Ruin problems in the generalized Erlang(n) risk model

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Abstract

For actuarial applications we consider the Sparre–Andersen risk model when the interclaim times are Generalized Erlang(n) distributed. Unlike the standard Erlang(n) case, the roots of the generalized Lundberg's equation with positive real parts can be multiple. This has a significant impact in the formulae for ruin probabilities that have to be found.

We start by addressing the problem of solving an integro–differential equation that is satisfied by the survival probability, as well as other probabilities related, and present a method to solve such equation. This is done by considering the roots with positive real parts of the generalized Lundberg's equation, and then establishing a one-one relation between them and the solutions of the integro–differential equation mentioned above. We first study the cases when all the roots are single and when there are roots with higher multiplicity. Secondly, we show that it is possible to have double roots but no higher multiplicity. Also, we show that the number of double roots depend on the choice of the parameters of the generalized Erlang(n) distribution, with a maximum number depending on n being even or odd.

Afterwards, we extend our findings above for the computation of the distribution of the maximum severity of ruin as well as, considering an interest force, to the study the expected discounted future dividends, prior to ruin. Our findings show an alternative and more general method to the one provided by Albrecher *et al.* (2005), by considering a general claim amount distribution.

Keywords: Sparre-Andersen risk model; Generalized Erlang(n) interclaim times; generalized Lundberg's equation; probability of reaching an upper barrier; maximum severity of ruin; expected discounted dividends prior to ruin.

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1 Introduction

In the present paper we work with the Sparre–Andersen model driven by equation

$$U(t)=u+ct-\sum_{i=0}^{N(t)}X_i,\ \ t\geq 0,$$

where $X_0\equiv 0$, u (≥ 0) is the initial capital, c (>0) is the premium income per unit time t, $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables (i.i.d.), each representing a single claim amount, with common distribution function $P(x)$, $P(0)=0$, and density $p(x)$. Its Laplace transform is denoted by $\hat{p}(.)$. We denote by $\mu_k=E[X_1^k]$ the k-th moment of X_i. The first moment μ_1 is assumed to exist as a general condition, higher moments are only required for moment calculations in Sections 4 and 5.

The sequence $\{X_i\}$ is independent of the counting process $\{N(t),t\geq 0\}$, with $N(t)=\max\{k: W_1+W_2+\cdots+W_k\leq t\}$ where the random variables W_i, $i\in\mathbb{N}^+$, are i.i.d. with common distribution generalized Erlang(n), with parameters $\lambda_1,\lambda_2,\ldots,\lambda_n$. The common probability density function of W_i, $i\in\mathbb{N}^+$, $k_n(t)$, is given by

$$k_n(t)=\sum_{i=1}^{n}\left(\prod_{j=1,j\neq i}^{n}\frac{\lambda_j}{\lambda_j-\lambda_i}\right)\lambda_i e^{-\lambda_i t},\quad n\in\mathbb{N}^+,$$

and the distribution function is

$$K_n(t)=1-\sum_{i=1}^{n}\left(\prod_{j=1,j\neq i}^{n}\frac{\lambda_j}{\lambda_j-\lambda_i}\right)e^{-\lambda_i t}.$$

We assume a positive loading factor, that is $cE(W_1)>E(X_1)$, equivalent to $c\sum_{i=1}^{n}\lambda_i^{-1}>\mu_1$.

Next, we set some definitions, notation and mathematical preliminaries regarding our main objects of interest in the Sparre–Andersen model. The time to ruin is denoted as $T=\inf\{t>0: U(t)<0|U(0)=u\}$, and $T=\infty$ if and only if $U(t)\geq 0\ \forall\, t>0$. The ultimate ruin probability is defined as $\Psi(u)=\Pr(T<\infty|U(0)=u)$ and the corresponding non–ruin probability (or survival probability) as $\Phi(u)=1-\Psi(u)$.

If we set an upper barrier $b\geq u$ regarding the payment of dividends, we denote by $\tau_b=\inf\{t>0: U(t)\geq b|U(0)=u\}$ the first time that the surplus upcrosses the level $b\geq u$. The probability that the surplus attains the level b from initial surplus u without first falling below zero is given by

$$\chi(u,b)=\Pr(T>\tau_b|U(0)=u),$$

with $\xi(u,b)=1-\chi(u,b)$ being the probability that ruin occurs from u before the surplus ever reaching b.

The probability that ruin occurs and that the deficit at ruin is at most y is given by $G(u,y)=\Pr(T<\infty,U(T)\geq -y|U(0)=u)$. For a given u, this is a defective distribution function, clearly $\lim_{y\to\infty}G(u,y)=\Psi(u)$. The corresponding (defective) density is denoted as $g(u,y)$. If we assume that the surplus process continues after ruin, we denote the time of the first upcross of the surplus through level "0" after ruin has occurred by $T'=\inf\{t:\ t>T,\ U(t)\geq 0\}$, for finite T. In the time interval where the surplus is at deficit, we define the maximum severity of ruin as

$$M_u=\sup\{|U(t)|:\ T\leq t\leq T'|U(0)=u\}.$$

We note that $M_u \geq |U(T)|$. The conditional distribution function of the maximum severity of ruin, given that ruin occurs, is defined as

$$J(z;u) = \Pr(M_u \leq z | T < \infty),\ u,z \geq 0.$$

We further consider the problem where an insurance portfolio is supposed to provide dividend income to the insurance company shareholders. Let constant $b \geq u$ be the dividend barrier, so that if the process upcrosses b dividends are payable continuously to the shareholders at rate c until a new claim occurs. Let the random variable D_u denote the present value, at force of interest $\delta(>0)$, of dividends payable to shareholders until ruin occurs, and denote the existing m-th moment of D_u by $V_m(u,b) = E[D_u^m]$, $m \geq 0$, where $V_0(u,b) \equiv 1$. For simplicity we will denote $V_1(u,b) = V(u,b)$. We assume the existence of $V_m(u,b)$.

This paper generalizes results previously presented by Bergel and Egídio dos Reis (2013), and therefore of Li (2008), who worked on the Erlang(n) renewal model. This generalization, from Erlang(n) to generalized Erlang(n) interarrival times, is by no means straightforward. This is an important generalization as unlike the former case, the roots of the generalized Lundgerg's equation can be multiple, as a consequence new formulae for ruin probabilitites have to be found. We start by showing two different theorems, the first proves the possible existence of double roots of the generalized Lundgerg's equation, the second proves the non existence of higher multiplicity. The number of double roots depends on the parameter choice and the parameter n of the generalized Erlang(n) distribution.

After, we use our findings and work further formulae for the probability that the surplus attains the upper level b ($\geq u$) from initial surplus u without first falling below zero, for the distribution of the maximum severity of ruin, and study formulae for the moments of discounted future dividends, when a dividend barrier is assumed.

