SOME SIMPLE AND CLASSICAL APPROXIMATIONS TO RUIN PROBABILITIES APPLIED TO THE PERTURBED MODEL

Miguel J. M. Seixas† **and Alfredo D. Egídio dos Reis**§[1]

†*AXA MedLa, Camino Fuente de La Mora nº, 28050 Madrid, Spain*
§*ISEG & CEMAPRE, Technical University of Lisbon, Rua do Quelhas 6, 1200-781 Lisboa, Portugal*
Email: `seixas.miguel@gmail.com`, `alfredo@iseg.utl.pt`

We study approximations to ultimate ruin probabilities under an extention to the classical Cramér-Lundberg risk model by adding a diffusion component. For the approximations, we adapt some simple, practical and well know methods used for the classical model. Under this approach, and for some cases, we are able to separate and compute the ruin probability either exclusively due to the oscillation, or due to a claim.

1. INTRODUCTION

We start by presenting **the model and the probability of ruin**. We study the **perturbed surplus process** as introduced by Dufresne and Gerber (1991) and defined for time t as:

$$V(t) = U(t) + \sigma W(t), \quad U(t) = u + ct - S(t), \quad t \geq 0\,,$$

where $U(t)$ defines the classical surplus process, c is the premium rate per unit time, $u = V(0) = U(0)$ is the initial surplus, $S(t) = \sum_{i=0}^{N(t)} X_i$, $X_0 \equiv 0$, are the aggregate claims up to time t, $N(t)$ is the number of claims received up to time t, X_i is the i-th individual claim, $W(t)$ is the diffusion component and σ^2 the variance parameter. $\{W(t), t \geq 0\}$ is a standard Wiener process, $\{N(t), t \geq 0\}$ is a Poisson process with parameter λ and $\{X_i\}_{i=1}^{\infty}$ is a sequence of $i.i.d.$ random variables, independent from $\{N(t)\}$ with common distribution function $P(.)$ with $P(0) = 0$. The corresponding density function is denoted as $p(.)$. Denote by $p_k = E[X^k]$. The existence of p_1 is basic and essential, only in some of our methods the existence of higher moments is needed. We assume that $\{S(t)\}$ and $\{W(t)\}$ are independent. We also assume that $c = (1+\theta)\lambda p_1$, where $\theta > 0$ is the premium loading coefficient.

[1] Support from FCT-Fundação para a Ciência e a Tecnologia (Programme FEDER/POCI 2010) is gratefully acknowledged.

The diffusion component introduces an additional uncertainty into the classical process, so that if ruin occurs it may be caused either from a claim or by an oscillation of the diffusion process. Let T be the time to ruin such that $T = \inf\{t : t \geq 0 \text{ and } V(t) \leq 0\}$, $T = \infty$ if $V(t) > 0$, $\forall t$. Ultimate ruin probability is given by

$$\psi(u) = \Pr(T < \infty | V(0) = u) = \psi_d(u) + \psi_s(u),$$

where $\psi_s(u)$ and $\psi_d(u)$ are the ruin probabilities due to a claim and to oscillation, respectively. Survival probability is $\delta(u) = 1 - \psi(u)$. We have that $\psi_d(0) = \psi(0) = 1$. Furthermore, $\delta(u)$, $\psi_s(u)$ and $\psi_d(u)$ follow defective renewal equations, respectively, for $u \geq 0$:

$$\begin{aligned}
\psi_s(u) &= (1-q)\left[H_1(u) - H_1 * H_2(u)\right] + +(1-q)\int_0^u \psi_s(u-x)h_1 * h_2(x)dx\,, \\
\psi_d(u) &= 1 - H_1(u) + (1-q)\int_0^u \psi_d(u-x)h_1 * h_2(x)dx\,, \\
\delta(u) &= qH_1(u) + (1-q)\int_0^u \delta(u-x)h_1 * h_2(x)dx\,, \qquad (1)
\end{aligned}$$

with $q = 1 - \lambda p_1/c$, h_1 and $h_2(.)$ given by ($H_1(.)$ and $H_2(.)$ are the corresponding d.f.):

$$\begin{aligned}
h_1(x) &= \zeta e^{-\zeta x}, x > 0,\ \ \zeta = 2c/\sigma^2\,, \\
h_2(x) &= p_1^{-1}\left[1 - P(x)\right], x > 0.
\end{aligned}$$

