho_i u}$, $i = 1, \ldots, L; j = 1, \ldots, \beta_i$ are linearly independent, any solution of $B(\mathcal{D})f(u) = q(\mathcal{D})W_f(u)$ can be expressed in the following way

$$f(u) = \sum_{i=1}^{L}\sum_{j=1}^{\beta_i} b_{ij}u^{\,j-1}e^{-\rho_i u}, \text{ for some constants } b_{ij}. \tag{4.4}$$

Then the ruin probability is

$$\psi(u) = \sum_{i=1}^{L}\sum_{j=1}^{\beta_i} a_{ij}u^{\,j-1}e^{-\rho_i u}.$$

$\square$

Using the boundary conditions (4.3) we can determine the constants a_{ij} that correspond to $\psi(u)$.

Example: For $n = 2$, the ruin probability in the Phase–Type(2) model has the expression

$$\begin{aligned}\psi(u) \quad &= \quad \frac{\rho_2 + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^\mathsf{T}(\hat{p}(\rho_2) - 1)}{\rho_2 - \rho_1 + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^\mathsf{T}(\hat{p}(\rho_2) - \hat{p}(\rho_1))}e^{-\rho_1 u} \\ &\quad - \frac{\rho_1 + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^\mathsf{T}(\hat{p}(\rho_1) - 1)}{\rho_2 - \rho_1 + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^\mathsf{T}(\hat{p}(\rho_2) - \hat{p}(\rho_1))}e^{-\rho_2 u},\end{aligned}$$

where $\rho_1, \rho_2 > 0$ are real and solutions of $B(-s) = q(-s)\hat{p}(s)$.

4.2 The Laplace transform of the time of ruin

The Laplace transform of the time of ruin $\psi(u,\delta) = E(e^{-\delta T_u}\mathbb{I}(T_u < \infty))$ for the Phase–Type(n) dual risk model satisfies the renewal equation

$$\psi(u,\delta) = \left(1 - K\left(\frac{u}{c}\right)\right)e^{-\delta\left(\frac{u}{c}\right)} + \frac{1}{c}\int_0^u k\left(\frac{u-s}{c}\right)e^{-\delta\left(\frac{u-s}{c}\right)}\int_0^\infty p(x)\psi(s+x,\delta)dx\, ds.$$

We can obtain a formula for the Laplace transform of the time of ruin $\psi(u,\delta)$ following an analogous approach to the previous section

Theorem 4.3. *The Laplace transform of the time of ruin $\psi(u,\delta)$ satisfies the integro–differential equation*

$$B_\delta(\mathcal{D})\psi(u,\delta) = q_\delta(\mathcal{D})W_\delta(u), \tag{4.5}$$

where $W_\delta(u) = \int_0^\infty p(x)\psi(u+x,\delta)dx$ and B_δ, q_δ are the same polynomials described before for the generalized Lundberg's equation (3.5).
The boundary conditions of (4.5) *are given by*

$$\begin{aligned}
\psi(0,\delta) &= 1, \\
\frac{d^i}{du^i}\psi(u,\delta)\bigg|_{u=0} &= \left(-\frac{\delta}{c}\right)^i - \sum_{j=0}^{i-1}\frac{1}{c^i}\binom{i}{j}(-\delta)^j k^{(i-1-j)}(0) \\
&+ \sum_{j=0}^{i-1}\left(\sum_{l=0}^{i-1-j}\frac{1}{c^{i-j}}\binom{i-1-j}{l}(-\delta)^l k^{(i-1-j-l)}(0)\right)W_\delta^{(j)}(0), \\
& i = 1,\ldots,n-1.
\end{aligned} \tag{4.6}$$

For the Phase–Type(n) dual risk model, we have found that the Laplace transform of the time of ruin can be written as follows

$$\psi(u,\delta) = \sum_{i=1}^{L}\sum_{j=1}^{\beta_i} a_{ij,\delta}u^{\,j-1}e^{-\rho_i u}, \tag{4.7}$$

where $\rho_1,\ldots,\rho_L$ are the only roots of the generalized Lundberg's equation which have positive real parts, and ρ_i has multiplicity β_i, with $\sum_{i=1}^{L}\beta_i = n$.

Using the boundary conditions (4.6) we can determine the constants $a_{ij,\delta}$ that correspond to $\psi(u,\delta)$.