The work flows as follows. In the next section we present some of the mathematical background on the model related to our problem. In Sections 3 and 4 we study the integro–differential equation and show explicit formulae for the maximum severity of ruin. On Section 5 we give attention to the dividends problem. Finally, in the last section we state some concluding remarks.

2 Mathematical background

In recent years the Sparre–Andersen risk model has been a major point of interest in risk and ruin theory. Many authors devoted their attention and did important advances in the topic. In this paper we present some new developments.

Following Gerber and Shiu (2003), we can prove that $\chi(u,b)$ satisfies an order n integro–differential equation with n boundary conditions and that can be written in the form

$$B(\mathcal{D})v(u) = \int_0^u v(u-y)p(y)dy, \quad u \geq 0, \tag{2.1}$$

where

$$B(\mathcal{D}) = \prod_{i=1}^{n} \left(I - \left(\frac{c}{\lambda_i} \right) \mathcal{D} \right),$$

and $\mathcal{D}$ is the differential operator. If we find n linearly independent particular solutions $v_j(u)$, $j = 1,\ldots,n$ for this equation, then we have

$$\chi(u,b) = \vec{v}(u)[\mathcal{V}(b)]^{-1}\vec{e}^{\mathsf{T}}, \tag{2.2}$$

where $\vec{v}(u) = (v_1(u), \ldots, v_n(u))$ is a $(1 \times n)$ vector of solutions, $\mathcal{V}(b)$ is a $(n \times n)$ matrix with entry (i, j) given by

$$(\mathcal{V}(b))_{ij} = \left. \frac{d^{i-1} v_j(u)}{du^{i-1}} \right|_{u=b},$$

and $\vec{e} = (1, 0, \ldots, 0)$ is a $(1 \times n)$ vector.

In this manuscript we will be seeking for those solutions, which in turn depend on the roots of the fundamental Lundberg's equation. Recall that for this case the fundamental Lundberg's equation is given by

$$\prod_{i=1}^{n} \left(1 - \frac{c}{\lambda_i} s\right) = \hat{p}(s). \tag{2.3}$$

We denote by the numbers $\rho_1, \rho_2, \ldots, \rho_{n-1} \in \mathbb{C}$, the roots of this equation which have positive real parts (there are of course other roots, among them is 0 and $-R$, where $R > 0$ is the adjustment coefficient, see Li and Garrido (2004a)).

On the other hand the generalized Lundberg's equation is given by

$$\prod_{i=1}^{n} \left(1 + \frac{\delta}{\lambda_i} - \frac{c}{\lambda_i} s\right) = \hat{p}(s), \tag{2.4}$$

where the constant $\delta > 0$ can be seen as the force of interest. This equation has exactly n roots with positive real parts (see Li and Garrido (2004a)) and will be considered in Section 5.

In a previous article, following Li (2008), Bergel and Egídio dos Reis (2013), found a vector of solutions $\vec{v}(u)$ for the case when the interclaim times follow an Erlang(n) distribution. Here, our work starts by giving the corresponding version of $\vec{v}(u)$ when we have generalized Erlang(n) interclaim times. This will be given in the next section. Then, we apply our results in order to find the corresponding expressions for the distribution of the maximum severity of ruin. Afterwards, we deal with the dividends problem, we mean the calculation of the m-th moment $V_m(u, b)$. For the compound Poisson model (or Erlang(1) model), an integro–differential equation for $V(u, b)$ can be found in Dickson (2005), and for $V_m(u, b)$ in Dickson and Waters (2004). For the generalized Erlang(n) model we give the respective integro–differential equations as well as a method to find their solutions, providing an alternative to the method given by Albrecher *et al.* (2005).

2.1 Multiplicity of the roots of the generalized (fundamental) Lundberg's equation

We study briefly the possibility that the Lundberg's equations (2.4) have multiple roots, more precisely double roots. We can rewrite equations (2.4) and (2.3) in the following form

$$B_\delta(s) = \hat{p}(s), \quad B(s) = \hat{p}(s),$$

where $B_\delta(s) = \prod_{i=1}^{n} \left(1 + \frac{\delta}{\lambda_i} - \frac{c}{\lambda_i} s\right)$ and $B(s) = \prod_{i=1}^{n} \left(1 - \frac{c}{\lambda_i} s\right)$ respectively.

Theorem 2.1. *Let s_1 and s_2 be two consecutive positive real zeros of $B_\delta(s)$. If $B_\delta(s)$ is positive in the interval (s_1, s_2) then the generalized Lundberg's equation has one of the following:*

- *Two real roots in the interval;*
- *A double root in the interval;*

- *No real roots on this interval.*

Proof: The proof is based on a comparison of both sides of the equation (2.4). We observe that for $s \in \mathbb{R}^+$, the Laplace transform $\hat{p}(s)$ is a positive and decreasing function of s, with $\hat{p}(0) = 1$ and $lim_{s\to\infty}\hat{p}(s) = 0$. In addition, $\hat{p}(s)$ is a convex function. Therefore $\hat{p}(s)$ has no zeros or poles in $s \in \mathbb{R}^+$. (In Figure 1 we show an example of $\hat{p}(s)$ when the interclaim arrivals are generalised Erlang(3) distributed, from Example 2.1).

Assume that $B_\delta(s_1) = B_\delta(s_2) = 0$ and $B_\delta(s) > 0$ in the interval (s_1, s_2). Define the distance between $\hat{p}(s)$ and $B_\delta(s)$ in the interval (s_1, s_2) as

$$d = d(\hat{p}, B_\delta) = \min_{x_1, x_2 \in (s_1, s_2)} \left\{ \sqrt{(x_1 - x_2)^2 + (\hat{p}(x_1) - B_\delta(x_2))^2} \right\}$$

If $d > 0$ there are no real roots of the generalized Lundberg's equation in (s_1, s_2), and if $d = 0$ we will have either two real roots or a double real root in (s_1, s_2). Additionally, in the case of a double real root, since $d = 0$ exactly for one point in the interval (s_1, s_2), the function $B_\delta(s)$ must be concave in the vicinity of this point.

□

A similar theorem can be stated for the fundamental Lundberg's equation (2.3), as follows.

Corollary 2.1. *Let s_1 and s_2 be two consecutive positive real zeros of $B(s)$. If $B(s)$ is positive in the interval (s_1, s_2) then the fundamental Lundberg's equation has one of the following:*

- *Two real roots in the interval.*
- *A double root in the interval.*
- *No real roots on this interval.*

Next example illustrates the three situations considered above.

