Dividend problems in the dual risk model

Lourdes B. Afonso, Rui M.R. Cardoso & Alfredo D. Egídio dos Reis*
CMA and FCT, New University of Lisbon & CEMAPRE and ISEG,
Technical University of Lisbon, Portugal

Abstract

We consider the compound Poisson dual risk model, dual to the well known *classical* risk model for insurance applications, where premiums are regarded as costs and claims are viewed as profits. The surplus can be interpreted as a venture capital like the capital of an economic activity involved in research and development. Like most authors, we consider an upper dividend barrier so that we model the gains of the capital and its return to the capital holders.

By establishing a proper and crucial connection between the two models we show and explain clearly the dividends process dynamics for the dual risk model, properties for different random quantities involved as well as their relations. Using our innovative approach we derive some already known results and go further by finding several new ones. We study different ruin and dividend probabilities, such as the calculation of the probability of a dividend, distribution of the number of dividends, expected and amount of dividends as well as the time of getting a dividend.

We obtain integro-differential equations for some of the above results and also Laplace transforms. From there we can get analytical results for cases where solutions and/or inversions are possible, in other cases we may only get numerical ones. We present examples under the two cases.

Keywords: Dual risk model; classical risk model; ruin probabilities; dividend probabilities; discounted dividends; dividend amounts; number of dividends.

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1 Introduction

We consider in this manuscript the dual risk model, as described, for instance, by Avanzi *et al.* (2007). The surplus or equity of a company at time t is given by the equation,

$$U(t) = u - ct + S(t), \ \ t \geq 0, \tag{1.1}$$

where u is the initial surplus, c is the constant rate of expenses, $\{S(t), t \geq 0\}$ is a compound Poisson process with parameter λ and density function $p(x)$, $x > 0$, of the positive gains, with mean p_1 (we therefore assume that it exists). Its distribution function is denoted as $P(x)$. The expected increase per unit time, given by $\mu = E[S(1)] - c = \lambda p_1 - c$, is positive, that is $c < \lambda p_1$. All these quantities have a corresponding meaning in the well known classical continuous time risk model, also known as the Cramér-Lundberg risk model, for insurance applications. For the remainder of our text we will refer this latter model as simply the standard risk model (shortly SM). For those used to work with it we note that the income condition, $c < \lambda p_1$, is reversed. A few authors have addressed the dual model (simply DM), we can go back to Gerber (1979, pp. 136-138) who called it the *negative claims model*, also see Bühlmann (1970). We can go even further back to authors like Cramér (1955), Takács (1967) and Seal (1969).

Avanzi *et al.* (2007), Section 1, explains well where applications of the dual model are said to be appropriate. We just retain a simple but illustrative interpretation, the surplus can be considered as the capital of an economic activity like research and development where gains are random, at random instants, and costs are certain. More precisely, the company pays expenses which occur continuously along time for the research activity and gets occasional profits according to a Poisson process. This model has been recently used by Bayraktar and Egami (2008) to model capital investments. Indeed, recently the model has been targeted with several developments, involving the present value of dividend payments and/or dividend strategies. We underline the cited work by Avanzi *et al.* (2007) and Avanzi (2009), an excellent review paper. Other works are of importance, some of which we briefly review below.

Important financial applications of the model ruled by (1.1) are the modeling of future dividends of the investments. So, we add an upper barrier, the dividend barrier, noted as $b\,(\geq u \geq 0)$. We refer to the upper graph in Figure 1 [see also Figure 1 of Avanzi *et al.* (2007)]. On the instant the surplus upcrosses the barrier a dividend is immediately paid and the process re-starts from level b. We can also consider the case $b < u$, however an immediate dividend is paid and the process starts from b, see Avanzi *et al.* (2007). This makes the situation less interesting from our point of view, so we will concentrate our work to the case $u \leq b$.

In this manuscript we are not interested on strategies of dividend payments, we focus on some key quantities, given a barrier level b. We will consider the payments either discounted or not. Several papers have been published recently using this model with an upper dividend barrier, where the calculation of expected amounts of the discounted paid dividends is targeted. Higher moments have also been considered. See Avanzi *et al.* (2007), Avanzi and Gerber (2008), Cheung and Drekic (2008), Gerber and Smith (2008), Ng (2009) and Ng (2010). Yang and Zhu (2008) compute bounds for the ruin probability. Song *et al.* (2008) consider Laplace transforms for the calculation of the expected duration of negative surplus. Cheung (2011) also deals with negative surplus excursions related problems.

For those works as well as in ours where the dividend barrier b is the key point, it is important to emphasize two aspects: we are going to consider two barriers, one reflecting and another absorbing, the dividend barrier b and the ruin level "0", respectively. In the case of the upper barrier b the process restarts at level b if this is overtaken by a gain. As mentioned above, this is because an immediate amount of surplus in excess of b is paid in the form of a dividend, it's a *pay-back* capital. It is not the case with the *ruin level* which makes the process to die down. Indeed, this happens with probability 1 (we'll come back to this issue later in the text). To achieve a payable dividend the process must not be ruined previously. Furthermore, under the conditions stated the process, *sooner or later*, will reach one of the two barriers, we mean, with probability one the process reaches a barrier.

In this paper we focus on the connection between the SM and the DM, and based on this we work on unknown problems, however having present some known results from a different viewpoint, which in some cases have interesting interpretations. We will underline these points apropriately. We base our research on the insights and ideas known from the classical risk model. This is a key point for our research. We first do a brief survey of the known results from the literature, then we make important connections between the classical and the dual model features. Afterwards, we make our own developments. We consider important that known results can be looked from our point of view so that our further developments are better taken and understood.

Let's now consider some of the basic definitions and notation for the dual risk model, those which we address throughout this paper. Some specific quantities we will define and denote on the appropriate section only. First, consider the process as driven by Equation (1.1), free of the dividend barrier. Let

$$\tau_x = \inf \{t > 0 : U(t) = 0 | U(0) = x\},$$

be the time to ruin, this is the usual definition for the model free of the dividend barrier ($\tau_x = \infty$ if $U(t) \geq 0 \ \forall t \geq 0$). Let

$$\psi(x, \delta) = E\left[e^{-\delta \tau_x} I(\tau_x < \infty) | U(0) = x\right],$$

where δ is a non negative constant. $\psi(u, \delta)$ is the Laplace transform of time to ruin τ_x. If $\delta = 0$ it reduces to the probability of ultimate ruin of the process free of the dividend barrier, when $\delta > 0$ we can see $\psi(u, \delta)$ *as the present value of a contingent claim of 1 payable at* τ_x*, evaluated under a given valuation force of interest* δ [see Ng (2010)].

Let's now consider an arbitrary upper level $\beta \geq u \geq 0$ in the model, see the upper graph of Figure 1, we don't call it yet a dividend barrier. Let

$$T_x = \inf \{t > 0 : U(t) > \beta | U(0) = x\}$$

be the time to reach an upper level $\beta \geq x \geq 0$ for the process which we allow to continue even if it crosses the ruin level "0". Due to the income condition T_x is a proper random variable since the probability of crossing β is one.

Let's now introduce into the model the barrier $\beta = b$ as a dividend barrier, and the ruin barrier "0", respectively reflecting and absorbing, such that if the process isn't ruined it will reach the level b. Here, an immediate dividend is paid by an amount in excess of b, the surplus is restored to level b and the process resumes. We will be mostly working the case $0 < u \leq b$.

Dividend will only be due if $T_x < \tau_x$ and ruin will only occur prior to that upcross otherwise. Whenever we refer to conditional random variables, or distributions, we will denote them by adding a "tilde", like $\tilde{T}_x$ for $T_x | T_x < \tau_x$.

Let $\chi(u, b)$ denote the probability of reaching b before ruin occurring, for a process with initial surplus u, and $\xi(u, b) = 1 - \chi(u, b)$ is the probability of ruin before reaching b. We have $\chi(u, b) = \Pr(T_u < \tau_u)$.

Because of the existence of the barrier b ultimate ruin has probability 1. The ruin level can be attained before or after the process is reflected on b. Then the probability of ultimate ruin is $\chi(u, b) + \xi(u, b) = 1$.

Let $D_u = \{U(T_u) - b \text{ and } T_u < \tau_u\}$ be the dividend amount and its distribution function be denoted as

$$G(u, b; x) = \Pr(T_u < \tau_u \text{ and } U(T_u) \le b + x) | u, b)$$

with density $g(u, b; x) = \frac{d}{dx} G(u, b; x)$. $G(u, b; x)$ is a defective distribution function, clearly $G(u, b; \infty) = \chi(u, b)$.