Example:
For $n=2$, the Laplace transform of the time of ruin in the Phase–Type(2) model has the expression

$$\begin{aligned}
\psi(u,\delta) &= \frac{\rho_2 - \frac{\delta}{c} + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^\mathsf{T}(\hat{p}(\rho_2)-1)}{\rho_2-\rho_1+\frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^\mathsf{T}(\hat{p}(\rho_2)-\hat{p}(\rho_1))}e^{-\rho_1 u} \\
&\quad - \frac{\rho_1 - \frac{\delta}{c} + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^\mathsf{T}(\hat{p}(\rho_1)-1)}{\rho_2-\rho_1+\frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^\mathsf{T}(\hat{p}(\rho_2)-\hat{p}(\rho_1))}e^{-\rho_2 u},
\end{aligned}$$

where $\rho_1,\rho_2>0$ are real and solutions of $B_\delta(-s) = q_\delta(-s)\hat{p}(s)$.

5 Expected Discounted Dividends

On this section we consider a barrier strategy for dividend calculation in terms of a dividend barrier b. Any time the surplus upcrosses b the excess is paid as dividend. From that payment instant the process restarts from level b and that repeats whenever it occurs in the future, until ruin.

Let $\{D_i\}_{i=1}^{\infty}$ be the sequence of the dividend payments and let $D(u,b)$ be the aggregate discounted dividends, at force of interest δ and from initial surplus u.

We denote by $V(u,b) = E[D(u,b)]$, the expected value of $D(u,b)$.

Note that

$$V(u,b) = E[u-b+D(b,b)] = u-b+V(b,b), \quad u \geq b. \tag{5.1}$$

The expected discounted dividends $V(u,b)$ satisfy the renewal equation

$$\begin{aligned} V(u,b) \quad = \quad & \int_0^{\frac{u}{c}} k(t)e^{-\delta t}\left[\int_0^{b-u+ct} V(u-ct+y,b)p(y)dy + \right. \\ & \left. \int_{b-u+ct}^{\infty} \widetilde{V}(u-ct+y,b)p(y)dy\right] dt, \quad u < b, \end{aligned}$$

with

$$\widetilde{V}(x,b) = E[D(x,b)] = E[x-b+D(b,b)] = x-b+V(b,b), \quad x \geq b.$$

Differentiating the renewal equation with respect to u we can obtain an integro–differential equation for $V(u,b)$

Theorem 5.1. *The expected discounted dividends $V(u,b)$ satisfy the integro–differential equation*

$$B_\delta(\mathcal{D})V(u,b) = q_\delta(\mathcal{D})W(u,b), \quad u < b, \tag{5.2}$$

where

$$W(u,b) = \int_u^b V(x,b)p(x-u)dx + \int_b^{\infty} \widetilde{V}(x,b)p(x-u)dx,$$

is the integral term.

The boundary conditions of (5.2) *are given by*

$$\begin{aligned} V(0,b) \quad &= \quad 0, \\ \frac{d^i}{du^i}V(u,b)\bigg|_{u=0} \quad &= \quad \sum_{j=0}^{i-1}\left(\sum_{l=0}^{i-1-j}\frac{1}{c^{i-j}}\binom{i-1-j}{l}(-\delta)^l k^{(i-1-j-l)}(0)\right)W^{(j)}(0,b), \\ & \quad\quad i = 1,\ldots,n-1. \end{aligned} \tag{5.3}$$

Because of the additional information of a barrier level b in $V(u,b)$, we can not solve the equation

$$B_\delta(\mathcal{D})V(u,b) = q_\delta(\mathcal{D})W(u,b),$$

to find an expression for $V(u,b)$ in the same way we did for the ruin probability $\psi(u)$ before, where we did not need to specify a particular density function $p(x)$ for the gain amounts.

Instead, we assume that the gain amounts follow a Phase–Type(m) distribution and we use the annihilator method to find $V(u,b)$.

Following the notation in 2, consider the case when the gains X_i follow a Phase-Type(m) distribution $P(x)$ with representation $(\boldsymbol{\alpha'}, \mathbf{B}', \mathbf{b}')$.

Let $\rho_1, \ldots, \rho_n$ be the roots of the generalized Lundberg's equation $B_\delta(-s) = q_\delta(-s)\hat{p}(s)$ with positive real parts, and $\rho_n, \ldots, \rho_{n+m}$ the roots with negative real parts.