Example 2.1. *Suppose that the interclaim times W_i follow a generalized Erlang(3) distribution, with parameters $\lambda_1 = 0.5, \lambda_2 = 1.5, \lambda_3 = 2.5$. Then $E[W_i] = 3.067$. Suppose that the claim amounts X_i are exponentially distributed with parameter $\beta \geq 0.5$. Then we choose $c = 1$ to satisfy the positive loading condition and let $\delta = 0.5$.*

Notice that the generalized Lundberg's equation becomes

$$B_\delta(s) = \frac{(1-s)(2-s)(3-s)}{1.875} = \frac{\beta}{\beta + s} = \hat{p}(s)$$

The function $B_\delta(s)$ has 3 zeros at $s = 1, 2, 3$, furthermore it is positive in the interval $(2,3)$. Then it is easy to verify that the fundamental Lundberg's equation has:

- *Two real roots in $(2,3)$ for $0.5 \leq \beta < 0.67003513333375991355$;*
- *A double root 2.61013 in $(2,3)$ for $\beta = 0.67003513333375991355$;*
- *Two complex conjugate roots, where the real part of them is in $(2,3)$ for $\beta > 0.67003513333375991355$.*

See Figure 1, with β rounded to 0.67. $*$

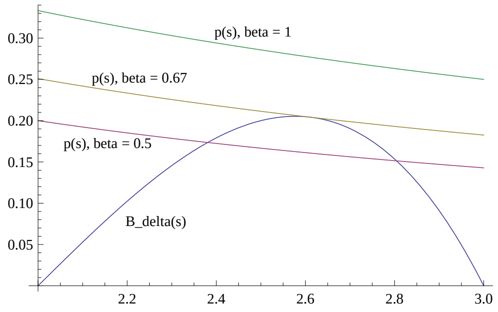

Figure 1: Example of the roots of the generalized Lundberg's equation

Next, we show that the fundamental Lundberg's equation can have more than one double positive real root. We look for the conditions that must be satisfied by the parameters of the Generalized Erlang distribution that would give the maximum possible number of double positive real roots in the fundamental Lundberg's equation. For this purpose we assume that the claim amounts X_i are exponentially distributed with parameter β.

Theorem 2.2. *The fundamental Lundberg's equation $B(s) = \hat{p}(s) = \beta/(\beta+s)$ can have at most $(n-1)/2$ double positive real roots for n odd, and $(n-2)/2$ double positive real roots for n even.*

Proof: Assume that n is odd. Since the fundamental Lundberg's equation has $n-1$ roots with positive real parts, the maximum possible number of double positive real roots would be $(n-1)/2$. Then the logic is as follows: we start assuming that the fundamental Lundberg's equation has $(n-1)/2$ double positive real roots and we find the conditions that the parameters λ_i of the generalized Erlang(n) distribution must satisfy to support this assumption.

Let $\rho_1, \rho_2, \ldots, \rho_{\frac{n-1}{2}}$ be the double positive real roots. Then

$$\begin{aligned} B(s) - \hat{p}(s) &= \prod_{i=1}^{n}\left(1-\frac{c}{\lambda_i}s\right) - \frac{\beta}{\beta+s} \\ &= \frac{\prod_{i=1}^{n}\left(1-\frac{c}{\lambda_i}s\right)(\beta+s)-\beta}{\beta+s} = \frac{(-c)^n}{\prod_{i=1}^{n}\lambda_i}\frac{s(s+R)\prod_{j=1}^{\frac{n-1}{2}}(s-\rho_j)^2}{\beta+s}, \end{aligned} \tag{2.5}$$

where R denotes the adjustment coefficient, $0 < R < \beta$.

Considering the parameters λ_i as unknowns we can compare the coefficients of s^i, for $i = 1, \ldots, n$ in Equation (2.5) to obtain a system of n equations on the variables λ_i, as follows

$$\begin{cases} \widetilde{\lambda}_1 &= c(\widetilde{\rho}_1 + (\beta - R)) \\ \widetilde{\lambda}_2 &= c^2(\widetilde{\rho}_2 + (\beta-R)\widetilde{\rho}_1 + \beta(\beta-R)) \\ \vdots & \\ \widetilde{\lambda}_{n-1} &= c^{n-1}(\widetilde{\rho}_{n-1} + (\beta-R)\sum_{k=0}^{n-2}\beta^k\,\widetilde{\rho}_{n-2-k}) \\ \widetilde{\lambda}_n &= c^n(\beta-R)\sum_{k=0}^{n-1}\beta^k\,\widetilde{\rho}_{n-1-k}, \end{cases} \tag{2.6}$$

where

$$\widetilde{\lambda}_i = \mathrm{SYM}_i(\lambda_1, \ldots, \lambda_n)$$

is the elementary symmetric polynomial of degree i on the n variables $\lambda_1, \lambda_2, \ldots, \lambda_n$ and

$$\widetilde{\rho_j} = \mathrm{SYM}_j(\rho_1, \rho_1, \rho_2, \rho_2 \ldots, \rho_{\frac{n-1}{2}}, \rho_{\frac{n-1}{2}})$$

is the elementary symmetric polynomial of degree j on the $n-1$ variables $\rho_1, \rho_1, \rho_2, \rho_2 \ldots, \rho_{\frac{n-1}{2}}, \rho_{\frac{n-1}{2}}$.

Therefore, the fundamental Lundberg's equation has $(n-1)/2$ double positive real positive roots if the system (2.6) has solution for the parameters λ_i such that these parameters are all positive and distinct.

If n is even, then $n-1$ is odd, and the maximum possible number of double positive real roots would be $(n-2)/2$. Proceeding with the same logic as above, we obtain a system like (2.6) on the variables λ_i. $\square$

A similar result can be obtained for the generalized Lundberg's equation. So far we have investigated the existence of double positive real roots in the Lundberg's equations for a Sparre–Andersen model with generalized Erlang(n) interclaim times. Now we study the possibility of higher order real roots.

Theorem 2.3. *The multiplicity of the positive real roots of the fundamental Lundberg's equation is at most two.*

Proof: We proceed by contradiction. Suppose that the fundamental Lundberg's equation $B(s) = \hat{p}(s)$ has a root $\rho > 0$ with multiplicity $k > 2$. This means that

$$\begin{cases} B(\rho) & = & \hat{p}(\rho) \\ B'(\rho) & = & \hat{p}'(\rho) \\ B''(\rho) & = & \hat{p}''(\rho) \\ \vdots & & \\ B^{(k-1)}(\rho) & = & \hat{p}^{(k-1)}(\rho). \end{cases}$$

Since $k > 2$ we have in particular $B(\rho) = \hat{p}(\rho)$ and $B'(\rho) = \hat{p}'(\rho)$. Then it is easy to verify that $B''(\rho) < 0$ for any root ρ satisfying these two conditions, because the function $B(s)$ is concave around ρ (see the proof of Theorem 2.1). On the other hand we have $\hat{p}''(\rho) > 0$ because $\hat{p}''(s) > 0\ \forall s \geq 0$ because the function $\hat{p}(s)$ is always convex. This implies that $B''(\rho) \neq \hat{p}''(\rho)$ and we arrive to a contradiction. $\square$