We refer now to the upper graph in Figure 1. If the process crosses b for the first time before ruin at a random instant, say $T_{(1)}$, then a random amount, denoted as $D_{(1)}$ is paid. The process repeats, now from level b. The random variables $D_{(i)}$ and $T_{(i)}$, $i = 1, 2, ...$, respectively dividend amount i and waiting time until that dividend, make a bivariate sequence of independent random variables $\left\{\left(T_{(i)}, D_{(i)}\right)\right\}_{i=1}^{\infty}$. We mean, $D_{(i)}$ and $T_{(i)}$ are dependent in general but $D_{(i)}$ and $T_{(j)}$, $i \neq j$, are independent. This follows from the Poisson process properties. This is well known from the *classical* Cramér-Lundberg risk model. Furthermore, if we take the subset $\left\{\left(T_{(i)}, D_{(i)}\right)\right\}_{i=2}^{\infty}$ we have now a sequence of independent and jointly identically distributed random variables (and independent of the $\left(T_{(1)}, D_{(1)}\right)$, the bivariate random variables only have the same joint distribution if $u = b$). To simplify notations we set that $\left(T_{(i)}, D_{(i)}\right)$ is distributed as (T_b, D_b), $i = 2, 3, ...$, and $\left(T_{(1)}, D_{(1)}\right)$ is distributed as (T_u, D_u).

Let M denote the number of dividends of the process. Total amount of discounted dividends at a force of interest $\delta > 0$ is denoted as $D(u, b, \delta)$ and $D(u, b) = D(u, b, 0^+)$ is the undiscounted total amount. Their n-th moments are denoted as $V_n(u; b, \delta)$ and $V_n(u; b)$, respectively. For simplicity denote as $V(u; b, \delta) = V_1(u; b, \delta)$. We have

$$\begin{aligned} D(u, b, \delta) &= \sum_{i=1}^{\infty} e^{-\delta\left(\sum_{j=1}^{i} T_{(j)}\right)} D_{(i)} \\ V_n(u; b, \delta) &= \mathbb{E}[D(u, b, \delta)^n] \end{aligned}$$

The purpose of this work is to find new results for the different quantities of interest around the dividend problem in the dual risk model as well as to provide new insights on already known results. A key in our work is the interface we establish between the SM and the DM. Despite the reversed income condition, many quantities can be characterized through features well known from the literature regarding the SM. For instance, the single dividend amount random variable can be viewed as the severity of ruin in the SM, although adapted to allow a second barrier.

Our contribution in this work can be split into three different levels. The first contribution is the new insight we give by connecting the DM and the SM. Although this is basically a tool for later developments it is of interest. This is done in Section 3. On a second stage, we develop a new insight and new formulae for an already known problem: moments for the discounted future dividends. The way we do allow us to find separate formulae for the different situations for dividend payments, unlike the known formulae from the literature where they appear as an aggregate. This is done in Section 4. On a 3rd level we develop expressions for new quantities such as probability of getting a dividend, number of dividends, amount of a single dividend. The first two quantities are studied in Section 5 whereas Section 6 deals with the single dividend amount. Before, in Section 2 we make an overview of the literature and results for the model. The last Section is devoted to work illustrative examples.

2 Paper review and results

We present here known results from the literature particularly those related to our developments. We are interested in working on the different random variables defined in the previous section and expectations on dividends. We are not interested in determining the optimal dividend strategy either in the DM or the SM, we only assume that a barrier strategy is applied whenever dividends are paid. So, we omit findings related to the former situation. Using a martingale argument Gerber (1979) showed that the ruin probability is given by

$$\psi(u) := \psi(u,0) = e^{-Ru}, \tag{2.1}$$

where R is the unique positive root of the equation

$$\lambda\left(\int_0^\infty e^{-Rx}p(x)dx - 1\right) + cR = 0. \tag{2.2}$$

We can use a standard probabilistic argument instead, we show it here as the method is going to be used later in the text for other purposes.

If there are no gains until $t_0 = u/c$ ruin level is crossed. By considering whether or not a gain occurs before time t_0, we have

$$\psi(u) = e^{-\lambda t_0} + \int_0^{t_0} \lambda\, e^{-\lambda\, t} \int_0^\infty p(x)\psi(u - ct + x)\, dxdt,$$

making $s = u - ct$ and rearranging we get

$$ce^{\lambda\,\frac{u}{c}}\psi(u) = c + \int_0^u \lambda\, e^{\lambda\,\frac{s}{c}} \int_0^\infty p(x)\psi(s+x)\, dxds.$$

Differentiating with respect to u we get

$$ce^{\lambda\,\frac{u}{c}}\psi(u)\,\frac{\lambda}{c} + ce^{\lambda\,\frac{u}{c}}\frac{d}{du}\,\psi(u) = \lambda e^{\lambda\,\frac{u}{c}} \int_0^\infty p(x)\psi(u+x)\, dx,$$

from which we get the following integro-differential equation

$$\lambda\psi(u) + c\,\frac{d}{du}\,\psi(u) = \lambda \int_0^\infty p(x)\psi(u+x)\, dx, \tag{2.3}$$

with the boundary conditions $\psi(0) = 1$ and $\psi(\infty) = 0$. Now, it's easy to set that $\psi(u+x) = \psi(u)\psi(x)$ since ruin to occur from initial level $u+x$ must first cross level u and from there get ruined (due to the independent increments property of the Poisson process). Hence,

$$\begin{aligned}
c\,\frac{d}{du}\,\psi(u) &= \lambda\psi(u)\left(\int_0^\infty p(x)\psi(x)dx - 1\right)\\
\frac{d}{du}\,\log\psi(u) &= \frac{\lambda}{c}\,(A-1)\\
\psi(u) &= e^{\frac{\lambda}{c}(A-1)u},
\end{aligned}$$

where $A = \int_0^\infty p(x)\psi(x)dx$ (A is not dependent on u). If we set $R = -\frac{\lambda}{c}\,(A-1)$, then we get (2.1).

Ng (2009) generalized the above probability for a positive δ, $\psi(u,\delta)$, which is given by

$$\psi(u,\delta) = e^{-R_\delta u}, \tag{2.4}$$

where R_δ is the unique positive root such that

$$\lambda\left(\int_0^\infty e^{-R_\delta x}p(x)dx - 1\right) + cR_\delta = \delta.$$

Results we show below concern moments of the total amount of discounted dividends by integro-differential equations, which in some cases shown can be solved analytically. As far as expectations of total dividends is concerned, using a direct and standard approach, by considering possible single gains/jumps events over a small time interval, Avanzi *et al.* (2007) found that

$$\begin{aligned}
V(u;b,\delta) &= u - b + V(b;b,\delta) \text{ if } u > b, \text{ and}\\
0 &= cV'(u;b,\delta) + (\lambda+\delta)V(u;b,\delta) - \lambda\int_0^{b-u} V(u+y;b,\delta)p(y)dy \qquad (2.5)\\
&\quad - \lambda\int_{b-u}^{\infty}(u-b+y)p(y)dy - \lambda V(b;b,\delta)\left[1 - P(b-u)\right], \text{ if } 0 < u < b,
\end{aligned}$$

noting that $V(0;b,\delta) = 0$, since ruin is immediate if $u = 0$. Besides, Avanzi *et al.* (2007) found solutions for equation (2.5) when exponential or mixtures of exponential gains size distributions are considered. Ng (2010) shows solutions when individual gains are phase-type distributed.

For higher moments of discounted dividends, Cheung and Drekic (2008) with a similar procedure show integro-differential equations similar to (2.5), for $n = 1, 2, ...,$

$$\begin{aligned}
V_n(u;b,\delta) &= \sum_{j=0}^{n}\binom{n}{j}(u-b)^{n-j}\,V_j(b;b,\delta) \text{ if } u > b, \text{ and}\\
0 &= cV_n'(u;b,\delta) + (\lambda+n\delta)V_n(u;b,\delta) - \lambda\int_0^{b-u} V_n(u+y;b,\delta)p(y)dy \qquad (2.6)\\
&\quad - \lambda\sum_{j=0}^{n}\binom{n}{j}V_j(b;b,\delta)\int_{b-u}^{+\infty}(y-b+u)^{n-j}\,p(y)dy, \text{ if } 0 < u < b.
\end{aligned}$$

as well as solutions for combinations of exponentials distributed gains size and for jump size distributions with rational Laplace transforms. Also, they work an approximation method. We also can get the above results using the approach used for getting (2.3).

