Performance and Predictive Power of Risk-Neutral Densities and Subjective Probability Density Functions

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Abstract

The main goal of this work is to extract the Risk Neutral Densities (RNDs) and Subjective Probability Density Functions (SPDFs) of the exchange rate USD/BRL (price of US Dollar in terms of Brazilian Real) for the period between June 2006 and September 2014. We have deepened previous studies, evaluating not only the market expectations provided by these densities, but also the performance of these densities in predicting the future realizations of this exchange rate.

The RNDs were estimated using two structural models and two non-structural models. In the first category, we included the Variance Gamma-OU (VG Gamma-OU) model and the CGMY Gamma-OU model. In the second category, we used the Density Functional Based on Confluent Hypergeometric function (DFCH) model and the Mixture of Lognormal Distributions (MLN) model.

In this study, the DFCH, the CGMY Gamma-OU and the VG Gamma-OU produced densities (RND and SPDF) with 1 month to maturity that exhibited a very good performance in forecasting the observed USD/BRL. The MLN, despite having and acceptable predictive power for 1 month term, underperformed substantially the other 3 models.

JEL Classification: G13; C13; C15; F31

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1 Introduction

An option is a contingent claim whose payoff depends on the future state of the underlying asset. Therefore, option prices are mainly influenced by the expectations of the future underlying asset prices and by its uncertainty. For a particular asset and maturity, the availability of option prices for various exercise prices allows the estimation of Risk-Neutral Densities (RND). These densities can provide important information to risk managers, policy makers and investors.

In the real world, there is only a range of discrete option prices, which makes it necessary to estimate the RNDs through a smoothing function. The RND depends on the model used on its estimation, which makes the choice of a reliable model very important. The use of the Black and Scholes model (B&S), the standard model in option pricing, is not recommended due to its limitations. This model is based on strong assumptions, such as modeling the asset price using a geometric Brownian Motion (GBM), with a constant expected return and a constant volatility. This constant volatility assumption contradicts the empirical evidence that we have different implied volatilities across maturities and across strike prices. The volatility shoud be stochastic over time.

In this work, we estimated RNDs for the period between June 2006 and September 2013 using option prices with the exchange rate USD/BRL as the underlying asset. These RNDs were estimated using two categories of models: structural and non-structural. A structural model assumes a specific dynamics for the price or volatility process. A non-structural model allows the estimation of a RND without describing any stochastic process for the price or volatility of the underlying asset. In the latter category we used the Density Functional Based on Confluent Hypergeometric function (DFCH) and the Mixture of Lognormal Distributions (MLN), two non-structural models that proved to be reliable in Santos and Guerra (2014). In the former category we used models based on Lévy processes: the Variance Gamma-OU (VG-OU) and the CGMY Gamma-OU (CGMY Gamma-OU), two rich and sophisticated stochastic volatility models that generate jumps for asset returns and volatility and, consequently, are able to generate plausible true RNDs, with statistical properties similar to the ones observed in the empirical data (leptokurtic distributions with fat tails).

As is generally known, the RND is not a direct measure of the subjective probabilites that investors attribute to the multiplicity of future underlying asset prices, since the option prices incorporate investors risk preferences. Theoretically, if we know the latter, we are able to estimate the Subjective Probability Density Function (SPDF). Bliss and Panigirtzoglou (2004) suggested the estimation of the risk aversion assuming that, if investors are rational, their SPDF is expected to correspond to the distribution of observed realizations (called objective distribution). This means that the SPDF should provide, on average, accurate forecasts and that the difference between the RND and the objective distribution arises from the risk aversion of the rational investor. Many authors estimated the SPDF imposing an assumption of stationarity over long periods (see Ait-Sahalia and Lo (2000) and Jackwerth (2000)). We believe that the Bliss and Panigirtzoglou (2004) approach is more suited to extract the SPDF, because it estimates probabilities that are more reactive to market changes and events, so we used this method to convert the RNDs into SPDFs. This work goes deeper than previous studies as it includes not only the analysis of the market sentiment, but also the analysis of the forecasting power of the RNDs and SPDFs.

This paper is organized into six sections. Section 2 describes the non-structural and structural models used to estimate the RNDs. Section 3 describes the methodology used to transform the RNDs into SPDFs. In section 4, we compared the prediction power of the RND and SPDF through the confrontation between the estimated densities and the realized outcomes of the exchange rate USD/BRL. In Section 5, we analyzed the historical probabilities implied on the RNDs and SPDFs for the period between June 2006 and September 2013. Finally, section 6 presents the conclusions.

2 Risk Neutral Density estimation

2.1 Option prices and the RND

The call option value is given by the discounted value of its expected payoff on the expiration date T.

$$C(X,T) = e^{-rT} \int_{X}^{\infty} f(S_T - X) dS_T, \tag{1}$$

where X is the exercise price, S_T is the price of the underlying asset at T and r is the risk-free interest rate. The expectation is taken under the risk neutral measure.

Breeden and Litzenberger (1978) deduced the relationship between the option prices and the RND, taking the second derivative of equation (1) with respect to X, that is

$$\frac{d^2 C(X,T)}{dX^2} = e^{-rT} f(S_T). \tag{2}$$

2.2 Structural models

In the next sections we introduce four structural models based on Lévy processes: the Variance Gamma, the CGMY, the CGMY Gamma-OU and the Variance Gamma-OU.

2.2.1 Lévy processes and the Lévy-Khintchine formula

A structural model should generate a flexible stochastic process. Therefore, we should consider processes that capture both small moves and large moves or

”jumps” in the asset returns. The non-Gaussian nature of the returns distribution and the fat-tails phenomenon is mainly influenced by ”jumps” in asset returns.

The class of Lévy processes includes many processes that allow small moves and large jumps in asset returns and. The Brownian motion, the Merton jump-diffusion, the Variance Gamma (VG), the Normal Inverse Gaussian (NIG), the CGMY and the Generalized Hyperbolic are examples of Lévy processes (see Schoutens (2003)).

A Lévy process $\{X_t, t \geq 0\}$ is a stochastic process with independent and stationary increments, which is also stochastically continuous. The well known Lévy-Khintchine formula provides a complete characterization of the distribution associated to a Lévy process (see Cont and Tankov (2003)). This formula states that the characteristic function of a Lévy process is given by

$$\Phi_X(u) = \exp(t\psi(u)), \ u \in \mathbb{R}. \tag{3}$$

where the characteristic exponent in equation (3) is given by

$$\psi(u) = i\gamma\mu - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} (e^{iux} - 1 - iux1_{|x|\leq 1})\nu(dx), \tag{4}$$

where $\gamma \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$ and ν is a positive measure on $\mathbb{R}\setminus\{0\}$, called the Lévy measure and satisfying:

$$\int_{\mathbb{R}\setminus\{0\}} \min\left\{1, x^2\right\} \nu(dx) < \infty.$$

A Lévy process characterized by the Lévy triplet (γ, σ^2, ν) can be decomposed into three independent parts: (i) the linear deterministic part given by the drift γt; (ii) the diffusion or Brownian part σW_t, where W_t is a standard Brownian motion; (iii) the pure jump part, which is characterized by the Lévy measure ν.

In financial applications, the price of an underlying asset S_t can be modelled by an exponential Lévy process and the risk-neutral dynamics is given by:

$$S_t = S_0 \exp(X_t), \tag{5}$$

where X_t is a Lévy process under the risk neutral measure $\mathbb{Q}$.

Variance Gamma process The Variance Gamma (VG) process is often used in financial applications, capturing a wide range of realistic stochastic properties, while remaining tractable (see Madan, Carr and Chang (1998) and Carr et al. (2002)). This process has no diffusion component (is a pure jump process) and captures both frequent small moves and rare large jumps. The process can be defined as a difference of two gamma processes or as a subordinating Brownian motion.