3 Solutions for the integro-differential equation

We look now for n linearly independent particular solutions $v_j(u),\ j = 1, \ldots, n$ of the integro-differential equation (2.1). For that purpose we need to use the roots of the fundamental Lundberg's equation that have positive real parts (denoted as $\rho_i,\ i = 1, \ldots, n-1$) and the non–ruin probability $\Phi(u)$ in the following manner,

Theorem 3.1. *If $\rho_1, \rho_2, \ldots, \rho_{n-1} \in \mathbb{C}$ are distinct, then the following functions are linearly independent particular solutions of the integro-differential equation (2.1):*

$$\begin{aligned} v_j(u) &= \int_0^u \Phi(u-y) e^{\rho_j y} dy, \quad j = 1, 2, \ldots, n-1, \\ v_n(u) &= \Phi(u). \end{aligned}$$

Proof:

It can be shown that any solution $v(u)$ of (2.1) has Laplace transform

$$\hat{v}(s) = \frac{d_v(s)}{B(s) - \hat{p}(s)},$$

where $d_v(s)$ is a polynomial of degree at most $n-1$ of the form

$$\begin{aligned} d_v(s) &= \sum_{i=0}^{n-1} \left(\sum_{k=i+1}^{n} \left(\sum_{i_1 < \cdots < i_k} \frac{(-1)^k}{\lambda_{i_1} \cdots \lambda_{i_k}} \right) v^{(k-1-i)}(0) \right) s^i \\ &= \sum_{i=0}^{n-1} \left(\sum_{k=i+1}^{n} B_k v^{(k-1-i)}(0) \right) s^i, \end{aligned}$$

where the coefficient B_k is given by $B_k = \displaystyle\sum_{i_1 < \cdots < i_k} \frac{(-1)^k}{\lambda_{i_1} \cdots \lambda_{i_k}}$.

It is known that $\Phi(u)$ is solution of (2.1), see Li (2008), its Laplace transform is given by

$$\hat{\Phi}(s) = -\Phi(0) \left(\frac{c^n}{\prod_{i=1}^{n} \lambda_i} \right) \frac{\prod_{i=1}^{n-1} (\rho_i - s)}{B(s) - \hat{p}(s)} = \frac{d_\Phi(s)}{B(s) - \hat{p}(s)},$$

therefore we have

$$d_\Phi(s) = -\Phi(0) \left(\frac{c^n}{\prod_{i=1}^{n} \lambda_i} \right) \prod_{i=1}^{n-1} (\rho_i - s).$$

Now, let's see that any function $v_j(u) = \int_0^u \Phi(u-y) e^{\rho_j y} dy$, with $j = 1, 2, \ldots, n-1$, is solution of (2.1). We can show that

$$B(\mathcal{D}) v_j(u) = d_\Phi(\rho_j) e^{\rho_j u} + \int_0^u (B(\mathcal{D}) \Phi(u-t)) e^{\rho_j t} dt$$

and that

$$\int_0^u v_j(u-y) p(y) dy = \int_0^u (B(\mathcal{D}) \Phi(u-t)) e^{\rho_j t} dt.$$

Since $d_\Phi(\rho_j) = 0$, $j = 1, 2, \ldots n-1$, we get the desired equality. It remains to prove that those $v_j(u)$'s are linearly independent.

Suppose that we have a linear combination such that $\sum_{j=1}^{n} c_j v_j(u) = 0$, $\forall u \geq 0$. Consider the cases (i) and (ii) below.

(i) $c_n = 0$:

Let $H(t) = \sum_{j=1}^{n-1} c_j e^{\rho_j t}$, then

$$\begin{aligned} \sum_{j=1}^{n} c_j v_j(u) &= \sum_{j=1}^{n-1} c_j \int_0^u \Phi(u-y) e^{\rho_j y} dy \\ &= \int_0^u \Phi(u-y) \sum_{j=1}^{n-1} c_j e^{\rho_j y} dy \\ &= \Phi * H(u) = 0. \end{aligned}$$

The fact that $\Phi * H(u) = 0$, $\forall u \geq 0$ with $\Phi(u) \not\equiv 0$, implies $H(u) \equiv 0$ almost everywhere. But $H(t)$ is a continuously differentiable function, this implies that $c_1 = c_2 = \cdots = c_n = 0$.

(ii) $c_n \neq 0$:

Define $G(t) = \sum_{j=1}^{n-1} (-c_j/c_n) e^{\rho_j t}$, so $\Phi * G(u) = \Phi(u)\ \forall u \geq 0$. Not all the remaining coefficients c_j's can be 0, otherwise $G(t) \equiv 0$. But then $\lim_{u\to+\infty} G(u) = \pm\infty$ depending on the sign of the non zero coefficients. As $\Phi(u)$ is a non–decreasing non–negative function with $\lim_{u\to+\infty} \Phi(u) = 1$, we will have that $\lim_{u\to+\infty} \Phi * G(u) = \pm\infty$, which is a contradiction.

This completes the proof. □

We have shown a set of n linearly independent particular solutions of the integro-differential equation (2.1) for the case when the roots $\rho_1, \rho_2, \ldots, \rho_{n-1} \in \mathbb{C}$ are distinct. Now we will show the corresponding particular solutions for the case of existing multiple roots.

First suppose that we have one root with multiplicity $n-1$. From Theorem 2.3 we know that there are no positive real roots with multiplicity higher than 2. However we decided to include the following theorem to illustrate in detail our method of finding the solutions of 2.1 in the case of multiplicities.

Theorem 3.2. *If $\rho_1 = \rho_2 = \ldots = \rho_{n-1} = \rho$ then the following functions are linearly independent particular solutions of the integro-differential equation (2.1):*

$$\begin{aligned} v_j(u) &= \int_0^u \Phi(u-y) y^{j-1} e^{\rho y} dy, \quad j = 1, 2, \ldots, n-1, \\ v_n(u) &= \Phi(u). \end{aligned}$$

Proof: In the same way we can prove by direct computation of the derivatives of the $v_j(u)$'s that

$$B(\mathcal{D}) v_j(u) = \int_0^u v_j(u-y) p(y) dy, \quad j = 1, 2, \ldots, n.$$

To see the linear independence of the $v_j(u)$'s we can proceed like in the proof of Theorem 3.1. □

Now, assume the most general case, when we have k ($1 \leq k \leq n-1$) different roots, $\rho_1, \rho_2, \ldots, \rho_k$, where the root ρ_i has multiplicity m_i and that $\sum_{i=1}^k m_i = n-1$.