3 Connecting the *classical* and the dual model

In our further developments the connection between the classical and the dual model is crucial as we want to translate methods and results from the first to the second, which has had extensive treatment. So, let's consider both models, at a first stage we consider the models free of barriers. As widely known, the *standard* Cramér-Lundberg risk model is ruled by the equation

$$U^*(t) = u^* + c^*t - S^*(t), \;\; t \geq 0, \tag{3.1}$$

where the quantities involved have similar characteristics (although different interpretations) to those corresponding to the dual model. To emphazise that we denote the corresponding quantities with the superscript "$*$". Apart from their application and interpretation, an important difference between the two models comes with the income condition as noted in the first paragraph of Section 1. We recall that in the SM model that condition comes expressed as $c^* > \lambda^* p_1^*$, which is reversed in our DM model. This condition assures that the surplus ultimately tends to infinity with probability one (if no barriers are added). The reversed condition in the DM is intended to achieve the same target, if it wasn't so the DM would be of difficult application, investors wouldn't get dividends as they could wish. To compare and relate both models we have to set the DM income condition on both now.

If we first consider the model without barriers we can relate the two models in the following way

$$U^*(t) = u^* + ct - S(t) = (\beta - u) + ct - S(t), \;\; t \geq 0, \beta > u. \tag{3.2}$$

Now, let's get back to our dividend problem and put the barriers back in order to establish the wanted relation between these two models. We refer to Figure 1. In the DM we consider it with an upper dividend barrier and a ruin barrier. The first is reflecting and the second is absorbing. In the corresponding standard model, the corresponding dividend barrier, $\beta = b$, can be seen as the ruin barrier of a surplus starting from initial surplus $\beta - u$. The other mentioned barrier usually is not considered in the SM, and it may just correspond to an upper line at level β. See again Figure 1.

As far as the DM is considered, we note that if the ruin level wasn't absorbing, i.e., the process would continue if the ruin level "0" was achieved, then the upper level b would be reached with probability 1, due to the income condition. However, we follow the model defined by Avanzi *et al.* (2007) where we should only pay dividends if the process isn't ruined. Perhaps we could work with negative capital, but that is out of scope in this work [that kind problem is addressed by Cheung (2011)]. We are only interested working over the set of the sample paths of the surplus process that do not lead to ruin. Hence, we need to calculate the probability of the surplus process reaches the barrier b before crossing the level zero. Note that this probability does not correspond to the survival probability, from initial level u.

Look at Figure 1, upper graph again. If we turn it upside down (rotate 180°) and look at it from right to left we get the *classical model shape*, where level "$\beta = b$" is the ruin level, "u" is the initial surplus, becoming $\beta - u$, and the level "0" is an upper barrier. $\left\{D_{(i)}\right\}_{i=2}^{\infty}$ is viewed as a sequence of *i.i.d.* severity of ruin random variables from initial surplus zero and $D_{(1)}$ the independent, but not identically distributed, severity of ruin random variable from initial surplus "$\beta - u$". Similarly, we have that $\left\{T_{(i)}\right\}_{i=2}^{\infty}$ can be viewed as a sequence of i.i.d. random variables meaning time of ruin from initial surplus 0, independent of $T_{(1)}$ which in turn represents the time of ruin from initial surplus $\beta - u$. The connection between

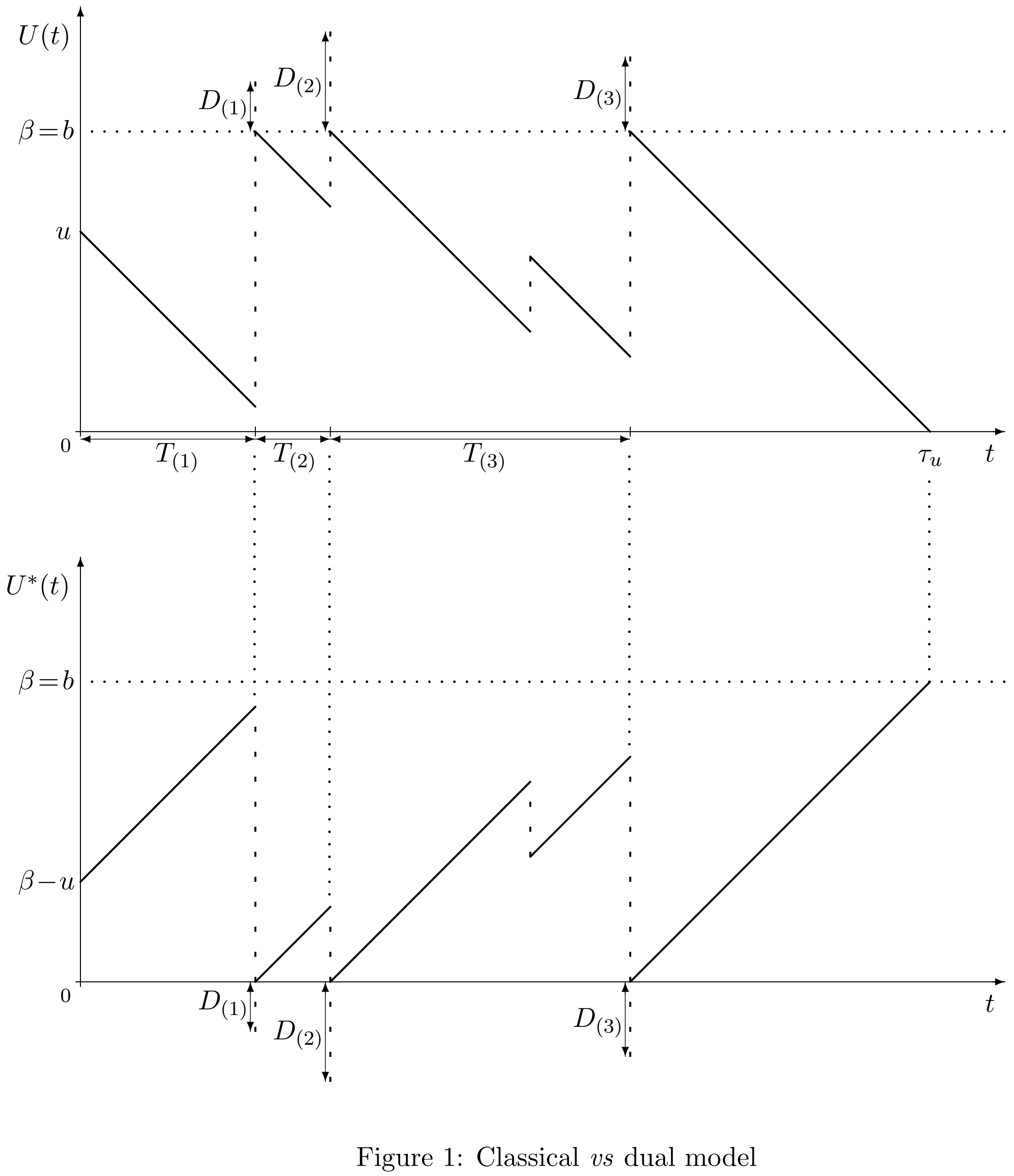

Figure 1: Classical *vs* dual model

the two models is briefly mentioned by Avanzi (2009) (Section 3.1), however not clearly. It is implicit here that in the case of the *classical* model whenever ruin occurs, the surplus is replaced at level "0".

We need to consider some results on the severity of ruin (expectations on the discounted severity of ruin) from the classical risk model adapted to allow an absorbing upper barrier $\beta > 0$. The following reasoning follows from Dickson and Waters (2004), Section 4, we adapted to consider the barrier β (we refer to Figure 1, $U^*(t)$ graph).