Using the subordinating approach, the VG is represented as a Brownian motion with drift θ defined at a random time given by a gamma process.

$$X_{VG}(t) = \theta G_t + \sigma W(G_t), \tag{6}$$

where G_t is a gamma process with mean t and variance rate ν, independent of the Brownian motion W. The characteristic function is given by

$$\Phi_{X_{VG}}(u,t) = \left(\frac{1}{1 - i\theta\nu u + \sigma^2\nu u^2/2}\right)^{\frac{t}{\nu}}, \tag{7}$$

The Lévy measure of the VG process is given by (see Madan, Carr and Chang (1998) and Carr et al. (2002)):

$$\nu_{VG}(dx) = \begin{cases} \frac{C\exp(G|x|)}{|x|} & \text{for } x < 0 \\ \frac{C\exp(-Mx)}{x} & \text{for } x > 0 \end{cases}, \tag{8}$$

where

$$\begin{aligned} C &= 1/\nu > 0, \\ G &= \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} - \frac{1}{2}\theta\nu\right)^{-1} > 0, \\ M &= \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} + \frac{1}{2}\theta\nu\right)^{-1} > 0. \end{aligned}$$

CGMY process The VG Lévy measure can be generalized (see Carr et al. (2002)), introducing a new parameter Y, and obtaining the new measure

$$f_{CGMY}(dx) = \begin{cases} \frac{C\exp(-G|x|)}{|x|^{1+Y}} & \text{for } x < 0 \\ \frac{C\exp(-Mx)}{x^{1+Y}} & \text{for } x > 0 \end{cases}, \tag{9}$$

where $Y < 2$. The Lévy process associated to this Lévy measure is known as the CGMY process.

The Y parameter controls the behavior of small jumps. When $Y = 0$, we obtain the previous VG process. The parameter C can be viewed as a measure of the global level of jump activity, scaling the expected number of jumps of all sizes. The parameters G and M control the rate of exponential decay on the right and left of the Lévy measure, respectively.

VG Gamma-OU and CGMY Gamma-OU process For short maturities, the introduction of jumps generates the appropriate implied volatility patterns seen in empirical data. However, the independence of log-returns in exponential-Lévy models has an unfortunate consequence: these models fail to reproduce the implied volatility smiles and skews observed in options market prices over a range of different maturities. This drawback is caused by the volatility clustering phenomenon. In order to surpass this problem, we consider a stochastic process for volatility, which should be a positive and mean-reverting process. This process can be defined, considering a stochastic time: in periods of high volatility time runs faster than in periods of low volatility. Therefore, random changes in volatility are captured by random changes in time. This stochastic time change can be modeled by a Gamma-OU process. The stochastic time concept in asset pricing was first used in Clark (1973).

The Ornstein-Uhlenbeck (OU) process was introduced by Barndorff and Shephard (2001) as a volatility model. It can be defined as the solution of the stochastic differential equation

$$dy_t = -\lambda y_t dt + dz_{\lambda t},\ y_0 > 0,$$

where $\{z_t, t \geq 0\}$ is a subordinator (strictly positive and increasing Lévy process) and $\lambda > 0$. The process z_t can also be called a Background Driving Lévy Process (BDLP).

The D-OU process $\{y_t, t \geq 0\}$ is a nonnegative nondecreasing Lévy process. This process is strictly stationary on the positive half-line, that is, there exists a law D, called the stationary law, such that y_t will follow the law D for every t if y_0 is has the distribution D. For a Gamma-OU process, the BDLP $\{z_t, t \geq 0\}$ is a Gamma process that has Lévy measure

$$w(x) = ab\exp(-bx)1_{(x>0)},$$

where a and b are positive parameters. The Gamma-OU process has the characteristic function

$$\Phi_{Y_{Gamma-OU}}(u,t,\lambda,a,b,y_0) = \exp(iuy_0\lambda^{-1}(1-\exp(-\lambda t)))\times \tag{10}$$

$$\exp\left(\frac{\lambda a}{iu-\lambda b}\left(b\log\left(\frac{b}{b-iu\lambda^{-1}(1-\exp(-\lambda t))}\right)-iut\right)\right). \tag{11}$$

The stochastic process CGMY Gamma-OU transforms the CGMY process X_t into a stochastic volatility process Z_t through a stochastic time change, modeled by a subordinator Y_t. In the CGMY Gamma-OU case, the CGMY process X_t is subordinates to the Gamma-OU process Y_t (see Carr et al. (2003)).

The characteristic function of the CGMY Gamma-OU process is given by

$$E(\exp(iuZ_t)) = E(\exp(\psi_{X_{CGMY}(Y_t)}(u))) \tag{12}$$

$$= \Phi_{Y_{Gamma-OU}}(-i\psi_{X_{CGMY}}(u),t,\lambda,a,b,y_0) \tag{13}$$

By a similar procedure, the VG process X_t can be transformed into a VG Gamma-OU process, that is:

$$E(\exp(iuZ_t)) = E(\exp(\psi_{X_{VG}(Y_t)}(u))) \tag{14}$$

$$= \Phi_{Y_{Gamma-OU}}(-i\psi_{X_{VG}}(u),t,\lambda,a,b,y_0) \tag{15}$$

If we know the characteristic function of the Lévy process, the RND can be extracted from option prices using a method based on the Fourier Transform (see Carr and Madan (1999)).

2.3 Non-Structural Models

2.3.1 Density Functional Based on Confluent Hypergeometric function

In this model, a specific functional form for the RND is not assumed. The model is based on a formula that encompasses various densities, such as the normal, gamma, inverse gamma, weibull, pareto and mixtures of all these probability densities.

Abadir and Rockinger (2003) proposed a functional based on the confluent hypergeometric function (${}_1F_1$). This function includes special cases of the incomplete gamma, normal and mixtures of the two distributions. Moreover, this method has the advantage of being more efficient than fully nonparametric estimation methods for small samples and more flexible than parametric methods.

The confluent hypergeometric function can be defined by:

$$ {}_1F_1(\alpha;\beta;z) \equiv \sum_{j=0}^{\infty} \frac{(\alpha)_j}{\beta_j}\frac{z^j}{j!} \equiv 1 + \frac{\alpha}{\beta}z + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{z^2}{2} + ..., \tag{16} $$

$$ (a)_j \equiv (a)(a+1)...(a+j-1) \equiv \frac{\Gamma(a+j)}{\Gamma(a)}, $$

with $\Gamma(v)$, for $v \in \mathbb{R}\backslash\{..., -2, -1, 0\}$, being the gamma function and $-\beta \notin \mathbb{N} \cup \{0\}$.

The option pricing functional is called density functional based on confluent hypergeometric function (DFCH) and gives the European call price as a mixture of two different confluent hypergeometric functions:

$$ \begin{aligned} C(X) = {} & c_1 + c_2X + 1_{X>m_1} a_1((X-m_1)^{b_1})\,{}_1F_1(a_2; a_3; b_2(X-m_1)^{b_3}) \\ & + (a_4)\,{}_1F_1(a_5; a_6; b_4(X-m_2)^2), \end{aligned} \tag{17} $$

where $-a_3, -a_6 \notin \mathbb{N} \cup \{0\}$, $b_2, b_4 \in \mathbb{R}^-$ and $a_1, a_2, a_4, a_5, b_1, b_3 \in \mathbb{R}$. In order to get the implied probability density function, the formula stated in equation (2) is applied to $C(X)$. Considering appropriate restrictions in order to guarantee that $f(X)$ integrates to 1 and assuming that $c_1 = -c_2m_2$, $b_1 = 1 + a_2b_3$, $a_5 = -\frac{1}{2}$, $a_6 = \frac{1}{2}$, formula (17) can be simplified (see Abadir and Rockinger (2003)) and the number of parameters to estimate in equation (17) is reduced to seven.

2.3.2 Mixture of lognormal distributions

Bahra (1997) and Melick and Thomas (1997) proposed the mixture of lognormal distributions (MLN) as a non-structural model. In their approach, the authors assumed a functional form for the RND that encompasses several stochastic processes for the underlying asset price.

The price of an European call option can be obtained as the discounted sum of all expected future payoffs:

$$C(X,\tau) = e^{-r\tau}\int_X^\infty f(S_T)(S_T - X)dS_T, \tag{18}$$

where $\tau = T - t$.