Theorem 3.3. *Under the conditions described above, the following functions are linearly independent particular solutions of the integro-differential equation (2.1)*

$$\begin{aligned} v_{00}(u) &= \Phi(u), \\ v_{ij}(u) &= \int_0^u \Phi(u-y) y^{j-1} e^{\rho_i y} dy, \quad i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, m_i. \end{aligned}$$

Proof: The proof of this theorem is based on the proofs of Theorems 3.1 and 3.2. Basically if we have k different roots $\rho_i, i = 1, \ldots, k$, and ρ_i has multiplicity m_i, with $\sum_{i=1}^k m_i = n-1$, then we combine the two previous theorems to obtain this result. □

3.1 A note on the survival probability

From the previous analysis we can recover some particularly interesting information regarding the survival probability and its derivatives when $u = 0$. Li and Garrido (2004b) showed that

$$\Phi(0) = \frac{\prod_{i=1}^n \lambda_i (c \sum_{i=1}^n \frac{1}{\lambda_i} - \mu_1)}{c^n \prod_{i=1}^{n-1} \rho_i}.$$

Using our findings we can do some extension by getting the $k-$th derivative $\Phi^{(k)}(u)\big|_{u=0}$. For simplicity we write $\Phi(0)=(\bar{\lambda}\,\bar{\mu}/(c^n\,\bar{\rho})$, where $\bar{\lambda}=\prod_{i=1}^n\lambda_i$, $\bar{\mu}=c\sum_{i=1}^n\frac{1}{\lambda_i}-\mu_1$ and $\bar{\rho}=\prod_{i=1}^{n-1}\rho_i$.

We have mentioned before that

$$\begin{aligned} d_\Phi(s) &= -\Phi(0)\left(\frac{c^n}{\prod_{i=1}^n\lambda_i}\right)\prod_{i=1}^{n-1}(\rho_i-s) \\ &= \left(-\frac{\bar{\mu}}{\bar{\rho}}\right)\prod_{i=1}^{n-1}(\rho_i-s)=\sum_{i=0}^{n-1}\tilde{a}_i s^i, \end{aligned} \tag{3.1}$$

where

$$\tilde{a}_i=\left(-\frac{\bar{\mu}}{\bar{\rho}}\right)\left((-1)^j\sum_{i_1<\cdots<i_{n-1-j}}\rho_{i_1}\cdots\rho_{i_{n-1-j}}\right).$$

On the other hand we have

$$d_\Phi(s)=\sum_{i=0}^{n-1}\left(\sum_{k=i+1}^{n}B_k\Phi^{(k-1-i)}(0)\right)s^i=\sum_{i=0}^{n-1}\tilde{b}_i s^i. \tag{3.2}$$

We compare the coefficient of s^i in (3.1) and (3.2) to get a system of n equations $\tilde{a}_i=\tilde{b}_i,\ 0\le i\le n-1$, for the unknowns $\Phi^{(k)}(0)$, $k=0,1,\ldots,n-1$ (we already know $\Phi(0)$). After solving that system we obtain the following

$$\Phi^{(k)}(0)=A_k\Phi(0),\quad k=0,1,\ldots,n-1, \tag{3.3}$$

where the constants A_k are given by $A_0=1$ and

$$A_k=\sum_{i_1<\cdots<i_k}(-1)^{k+1}\left[\frac{\lambda_{i_1}\cdots\lambda_{i_k}}{c^k}-\rho_{i_1}\cdots\rho_{i_k}\right]+\sum_{j=1}^{k-1}(-1)^{k+1-j}\left[\frac{\lambda_{i_1}\cdots\lambda_{i_{k-j}}}{c^{k-j}}\right],$$

with $k=1,\ldots,n-1$. We notice that the higher derivatives of $\Phi(u)$ at $u=0$ are just multiples of $\Phi(0)$.

Li and Garrido (2004a) found a defective renewal equation for the survival probability $\Phi(u)$

$$\Phi(u)=\int_0^u\Phi(u-y)\eta_0(y)dy+\Phi(0), \tag{3.4}$$

where $\eta_0(y)=\frac{\bar{\lambda}}{c^n}T_0T_{\rho_{n-1}}\cdots T_{\rho_1}p(y)$ is a “defective density”, and T_r is the following complex operator of an integrable real-valued function f, with $Re(r)\ge 0$,

$$T_rf(x)=\int_x^\infty e^{-r(u-x)}f(u)du\,,\quad r\in\mathbb{C},\quad x\ge 0\,.$$

We compute the derivatives of $\Phi(u)$ at $u=0$ using equation (3.4) and obtain

$$\Phi^{(k)}(0)=\Phi(0)\left[\eta_0^k(0)+\sum_{i=1}^{k-1}\binom{k-1}{i}\eta_0^{k-1-i}(0)\eta_0^{(i)}(0)\right],\ k=1,\ldots,n-1, \tag{3.5}$$

Thus, comparing the expressions for $\Phi^{(k)}(0)$ in (3.3) and (3.5) we get

$$A_k = \eta_0^k(0) + \sum_{i=1}^{k-1} \binom{k-1}{i} \eta_0^{k-1-i}(0)\eta_0^{(i)}(0).$$

Hence, from the equation above we obtain expressions for the derivatives of $\eta_0(y)$ at $y = 0$

$$\begin{aligned} \eta_0(0) &= A_1, \\ \eta_0^{(k-1)}(0) &= \sum_{j=0}^{k} (-1)^{j+1} \left(\sum_{i_1 < \cdots < i_j} \frac{\lambda_{i_1} \cdots \lambda_{i_j}}{c^j} \right) \times \\ & \quad \left(\sum_{i_1 \le \cdots \le i_{k-j}} \rho_{i_1} \cdots \rho_{i_{k-j}} \right), \end{aligned} \tag{3.6}$$

for $k = 1, \ldots, n-1$. On the other hand, we compute directly the derivatives of $\eta_0(y)$ at $y = 0$ to get the expression

$$\begin{aligned} \eta_0^{(k-1)}(0) &= -\sum_{i=n-k}^{n} \left(\sum_{1 \le j_1 < \cdots < j_{n-i} \le n} \left(\prod_{m=1}^{n-i} \left(\rho_{n-(k-1)} - \frac{\lambda_{j_m}}{c} \right) \right) \right) \times \\ & \quad \left(\sum_{1 \le j_1 \le \cdots \le j_{i-n+k} \le n-k} \left(\prod_{m=1}^{i-n+k} \left(\rho_{j_m} - \rho_{n-(k-1)} \right) \right) \right). \end{aligned} \tag{3.7}$$

Both expressions for $\eta_0^{(k-1)}(0)$ given in (3.6) and (3.7) are equivalent. From this equivalence we obtain many combinatorial identities, but that belongs to the field of Combinatorics and goes beyond the scope of this work.