Hence, considering the SM until the end of this section, we present some new definitions valid here only. Let Y_u^* denote the deficit at ruin and T_u^* the time of ruin given an initial surplus u. We denote the defective distribution of the deficit Y_u^* as $G^*(u;x)$ with density $g^*(u;x)$ $[G^*(u;\infty) = \psi^*(u)$, no barrier β considered]. Now, define $\phi_n(u^*,\beta,\delta) = \mathbb{E}[e^{-\delta T_u^*}(Y_u^*)^n]$, $n = 0,1,2,...$, as an expectation of a discounted power of the severity of ruin ($\phi_1(u^*,\beta,\delta)$ is the expected discounted severity of ruin). We note that $\phi_n(\beta,\beta,\delta) = 0$, since the process is immediately absorbed at β. Let t_0 denote the time that the surplus takes to reach β if there are no claims, so that $u^* + ct_0 = \beta$. Using the same approach to set Formula (2.1), by conditioning on the time and the amount of the first claim,

$$\begin{aligned}\phi_n(u^*,\beta,\delta) = &\int_0^{t_0} \lambda e^{-\lambda t} \int_0^{u^*+ct} e^{-\delta t}\phi_n(u^*+ct-y,\beta,\delta)p(y)dydt \\ &+ \int_0^{t_0} \lambda e^{-\lambda t} \int_{u^*+ct}^{\infty} e^{-\delta t}(y-u^*-ct)^n p(y)dydt\end{aligned} \tag{3.3}$$

and by substituting $s = u^* + ct$,

$$\begin{aligned}\phi_n(u^*,\beta,\delta) = &\frac{1}{c}\int_{u^*}^{\beta} \lambda e^{-(\lambda+\delta)\frac{s-u^*}{c}} \int_0^{s} \phi_n(s-y,\beta,\delta)p(y)dyds \\ &+ \frac{1}{c}\int_{u^*}^{b} \lambda e^{-(\lambda+\delta)\frac{s-u^*}{c}} \int_s^{\infty} (y-s)^n p(y)dyds.\end{aligned}$$

Differentiating and rearranging, we get the following integro-differential equation, with boundary condition $\phi_n(\beta,\beta,\delta) = 0$,

$$\frac{d}{du^*}\phi_n(u^*,\beta,\delta) = \frac{\lambda+\delta}{c}\phi_n(u^*,\beta,\delta) - \frac{\lambda}{c}\int_0^{u^*} \phi_n(u^*-y,\beta,\delta)p(y)dy - \frac{\lambda}{c}\int_{u^*}^{\infty}(y-u^*)^n p(y)dy. \tag{3.4}$$

If we denote by $\bar{\phi}_n(s;\beta,\delta)$ the Laplace transform of $\phi_n(u^*,\beta,\delta)$ with respect to u^* after extending the domain of u^* from $0 \le u^* \le \beta$ to $u^* \ge 0$, i.e.,

$$\bar{\phi}_n(s;\beta,\delta) = \int_0^{\infty} e^{-su}\phi_n(u;\beta,\delta)\,du$$

we have

$$\begin{aligned}&c\left[s\bar{\phi}_n(s;\beta,\delta) - \phi_n(0;\beta,\delta)\right] = \\ &\qquad (\lambda+\delta)\bar{\phi}_n(s;\beta,\delta) - \lambda\bar{\phi}_n(s;\beta,\delta)\bar{p}(s) - \lambda\int_0^{\infty} e^{-su}\int_u^{\infty}(y-u)^n p(y)\,dy\,du\,,\end{aligned} \tag{3.5}$$

where $\bar{p}(s)$ is the Laplace transform of the density $p(x)$. The double integral can be re-expressed by using results in Section 1 of Willmot *et al.* (2005), which are also used in

Cheung and Drekic (2008), involving the link between stop-loss moments and equilibrium distributions. So, let $P_n(y)$ be the n-th equilibrium function of $P(y)$ (if it exists) such that

$$P_n(y) = \frac{\int_0^y [1 - P_{n-1}(t)]\, dt}{\int_0^\infty [1 - P_{n-1}(t)]\, dt}, \quad \text{for } y \geq 0, \; n = 1, 2, \ldots$$

where $P_0(y) = P(y)$. The corresponding density and Laplace transform are given by, denoted as $p_n(y)$ and $\bar{p}_n(s)$ respectively,

$$p_n(y) = \tfrac{1-P_{n-1}(y)}{\int_0^\infty [1-P_{n-1}(t)]\, dt}, \qquad \bar{p}_n(s) = \tfrac{1-\bar{p}_{n-1}(s)}{s\int_0^\infty [1-P_{n-1}(t)]\, dt}.$$

Using Equation (1.5) of Willmot *et al.* (2005), we obtain

$$\int_u^\infty (y-u)^n p(y)\, dy = p_n(1 - P_n(u)) = p_n \times p_{n+1}(u) \int_0^\infty [1 - P_n(t)]\, dt,$$

where p_n is the n-th moment of the single gain amount, provided it exists. Since

$$\int_0^\infty e^{-su} \int_u^\infty (y-u)^n p(y)\, dy\, du = p_n \bar{p}_{n+1}(s) \int_0^\infty [1 - P_n(t)\,]\, dt = p_n \frac{1 - \bar{p}_n(s)}{s},$$

then, from (3.5) we have,

$$\bar{\phi}_n(s, \beta, \delta) = \frac{c\phi_n(0, \beta, \delta) - \lambda p_n \frac{1-\bar{p}_n(s)}{s}}{cs - (\lambda + \delta) + \lambda \bar{p}(s)}. \tag{3.6}$$

Expectation $\phi_n(0, \beta, \delta)$ can be found using the boundary condition above.

These results are going to be used for computation of expectations of times and dividends in the dual model, it will become clear in the next section. Finally, we recall the probability that the surplus attains the level β without first falling below zero, $\chi^*(u^*, \beta) = (1 - \psi^*(u^*))/(1 - \psi^*(\beta))$, where $\psi^*(.)$ is the ultimate ruin probability in the SM. Its complementary comes $\xi^*(u^*, \beta) = 1 - \chi^*(u^*, \beta)$ [please see Dickson (2005), Section 8.2, for details]. We will be using similar probabilities in the dual model.

4 A new approach for the expected discounted dividends

For $0 \leq u \leq b$, looking at Figure 1 it's easy to set the present value of the total dividends in infinite time, or the total discounted dividends amounts, $D(u, b, \delta)$. Its expected value becomes,

$$\begin{aligned} V(u; b, \delta) &= \mathbb{E}[D(u, b, \delta)] = \mathbb{E}\left[\sum_{i=1}^{\infty} e^{-\delta\left(\sum_{j=1}^{i} T_{(j)}\right)} D_{(i)}\right] = \sum_{i=1}^{\infty} \mathbb{E}\left[e^{-\delta\left(\sum_{j=1}^{i} T_{(j)}\right)} D_{(i)}\right] \\ &= \mathbb{E}\left(e^{-\delta T_{(1)}} D_{(1)}\right) + \mathbb{E}\left(e^{-\delta T_{(1)}}\right) \mathbb{E}\left(e^{-\delta T_{(2)}} D_{(2)}\right) \\ &\quad + \mathbb{E}\left(e^{-\delta T_{(1)}}\right) \mathbb{E}\left(e^{-\delta T_{(2)}}\right) \mathbb{E}\left(e^{-\delta T_{(3)}} D_{(3)}\right) + \ldots \end{aligned}$$

because the pairs of random variables are independent $\left(T_{(i)}, D_{(i)}\right)$, $i = 1, 2, \ldots$, two by two. Note that $T_{(i)}$ and $D_{(i)}$ are dependent in general, they have similar properties as time to ruin and its severity in the classical case. Besides, $\left(T_{(i)}, D_{(i)}\right)$, $i = 2, 3, ...$, are also identically distributed, i.e. $(T_b, D_b) \overset{d}{=} \left(T_{(i)}, D_{(i)}\right)$, $i = 2, 3, ...$, also $T_{(i)} \overset{d}{=} T_b$, $i = 2, 3, ...$ Set $(T_u, D_u) \overset{d}{=} \left(T_{(1)}, D_{(1)}\right)$, we can write

$$\begin{aligned} V(u; b, \delta) &= \mathbb{E}\left(e^{-\delta T_u} D_u\right) + \mathbb{E}\left(e^{-\delta T_u}\right) \mathbb{E}\left(e^{-\delta T_b} D_b\right) + \mathbb{E}\left(e^{-\delta T_u}\right) \mathbb{E}\left(e^{-\delta T_b}\right) \mathbb{E}\left(e^{-\delta T_b} D_b\right) \\ &\quad + \mathbb{E}\left(e^{-\delta T_u}\right) \mathbb{E}\left(e^{-\delta T_b}\right)^2 \mathbb{E}\left(e^{-\delta T_b} D_b\right) + ... \\ &= \mathbb{E}\left(e^{-\delta T_u} D_u\right) + \mathbb{E}\left(e^{-\delta T_u}\right) \mathbb{E}\left(e^{-\delta T_b} D_b\right) \sum_{i=0}^{\infty} \mathbb{E}\left(e^{-\delta T_b}\right)^i \end{aligned} \tag{4.1}$$

Hence

$$V(u; b, \delta) = \mathbb{E}\left(e^{-\delta T_u} D_u\right) + \mathbb{E}\left(e^{-\delta T_u}\right) \frac{\mathbb{E}\left(e^{-\delta T_b} D_b\right)}{1 - \mathbb{E}\left(e^{-\delta T_b}\right)}. \tag{4.2}$$

A similar expression can be found in Dickson and Waters (2004) relating discounted time and severity of ruin in the classical model with a dividend strategy. We *only* need to evaluate $\mathbb{E}\left(e^{-\delta T_u} D_u\right)$, $\mathbb{E}\left(e^{-\delta T_b} D_b\right)$, $\mathbb{E}\left(e^{-\delta T_u}\right)$ and $\mathbb{E}\left(e^{-\delta T_b}\right)$.