Bahra (1997) considered that $f(S_t)$ can be represented by a mixture of two lognormal density functions:

$$f(S_t) = wL(\alpha_1, \beta_1, S_t) + (1-w)\,L(\alpha_2, \beta_2, S_t), \tag{19}$$

where $L(\alpha_i, \beta_i, S_t)$ is the i-th lognormal density with parameters α_i and β_i:

$$\begin{aligned} L(\alpha_i, \beta_i, S_t) &= \frac{1}{S_t\beta_i\sqrt{2\pi}}e^{[-(\ln(S_t)-\alpha_i)^2/2\beta_i^2]}, \\ \alpha_i &= \ln(S_t) + (\mu_i - \frac{1}{2}\sigma_i^2)(T-t), \\ \beta_i &= \sigma_i\sqrt{(T-t)}. \end{aligned} \tag{20}$$

The price of the call option is given by the formula (see Jondeau et al. (2006))

$$\begin{aligned} c(X,\tau) = e^{-r\tau}\{&w[e^{\alpha_1+\frac{1}{2}\beta_1^2}N(d_1) - XN(d_2)] \\ &+ (1-w)[e^{\alpha_2+\frac{1}{2}\beta_2^2}N(d_3) - XN(d_4)]\}, \end{aligned} \tag{21}$$

where

$$\begin{aligned} d_1 &= \frac{-\ln(X) + \alpha_1 + \beta_1^2}{\beta_1}, \\ d_2 &= d_1 - \beta_1, \\ d3 &= \frac{-\ln(X) + \alpha_2 + \beta_2^2}{\beta_2}, \\ d_4 &= d_3 - \beta_2. \end{aligned} \tag{22}$$

3 Subjective PDF estimation

The SPDF was estimated following the approach of Bliss and Panigirtzoglou (2004), which is based on the assumption that investors are rational and risk-averse. The first assumption implies that investors can predict, on average, the future outcome of the realized returns on the underlying asset (Objective

Distribution). The latter means that investors are willing to pay a premium as a way to avoid undesirable outcomes. This means that, if the RND is an unbiased predictor (equal to the Objective Distribution), then there is no evidence of risk premia in option prices and the RND corresponds to the SPDF; on the other hand, if the RND fails to predict the future outcomes, then the bias is measured by the degree of the investor risk aversion. Therefore, the SPDF corresponds to the probability density function after deducting the rational risk preferences from the RND. The rational risk preferences are determined by the degree of risk aversion that minimizes the bias between the RND and the realized returns of the underlying asset. This bias was measured by the Berkowitz test and the exponential utility function was used as representing the investors risk preferences.

3.1 Testing RND forecast performance with Berkowitz Test

For each cross section of option prices, we have one RND which compares with only one realization (e.g. the one month RND estimated on a certain date can only be compared with the observed return on the underlying asset after one month).

Given that our predictors are densities, the RND's accuracy has to be measured through a statistical test that measures the overall fit of the distribution instead of point estimates. Therefore, we used the Berkowitz test, proposed by Berkowitz (2001) as an alternative way to measure the performance of a probability distribution of credit losses. This statistical test is an alternative to the standard methods whose density performance is given by the number of realized returns that exceed a limit, usually defined by the VAR (Value-at-Risk - loss that is predicted not to be exceeded with a certain probability, frequently 99%). These tests, based on the number of exceedances, are considered to be less reliable because they are confined to part of the distribution and focus the attention on rare violations. The Berkowitz test, whose objective is to evaluate the entire distribution, involves the following transformation of the realized returns on the underlying asset:

$$y_t = \int_{-\infty}^{X_t} \hat{f}(u)du = \hat{F}(X_t). \tag{23}$$

where X_t refers to the post realizations on t, $\hat{f}$ is the forecast density (in this case represented by the RND) and $\hat{F}$ corresponds to the cumulative distribution of $\hat{f}$. Under the null hypothesis that the forecast density is equal to the true probability density function (PDF), the y_t is i.i.d. and uniformly distributed on (0,1). This result is very interesting because it holds regardless of the distribution of the realized returns and even if the forecast density changes over time (which is clearly our case).

The previous transformation is made in two steps:

1. Convert X_t into the predicted probability of observing this return or a smaller one. This probability is estimated by inserting X_t in the RND,

$$y_t = \int_{-\infty}^{X_t} RND(u)du, \tag{24}$$

2. Apply the inverse cumulative standard normal distribution function to y_t,

$$z_t = \Phi^{-1}(y_t). \tag{25}$$

We could test directly the uniformity of the transformed variable y_t using statistical tests employed for this purpose: Kolmogorov-Smirnov, Chi-squared, and Kupier tests. However, Berkowitz (2001) suggests a further transformation from y_t to z_t in order to make use of a more powerful test based on the normality. He defined z_t as a first-order autoregressive model:

$$z_t - \mu = \rho(z_{t-1} - \mu) + \varepsilon_t. \tag{26}$$

with μ and $var(\varepsilon_t)$ being the mean and variance of z_t respectively, and ρ being being the correlation between z_{t-1} and z_t. Under the null hypothesis that $\hat{f}$ is the true PDF, z_t should be independent across observations and standard normal ($\mu = 0$, $var(\varepsilon_t) = 1$ and $\rho = 0$). In the alternative hypothesis we have a first-order autoregressive model with mean different from 0 and variance different from 1. Berkowitz (2001) measured the difference between these two models using the likelihood ratio test:

$$LR_3 = -2\left[L(0,1,0) - L(\hat{\mu}, \hat{\sigma}^2, \hat{\rho})\right], \tag{27}$$

where $L(\mu, \sigma^2, \rho)$ denotes the log-likelihood-function and the hats denote estimated values. Under the null hypothesis LR_3 follows a $\chi^2(3)$ distribution.

$$L(\mu, \sigma^2, \rho) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log\left[\frac{\sigma^2}{1-\rho^2}\right] - \frac{(z_t - \mu/(1-\rho))^2}{2\sigma^2/(1-\rho^2)} \tag{28}$$

$$-\frac{T-1}{2}\log(2\pi) - \frac{T-1}{2}\log(\sigma^2) - \sum_{t=2}^{T}\frac{(z_t - \mu - \rho z_{t-1})^2}{2\sigma^2}. \tag{29}$$

3.2 Risk Aversion and Utility function

It is known that the probabilities provided by the RND are not subjective probabilities, but instead are risk neutral probabilities. A risk-neutral investor is indifferent in terms of risk, which means that he does not take any investment decision based on its uncertainty. If two investments have the same expected payoff he will not give preference to one over the other, even if one is riskier. However, in the real world there are a multiplicity of investors with different preferences: investors can be more attracted to risk or can be more risk averse and are willing to pay a premium to avoid undesirable outcomes. This means that, the price of an option does not depend only on the probability of future outcomes, but also on the utility it yields.

Various methods have been developed in economics to define the relationship between utility and wealth and, consequently, to measure the degree of risk aversion. In this paper, we assumed a parametric form for the utility function using the exponential utility function:

$$U(S_t) = -\frac{e^{-\gamma S_t}}{\gamma}, \tag{30}$$

where γ is the only parameter of the function. Pratt and Arrow proposed the calculation of the risk aversion coefficient using the second derivative of the utility function (measures how the change in utility itself changes as a function of wealth) and dividing it by the first derivative.

$$ARA = -\frac{U''(S_t)}{U'(S_t)} = \gamma, \tag{31}$$

$$RRA = -\frac{S_t U''(S_t)}{U'(S_t)} = S_t\gamma, \tag{32}$$

Equation (31) corresponds to the absolute risk aversion (ARA), which in this case is constant (the amount of wealth that we are willing to risk remains the same as the wealth increases). Instead of measuring absolute changes in wealth, we can measure proportional changes, which can be done through Equation (32) that determines the relative risk aversion (RRA). In the exponential utility function, the RRA is dependent on both gamma and the realization S_t, which is time varying.

According to Ait-Sahalia and Lo (2000), given the RND and the utility function it is possible to derive the SPDF through the following relations:

$$\frac{SPDF}{RND} = \lambda\frac{U'(S_T)}{U'(S_t)} = \zeta(S_T, S_t), \tag{33}$$

$$SPDF = \frac{\frac{RND}{\zeta(S_T,S_t)}}{\int \frac{RND}{\zeta(x,S_t)}dx} = \frac{\frac{U'(S_t)}{\lambda U'(S_T)}RND}{\int \frac{U'(S_t)}{\lambda U'(S_T)}RNDdx} = \frac{\frac{RND}{U'(S_T)}}{\int \frac{RND}{U'(x)}dx} \tag{34}$$

where λ is constant and $\zeta(S_T, S_t)$ is the pricing kernel. The SPDF must be normalized to integrate to 1.

4 Data

For this work, we used the currency OTC options with the underlying USD/BRL. The quotes were taken from the daily settlement bid prices in Bloomberg for Offshore USDBRL FX Options. The collected data covers the period from May

2005 to September 2013 and comprises the end of month prices. We only considered out-of-the money options (calls and puts) and at-the-money options, due to the general understanding that out-of-the-money options tend to be more liquid than in-the-money options. In this work, we estimate the RNDs using one, three and six months to maturity options.