Remark 3.1. *The defective density $\eta_0(y)$ is a special case of the function $\eta_\delta(y)$ for a force of interest $\delta \ge 0$. This function appears in Li and Garrido (2004a) for the study of the Gerber–Shiu penalty functions.*

4 The maximum severity of ruin

In the previous section we have shown how to obtain the solutions of the integro–differential equation. Now, we will use these results to obtain the corresponding expressions for the distribution of the maximum severity of ruin. We will find an expression for that distribution which only depends on the non-ruin probability $\Phi(u)$ and on the claim amounts distribution.

From Dickson (2005) and (2.2) we know that the distribution of the maximum severity of ruin $J(z;u)$ can be expressed as:

$$J(z;u) = \frac{1}{1-\Phi(u)} \int_0^z g(u,y)(v_1(z-y), \ldots, v_n(z-y))dy[V(z)]^{-1}\vec{e}^T. \tag{4.1}$$

If we denote by

$$\begin{aligned} \vec{h}(z,u) &= \int_0^z g(u,y)(v_1(z-y), \ldots, v_n(z-y))dy \\ &= \left(\int_0^z g(u,y)v_1(z-y)dy, \ldots, \int_0^z g(u,y)v_n(z-y)dy \right) \\ &= (h_1(z,u), \ldots, h_n(z,u)), \end{aligned}$$

then we only have to find an expression for every component of $\vec{h}(z,u)$.

Considering the case of the Theorem 3.1 in the previous section we have that for $j=1,2,\dots,n-1$,

$$\begin{aligned}\int_0^z g(u,y)v_j(z-y)dy &= \int_0^z g(u,y)\int_0^{z-y}\Phi(z-y-x)e^{\rho_j x}dxdy \\ &= \int_0^z e^{\rho_j x}[\Phi(u+(z-x))-\Phi(u)]dx,\end{aligned}$$

and for $j=n$,

$$\int_0^z g(u,y)v_n(z-y)dy = \int_0^z g(u,y)\Phi(z-y)dy = \Phi(u+z)-\Phi(u).$$

In a similar way, when we consider the case of Theorem 3.3, we have that for $i=j=0$,

$$\int_0^z g(u,y)v_{00}(z-y)dy = \int_0^z g(u,y)\Phi(z-y)dy = \Phi(u+z)-\Phi(u),$$

and for $i=1,\dots,k;\quad j=1,\dots,m_i$,

$$\begin{aligned}\int_0^z g(u,y)v_{ij}(z-y)dy &= \int_0^z g(u,y)\int_0^{z-y}\Phi(z-y-x)x^{j-1}e^{\rho_i x}dxdy \\ &= \int_0^z x^{j-1}e^{\rho_i x}\left[\int_0^{z-x} g(u,t)\Phi((z-x)-t)dt\right]dx \\ &= \int_0^z x^{j-1}e^{\rho_i x}[\Phi(u+(z-x))-\Phi(u)]dx.\end{aligned}$$

Example 4.1. *Generalized Erlang(3) – Exponential. We work explicit formulae for the particular case when interclaim arrivals are generalized Erlang(*$3,\lambda_1,\lambda_2,\lambda_3$*) distributed, with* $\lambda_1\neq\lambda_2\neq\lambda_3$*, and claim amounts are exponential(*β*) distributed.*

Considering the safety loading $c=(1+\theta)\lambda_1\lambda_2\lambda_3/\beta(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)$ *with* $\theta>0$*, the generalized Lundberg's equation* (2.2) *takes the form*

$$\prod_{i=1}^{3}\left(1-\left(\frac{c}{\lambda_i}\right)s\right)-\frac{\beta}{(s+\beta)}=0.$$

It has four roots: $0,\rho_1,\rho_2$ *and* $-R$*, where* $0<R<\beta$ *is the adjustment coefficient. Assume that* $\rho_1=\rho_2=\rho$ *is a double (real) root (therefore* $\rho>0$*).*

After applying Theorem 3.2, the three solutions of the integro – differential equation (2.1) *come*

$$\begin{aligned}\Phi(u) &= 1-\left(1-\frac{R}{\beta}\right)e^{-Ru} \\ v_2(u) &= \frac{-1}{\rho}+\frac{\beta-R}{\beta(R+\rho)}e^{-Ru}+\frac{R(\beta+\rho)}{\rho\beta(R+\rho)}e^{\rho u} \\ v_3(u) &= \frac{1}{\rho^2}-\frac{\beta-R}{\beta(R+\rho)^2}e^{-Ru}-\frac{R(2\beta\rho+R\beta+\rho^2)}{\rho^2\beta(R+\rho)^2}e^{\rho u}+\frac{R(\beta+\rho)}{\rho\beta(R+\rho)}ue^{\rho u}\end{aligned}$$

Calculating (4.1) *we get*

$$J(z;u)=1-\frac{\alpha e^{-Rz}}{1-\gamma e^{-(\rho+R)z}-\delta e^{-(\rho+R)z}z-\eta e^{-Rz}},$$

where

$$\begin{aligned}
\alpha &= \frac{R(R+\rho)^2}{\beta(\beta+\rho)^2}, \qquad \delta = -\frac{R(\rho+R)(\beta-R)}{\rho(\beta+\rho)}, \\
\gamma &= -\frac{R(\beta-R)}{\rho^2(\beta+\rho)^2}((R+\rho)(\beta+\rho)+\rho(2\rho+R+\beta)), \\
\eta &= 1-\frac{R}{\beta+\rho}\left[\frac{R+\rho}{\beta}-\frac{(\beta-R)(R+\rho)}{\rho\beta}-\frac{(\beta-R)(R+2\rho)}{\rho^2}\right],
\end{aligned}$$

with $\eta = 1-\alpha-\gamma$. *Note that this expression for* $J(z;u)$ *is independent from* u.