We further introduce the **maximal aggregate loss** defined as $L = \max\{t \geq 0, L(t) = u - V(t)\}$. It can be decomposed as

$$L = L_0^{(1)} + \sum_{i=1}^{M}\left(L_i^{(1)} + L_i^{(2)}\right), \qquad (2)$$

$$L_i^{(1)} = \max\{L(t), t < t_{i+1}\} - L(t_i),\ i = 0, 1, \ldots, M, \qquad (3)$$

$$L_i^{(2)} = L(t_i) - L(t_{i-1}) - L_{i-1}^{(1)},\ i = 1, \ldots, M, \qquad (4)$$

where M is the number of records of $L(t)$ that are caused by a claim, $L_i^{(1)}$ and $L_i^{(2)}$ are the *record highs* due to oscillation and a claim. $\{L_i^{(1)}\}_{i=0}^{\infty}$ and $\{L_i^{(2)}\}_{i=1}^{\infty}$ are independent sequences of $i.i.d$ random variables, with common d.f. $H_1(.)$, and $H_2(.)$, respectively. Also, $\delta(x) = \Pr\{L \leq x\}$ is a compound geometric d.f. and existing moments can be found easily.

We work different **approximation methods adapted from the classical model**, simple and of *classical use* in many risk theory manuals. We start with the method by **De Vylder (1978)**, that relies on the use of the exact ruin formula when the individual claim amount is exponential. We follow with a method by **Dufresne and Gerber (1989)** that produces upper and lower limits for the ruin probability and it is very useful to test the accuracy of the other methods presented, often simpler, for the cases where we do not have exact figures for the ruin probability. These two methods were already tried by Silva (2006), who presented no figures. After, we adapt an approximation known as **Beekman and Bowers'**, presented in Beekman (1969). It uses an appropriate gamma distribution in the defective renewal equation for $\delta(u)$. Jacinto (2008) also did some work on the previous methods. We further work two other models, **Tijms'** and the **Fourier transform** methods. The former was originally presented in the context of queueing theory by Tijms (1994), the latter is an adaptation of the work by Lima et al. (2002).

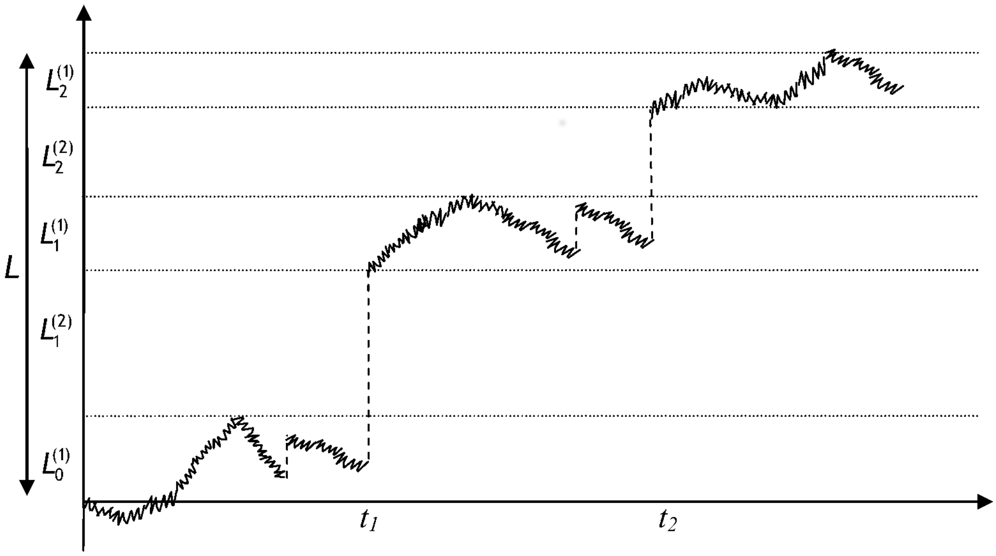

Figure 1: Decomposition of the maximal aggregate loss.

2. APPROXIMATIONS IN THE PERTURBED MODEL

We follow the order presented in the previous section and start with the **De Vylder's approximation**. Following De Vylder (1978), the original process, $V(t)$, is replaced by another process

$$V^*(t) = u + c^*t - S^*(t) + \sigma^* W(t),$$

where the individual claims follow an $exponential(\beta)$, and parameters β, c^*, λ^* and $\sigma^{*\,2}$ are calculated so that the existing lower four moments of $V(t)$ and $V^*(t)$ match:

$$\beta = 4\tfrac{p_3}{p_4}; \qquad \lambda^* = 32\lambda\tfrac{p_3^4}{3p_4^3}; \qquad c^* = 8\lambda\tfrac{p_3^3}{3p_4^2} + c - \lambda p_1; \qquad \sigma^{*\,2} = \sigma^2 + \lambda p_2 - 4\lambda\tfrac{p_3^2}{3p_4}.$$