To compute the above quantitives, and therefore $V(u; b, \delta)$, we can make use of Expression (3.3) and Equation (3.4) at the end of Section 3, $\phi_n(u^*, b, \delta) = \mathbb{E}\left(e^{-\delta T_u} D_u^n\right)$ for $n = 0, 1$, with $u^* = b - u$ and $u^* = 0$ $(u = b)$. As an alternative for that calculation, we can invert (3.6).

In the simpler case $u = b$, we have $\left(T_{(1)}, D_{(1)}\right) \overset{d}{=} (T_b, D_b)$ and $T_{(1)} \overset{d}{=} T_b$, and the above formula simplifies to

$$V(b; b, \delta) = \frac{\mathbb{E}\left(e^{-\delta T_b} D_b\right)}{1 - \mathbb{E}\left(e^{-\delta T_b}\right)}. \tag{4.3}$$

Then we have

$$V(u; b, \delta) = u - b + \frac{\mathbb{E}\left(e^{-\delta T_b} D_b\right)}{1 - \mathbb{E}\left(e^{-\delta T_b}\right)} \text{ if } u \geq b,$$

because $V(b; b, \delta)$ is (4.3). Note that Formula (4.2) can be written as

$$V(u; b, \delta) = \mathbb{E}\left(e^{-\delta T_u} D_u\right) + \mathbb{E}\left(e^{-\delta T_u}\right) V(b; b, \delta), \tag{4.4}$$

which means that the expected value of (discounted) future dividends equals the expected value of the (discounted) first dividend plus the discounted dividends from the process re-starting from b.

Unlike Avanzi *et al.* (2007), $V(u; b, \delta)$ comes as a function of the expected disconted first dividend amount, of the Laplace transform of the time to get a first dividend and similar quantities for other future dividends and respective times. Then we have to find solutions for all these. For instance, with these formulae we can see the contribution of the first dividend amount (discounted) to the global future dividends. In the last section we can see some numerics.

Using the same method we can compute higher moments. For instance, if we want to compute the variance of the accumulated discounted dividends we need to compute $V_2(u; b, \delta)$. Let Z_i be the discounted dividend i so that

$$Z_i = e^{-\delta\left(\sum_{j=1}^{i} T_{(j)}\right)} D_{(i)}.$$

Then,

$$V_2(u;b,\delta) = \mathbb{E}[D(u,b,\delta)^2] = \sum_{i=1}^{\infty}\mathbb{E}[Z_i^2] + 2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\mathbb{E}[Z_iZ_j]\ .$$

Using the above properties on the distributions, $(T_b, D_b) \stackrel{d}{=} \left(T_{(i)}, D_{(i)}\right)$, $i = 2,3,...$, and $(T_u, D_u) \stackrel{d}{=} \left(T_{(1)}, D_{(1)}\right)$ as well as independence in the sequence, we can write

$$\begin{aligned}
\mathbb{E}[Z_i^2] &= \mathbb{E}\left(e^{-2\delta T_u}\right)\mathbb{E}\left(e^{-2\delta T_b}\right)^{i-2}\mathbb{E}\left(e^{-2\delta T_b}D_b^2\right),\ i = 2,3,4,...\\
\mathbb{E}[Z_1^2] &= \mathbb{E}\left(e^{-2\delta T_u}D_u^2\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{\infty}\mathbb{E}[Z_i^2] &= \mathbb{E}\left(e^{-2\delta T_u}D_u^2\right) + \mathbb{E}\left(e^{-2\delta T_u}\right)\mathbb{E}\left(e^{-2\delta T_b}D_b^2\right)\sum_{i=2}^{\infty}\mathbb{E}\left(e^{-2\delta T_b}\right)^{i-2}\\
&= \mathbb{E}\left(e^{-2\delta T_u}D_u^2\right) + \frac{\mathbb{E}\left(e^{-2\delta T_u}\right)\mathbb{E}\left(e^{-2\delta T_b}D_b^2\right)}{1-\mathbb{E}\left(e^{-2\delta T_b}\right)}\ .
\end{aligned}$$

Now,

$$\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\mathbb{E}[Z_iZ_j] = \sum_{j=2}^{\infty}\mathbb{E}[Z_1Z_j] + \sum_{i=2}^{\infty}\sum_{j=i+1}^{\infty}\mathbb{E}[Z_iZ_j],$$

with

$$\begin{aligned}
\mathbb{E}[Z_1Z_j] &= \mathbb{E}\left(e^{-2\delta T_u}D_u\right)\mathbb{E}\left(e^{-\delta T_b}D_b\right)\mathbb{E}\left(e^{-\delta T_b}\right)^{j-2},\ j \geq 2\\
\mathbb{E}[Z_iZ_j] &= \mathbb{E}\left(e^{-2\delta T_u}\right)\mathbb{E}\left(e^{-2\delta T_b}D_b\right)\mathbb{E}\left(e^{-\delta T_b}D_b\right)\mathbb{E}\left(e^{-2\delta T_b}\right)^{i-2}\mathbb{E}\left(e^{-\delta T_b}\right)^{j-i-1},
\end{aligned}$$

for $i < j$, $i = 2,3,...$ Then

$$\begin{aligned}
\sum_{j=2}^{\infty}\mathbb{E}[Z_1Z_j] &= \frac{\mathbb{E}\left(e^{-2\delta T_u}D_u\right)\mathbb{E}\left(e^{-\delta T_b}D_b\right)}{1-\mathbb{E}\left(e^{-\delta T_b}\right)},\\
\sum_{j=i+1}^{\infty}\mathbb{E}\left(e^{-\delta T_b}\right)^{j-(i+1)} &= \sum_{k=0}^{\infty}\mathbb{E}\left(e^{-\delta T_b}\right)^{k} = \frac{1}{1-\mathbb{E}\left(e^{-\delta T_b}\right)},\\
\sum_{i=2}^{\infty}\sum_{j=i+1}^{\infty}\mathbb{E}[Z_iZ_j] &= \frac{\mathbb{E}\left(e^{-2\delta T_u}\right)\mathbb{E}\left(e^{-2\delta T_b}D_b\right)\mathbb{E}\left(e^{-\delta T_b}D_b\right)}{1-\mathbb{E}\left(e^{-\delta T_b}\right)}\sum_{i=2}^{\infty}\mathbb{E}\left(e^{-2\delta T_b}\right)^{i-2}\\
&= \frac{\mathbb{E}\left(e^{-2\delta T_u}\right)\mathbb{E}\left(e^{-2\delta T_b}D_b\right)\mathbb{E}\left(e^{-\delta T_b}D_b\right)}{\left[1-\mathbb{E}\left(e^{-\delta T_b}\right)\right]\left[1-\mathbb{E}\left(e^{-2\delta T_b}\right)\right]}.
\end{aligned}$$

Function $V_2(u;b,\delta)$ can be expressed in terms of $\phi_n(u^*,b,m\delta)$ $(=\phi_n(b-u,b,m\delta) = \mathbb{E}\left(e^{-m\delta T_u}D_u^n\right))$, $m = 1,2$, in the following way:

$$\begin{aligned}
V_2(u;b,\delta) = \phi_2(b-u,b,2\delta) + \frac{\phi_0(b-u,b,2\delta)\phi_2(0,b,2\delta)}{1-\phi_0(0,b,2\delta)} \qquad (4.5)\\
+ 2\left[\frac{\phi_1(b-u,b,2\delta)\phi_1(0,b,\delta)}{1-\phi_0(0,b,\delta)} + \frac{\phi_0(b-u,b,2\delta)\phi_1(0,b,2\delta)\phi_1(0,b,\delta)}{(1-\phi_0(0,b,\delta))(1-\phi_0(0,b,2\delta))}\right].
\end{aligned}$$

We see that we need to evaluate the following six different quantities $\mathbb{E}\left(e^{-2\delta T_u}\right)$, $\mathbb{E}\left(e^{-2\delta T_b}\right)$, $\mathbb{E}\left(e^{-2\delta T_u} D_u\right)$, $\mathbb{E}\left(e^{-2\delta T_b} D_b\right)$, $\mathbb{E}\left(e^{-2\delta T_b} D_b^2\right)$, $\mathbb{E}\left(e^{-2\delta T_u} D_u^2\right)$, apart from those four needed for the first moment $\mathbb{E}\left(e^{-\delta T_u} D_u\right)$, $\mathbb{E}\left(e^{-\delta T_b} D_b\right)$, $\mathbb{E}\left(e^{-\delta T_u}\right)$ and $\mathbb{E}\left(e^{-\delta T_b}\right)$.