Now we compute the conditional moments of the maximum severity M_u *given that ruin occurs, by formula, for* $r \geq 1$,

$$\begin{aligned}
E(M_u^r | T<\infty) &= r\int_0^\infty z^{r-1}(1-J(z;u))dz \\
&= r\alpha\int_0^\infty \frac{z^{r-1}e^{-Rz}}{1-\gamma e^{-(\rho+R)z}-\delta z e^{-(\rho+R)z}-\eta e^{-Rz}}\,dz.
\end{aligned}$$

Now, choosing $\beta = 1$, $\lambda_1 = 6.098$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\theta = 0.1$ *and* $c = 1.103$, *we obtain a double root* $\rho = 4.596$, *the adjustment coefficient* $R = 0.129$ *and*

$$J(z;u) = 1-\frac{0.092e^{-0.129z}}{1+0.012e^{-4.724z}+0.021e^{-4.724z}z-0.921e^{-0.129z}}.$$

The expected value and the standard deviation of the maximum severity of ruin are $E(M_u) = 1.932$ *and* $s.d.(M_u) = 3.528$. $*$

5 Dividends

In this section we consider the dividends problem. We can follow Dickson and Waters (2004) to generalize an equation for $V_m(u,b)$ in a Generalized Erlang(n) risk process. So, conditioning on the time and the amount of the first claim we get, for $0 \leq u < b$ and $m = 1,2,...,$

$$\begin{aligned}
V_m(u,b) &= \int_{\frac{b-u}{c}}^\infty k_n(t)e^{-m\delta t}\left[\left(c\,\bar{s}_{\overline{t-\frac{b-u}{c}|}}\right)^m + \right. \\
&\left. +\sum_{j=1}^m \binom{m}{j}\left(c\,\bar{s}_{\overline{t-\frac{b-u}{c}|}}\right)^{m-j}\int_0^b p(x)V_j(b-x,b)dx\right]dt + \\
&+\int_0^{\frac{b-u}{c}} e^{-m\delta t}k_n(t)\int_0^{u+ct}V_m(u+ct-x,b)p(x)dx\,dt, \quad m \geq 1. \qquad (5.1)
\end{aligned}$$

In particular, for $m = 1$

$$\begin{aligned}
V(u,b) &= \int_{\frac{b-u}{c}}^\infty k_n(t)e^{-\delta t}\left(c\,\bar{s}_{\overline{t-\frac{b-u}{c}|}}+\int_0^b p(x)V(b-x,b)dx\right)dt + \\
&+\int_0^{\frac{b-u}{c}} e^{-\delta t}k_n(t)\int_0^{u+ct}V(u+ct-x,b)p(x)dx\,dt, \qquad (5.2)
\end{aligned}$$

where $\bar{s}_{\overline{t|}} = (e^{\delta t}-1)/\delta$ in standard actuarial notation.

Following Albrecher *et al.* (2005), we derived for a Generalized Erlang(n) risk process the integro–differential equations satisfied by the discounted expected dividends

$$\begin{aligned} \prod_{i=1}^{n}\left(\left(1+\frac{\delta}{\lambda_i}\right)I-\frac{c}{\lambda_i}\mathcal{D}\right)V(u,b) &= \int_0^u V(u-x,b)p(x)dx \\ \left.\frac{d^kV(u,b)}{du^k}\right|_{u=b} &= \left(\frac{\delta}{c}\right)^{k-1},\ 1\le k\le n, \end{aligned} \tag{5.3}$$

and for a general $m=1,2,\ldots,$

$$\begin{aligned} \prod_{i=1}^{n}\left(\left(1+\frac{\delta}{\lambda_i}\right)I-\frac{c}{\lambda_i}\mathcal{D}\right)V_m(u,b) &= \int_0^u V_m(u-x,b)p(x)dx \\ \left.\frac{d^kV_m(u,b)}{du^k}\right|_{u=b} &= \sum_{j=1}^{k}\frac{m!}{(m-j)!}\left\{ {k \atop j} \right\}\left(\frac{\delta}{c}\right)^{k-j}V_{m-j}(b,b),\ 1\le k\le n, \end{aligned} \tag{5.4}$$

where $\left\{ {k \atop j} \right\}=\frac{1}{j!}\sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}i^k$ denotes the Stirling numbers of the second kind. We define for convenience $V_{m-j}(u,b)\equiv 0$, for $m<j$ in the formula above.

These equations generalize those proposed by Dickson (2005) and Dickson and Waters (2004) for the classical compound Poisson risk model, and are not only more explicit than those proposed by Albrecher *et al.* (2005), but also applicable for a general claim amount distribution instead (see their Equation (10) of Section 4).

Assume that the Generalized Lundberg's equation (2.4) has k different roots with positive real parts, $\rho_1,\rho_2,\ldots,\rho_k$, and the root ρ_i has multiplicity $m_i\ge 1$, $i=1,2,\ldots,k$. Following an argument originally proposed by Bühlman (1970), Section 6.4.9 for a Poisson risk model, we propose for a Generalized Erlang(n) risk model that $V(u,b)$ can be written in the form

$$V(u,b)=\sum_{i=1}^{k}\left(\sum_{j=1}^{m_i}C_{ij}(b)\beta_{ij}(u)\right)e^{\rho_i u}, \tag{5.5}$$

where C_{ij}'s are constants (that depend on the parameter b), and the functions $\beta_{ij}(u)$ are solutions of the integro–differential equations

$$\prod_{t=1}^{n}\left(I-\frac{c}{\lambda_{ti}}\mathcal{D}\right)v(u)=\int_0^u v(u-x)p_i(x)dx, \tag{5.6}$$

with $\lambda_{ti}=\lambda_t+\delta-c\rho_i$ and $p_i(x)=e^{-\rho_i x}p(x)/\hat{p}(\rho_i)$, for $t=1,\ldots,n$, $i=1,\ldots,k$.

Thus, the functions $\beta_{ij}(u)$ can be computed solving an equation of the same kind as Equation (2.1) but with different "parameters" and a different "density". Constants C_{ij}'s are determined using the boundary conditions given in (5.3), which gives a system of n equations with n unknowns

$$\left.\frac{d^kV(u,b)}{du^k}\right|_{u=b}=\sum_{i=1}^{k}\sum_{j=1}^{m_i}C_{ij}(b)\left.\frac{d^k(\beta_{ij}(u)e^{\rho_i u})}{du^k}\right|_{u=b}=\left(\frac{\delta}{c}\right)^{k-1},\ 1\le k\le n, \tag{5.7}$$

We summarize this in the following theorem

Theorem 5.1. *The solutions of integro-differential equation* (5.3) *can be written on the form*

$$V(u,b) = \sum_{i=1}^{k} \left(\sum_{j=1}^{m_i} C_{ij}(b)\beta_{ij}(u) \right) e^{\rho_i u},$$

where ρ_i*'s are the roots with positive real parts of the generalized Lundberg's equation* (2.4)*,* $\beta_{ij}(u)$*'s are defined in* (5.6) *and the constants* C_{ij}*'s are defined in* (5.7)*.*

Proof:

The proof is long but straightforward, it follows by taking derivatives of $V(u,b)$ and finding out which conditions must be satisfied by the ρ_i's and $\beta_{ij}(u)$'s to get the equality in (5.3). □

This method generalizes the results by Albrecher *et al.* (2005), since it works for any kind of claim amounts distribution, and not only for the distributions with rational Laplace transforms. Special care should be taken in the case when some of the roots (ρ_i's) of the generalized Lundberg's equation are complex, by using standard techniques of the theory of differential equations. The same approach can be implemented to find a general $V_m(u,b)$, $m \geq 2$, writing it in the form (5.5) and using the corresponding boundary conditions given in (5.4).