For simplicity, assume that all those roots are distinct (although this is not the case in general).

Because of condition (5.1), we can not write the solutions of (5.2) as a linear combination of n exponential functions as we did before in the cases of the ruin probability and the Laplace transform of the time of ruin. We will need instead more than n exponential functions; the exact number needed will depend on the nature of the distribution of the single gains, $P(x)$. However, we can apply the annihilator approach known from the theory of ordinary differential equations to find the appropriate solutions.
We can rewrite $W(u,b)$ as

$$\begin{aligned} W(u,b) &= \int_u^b V(x,b)p(x-u)dx + \int_b^\infty (x-b+V(b,b))p(x-u)dx \qquad (5.4) \\ &= \int_u^b V(x,b)p(x-u)dx + \int_b^\infty \widetilde{V}(x,b)p(x-u)dx, \end{aligned}$$

with $\widetilde{V}(x,b) = x - b + V(b,b)$. The idea is to find a linear differential operator that will annihilate $p(x-u)$ (where the variable is u), so that when we apply this operator to the integro–differential equation (5.2) we obtain a linear homogeneous differential equation of a higher degree.
We apply the annihilator operator $A(\mathcal{D}) = Det(\mathbf{I_m}\mathcal{D} + \mathbf{B}')$ at both sides of the integro–differential equation

$$B_\delta(\mathcal{D})V(u,b) = q_\delta(\mathcal{D})W(u,b),$$

where $\mathbf{I_m}$ is the identity $m \times m$ matrix, and we obtain an homogeneous integro–differential equation of degree $m+n$. We look for solutions of this equation of the form

$$V(u,b) = \sum_{l=1}^{n+m} a_l(b)e^{-\rho_l u}, \quad u < b. \qquad (5.5)$$

Using the boundary n conditions (5.3), and the identity

$$\alpha' \left[\sum_{k=1}^{n+m} a_k e^{-\rho_k b} \left((\rho_k \mathbf{I_m} - \mathbf{B}')^{-1}\mathbf{B}' + \mathbf{I_m} \right) - \mathbf{B}'^{-1} \right] = \mathbf{0}. \qquad (5.6)$$

which gives another m conditions, we obtained a system of $m+n$ equations on the $m+n$ unknowns $a_l(b)$. Thus, solving this system gives us the exact expression of $V(u,b)$.

Example: We want to compute $V(u,b)$. Assume that the times between gains are Erlang$(2,\lambda)$ distributed and the gain amounts are Erlang$(2,\beta)$ distributed.

Then the net profit condition is $c < \frac{\lambda}{\beta}$ and the generalized Lundberg's equation becomes

$$(\lambda + \beta - cs)^2(\beta + s)^2 = \lambda^2\beta^2 \tag{5.7}$$

Let

$$V(u,b) = \sum_{l=1}^{4} a_l(b)e^{-\rho_l u}$$

The exponents ρ_l's are the four roots of (5.7). Assume that ρ_1, ρ_2 have positive real parts and ρ_3, ρ_4 have negative real parts.

The coefficients a_l's are obtained using the corresponding boundary conditions $V(0,b) = V'(0,b) = 0$

$$\sum_{l=1}^{4} a_l = 0, \text{ and } \sum_{l=1}^{4} a_l\rho_l = 0,$$

and the additional constrains

$$\sum_{l=1}^{4} a_l e^{-\rho_l b}\frac{\rho_l}{\rho_l+\beta} = -\frac{1}{\beta}, \quad \sum_{l=1}^{4} a_l e^{-\rho_l b}\frac{\rho_l\beta}{(\rho_l+\beta)^2} = -\frac{1}{\beta},$$

Set the values for the parameters $\lambda = \beta = 1$, $c = 0.75$, $\delta = 0.02$. Then $\rho_1 = 0.423$, $\rho_2 = 1.831$, $\rho_3 = -0.063$ and $\rho_4 = -1.471$.