For each maturity, we estimated 101 RNDs between May 2005 and September 2013 (one for each month). Using these RNDs, we estimated 88 SPDFs. For each month, the SPDF was determined by the degree of risk aversion that minimizes the bias between the RND and the realized returns of the underlying asset in the previous twelve months (e.g. the one month SPDF of June 2006 was estimated using the RNDs between May 2005 and May 2006 and the observed exchange rates one month after these RNDs).

5 Performance of non-structural and structural models

In this section, we analyze the performance of the RND and SPDF estimators in predicting the future changes of the exchange rate USD/BRL. For these estimations we used the CGMY Gamma-OU, VG Gamma-OU, DFCH and MLN. The three objectives of this analysis were: to investigate whether the estimated SPDF exceeds the RND in terms of predicting power (an expected result since the RND is not a direct measure of the subjective probabilities), to check if these densities provide any information about the future outcomes and to analyze which method has the highest forecast power.

In order to test the predictive power of the estimated RND, we compared the one month RNDs of the period between June 2006 and September 2013 with the USD/BRL outcomes in the following months. The same backtesting analysis was done for the 88 SPDFs estimated for the same period.

The results of these comparisons (reflected in the transformed variables of section 3.1) were aggregated according to the sub-periods considered in the test and the predictive power was measured using the Berkowitz test. In the 12 months Berkowitz test (Table A.1), each backtesting set was composed by 12 consecutive months. In the sub-periods Berkowitz test (Table A.2), each backtesting set was composed by the months considered in each sub-period of section 6.

The results in Tables A.1 and A.2 show the worst performance of the MLN model in comparison with the other models. However, it maintains, like the other models, average p-values well above 5% (average p-values of 25,56% and 28,57% for the RND and SPDF respectively). It should be noted that the DFCH and the CGMY Gamma-OU models had, on average, the highest p-values for both the RND and SPDF, as can be seen in table A.1. The VG Gamma-OU model has also demonstrated to have a good predictive power with an average p-value closer to the values of the previous two models (average p-values of 35,63% and 43,5% for the RND and SPDF respectively). If we look to the number of times in which the p-value does not exceeds 5% (rejecting the null hypothesis with a confidence level of 95%), we can even see the superiority of the VG Gamma-OU over the CGMY Gamma-OU in terms of SPDF estimation.

In addition, we observed that, in average, the SPDF outperforms the RND (in almost all cases, the mean of the p-values of the SPDF is higher). However, it should be noted that the SPDF does not always improve the predictive power of the RND. This is most evident on the number of times that the p-values does not exceed 5% in table A.1. In fact, for all models the null hypothesis is more times rejected with the SPDF than with the RND.

We also executed the backtesting analysis with the 3 months RND and SPDF, taking care to cancel the correlation effects (for a 12 months period we only considered 4 non-overlapping months - e.g. between June 2006 and June 2007 we included the 3 months RND of June 2006, September 2006, December 2006, March 2007) and, this way, avoid the incorrect rejection of the null hipotheses. We noticed that the forecasting power for the 3 months term was very low, as shown in table A.3 (see Appendix 8) where the lower p-values are evident.

The good predictive power of the 1 month RND lead us to different conclusions from the work of Bliss and Panigirtzoglou (2004) and Anagnou et al (2002). We also noted that, on average, the SPDF improved the forecast power, which is in line with the conclusion of the previous authors. However, contrary to what we may expect, the SPDFs were rejected as good forecasts of the objective densities more times than the RND. We stress, however, that our work is not directly comparable. Firstly, these authors considered different options on different underlying assets. Secondly, they extracted the RNDs using a different methodologies.

6 Historical behavior of the True and Risk Neutral Probabilities

In this section, we analyzed the information provided by the estimated RNDs and SPDFs for the period between June 2006 (half a year before the problems regarding the subprime crisis started to worsen) and September 2013 (after the peak of the sovereign crisis). The RNDs were obtained using the DFCH model (we only considered the 1 month term). The economic information was obtained in the Weekly Global Economic Brief, published by the World Bank.

Before continuing, a few concepts should be explained. An increase/decrease in the exchange rate USD/BRL means a depreciation/appreciation of the Brazilian Real (hereafter called BRL) in respect to the US Dollar. The probability of a large depreciation/appreciation of the BRL means the likelihood of an increase/decrease of the USD/BRL above 25%. The probability of a medium and small depreciation/appreciation of the BRL means the likelihood of an increase/decrease above 5% and 2% respectively.

Figure 1: DFCH model - Subjective Probability and RND of an increase in USD/BRL above 25%.

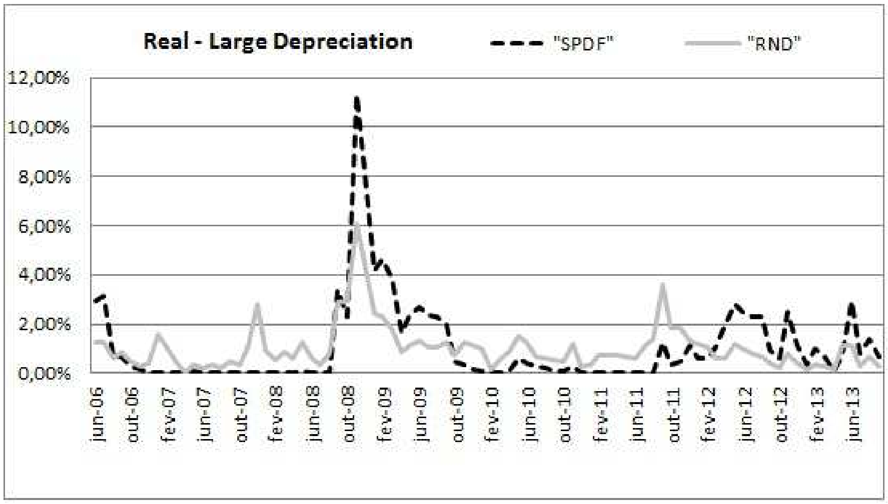


Figure 2: DFCH model - Subjective Probability and RND of a drop in USD/BRL above 25%.

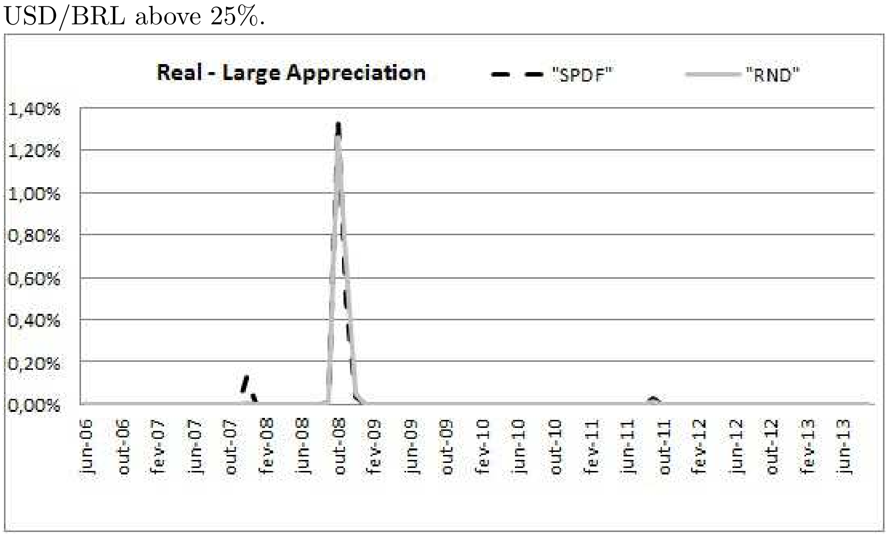


Figure 3: DFCH model - Subjective Probability and RND of an increase in USD/BRL above 5%.