Example 5.1. *Consider that the interclaim times are generalized Erlang*(3) *distributed, with parameters* $\lambda_1, \lambda_2, \lambda_3$*, and the claim amounts are exponentially distributed with parameter* α*. Let the force of interest be* $\delta > 0$*.*

The Generalized Lundberg's equation (2.4) *is*

$$(\lambda_1 + \delta - cs)(\lambda_2 + \delta - cs)(\lambda_3 + \delta - cs) = \frac{\lambda_1\lambda_2\lambda_3\alpha}{\alpha + s},$$

where $c = (1+\theta)\lambda_1\lambda_2\lambda_3/\alpha(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)$ *for some* $\theta > 0$*. There are 3 roots with positive real parts. Let* $R > 0$ *be the adjustment coefficient and assume that we have a single root* $\rho_1 > 0$ *and a double root* $\rho_2 > 0$*.*

Applying Theorem 5.1 we can write

$$V(u,b) = C_{11}v_{11}(u)e^{\rho_1 u} + (C_{21}v_{21}(u) + C_{22}v_{22}(u))e^{\rho_2 u}.$$

We already know ρ_1 *and* ρ_2*. We must find the constants* C_{11}, C_{21}, C_{22} *and the functions* $v_{11}(u), v_{21}(u), v_{22}(u)$*.*

To find the functions $v_{11}(u), v_{21}(u)$ *and* $v_{22}(u)$ *we proceed as follows:*

-
- $v_{11}(u)$ *is a solution of the integro–differential equation*

$$\prod_{t=1}^{3} \left(I - \frac{c}{\lambda_{t1}}\mathcal{D} \right) v(u) = \int_0^u v(u-x)p_1(x)dx, \tag{5.8}$$

where $\lambda_{t1} = \lambda_t + \delta - c\rho_1$ *and* $p_1(x) = (\alpha + \rho_1)e^{-(\alpha+\rho_1)x} = \alpha_1 e^{-\alpha_1 x}$*, for* $t = 1,2,3$*.*

Let

$$(\lambda_{11} - cs)(\lambda_{21} - cs)(\lambda_{31} - cs) = \frac{\lambda_{11}\lambda_{21}\lambda_{31}\alpha_1}{\alpha_1 + s}$$

be the associated fundamental Lundberg's equation and $R_1 > 0$ *the corresponding adjustment coefficient. Then, we can choose* $v_{11}(u) = 1 - (1 - R_1/\alpha_1)e^{-R_1 u}$*, a "survival probability", which is a well known solution of* (5.8)*.*

- $v_{21}(u)$ *and* $v_{22}(u)$ *are both solutions of*

$$\prod_{t=1}^{3}\left(I-\frac{c}{\lambda_{t2}}\mathcal{D}\right)v(u)=\int_0^u v(u-x)p_2(x)dx, \tag{5.9}$$

where $\lambda_{t2}=\lambda_t+\delta-c\rho_2$ *and* $p_2(x)=(\alpha+\rho_2)e^{-(\alpha+\rho_2)x}=\alpha_2e^{-\alpha_2x}$*, for* $t=1,2,3$.
Let

$$(\lambda_{12}-cs)(\lambda_{22}-cs)(\lambda_{32}-cs)=\frac{\lambda_{12}\lambda_{22}\lambda_{32}\alpha_2}{\alpha_2+s}$$

be the associated Fundamental Lundberg's equation, $R_2>0$ *the corresponding adjustment coefficient and* ρ_{21},ρ_{22} *the two roots with positive real parts.*
Let $\tilde{v}(u)=1-(1-R_2/\alpha_2)e^{-R_2u}$.
Then, using Theorem 3.1 (if $\rho_{21}\neq\rho_{22}$ *we use Theorem 3.2, otherwise), we can choose*

$$\begin{aligned}
v_{21}(u) &= \int_0^u \tilde{v}(u-y)e^{\rho_{21}y}dy\\
&= \frac{-1}{\rho_{21}}+\frac{\alpha_2-R_2}{\alpha_2(R_2+\rho_{21})}e^{-R_2u}+\frac{R_2(\alpha_2+\rho_{21})}{\rho_{21}\alpha_2(R_2+\rho_{21})}e^{\rho_{21}u}\\
v_{22}(u) &= \int_0^u \tilde{v}(u-y)e^{\rho_{22}y}dy\\
&= \frac{-1}{\rho_{22}}+\frac{\alpha_2-R_2}{\alpha_2(R_2+\rho_{22})}e^{-R_2u}+\frac{R_2(\alpha_2+\rho_{22})}{\rho_{22}\alpha_2(R_2+\rho_{22})}e^{\rho_{22}u}
\end{aligned}$$

To find the constants C_{11},C_{21},C_{22} *we use the boundary conditions given in* (5.3) *and we obtain*

$$\begin{pmatrix} C_1\\ C_2\\ C_3\end{pmatrix}=\begin{pmatrix}
\left.\frac{d(e^{\rho_1u}v_{11}(u))}{du}\right|_{u=b} & \left.\frac{d(e^{\rho_2u}v_{21}(u))}{du}\right|_{u=b} & \left.\frac{d(e^{\rho_2u}v_{22}(u))}{du}\right|_{u=b}\\
\left.\frac{d^2(e^{\rho_1u}v_{11}(u))}{du^2}\right|_{u=b} & \left.\frac{d^2(e^{\rho_2u}v_{21}(u))}{du^2}\right|_{u=b} & \left.\frac{d^2(e^{\rho_2u}v_{22}(u))}{du^2}\right|_{u=b}\\
\left.\frac{d^3(e^{\rho_1u}v_{11}(u))}{du^3}\right|_{u=b} & \left.\frac{d^3(e^{\rho_2u}v_{21}(u))}{du^3}\right|_{u=b} & \left.\frac{d^3(e^{\rho_2u}v_{22}(u))}{du^3}\right|_{u=b}
\end{pmatrix}^{-1}\begin{pmatrix}1\\ \left(\frac{\delta}{c}\right)\\ \left(\frac{\delta}{c}\right)^2\end{pmatrix}.$$

6 Some concluding remarks

In this work we have shown a method to find expressions for the distribution of the maximum severity of ruin in the Sparre–Andersen model with generalized Erlang(n) interclaim times in the cases when the generalized Lundberg's equation has single or multiple roots. Those expressions depend exclusively on the non–ruin probability and the claim amounts distribution.

Multiple roots do not arise in the (single) Erlang(n) model. Our work shows that the existence of multiple roots brings the need for further developments or a different approach. Also, we generalized the results obtained by Albrecher *et al.* (2005) for the expected discounted future dividends considering an arbitrary claim amount distribution and for the case of multiple roots.

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