$u \backslash b$	3	5	6	7	8	10	15	20
2	3.079	4.107	4.390	4.507	4.489	4.212	3.187	2.333
3	4.533	6.033	6.450	6.621	6.595	6.188	4.682	3.428
5	6.533	8.773	9.374	9.622	9.584	8.993	6.805	4.981
10	11.533	13.773	14.501	14.825	14.770	13.829	10.468	7.663
15	16.533	18.773	19.501	19.825	19.770	18.829	14.478	10.603
20	21.533	23.773	24.501	24.825	24.770	23.829	19.478	14.537

Table 5.1: Values of $V(u,b)$

5.1 Optimal Dividends

We noticed that for a fixed u the value of $V(u,b)$ increases until a certain value of b and then decreases.

For a given initial capital u, let b^* denote the optimal value of the barrier b that maximizes the expected discounted dividends $V(u,b)$.

Avanzi *et al* (2007) shows that for a dual model with exponentially distributed inter-gain times the value of b^* is independent of u.
We have observed that the same situation occur for a dual model with Phase–type(n) distributed inter-gain times and Phase–Type(m) distributed gain amounts. Let b^* be the value that maximizes $V(u,b)$.

For a dividend barrier strategy, the optimal level is independent of the initial surplus.

Theorem 5.2. *b^* is independent of the initial surplus u.*

Proof. For a given initial surplus $u_0 \geq 0$ let b_0^* be the optimal barrier level that maximizes the expected discounted dividends.

This means that $V(u_0,b)$ is maximal at $b = b_0^*$ and

$$\frac{\partial}{\partial\, b} V(u_0, b)\Big|_{b=b_0^*} = 0, \quad \text{for} \quad u = u_0.$$

The idea of this proof is to show that

$$\frac{\partial}{\partial\, b} V(u, b)\Big|_{b=b_0^*} = 0, \quad \forall u \geq 0.$$

From (5.1), we have

$$\frac{\partial}{\partial\, b} V(u, b)\Big|_{b=b_0^*} = 0 = -1 + \frac{d}{d\, b} V(b, b)\Big|_{b=b_0^*}, \quad \forall u \geq b_0^*,$$

and we obtain

$$\frac{d}{d\, b} V(b, b)\Big|_{b=b_0^*} = 1.$$

Since we have $V(0,b) \equiv 0$ then clearly

$$\frac{\partial}{\partial\, b} V(0, b)\Big|_{b=b_0^*} = 0, \quad \text{for} \quad u = 0.$$

It only remains to show that

$$\frac{\partial}{\partial\, b} V(u, b)\Big|_{b=b_0^*} = 0, \quad 0 < u < b_0^*.$$

Previously in Theorem (5.1) we have found that in the Phase − Type(n) dual risk model the expected discounted dividends $V(u,b)$ satisfy the integro–differential equation

$$B_\delta(\mathcal{D})V(u,b) = q_\delta(\mathcal{D})W(u,b),$$

where

$$W(u,b) = \int_u^b V(y,b)p(y-u)dy + \int_b^\infty (y - b + V(b,b))p(y-u)dy,$$

Moreover, assuming that the gain amounts follow another Phase – Type(m) distribution, with density function $p(x) = \boldsymbol{\alpha'} e^{\mathbf{B}' x} \mathbf{b'}^{\mathsf{T}}$, we were able to write an expression of $V(u, b)$ of the form (5.5)

$$V(u,b) = \sum_{l=1}^{n+m} a_l(b) e^{-\rho_l u}.$$

Since

$$\begin{aligned} \frac{\partial}{\partial b} W(u,b)\Big|_{b=b_0^*} &= \int_u^{b_0^*} \frac{\partial}{\partial b} V(y,b)\Big|_{b=b_0^*} p(y-u)dy + \\ & \underbrace{\left(-1 + \frac{d}{d\,b} V(b,b)\Big|_{b=b_0^*}\right)}_{=0} \int_{b_0^*}^{\infty} p(y-u)dy \\ &= \int_u^{b_0^*} \frac{\partial}{\partial b} V(y,b)\Big|_{b=b_0^*} p(y-u)dy, \end{aligned}$$

then for $0 < u < b_0^*$ we have

$$B_\delta(\mathcal{D}) \, \frac{\partial}{\partial b} V(u,b)\Big|_{b=b_0^*} = q_\delta(\mathcal{D}) \, \frac{\partial}{\partial b} W(u,b)\Big|_{b=b_0^*},$$

or equivalently

$$B_\delta(\mathcal{D}) \, \frac{\partial}{\partial b} V(u,b)\Big|_{b=b_0^*} = q_\delta(\mathcal{D}) \left[\int_u^{b_0^*} \frac{\partial}{\partial b} V(y,b)\Big|_{b=b_0^*} p(y-u)dx \right], \quad 0 < u < b_0^* \tag{5.8}$$