When $u = b$ moment (4.5) simplifies to

$$\begin{aligned} V_2(b; b, \delta) &= \frac{\phi_2(0, b, 2\delta)}{1 - \phi_0(0, b, 2\delta)} + 2\frac{\phi_1(0, b, 2\delta)\phi_1(0, b, \delta)}{(1 - \phi_0(0, b, \delta))(1 - \phi_0(0, b, 2\delta))} \\ &= \frac{\mathbb{E}\left(e^{-2\delta T_b} D_b^2\right)}{1 - \mathbb{E}\left(e^{-2\delta T_b}\right)} + 2\frac{\mathbb{E}\left(e^{-2\delta T_b} D_b\right)\mathbb{E}\left(e^{-\delta T_b} D_b\right)}{(1 - \mathbb{E}\left(e^{-\delta T_b}\right))(1 - \mathbb{E}\left(e^{-2\delta T_b}\right))}\,. \end{aligned} \tag{4.6}$$

We deduct an easy recursive formula for the computation of the n-th order moment $V_n(u; b, \delta)$, valid for $n = 1, 2, ...$ We can use the idea implicit in Formula (4.4) and write that

$$D(u; b, \delta) = e^{-\delta T_u}\left(D_u + D(b; b, \delta)\right),$$

so that

$$\begin{aligned} D(u; b, \delta)^n &= e^{-n\delta T_u}\left(D_u + D(b; b, \delta)\right)^n \\ &= e^{-n\delta T_u}\sum_{k=0}^{n}\binom{n}{k}{D_u}^k D(b; b, \delta)^{n-k}\,. \end{aligned}$$

Therefore,

$$V_n(u; b, \delta) = \sum_{k=0}^{n}\binom{n}{k}\mathbb{E}\left[e^{-n\delta T_u}{D_u}^k\right]V_{n-k}(b; b, \delta)\,, \tag{4.7}$$

with $V_0(b; b, \delta) = 1$. We see that the n-th moment $V_n(u; b, \delta)$ depend on the evaluation of the $n + 1$ quantities $V_k(b; b, \delta)$, $k = 0, 1, 2, ..., n$. For these latter we can find a recursion as follows. Setting on the former $u = b$ we get

$$V_n(b; b, \delta) = \sum_{k=1}^{n}\binom{n}{k}\mathbb{E}\left[e^{-n\delta T_b}{D_b}^k\right]V_{n-k}(b; b, \delta) + \mathbb{E}\left[e^{-n\delta T_b}\right]V_n(b; b, \delta)\,,$$

and so we get the starting recursion

$$V_n(b; b, \delta) = \frac{\sum_{k=1}^{n}\binom{n}{k}\mathbb{E}\left[e^{-n\delta T_b}{D_b}^k\right]V_{n-k}(b; b, \delta)}{1 - \mathbb{E}\left[e^{-n\delta T_b}\right]}\,. \tag{4.8}$$

We can do some sensibility analysis in our examples section (Section 7) to evaluate the contribution of $V_n(b; b, \delta)$ on $V_n(u; b, \delta)$, by giving different sets of values of (u, b), with $u \leq b$.

If we consider a finite number of future dividends, we can compute moments of discounted future dividends so that we can have some understanding of their change as $n \to \infty$. For instance considering the first moment, let $V(u; b, \delta, n)$ denote the expected discounted finite future dividends. We can compute $V(u; b, \delta, n)$ using an analogous approach to that used in (4.1), as follows. For $n = 2, 3, ...$,

$$\begin{aligned} V(u; b, \delta, n) &= \mathbb{E}\left[\sum_{i=1}^{n} e^{-\delta\left(\sum_{j=1}^{i} T_{(j)}\right)} D_{(i)}\right] \\ &= \mathbb{E}\left(e^{-\delta T_u} D_u\right) + \mathbb{E}\left(e^{-\delta T_u}\right)\mathbb{E}\left(e^{-\delta T_b} D_b\right)\sum_{i=0}^{n-2}\mathbb{E}\left(e^{-\delta T_b}\right)^i \\ &= \mathbb{E}\left(e^{-\delta T_u} D_u\right) + \mathbb{E}\left(e^{-\delta T_u}\right)\mathbb{E}\left(e^{-\delta T_b} D_b\right)\frac{1 - \mathbb{E}\left(e^{-\delta T_b}\right)^{n-1}}{1 - \mathbb{E}\left(e^{-\delta T_b}\right)}, \end{aligned} \tag{4.9}$$

and $V(u;b,\delta,1) = \mathbb{E}\left(e^{-\delta T_u} D_u\right)$. The above expression enhances the utility of our approach with new formulae, because it allows to compute the contribution of each dividend to the aggregate expectation as given by Avanzi *et al.* (2007).

5 On the probability of getting a dividend

As we said at the end of Section 1, to get a dividend is necessary that the process reaches/crosses the level b before ruin. That occurs with probability $\chi(u,b)$. The complementary probability is $\xi(u,b)$. Finding closed forms for $\xi(u,b)$, or $\chi(u,b)$, isn't as straightfoward as the similar quantities in the classical model referred at the end of Section 3. We are developing two different methods to find those probabilities: First, we find integro-differential equations, second we compute Laplace transforms. We note that $\xi(u,b)$ and $\chi(u,b)$, appear to be related to Ng(2010)'s definitions for discounted upcrossing and downcrossing probabilities (pp. 285 and 287). However, those are specifically defined for the particular case when the single amount gains are Phase-type distributed.

Using the usual approach, to reach ruin level prior to dividend level is possible with or without a gain (at time $t_0 : u - ct_0 = 0$). Then, for $0 < u < b$:

$$\xi(u,b) = e^{-\lambda t_0} + \int_0^{t_0} \lambda e^{-\lambda t} \int_0^{b-(u-ct)} p(x)\xi(u-ct+x,b)\,dxdt,$$

from which we find

$$\lambda\xi(u,b) + c\frac{d}{du}\xi(u,b) = \lambda\int_0^{b-u} p(x)\xi(u+x,b)\,dx, \tag{5.1}$$

where the boundary conditions is $\xi(0,b) = 1$. Setting $\xi(u+x,b) = \xi(u,b-x)\xi(x,b) = \xi(x,b-u)\xi(u,b)$ we get

$$\begin{aligned}
\lambda\xi(u,b) + c\frac{d}{du}\xi(u,b) &= \lambda\xi(u,b)\int_0^{b-u} p(x)\xi(x,b-u)\,dx \\
\frac{c}{\lambda}\frac{d}{du}\xi(u,b) &= \xi(u,b)\left(\int_0^{b-u} p(x)\xi(x,b-u)\,dx - 1\right) \\
\frac{d}{du}\log\xi(u,b) &= \frac{\lambda}{c}\left(\int_0^{b-u} p(x)\xi(x,b-u)\,dx - 1\right).
\end{aligned}$$

Likewise, we can get

$$\lambda\chi(u,b) + c\frac{d}{du}\chi(u,b) = \lambda\int_0^{b-u} p(x)\chi(u+x,b)\,dx + \lambda\left[1 - P(b-u)\right],$$

where the boundary conditions is $\chi(0,b) = 0$.