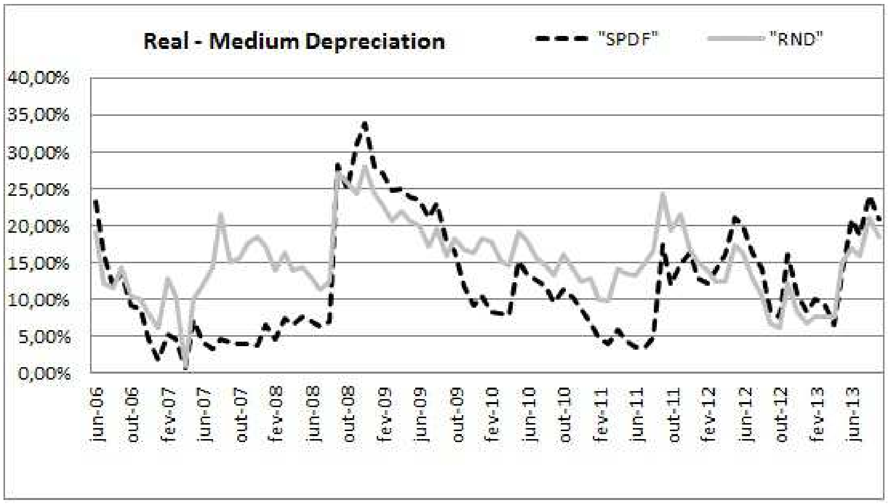
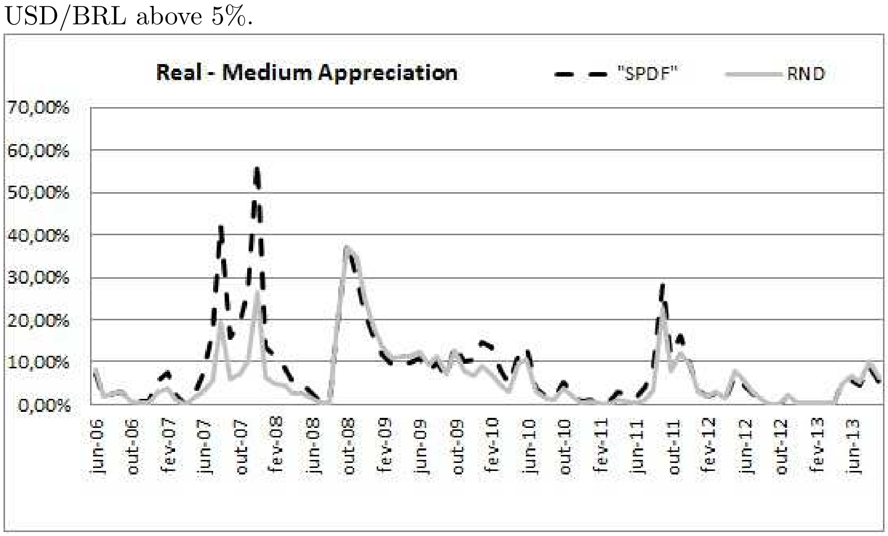


Figure 4: DFCH model - Subjective Probability and RND of a drop in USD/BRL above 5%.



6.1 From June 2006 to July 2007

The second half of 2006 and first half of 2007 was a period of BRL appreciation and volatility reduction. In order to avoid the BRL appreciation and protect Brazil exports, the COPOM (Brazil Monetary Policy Committee) proceeded consecutive SEDIC (overnight reference rate of the Brazilian inter-bank money market) rate cuts. This falling SEDIC led to the temporary decrease of the probabilities of a small/medium BRL appreciation. However, the probabilities of a small/medium BRL depreciation decreased more strongly during this period, which shows the lack of effectiveness of this FX intervention. In fact, the BRL continued to appreciate (Figure 8) as a result of the healthier Brazilian macroeconomic conditions when compared with US data and the increasing credit and mortgage issues (the subprime crisis).

6.2 From August 2007 to July 2008

The trend described in the previous paragraph (BRL appreciation and decrease in volatility) was temporarily interrupted between July and September 2007, with a peak in volatility and increase in USD/BRL. This BRL weakening was related to heightened fears that the subprime and credit crisis in US would potentially reduce the global risk appetite for the emerging markets. It was the first shock concerning the subprime crisis, with shortages and lack of liquidity in the money market (rumors about some financial institutions experiencing liquidity difficulties, such as Northern Rock).

After September 2007 and until July 2008, the BRL returned do its upward trend, but this time, this movement was also supported by an increase of the rate's differential between the Brazilian and US interest rates (In April 2008 COPOM started a series of four consecutive rises and in June 2006 the FED began a cycle of Federal Funds rate lowering). During this period the volatility slightly decrease and the investor's expectations revealed an asymmetry towards the BRL appreciation.

6.3 From August 2008 to February 2009

In August 2008 the BRL appreciation came to an end. Two reasons might have contributed to the end of the USD depreciation: the end of the rises in oil prices (historically there is a negative correlation between oil prices and the dollar) and commodity prices (as a commodity exporter Brazil's trade surplus would be negatively affected), and the improvement in the US Balance of Payments. There was also a huge increase in uncertainty, which could be seen in the abrupt rise of the probabilities of large USD/BRL fluctuations (BRL appreciation or depreciation above 25%), namely the probability of a large BRL depreciation. This upward movement in volatility reached its maximum in November 2008 after a sequence of negative events (Government-sponsored enterprises Fannie Mae and Freddie Mac which owned or guaranteed about half of the U.S mortgage market were being placed into conservatorship of the FHFA [1], Lehman Brothers led for bankruptcy, the Bank of America purchased Merril Lynch and the US government bailed out Goldman Sachs and Morgan Stanley) that increased the risk aversion and the fears that the capital inflows for the emerging economies such as Brazil would be reduced, which would depreciate its exchange rate.

In December 2008 the USD stopped its rally and the volatility started to decrease. This new trend was partially caused by the increase in the US quantitative easing[2] and by the decrease of the Fed Reserve Target Rate to 0.25% in December 2008.

6.4 From March 2009 to October 2009

In the period between March 2009 and October 2009 the BRL appreciation and the volatility's downward trend continued and come closer to the levels prior to the turbulent period that started in August 2008, which could indicate the perception in the financial markets that the worst of the global recession was over. Nevertheless, the volatility remained at higher levels than before the peak of the crisis.

6.5 After November 2009 and up to June 2011:

Although the fears of overheating in the Brazil economy and the first signs of economic recovery in the US, this period was characterized by a small BRL appreciation. However, the sharp decrease of the previous period ended and we even witnessed two strong USD/BRL corrections in the first semester of 2010: this pair reached a level close to 1,9 in February and May 2010.

In general terms, the probabilities of a real appreciation decreased, which indicates a diminishing probability of the declining trend in the USD/BRL. This probability's trend was temporary interrupted in April/May 2010 and October 2010 by events that could contribute to the BRL appreciation:

- April/May 2010 had seen the start of the cycle of rises in SELIC rates and the disclosure of a strong GDP growth rate in Brazil in the first quarter of 2010,
- October 2010. The RND was taken on the 29th October which was the eve of the second round of Brazilian Presidential elections in 31th October. There was also uncertain in the market regarding the size of the Fed's QE2 program that gave rise to an appreciation of the emerging market currencies against the dollar.

[1] The Federal Housing Finance Agency is an independent federal agency created on July 30, 2008, when the President George Bush signed into law the Housing and Economic Recovery Act of 2008. The Act objective was to create a world-class, empowered regulator with all of the authorities necessary to oversee vital components of US.s secondary mortgage markets. Fannie Mae, Freddie Mac, and the Federal Home Loan Banks.

[2] Quantitative easing was used by the FED to increase the supply of money by increasing the excess reserves of the banking system, through buying not only government bonds, but also troubled assets in order to improve the liquidity of these assets.

The volatility followed the decreasing trend from the previous period and in June 2011 reached the lowest level since the start of the subprime crisis.

6.6 After July 2011 and up to May 2012:

The period between July 2011 and September 2011 was marked by an abrupt rise in volatility and probability of large movements in the exchange rate USD/BRL, up to the highest level after the peak of the subprime crisis. The main events that occurred during these 3 months were:

- Intensification of the negotiations regarding the increase of the U.S. debt ceiling. From the date in which the debt ceiling was surpassed (May 16, 2011) until the date where the final agreement between republicans and Democrats passed on the Senate (August 2, 2011), the US debt issuance was substituted by "extraordinary measures" to fund federal obligations,
- Disappointing figures from the US and Global Growth pointed to a weakening of the economic activity. This situation triggered a strong rise of uncertainty and substantially increased the fear of a downturn in the world economy,
- Intensification of the sovereign debt crisis in the Euro Area. Unlike previous episodes of volatility in the Euro Area, this time there was contagion worldwide, partly prompted by the apparent inability of politicians to get in front of the crisis.