When we replace

$$\frac{\partial}{\partial b} V(u,b)\Big|_{b=b_0^*} = \sum_{l=1}^{n+m} a_l'(b_0^*) e^{-\rho_l u}$$

in (5.8) we obtained an identity of exponential functions in terms of the coefficients $a_l'(b_0^*)$ which is valid for all u in $(0, b_0^*)$, therefore $a_l'(b_0^*) = 0$ for all $l = 1, \ldots m+n$.

This proves that

$$\frac{\partial}{\partial b} V(u,b)\Big|_{b=b_0^*} = 0, \quad 0 < u < b_0^*.$$

Therefore we have proven that the optimal barrier level is independent of u.

□

6 Appendix

In this section we prove the theorems that we stated in this manuscript.

Proof of Theorem (3.1)

Proof. We prove that $(s\mathbf{I}-\mathbf{B})^{-1}(s\mathbf{I}-\mathbf{B})=\mathbf{I}$ or, equivalently, that

$$(s\mathbf{I}-\mathbf{B})N(s,\mathbf{B}) = det(s\mathbf{I}-\mathbf{B})\mathbf{I}.$$

If we denote by

$$a_i = \sum_{j=0}^{n-1-i}(-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-1-i-j},$$

then

$$\begin{aligned}(s\mathbf{I}-\mathbf{B})N(s,\mathbf{B}) &= (s\mathbf{I}-\mathbf{B})\sum_{i=0}^{n-1}\left(\sum_{j=0}^{n-1-i}(-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-1-i-j}\right)s^i \\ &= (s\mathbf{I}-\mathbf{B})\sum_{i=0}^{n-1}a_i s^i \\ &= a_{n-1}s^n + \sum_{i=1}^{n-1}(a_{i-1}-a_i\mathbf{B})s^i - a_0\mathbf{B}.\end{aligned}$$

Now we can easily verify that $a_{n-1}=\mathbf{I}$. Since

$$det(s\mathbf{I}-\mathbf{B}) = \sum_{i=0}^{n}(-1)^{n-i}tr_{n-i}(\mathbf{B})s^i,$$

we get $-a_0\mathbf{B} = (-1)^n det(\mathbf{B})\mathbf{I}$ and

$$\begin{aligned}a_{i-1}-a_i\mathbf{B} &= \sum_{j=0}^{n-i}(-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-i-j} - \left(\sum_{j=0}^{n-1-i}(-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-1-i-j}\right)\mathbf{B} \\ &= (-1)^{n-i}tr_{n-i}(\mathbf{B})\mathbf{I}.\end{aligned}$$

Therefore,

$$\begin{aligned}(s\mathbf{I}-\mathbf{B})N(s,\mathbf{B}) &= \mathbf{I}s^n + \sum_{i=1}^{n-1}((-1)^{n-i}tr_{n-i}(\mathbf{B})\mathbf{I})s^i + (-1)^n det(\mathbf{B})\mathbf{I} \\ &= \sum_{i=0}^{n}((-1)^{n-i}tr_{n-i}(\mathbf{B})\mathbf{I})s^i = det(s\mathbf{I}-\mathbf{B})\mathbf{I}.\end{aligned}$$

□

Proof of Theorem (4.1)

Proof. We proceed taking successive derivatives of the ruin probability using the renewal equation (4.1). Changing the variable, $u - ct = s$, the renewal equation can be rewritten in the form

$$\psi(u) = 1 - K\left(\frac{u}{c}\right) + \frac{1}{c}\int_0^u k\left(\frac{u-s}{c}\right) W(s)ds,$$

where $W(s) = \int_0^\infty \psi(s+x)p(x)dx$.