Then, we use the exact ruin probability formula from Dufresne and Gerber (1991), so that approximation comes

$$\psi_{DV}(u) = C_1 e^{-r_1} + C_2 e^{-r_2}, \qquad C_1 = \tfrac{r_1-\beta}{\beta}\tfrac{r_2}{r_1-r_2}, \qquad C_2 = \tfrac{r_2-\beta}{\beta}\tfrac{r_1}{r_2-r_1}\ ,$$

where r_1 and r_2 are the solutions of equation, $r\sigma^{*2}/2 + \lambda^*/(\beta - r) = c^*$. Furthermore, we can obtain approximations for the decomposed probabilities $\psi_s(u)$ and $\psi_d(u)$, simply using the exact result for the case where the individual losses are exponential.

The second method is **Dufrene & Gerber's upper and lower bounds**. It is based on getting appropriate discrete distributions to replace on the convolution formula for the survival probability, Formula (7) in Dufresne and Gerber (1989). For the perturbed model, we use a similar method, now based on Formula (5.8) of Dufresne and Gerber (1991). Discrete random variables are defined followed by bounds computation for the ruin probabilities [see Sections 2.3 and 2.4 of Dufresne and Gerber (1989)]. We have

$$L^j = L_0^{j,(1)} + \sum_{i=1}^{M}\left(L_i^{j,(1)} + L_i^{j,(2)}\right),$$

with $L^j = L_0^{j,(1)}$ if $M = 0$ and $j = l, u$, $L_i^{l,(k)} = \vartheta \left[L_i^{(k)}/\vartheta \right]$, $L_i^{u,(k)} = \vartheta \left[(L_i^{(k)} + \vartheta)/\vartheta \right]$ for $\{k = 1, i = 0, ..., M\}$, $\{k = 2, i = 1, ..., M\}$, $\vartheta \epsilon (0, 1)$ and $[x]$ is the integer part of x. Each summand of L, $L_i^{(k)}$, in (2), is correspondingly approximated by both the next lower and higher multiples of ϑ. We have then,

$$L^l \leq L \leq L^u \Rightarrow \Pr(L^l \geq v) \leq \psi(v) \leq \Pr(L^u \geq v).$$

We need the p.f. of the discrete r.v.'s $L_i^{l,(1)}$, $L_i^{l,(2)}$, $L_i^{u,(1)}$ and $L_i^{u,(2)}$, they are given by, respectively,

$$\begin{aligned} h_{n,k}^l &= \Pr\left(L_i^{l,(n)} = k\vartheta\right) = H_n(k\vartheta + \vartheta) - H_n(k\vartheta),\ n = 1, 2;\ k = 0, 1, ..., \\ h_{n,k}^u &= \Pr\left(L_i^{u,(n)} = k\vartheta\right) = H_n(k\vartheta + \vartheta) - H_n(k\vartheta),\ n = 1, 2;\ k = 0, 1, ... \end{aligned}$$

The following probability functions of L^l and L^u, f_k^l and f_k^u, can be computed using Panjer's recursion (for the compound geometric distribution)

$$f_k^j = \Pr\left(L^j = k\vartheta\right), k = 0, 1, ... \text{ for } j = l, u\,.$$

We arrive to the following bounds for $\psi(.)$, where

$$1 - \sum_{k=0}^{m-1} f_k^l \leq \psi(m\vartheta) \leq 1 - \sum_{k=0}^{m} f_k^u, \quad m = 0, 1, ..., v/\vartheta, \quad v = 0, 1, ...$$

We consider now **Beekman-Bowers' approximation**. We replace $\delta * h_2(.)$ in the renewal equation (1), $\delta(u) = qH_1(u) + (1-q)h_1 * \delta * h_2(u)$, by a d.f. of a gamma($\alpha, \beta$), denoted as $H_3(u)$. We arrive to the approximation

$$\delta_{BB}(u) = qH_1(u) + (1-q)h_1 * H_3(u),$$

Parameters α and β are got by equating the moments of $\delta_{BB}(u)$ with those of $\delta(u)$, respectively.