We can compute Laplace transforms on Equation (5.1) as an alternative method to find $\xi(u,b)$. We can a use a method of change of variable already used by Avanzi *et al.* (2007), Section 6, retrieved by Cheung and Drekic (2008) and mentioned in the review paper by Avanzi (2009). In that equation replace u by $z = b-u$ and define $\mathcal{E}(z,b) = \xi(b-z,b) = \xi(u,b)$. This change of variable analytically is like setting the relation between the two models, classical and dual. Note that $\mathcal{E}(b,b) = \xi(0,b) = 1$. The corresponding integro-differential equation for $\mathcal{E}(z,b)$ is

$$\lambda\mathcal{E}(z,b) - c\,\frac{\partial}{\partial z}\,\mathcal{E}(z,b) - \lambda\int_0^z p(z-y)\mathcal{E}(y,b)\,dy = 0.$$

In function $\mathcal{E}(z,b)$ extend the range of z from $0 \le z \le b$ to $0 \le z \le \infty$ and denote the resulting function by $\epsilon(z)$, then compute its Laplace transform, denoted as $\bar{\epsilon}(s)$, so that

$$\lambda\bar{\epsilon}(s) - c\,[s\bar{\epsilon}(s) - \epsilon(0)] - \lambda\bar{\epsilon}(s)\bar{p}(s) = 0.$$

Hence,

$$\bar{\epsilon}(s) = \frac{c\epsilon(0)}{cs - \lambda + \lambda\bar{p}(s)}, \tag{5.2}$$

where $\epsilon(0) = \xi(b,b)$ (note that $\epsilon(b) = \mathcal{E}(b,b) = \xi(0,b) = 1$). When $\bar{p}(s)$ is a rational function we can invert $\bar{\epsilon}(s)$ to find a solution for $\epsilon(z)$. Finally $\xi(u,b) = \epsilon(b-u)$ for $0 \le u \le b$.

Now let's consider the multiple dividend situations and let M, the number of dividends to be claimed. Then, it's easy to find that

$$\begin{aligned} \Pr[M &= 0] = \xi(u,b) \\ \Pr[M &= k] = \chi(u,b)\chi(b,b)^{k-1}\xi(b,b),\; k = 1,2,... \end{aligned}$$

M follows a zero-modified geometric distribution (if $u = b$ we get a geometric distribution with $\Pr[M = k] = \chi(b,b)^k\xi(b,b)$, $k = 0,1,2,...$). The total amount of dividend gains (not discounted, $D(u,b)$) follows a compound zero-delayed geometric distribution.

6 On the dividend amount distribution

In this section we deal with the distribution of the random variable D_u representing the amount of a single dividend, non discounted. The distribution function of the joint dividend amount and the fact that it occurs was denoted in Section 1 as $G(u,b;x)$. Its density is $g(u,b;x)$. We develop a set of different results connected to the distribution of D_u and different ways to provide its computation. We start by relating it to the probability $\xi(u,\beta)$, or $\chi(u,\beta)$, then using a standard method in ruin theory within the standard model, we find integro-differential and integral equations for $G(u,b;x)$ and $g(u,b;x)$. From the former we compute Laplace transforms. To finish, we relate the distribution of a single amount distribution with the distribution of the severity of ruin of the standard risk model, so that we can use well known results to help the computation of the distribution $G(u,b;x)$.

First, consider the process free of the barrier, setting the general fixed value $\beta \ge u$. Considering that ruin can occur before or after crossing β, and considering the quantities $\xi(u,\beta)$ and $g(u,\beta;x)$, we can write

$$\begin{aligned} \psi(u) &= \xi(u,\beta) + \int_0^\infty g(u,\beta;x)\psi(\beta+x)dx = \xi(u,\beta) + \psi(\beta)\int_0^\infty g(u,\beta;x)\psi(x)dx \\ \xi(u,\beta) &= e^{-Ru} - e^{-R\beta}\bar{g}(u,\beta;R) \end{aligned} \tag{6.1}$$

noting $\psi(x) = e^{-Rx}$ and rearranging. $\bar{g}(u,\beta;R)$ represents the Laplace transform of the density $g(u,\beta;x)$ evaluated at R.

Second, now we get back to the usual model with a dividend barrier $\beta = b$, we can compute an integro-differential equation for $G(u, b; x)$ using the standard procedure. Conditioning on the first gain we get, where t_0 is such that $u - ct_0 = 0$,

$$G(u, b; x) = \int_0^{t_0} \lambda e^{-\lambda t} \left[\int_0^{b-(u-ct)} p(y) G(u - ct + y, b; x)\, dy + \int_{b-(u-ct)}^{b-(u-ct)+x} p(y) dy \right] dt.$$

Rearranging and differentiating with respect to u, we obtain the following integro-differential equation

$$\lambda G(u, b; x) + c \frac{\partial}{\partial u} G(u, b; x) = \lambda \int_u^b p(y - u) G(y, b; x)\, dy + \lambda \left[P(b - u + x) - P(b - u) \right], \quad (6.2)$$

with boundary condition $G(0, b; x) = 0$. We get, differentiating with respect to x,

$$\lambda g(u, b; x) + c \frac{\partial}{\partial u} g(u, b; x) = \lambda \int_0^{b-u} p(y) g(u + y, b; x)\, dy + \lambda p(b - u + x). \quad (6.3)$$

From the above integro-differential equations we compute Laplace transforms, for $G(u, b; x)$ or $g(u, b; x)$, whose inversion can lead easier to either close formulae or numerical calculation of the distribution. We use methods similar to those used for (5.2). Let $\mathcal{G}(z, b; x) = G(b - z, b; x)$. Then $\mathcal{G}(b, b; x) = G(0, b; x) = 0$. Then, from (6.2) we get

$$\lambda \mathcal{G}(z, b; x) - c \frac{\partial}{\partial z} \mathcal{G}(z, b; x) - \lambda \int_0^z p(z - y) \mathcal{G}(y, b; x)\, dy - \lambda \left[P(z + x) - P(z) \right] = 0$$

Let $\rho(z, x)$ be the corresponding function (to $\mathcal{G}(z, b; x)$) arising from extending the range of z. Taking Laplace transforms we get,

$$\lambda \bar{\rho}(s, x) - c \left[s \bar{\rho}(s, x) - \rho(0, x) \right] - \lambda \bar{\rho}(s, x) \bar{p}(s) + \lambda \left[\frac{\bar{p}(s)}{s} - \hat{p}(s, x) \right] = 0,$$

where

$$\begin{aligned} \bar{\rho}(s, x) &= \int_0^\infty e^{-sz} \rho(z, x)\, dz \\ \hat{p}(s, x) &= e^{sx} \int_x^\infty e^{-sy} P(y)\, dy. \end{aligned} \quad (6.4)$$

Hence,

$$\bar{\rho}(s, x) = \frac{c\rho(0, x) + \lambda \left[\bar{p}(s)/s - \hat{p}(s, x) \right]}{cs - \lambda + \lambda \bar{p}(s)} \quad (6.5)$$

Likewise, for the density $g(u, b; x)$ from (6.3) and noting that $g(0, b; x) = 0$, setting $\gamma(z, x) = g(b - z, b, x)$ we get

$$\lambda \gamma(z, x) - c \frac{\partial}{\partial z} \gamma(z, x) - \lambda \int_0^z p(z - y) \gamma(y, x)\, dy - \lambda p(z + x) = 0$$

from which we get the Laplace transform for $\gamma(z, x)$

$$\bar{\gamma}(s, x) = \frac{c\gamma(0, x) - \lambda \tilde{p}(s, x)}{cs - \lambda + \lambda \bar{p}(s)}$$

with

$$\tilde{p}(s,x) = e^{sx}\int_x^{\infty} e^{-sy}p(y)\,dy.$$

Note that

$$\frac{\partial}{\partial x}\bar{\rho}(s,x) = \bar{\gamma}(s,x).$$

Like in the case of the SM, using a probability argument we can find integral equations for $G(u,b;x)$ and $g(u,b;x)$, similar to those found by Gerber *et al.* (1987) for the severity of ruin:

$$\begin{aligned} G(u,b;x) &= \int_0^{b-u} g(u,u;y)G(u+y,b;x)\,dy + \int_{b-u}^{b-u+x} g(u,u;y)dy \\ g(u,b;x) &= \int_0^{b-u} g(u,u;y)g(u+y,b;x)\,dy + g(u,u;b-u+x)\ . \end{aligned}$$

Since the distribution of the severity of ruin in the SM is largely studied, we can relate that distribution with the one of the amount of a single dividend in this dual model, so that we can use that relation to calculate $G(u,b;x)$. For that, first consider the process continuing even if ruin occurs. The process can cross for the first time the upper dividend level before or after having ruined. Then we can write the (proper) distribution of the amount by which the process first upcrosses b, denoted as $H(u,b;x) = \Pr\left[U(T_u) \le b+x\right]$. As we denoted in Section 3, $G^*(\cdot;\cdot)$ and $g^*(\cdot;\cdot)$ refer to the distribution and density function of the severity of ruin in the classical model, respectively. We have

$$\begin{aligned} H(u,b;x) &= H(u,b;x|T_u<\tau_u)\chi(u,b) + H(u,b;x|T_u>\tau_u)\xi(u,b) \\ &= G(u,b;x) + \xi(u,b)H(0,b;x). \end{aligned}$$

The second equation above simply means that the probability of the amount by which the process first upcrosses b is less or equal than x, equals the probability of a dividend claim less or equal than x plus the probability of a similar amount but in that case it cannot be a dividend. This second probability can be computed through the probability of first reaching the level "0", $\xi(u,b)$, times the probability of an upcrossing of level b by the same amount ($\le x$) but restarting from 0, $H(0,b;x)$.