We want to prove the equation $B(\mathcal{D})\psi(u) = q(\mathcal{D})W(u)$. Notice that the $B(\mathcal{D})$ has the following property with the Phase–Type density:

$$\begin{aligned}
B(\mathcal{D})k\left(\frac{u-s}{c}\right) &= \sum_{k=0}^{n} B_k \mathcal{D}^k[\boldsymbol{\alpha} e^{\mathbf{B}(\frac{u-s}{c})}\mathbf{b}^\mathsf{T}] = \boldsymbol{\alpha}\left[\sum_{k=0}^{n} B_k \mathcal{D}^k (e^{\mathbf{B}(\frac{u-s}{c})})\right]\mathbf{b}^\mathsf{T} \\
&= \boldsymbol{\alpha}\left[\sum_{k=0}^{n} B_k \left(\frac{1}{c}\right)^k \mathbf{B}^k e^{\mathbf{B}(\frac{u-s}{c})}\right]\mathbf{b}^\mathsf{T} \\
&= \boldsymbol{\alpha}\left[\sum_{k=0}^{n} B_k \left(\frac{1}{c}\mathbf{B}\right)^k\right] e^{\mathbf{B}(\frac{u-s}{c})}\mathbf{b}^\mathsf{T} \\
&= \boldsymbol{\alpha}\left[B\left(\frac{1}{c}\mathbf{B}\right)\right] e^{\mathbf{B}(\frac{u-s}{c})}\mathbf{b}^\mathsf{T} \\
&= \boldsymbol{\alpha}\left[\frac{det(\mathbf{B} - c\mathbf{I}\left(\frac{1}{c}\mathbf{B}\right))}{det(\mathbf{B})}\right] e^{\mathbf{B}(\frac{u-s}{c})}\mathbf{b}^\mathsf{T} = 0,
\end{aligned}$$

Analogously, we can see $B(\mathcal{D})\left(1 - K\left(\frac{u}{c}\right)\right) = 0$.

The j derivative of the ruin probability $\psi(u)$ with respect to u is given by the expression

$$\begin{aligned}
\frac{d^j}{du^j}\psi(u) &= -\left(\frac{1}{c}\right)^j k^{(j-1)}\left(\frac{u}{c}\right) + \sum_{i=0}^{j-1}\left(\frac{1}{c}\right)^{i+1} k^{(i)}(0)W^{(j-1-i)}(u) \\
&\quad + \left(\frac{1}{c}\right)^{j+1}\int_0^u k^{(j)}\left(\frac{u-s}{c}\right)W(s)ds
\end{aligned}$$

for $j = 1, \ldots, n-1$. Hence, we obtain

$$\left.\frac{d^j}{du^j}\psi(u)\right|_{u=0} = -\left(\frac{1}{c}\right)^j k^{(j-1)}(0) + \sum_{i=0}^{j-1}\left(\frac{1}{c}\right)^{i+1} k^{(i)}(0)W^{(j-1-i)}(0),\ j = 1, \ldots, n-1.$$

Now we apply the apply the differential operator $B(\mathcal{D})$ to the ruin probability $\psi(u)$

$$
\begin{aligned}
B(\mathcal{D})\psi(u) &= \underbrace{B(\mathcal{D})\left(1-K\left(\frac{u}{c}\right)\right)}_{=0} + B(\mathcal{D})\left(\frac{1}{c}\int_0^u k\left(\frac{u-s}{c}\right)W(s)ds\right) \\
&= \sum_{j=0}^{n} B_j \mathcal{D}^j \left(\frac{1}{c}\int_0^u k\left(\frac{u-s}{c}\right)W(s)ds\right) \\
&= \sum_{j=0}^{n} B_j \left(\sum_{i=0}^{j-1}\left(\frac{1}{c}\right)^{i+1} k^{(i)}(0)W^{(j-1-i)}(u) + \left(\frac{1}{c}\right)^{j+1}\int_0^u k^{(j)}\left(\frac{u-s}{c}\right)W(s)ds\right) \\
&= \sum_{j=1}^{n} B_j \sum_{i=0}^{j-1}\left(\frac{1}{c}\right)^{i+1} k^{(i)}(0)W^{(j-1-i)}(u) + \left(\frac{1}{c}\right)^{j+1}\int_0^u \underbrace{B(\mathcal{D})k\left(\frac{u-s}{c}\right)}_{=0}W(s)ds \\
&= \sum_{j=0}^{n-1}\left(\sum_{i=j+1}^{n} B_i \left(\frac{1}{c}\right)^{i-j} k^{(i-1-j)}(0)\right)W^{(j)}(u) = \sum_{j=0}^{n-1}\tilde{B}_j W^{(j)}(u) = q(\mathcal{D})W(u).
\end{aligned}
$$

□

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