We address now **Tijms' approximation**. This method relies on the existence of the adjustment coefficient and an asymptotic formulae for $\psi(u)$, $\psi_d(u)$, and $\psi_s(u)$. Similarly to Tijms (1994) we consider the approximating expression

$$\psi_T(u) = Ce^{-Ru} + Ae^{-Su}, u \geq 0\,,$$

where A is chosen such that $\psi(0) = \psi_T(0)$. As $\psi(0) = 1$, then $A = (1 - C)$. As $\psi(.)$ is the survival function of L, S is chosen in order that $\int_0^\infty \psi_T(u)du = E[L]$. Hence,

$$E[L] = \frac{C}{R} + \frac{(1-C)}{S} \Leftrightarrow S = \frac{R\,(1-C)}{RE[L] - C}\,.$$

The method we work and simply name as **Fourier** transform is not quite an approximation method but an exact formula that allows to compute numerically the ruin probability. This method uses the Fourier transform,

$$\phi_{f(x)}(s) = \int_0^{+\infty} e^{isx} f(x)dx = \underbrace{\int_0^{+\infty} \cos(sx)f(x)dx}_{\phi^r_{f(x)}(s)} + i\underbrace{\int_0^{+\infty} \sin(sx)f(x)dx}_{\phi^c_{f(x)}(s)},$$

so that for $F'(x) = f(x)$ we have

$$F(x) = F(0) + \frac{2}{\pi}\int_0^\infty \frac{\sin(xs)}{s}\phi^r_{f(x)}(s)ds\,. \tag{5}$$

From the integro-differential equation for $\psi(u)$ we get

$$\psi'(u) = -qh_1(u) + (1-q)\int_0^u \psi'(u-x)h_1 * h_2(x)dx\,,$$

and the transform can be written as

$$\phi_{\psi'(u)}(s) = \frac{A+iB}{C-iD} = \frac{AC-BD+i(BC+AD)}{C^2+D^2}\,,$$

with $A = -q\phi^r_{h_1(u)}(s)$, $B = -q\phi^c_{h_1(u)}$, $C = 1 - J(1-q)/sp_1$ and $D = I(1-q)/sp_1$. I and J depend only on the real and the complex part of $\phi_{h_1(u)}(s)$ and $\phi_{p(u)}(s)$:

$$\begin{aligned} I &= \phi^r_{h_1(u)}(s) - \phi^r_{h_1(u)}(s)\phi^r_{p(u)}(s) + \phi^c_{h_1(u)}(s)\phi^c_{p(u)}(s) \\ J &= \phi^r_{h_1(u)}(s)\phi^c_{p(u)}(s) - \phi^c_{h_1(u)}(s) + \phi^c_{h_1(u)}(s)\phi^r_{p(u)}(s)\,. \end{aligned}$$

Approximation $\psi_F(u)$ is then got computing numerically the inversion integral (5). Similar results can be derived for $\psi_{d,F}(u)$ and $\psi_{s,F}(u)$ (the index $_F$ refers to this method).

3. NUMERICAL ILLUSTRATIONS

For illustration we show figures for three examples: when single amounts follow Exponential(1), Gamma$(2,2)$ and Pareto$(5,4)$, all with mean one. Other parameters are: $c = 2$, $\lambda = 1$, $\sigma = 1$ and $\vartheta = 0.01$. Tables 1 and 2 show figures concerning the first example (De Vylder's is exact in this case). Table 3 shows figure for the Gamma$(2,2)$ case. Table 4 shows figures for the Pareto$(5,4)$ case and all methods except Tijms', as it doesn't apply. Table 5 shows the percentage of ruin due to oscillation for the worked cases.

u	$\psi(u)\ (I)$	$\psi_{BB}(u)(II)$	$(I)/(II)$	$\psi_T(u)\ (III)$	$(I)/(III)$
1	0.40470	0.39819	1.01633	0.40470	1.00000
3	0.16674	0.17096	0.97529	0.16674	1.00000
5	0.06938	0.07089	0.97866	0.06938	1.00000
10	0.00775	0.00731	1.06010	0.00775	1.00000
15	0.00087	0.00072	1.19580	0.00087	1.00000

Table 1: Exact figures, Beekman-Bowers' and Tijms' approximations for $Exponential(1)$

u	$\psi(u)$ (I)	$\psi_F(u)$ (II)	$(I)/(II)$	$\psi_d(u)$ (III)	$\psi_{d,F}(u)$ (IV)	$(III)/(IV)$
1	0.40470	0.40470	1.00000	0.09688	0.09688	0.99999
3	0.16674	0.16674	1.00000	0.03655	0.03655	1.00000
5	0.06938	0.06937	1.00000	0.01521	0.01521	1.00000
10	0.00775	0.00775	1.00000	0.00170	0.00170	1.00000
15	0.00087	0.00087	1.00000	0.00019	0.00019	1.00002