After a peak in volatility in September 2011, the uncertain decreased again in October 2011, as demonstrated in the decrease of both the standard deviation and probability of large movements (figures 7, 1 and 2). The positive reaction in the last week of October 2011 is partly related with an agreement in the Euro Area to strengthen the European Financial Stability Facility (EFSF) and to write down 50% of Greek debt.

6.7 June 2012 to April 2013

Decrease in uncertainty due to signs of improvement on Market sentiment after the agreement between Euro Area finance ministers in order to provide Euros 100bn package to recapitalized troubled Spanish banks (Bankia, Catalunya Banc, Banco de Valencia and Novagalicia Banco). The ratification of the Euros 700bn European Stability Mechanism (ESM) by the German Constitutional Court, which removed significant hurdles from deeper European integration, also helped to reduce the May/June high volatility.

The continuation of interest rate cuts in Brazil until October 2012 and to the release of better economic indicators in US (higher than expected US GDP growth in Q3 2012, recovering in housing Market – housing market reached a two year high- and increase in payroll jobs – 236 000 jobs were added in February) may have led to the considerable decrease in the probabilities of a BRL appreciation.

6.8 May 2013 to September 2013

Ben Bernanke announced a "tapering" of some Quantitative Easing measures conditionally upon continued positive economic data. The possibility that US could end its quantitative easing programs lead to an expectation of rise in long-term interest rates , which caused capital flows into US and an substantially increase of the likelihood of a large BRL depreciation.

7 Conclusion

With the aim to test if the RNDs and SPDFs are good predictors of the future asset values, we extracted the RNDs from the option prices on the exchange rate USD/BRL and analyzed its predictive power. Assuming that investors are rational, we also estimated the SPDF in order to check if the subjective probabilities provided any additional information. The probabilities supplied by these densities were also analyzed for the period between June 2006 and September 2013 and related with the main events that affected the options market during this period. The RNDs were estimated using two non-structural models (DFCH and MLN) and two structural models with stochastic volatility (CGMY Gamma-OU and VG Gamma-OU).

In general terms, we found that all the tested models produced RNDs and SPDF capable of predicting the future outcomes for the 1 month term. We stress, however, that the MLN underperformed substantially the other models and that all the models failed to predict the future outcomes for the 3 months term.The DFCH model slightly outperformed the two structural models.

The good performance of the DFCH, CGMY Gamma-OU and VG Gamma-OU for the shorter terms places them has potential good alternatives to the standard methods used in market and credit risk and pricing (e.g. Black and Scholes and Merton models - simplistic models without capacity to capture fat tails). In this context, we emphasize the importance of the CGMY Gamma-OU and VG Gamma-OU due to the advantages in using structural models for this purpose (due to their tractability).

Acknowledgements

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Figure 5: DFCH model - Subjective Probability and RND of an increase in USD/BRL above 2%.

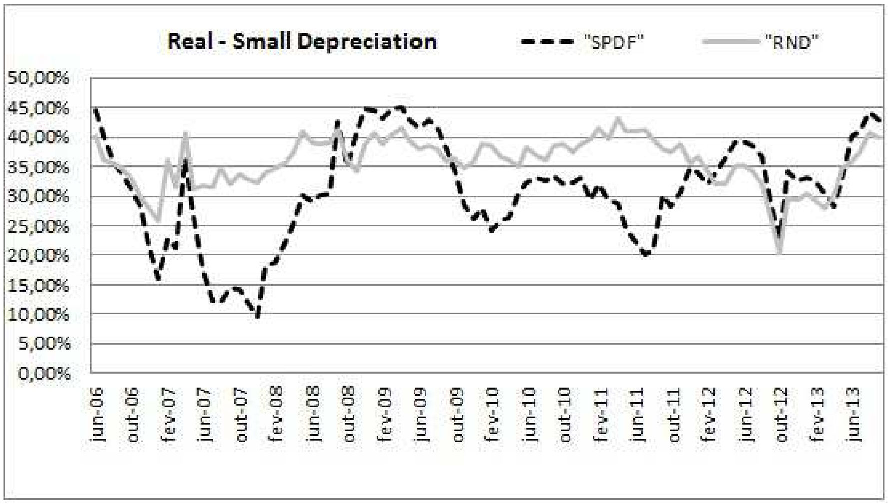
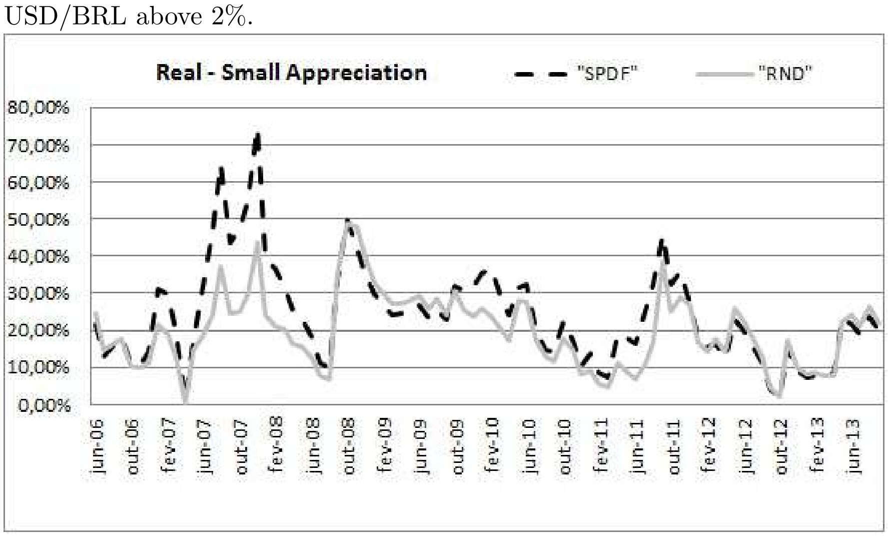


Figure 6: DFCH model - Subjective Probability and RND of a drop in USD/BRL above 2%.



8 Appendix

Table A.1: P-Value Berkowitz Test of 1 month term RND and SPDF. Outcome period of one year.