Table 2: Exact figures and Fourier method for $Exponential(1)$

u	Lower Bound	$\psi_{DV}(u)$	$\psi_{BB}(u)$	$\psi_T(u)$	$\psi_F(u)$	Upper Bound
1	0.38643	0.39199	0.38231	0.39394	0.38867	0.39092
3	0.12024	0.12155	0.12660	0.12198	0.12196	0.12369
5	0.03696	0.03775	0.03825	0.03780	0.03780	0.03865
10	0.00194	0.00203	0.00167	0.00202	0.00202	0.00211
15	0.00010	0.00011	0.00007	0.00011	0.00011	0.00012

Table 3: Dufresne-Gerber's Bounds, De Vylder's, Beekman-Bowers', Tijms' & Fourier, $Gamma$.

u	Lower Bound	$\psi_{DV}(u)$	$\psi_{BB}(u)$	$\psi_F(u)$	Upper Bound
1	0.40867	0.45521	0.38282	0.41036	0.41206
3	0.19577	0.15464	0.20096	0.19707	0.19838
5	0.10339	0.08437	0.11286	0.10423	0.10509
10	0.02511	0.02879	0.02824	0.02537	0.02564
15	0.00727	0.01032	0.00730	0.00736	0.00744

Table 4: Dufresne-Gerber's Bounds, De Vylder's, Beekman-Bowers' & Fourier; $Pareto(5, 4)$

	Exponential	Gamma			Pareto		
u	$\psi_d(u)/\psi(u)$	$\psi_{d,F}(u)$	$\psi_{s,F}(u)$	$\psi_{d,F}(u)/\psi_F(u)$	$\psi_{d,F}(u)$	$\psi_{s,F}(u)$	$\psi_{d,F}(u)/\psi_F(u)$
1	24%	0.11221	0.27647	29%	0.09042	0.31994	22%
3	22%	0.03570	0.08626	29%	0.03296	0.16411	17%
5	22%	0.01107	0.02673	29%	0.01590	0.08833	15%
10	22%	0.00059	0.00143	29%	0.00334	0.02203	13%
15	22%	0,00003	0,00008	29%	0.00085	0.00650	12%

Table 5: Weight of $\psi_d(u)$ for $Exponential(1)$, $Gamma(2, 2)$ and $Pareto(5, 4)$

4. CONCLUDING REMARKS

We underline the poor fit of the Beekman-Bowers' method no matter the distribution examples we worked on. De Vylder's and Tijms' look capable of producing good results for light tail claims size distributions. On any case, Dufresne & Gerber's bounds method produces good approximations. The same happens with Fourier transform method, as said it produces numerically exact figures. A final remark deals with the contribution of the oscillation component which plays a substantial role in the ruin probability, especially in the light tails cases. We chose a volatility of one, equal to the mean claim size in all examples, a deeper study can be done choosing different values. For more details on the work please see Seixas (2012).

References

J. Beekman. A ruin function approximation. *Trans. Soc. Actuaries*, 21:41–48 and 275–279, 1969.

F. De Vylder. A practical solution to the problem of ultimate ruin probability. *Scandinavian Actuarial Journal*, 1978(2):114–119, 1978.

F. Dufresne and H. U. Gerber. Three methods to calculate the probability of ruin. *ASTIN Bulletin*, 19(1):71–90, 1989.

F. Dufresne and H. U. Gerber. Risk theory for the compound poisson process that is perturbed by diffusion. *Insurance: Mathematics and Economics*, 10:51–59, 1991.

A. C. F. S. Jacinto. Aproximações à probabilidade de ruína no modelo perturbado. Master's thesis, ISEG Technical University of Lisbon, 2008.

R. Kaas, M. Goovaerts, J. Dhaene, and M. Denuit. *Modern Actuarial Risk Theory: Using R*. Springer, Chichester, 2nd edition edition, 2008.

F. D. P. Lima, J. M. A. Garcia, and A. D. Egídio dos Reis. Fourier/laplace transforms and ruin probabilities. *ASTIN Bulletin*, 32(1):91–105, 2002.

M. J. M. Seixas. Some simple and classical approximations to ruin probabilities applied to the perturbed model. Master's thesis, ISEG Technical University of Lisbon, 2012.

M. D. V. Silva. Um processo de risco perturbado: Aproximações numéricas à probabilidade de ruína. Master's thesis, Faculty of Science University of Oporto, 2006.

H. Tijms. *Stochastic Models - An Algorithmic Approach*. John Wiley & Sons, Chichester, 1994.