	DFCH		CGMY Gamma-OU		MLN		VG	
# p-value <5%	5	4	15	5	29	25	7	5
Average	45,17%	43,39%	46,12%	38,43%	28,57%	25,56%	43,50%	35,63%
Date	p-value SPDF	p-value RND	p-value SPDF	p-value RND	p-value SPDF	p-value RND	p-value SPDF	p-value RND
30-06-2006	52,49%	26,49%	15,74%	4,46%	28,28%	6,06%	10,51%	2,80%
31-07-2006	69,43%	46,83%	82,80%	30,55%	82,30%	54,46%	8,86%	20,14%
31-08-2006	60,51%	24,19%	46,03%	6,17%	38,13%	11,85%	6,06%	3,65%
29-09-2006	56,85%	16,45%	66,33%	10,60%	42,08%	13,26%	73,28%	8,70%
31-10-2006	50,40%	15,54%	57,53%	7,97%	33,64%	10,97%	57,03%	5,56%
30-11-2006	57,52%	23,61%	93,99%	17,18%	59,51%	23,12%	84,56%	16,01%
29-12-2006	52,28%	24,40%	94,38%	14,24%	60,74%	18,76%	84,21%	13,09%
31-01-2007	48,02%	12,72%	94,59%	8,62%	63,83%	15,93%	86,98%	8,91%
28-02-2007	34,22%	20,48%	88,80%	17,67%	38,78%	21,89%	90,38%	17,35%
30-03-2007	32,60%	17,48%	84,45%	15,00%	34,49%	17,90%	82,09%	13,95%
30-04-2007	86,97%	45,83%	84,67%	19,54%	42,41%	28,05%	68,12%	16,45%
31-05-2007	83,19%	38,26%	91,82%	17,71%	49,98%	25,37%	65,09%	14,60%
29-06-2007	89,20%	40,69%	93,94%	18,68%	54,30%	26,86%	65,58%	15,39%
31-07-2007	85,80%	47,21%	83,16%	21,80%	70,30%	27,04%	67,40%	17,15%
31-08-2007	22,71%	70,77%	3,83%	34,32%	0,24%	1,51%	16,60%	35,62%
28-09-2007	10,79%	55,40%	3,86%	39,83%	0,14%	1,07%	16,29%	44,24%
31-10-2007	7,96%	27,51%	3,86%	20,81%	0,28%	0,51%	14,56%	24,50%
30-11-2007	9,66%	21,46%	5,87%	18,74%	0,39%	0,45%	16,21%	21,59%
31-12-2007	13,95%	21,26%	5,14%	18,20%	0,30%	0,45%	15,32%	20,94%
31-01-2008	7,17%	13,42%	4,01%	17,43%	0,30%	0,43%	10,91%	19,91%
29-02-2008	13,68%	16,55%	8,79%	18,30%	0,52%	0,69%	21,05%	19,55%
31-03-2008	4,76%	5,80%	3,60%	7,32%	0,14%	0,29%	10,42%	6,91%
30-04-2008	1,61%	3,35%	0,81%	3,35%	0,02%	0,14%	3,22%	3,41%
30-05-2008	2,74%	7,75%	1,17%	7,96%	0,02%	0,22%	3,17%	8,86%
30-06-2008	2,77%	9,79%	1,00%	9,63%	0,01%	0,26%	2,13%	11,72%
31-07-2008	1,84%	4,02%	1,41%	4,52%	0,03%	0,33%	2,20%	5,45%
29-08-2008	7,91%	11,12%	10,24%	12,16%	0,90%	6,21%	8,99%	12,67%
30-09-2008	15,16%	20,29%	17,05%	20,63%	1,97%	19,62%	15,82%	20,72%
31-10-2008	16,95%	24,23%	18,18%	21,97%	2,04%	23,64%	17,64%	22,68%
28-11-2008	17,29%	21,86%	17,87%	19,58%	4,25%	20,96%	18,37%	19,69%
31-12-2008	46,68%	51,31%	41,80%	45,77%	19,14%	66,26%	36,82%	41,84%
30-01-2009	35,50%	26,20%	31,80%	24,28%	15,23%	39,18%	25,58%	19,89%
27-02-2009	59,78%	41,41%	55,87%	38,72%	32,83%	56,49%	46,45%	34,91%
31-03-2009	70,69%	40,81%	69,31%	39,63%	54,95%	56,22%	57,45%	34,66%
30-04-2009	87,38%	31,58%	88,74%	34,61%	92,10%	57,92%	82,60%	27,40%
29-05-2009	77,79%	25,69%	78,22%	27,29%	78,56%	50,00%	75,23%	26,37%
30-06-2009	64,24%	21,88%	64,93%	23,64%	75,80%	50,08%	62,67%	23,68%
31-07-2009	58,89%	21,30%	59,50%	23,28%	70,54%	49,59%	57,63%	23,38%
31-08-2009	57,13%	27,19%	67,72%	37,44%	69,17%	54,35%	60,03%	32,94%
30-09-2009	33,78%	19,59%	52,14%	34,07%	53,27%	47,47%	38,31%	25,63%
30-10-2009	34,56%	25,24%	54,59%	40,47%	52,14%	49,92%	44,71%	33,27%
30-11-2009	38,28%	23,55%	53,45%	33,05%	47,72%	41,06%	41,65%	25,70%
31-12-2009	15,38%	5,22%	26,72%	8,52%	36,98%	21,47%	14,95%	6,78%
29-01-2010	18,78%	7,38%	33,52%	12,70%	44,55%	30,40%	21,01%	11,02%
26-02-2010	19,68%	7,66%	34,41%	12,32%	45,78%	30,89%	22,22%	10,95%
31-03-2010	48,24%	25,68%	56,50%	25,90%	56,59%	42,79%	29,12%	21,44%
30-04-2010	14,15%	7,96%	26,18%	10,43%	13,54%	9,88%	12,63%	9,94%
31-05-2010	5,43%	2,56%	10,85%	3,08%	4,99%	2,24%	3,59%	2,44%
30-06-2010	7,56%	2,69%	13,75%	3,26%	8,38%	2,58%	3,79%	2,48%
30-07-2010	65,86%	14,47%	61,76%	13,97%	68,02%	15,55%	28,76%	10,82%
31-08-2010	42,64%	96,07%	24,46%	96,98%	0,41%	5,06%	80,53%	98,05%
30-09-2010	67,56%	88,93%	25,24%	71,34%	0,56%	4,43%	61,32%	47,73%
29-10-2010	47,07%	83,37%	13,24%	56,77%	0,21%	1,95%	29,18%	30,99%
30-11-2010	36,26%	99,25%	11,95%	85,52%	0,24%	2,61%	37,71%	65,90%
31-12-2010	35,99%	90,87%	5,78%	45,24%	0,08%	1,06%	20,46%	28,80%
31-01-2011	36,51%	93,42%	6,24%	49,01%	0,08%	1,06%	27,15%	34,33%
28-02-2011	18,26%	83,99%	2,32%	38,19%	0,01%	0,12%	11,02%	26,74%
31-03-2011	12,03%	81,12%	2,59%	62,64%	0,01%	0,14%	9,26%	51,59%
29-04-2011	7,92%	60,68%	1,54%	43,13%	0,00%	0,01%	5,48%	38,79%
31-05-2011	7,78%	54,21%	1,48%	42,39%	0,00%	0,01%	5,13%	39,83%
30-06-2011	8,14%	43,78%	1,55%	37,27%	0,00%	0,01%	4,85%	35,39%
29-07-2011	22,45%	63,56%	3,71%	47,93%	0,00%	0,01%	8,99%	43,10%
31-08-2011	78,31%	87,69%	67,63%	89,04%	15,74%	16,35%	45,96%	75,01%
30-09-2011	80,32%	74,90%	87,47%	93,62%	32,98%	24,42%	79,29%	96,50%
31-10-2011	75,00%	69,92%	83,35%	82,92%	32,84%	19,68%	83,57%	89,80%
30-11-2011	97,44%	96,55%	90,59%	94,72%	23,50%	24,47%	94,11%	98,18%
30-12-2011	92,09%	87,87%	94,29%	90,54%	37,43%	29,58%	90,32%	90,39%
31-01-2012	92,86%	87,71%	96,32%	93,56%	38,08%	31,18%	92,50%	94,45%
29-02-2012	74,41%	77,09%	89,68%	87,58%	71,84%	65,06%	76,43%	81,36%
30-03-2012	75,89%	85,43%	90,46%	93,60%	34,64%	95,17%	85,52%	87,92%
30-04-2012	67,23%	64,85%	75,92%	76,35%	7,72%	72,30%	77,79%	73,37%
31-05-2012	98,94%	94,79%	99,55%	98,31%	49,18%	91,86%	99,68%	99,31%
29-06-2012	97,92%	94,16%	98,39%	[illegible]	55,34%	90,55%	98,37%	99,02%
31-07-2012	89,25%	81,47%	90,05%	86,90%	60,56%	70,53%	90,33%	92,67%
31-08-2012	97,38%	94,96%	88,63%	91,21%	29,67%	69,58%	92,72%	96,02%
28-09-2012	97,21%	96,73%	86,30%	91,12%	29,68%	72,53%	90,15%	95,12%

Table A.2: P-Value Berkowitz Test of 1 month term RND and SPDF. Outcome periods correspond to the considered sub-periods between June 2006 and September 2013.

Average	55,77%	55,80%	54,02%	49,09%	44,05%	44,47%	46,62%	43,86%
	HYPER		CGMY Gamma-OU		MLN		VG	
Period	p-value SPDF	p-value RND	p-value SPDF	p-value RND	p-value SPDF	p-value RND	p-value SPDF	p-value RND
Jun2006-Jul2007	68,89%	43,42%	71,02%	19,31%	81,72%	42,94%	9,71%	11,81%
Aug2007-Jul2008	88,93%	77,15%	89,77%	64,27%	91,70%	74,08%	87,11%	54,46%
Aug2008-Feb2009	25,09%	19,70%	29,88%	19,49%	18,55%	14,50%	29,17%	24,94%
Mar2009-Oct2009	22,08%	25,55%	24,59%	26,05%	16,42%	23,78%	21,28%	24,35%
Nov2009-Jun2011	19,19%	7,77%	27,50%	9,39%	39,17%	24,71%	17,47%	5,69%
Jul2011-May2012	45,48%	95,00%	14,53%	75,38%	0,04%	0,52%	30,21%	53,00%
Jun2012-Apr2013	81,72%	80,22%	84,01%	85,16%	11,29%	76,67%	84,49%	82,14%
May2013-Sep2013	94,79%	97,56%	90,84%	93,66%	93,51%	98,57%	93,49%	94,50%

Table A.3: P-Value Berkowitz Test of 3 months term RND and SPDF. Non-overlapping data during and outcome period of one year.

Average	7,56%	3,78%	5,32%	3,10%
	HYPER		CGMY Gamma-OU	
Date	p-value	p-value RND	p-value	p-value RND
30-06-2006	5,91%	0,23%	0,07%	0,00%
29-09-2006	13,63%	0,03%	0,00%	0,00%
29-12-2006	12,21%	0,14%	0,00%	0,00%
30-03-2007	72,01%	1,31%	0,08%	0,01%
29-06-2007	10,38%	13,00%	0,68%	0,14%
28-09-2007	0,00%	0,03%	0,02%	0,04%
31-12-2007	0,00%	0,02%	0,02%	0,04%
31-03-2008	0,00%	0,01%	0,00%	0,00%
30-06-2008	0,00%	0,00%	0,00%	0,00%
30-09-2008	0,19%	0,02%	0,01%	0,01%
31-12-2008	0,12%	0,23%	0,04%	0,06%
31-03-2009	5,46%	5,94%	5,20%	3,39%
30-06-2009	22,09%	12,53%	33,43%	10,68%
30-09-2009	24,14%	11,96%	35,78%	9,82%
31-12-2009	4,93%	0,35%	5,11%	0,30%
31-03-2010	0,01%	0,00%	0,54%	0,02%
30-06-2010	2,34%	1,52%	4,73%	0,58%
30-09-2010	0,31%	1,64%	0,50%	0,45%
31-12-2010	0,00%	4,28%	0,62%	0,92%
31-03-2011	0,00%	4,61%	2,72%	8,66%
30-06-2011	0,00%	1,85%	1,44%	3,25%
30-09-2011	0,00%	9,80%	13,92%	9,64%
30-12-2011	0,00%	8,25%	9,50%	9,65%
30-03-2012	0,00%	12,33%	14,78%	14,48%
29-06-2012	18,94%	4,70%	5,25%	4,81%
28-09-2012	3,92%	3,45%	3,85%	3,62%

Figure 7: DFCH model - RND Standard Deviation.

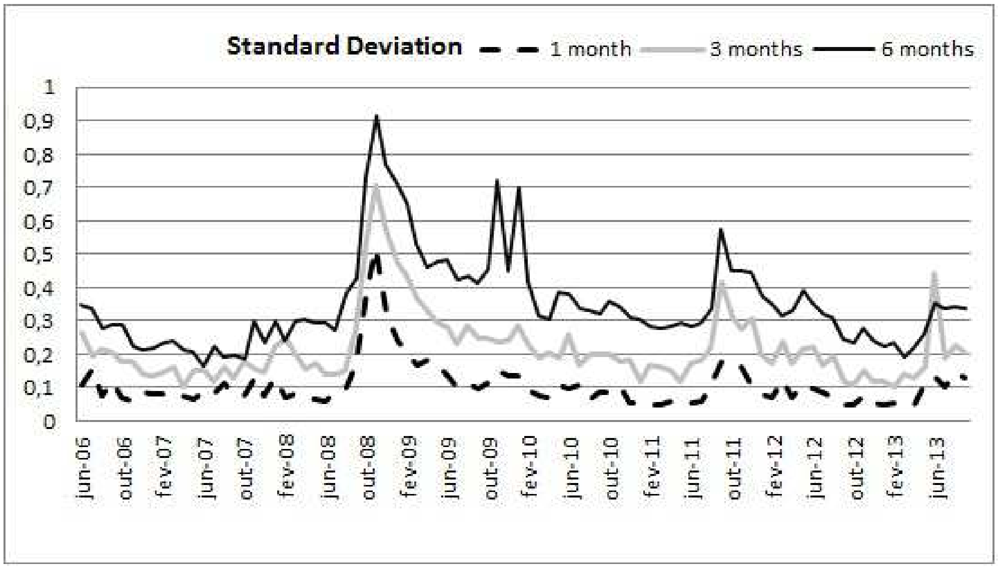


Figure 8: Exchange rate USD/BRL between June 2006 and September 2013.

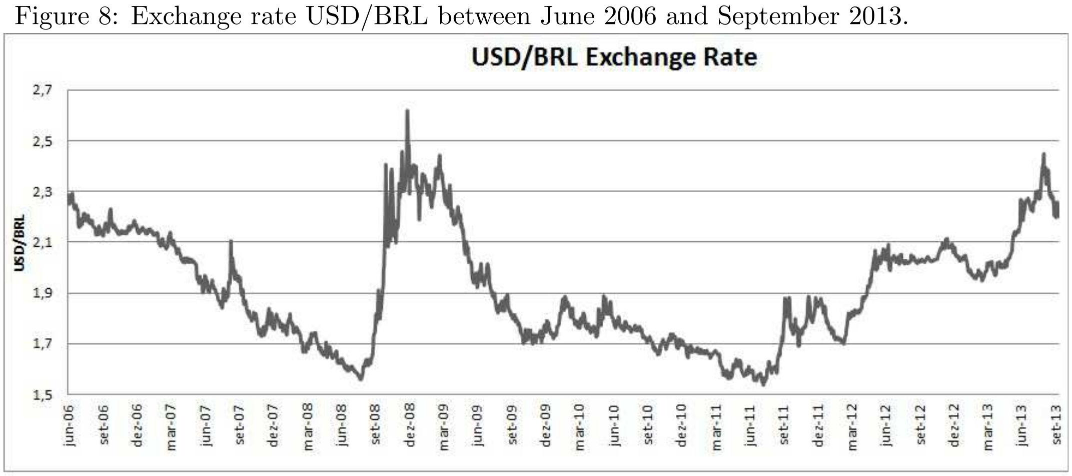


Figure 9: SELIC reference rate between June 2006 and September 2013.

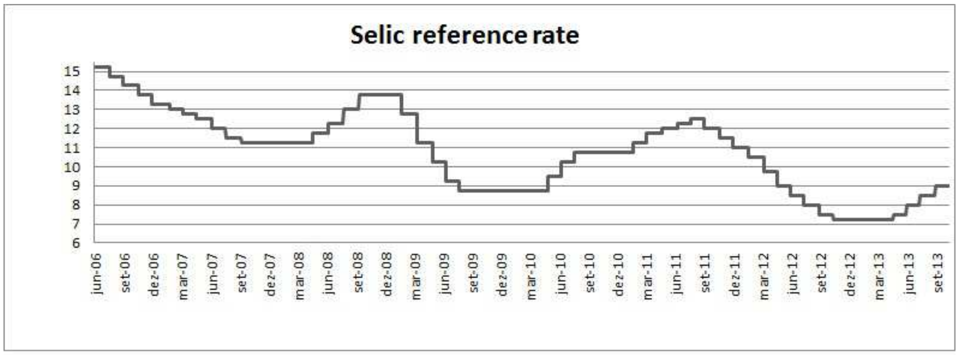
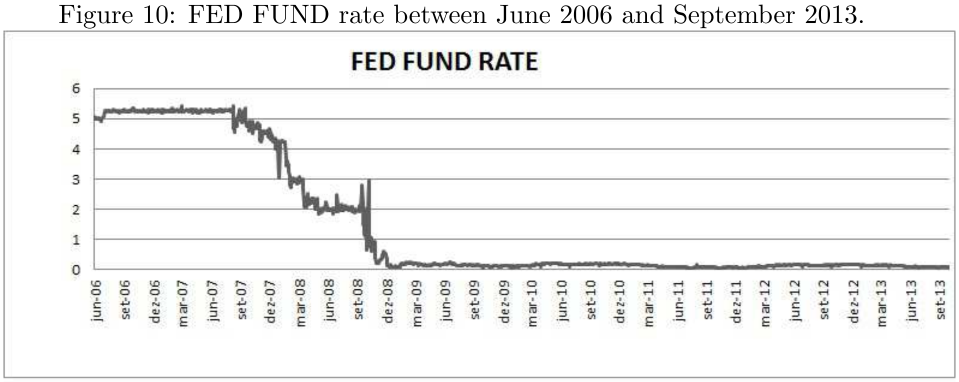


Figure 10: FED FUND rate between June 2006 and September 2013.



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