We can compute $H(u,b;x)$ through expressions known for the distribution of the severity of ruin obtained from the standard risk model (recall that the income condition is reversed, making it a proper distribuition function). Then we get

$$G^*(b-u;x) = G(u,b;x) + \xi(u,b)G^*(b;x)$$

equivalent to

$$G(u,b;x) = G^*(b-u;x) - \xi(u,b)G^*(b;x). \tag{6.6}$$

From here we can express differently the probability $\xi(b,b)$, other than the one that can be given from formula (5.2). First set $u=b$, we have then

$$\begin{aligned} H(b,b;x) &= G(b,b;x) + \xi(b,b)H(0,b;x) \Leftrightarrow \\ G(b,b;x) &= G^*(0;x) - \xi(b,b)G^*(b;x). \end{aligned}$$

Now, differentiate, take Laplace transforms and evaluate at R we get

$$\begin{aligned} g(b,b;x) &= g^*(0;x) - \xi(b,b)g^*(b;x) \\ \bar{g}(b,b;R) &= \overline{g^*}(0;R) - \xi(b,b)\overline{g^*}(b;R), \end{aligned}$$

then use (6.1) to get

$$\xi(b,b) = \frac{\left[1-\overline{g^*}(0;R)\right]e^{-Rb}}{1-\overline{g^*}(b;R)e^{-Rb}}.$$

$g^*(0;x) = p_1^{-1}\left[1-P(x)\right]$ is the severity density in SM whose Laplace transform is

$$\overline{g^*}(0;R) = \frac{1}{Rp_1}\left(1-\int_0^\infty e^{-Rx}p(x)dx\right) = \frac{c}{\lambda p_1}$$

using (2.2). We still need to compute $\overline{g^*}(b;R)$, clearly, it is not a trivial calculation since it's a Laplace transform of the severity of ruin with a positive initial surplus in the SM. If $p(x)$ is exponential then $\overline{g^*}(0;R) = \overline{g^*}(u;R)$, but that is *the trivial* example.

7 Illustrations

We worked three examples with quite different features: when individual gain amounts follow an exponential distribution, when they follow a particular combination of exponentials and when we consider a damped sine distribution [see Cheung and Drekic (2008)]. In the first case it is possible to find closed formulae for most of the quantities under study. Besides, we can identify the (conditional) distribution of the individual dividends amount. In the other two, since closed formulae were not possible, we worked out numerical quantities with the help of the Maple software. We found the same sort of quantities worked in the previous example and also additional ones related with the distribution of the single amount of a dividend. Additional tables of figures are shown.

7.1 Exponential jumps

We consider the case when gain amounts are exponentially distributed, that is $p(y) = \alpha e^{-\alpha y}$, $y > 0$, with $\alpha > 0$. From Section 3 we know that $\mathbb{E}\left(e^{-\delta T_u}D_u^n\right) = \phi_n\left(b-u,b,\delta\right)$. We solve the integro-differential equation (3.4) and then we find $V(u;b,\delta)$ simply using (4.2). Therefore,

$$\begin{aligned} \mathbb{E}\left(e^{-\delta T_u}D_u^n\right) &= \phi_n\left(b-u,b,\delta\right) = \frac{n!}{\alpha^n}\frac{\lambda}{c}\frac{e^{-r_2u}-e^{-r_1u}}{(r_1+\alpha)e^{-r_2b}-(r_2+\alpha)e^{-r_1b}} \qquad (7.1) \\ V(u;b,\delta) &= \frac{\lambda}{\alpha}\frac{e^{-r_2u}-e^{-r_1u}}{(c(r_1+\alpha)-\lambda)e^{-r_2b}-(c(r_2+\alpha)-\lambda)e^{-r_1b}}, \end{aligned}$$

where $r_1 < 0$ e $r_2 > 0$ are solutions of the equation

$$s^2 + \left(\alpha - \frac{\lambda+\delta}{c}\right)s - \frac{\alpha\delta}{c} = 0.$$

Expression above for $V(u;b,\delta)$ is equivalent to the one by Avanzi *et al.* (2007). Function $V_2(u;b,\delta)$ can be evaluated using (4.5), after some simplification we get

$$\begin{aligned} V_2(u,b,\delta) &= 2\,\frac{c\lambda}{\alpha^2} \\ &\times \frac{((r_1+\alpha)e^{r_1 b}-(r_2+\alpha)e^{r_2 b})(e^{-s_2 u}-e^{-s_1 u})}{((c(r_1+\alpha)-\lambda)e^{r_1 b}-(c(r_2+\alpha)-\lambda)e^{r_2 b})((c(s_1+\alpha)-\lambda)e^{-s_2 b}-(c(s_2+\alpha)-\lambda)e^{-s_1 b})} \end{aligned}$$

where s_1 and s_2 are the roots of the equation

$$s^2+\left(\alpha-\frac{\lambda+2\delta}{c}\right)s-\frac{2\alpha\delta}{c}=0$$

For the computation of $\chi(u,b)$ and $\xi(u,b)$, from (5.1) we get, where $R=\lambda/c-\alpha$,

$$\begin{aligned} \chi(u,b) &= \frac{\lambda-\lambda e^{-Ru}}{\lambda-\alpha c e^{-Rb}} \\ \xi(u,b) &= \frac{\lambda e^{-Ru}-\alpha c e^{-Rb}}{\lambda-\alpha c e^{-Rb}}. \end{aligned}$$

It is much easier to find $\xi(u,b)$ by using (5.2). We get the auxiliary function, see Section 5,

$$\epsilon(u)=\frac{(\alpha+R)e^{Ru}-\alpha}{(\alpha+R)e^{Rb}-\alpha},.$$

and then we get the final result using $\xi(u,b)=\epsilon(b-u)$.

To find the solution for the distribution of a single amount of dividend, $G(u,b;x)$, we can use the Laplace transforms dealt in Section 6. After some algebra, we get

$$\rho(u,x)=(1-e^{-\alpha x})\left[1+\frac{\alpha-(\alpha+R)e^{Ru}}{(\alpha+R)e^{Rb}-\alpha}\right].$$

Finally we get,

$$G(u,b;x)=\rho(b-u,x)=(1-e^{-\alpha x})\,\frac{\lambda-\lambda e^{-Ru}}{\lambda-\alpha c e^{-Rb}}.$$

Note that we have, for the conditional d.f., denoted as $\tilde{G}(u,b;x)$,

$$\tilde{G}(u,b;x)=\frac{G(u,b;x)}{\chi(u,b)}=1-e^{-\alpha x}.$$

We could get the same using (6.6). There is a correspondence to the classical model. Due to the memoryless property of the exponential distribution, the conditional distribution of the single dividend amount has the same distribution of the single gains amount, conditional on the upcross of level b prior to ruin.

Consider now the conditional distributions, given that $T_u<\tau_u$. Looking back at the beginning of this section we have that

$$\mathbb{E}\left(e^{-\delta T_u}D_u^n|T_u<\tau_u\right)=\frac{\phi_n(b-u,b,\delta)}{\chi(u,b)}.$$

Since the conditional distribution of D_u is exponential we know that $\mathbb{E}\left(D_u^n | T_u < \tau_u\right) = n!/\alpha^n$. Furthermore, we have that $\mathbb{E}\left(e^{-\delta T_u} | T_u < \tau_u\right) = \phi_0(b-u, b, \delta)\chi(u,b)^{-1}$. Thus

$$\mathbb{E}\left(e^{-\delta T_u} D_u^n | T_u < \tau_u\right) = \mathbb{E}\left(D_u^n | T_u < \tau_u\right) \mathbb{E}\left(e^{-\delta T_u} | T_u < \tau_u\right) \tag{7.2}$$

where

$$\mathbb{E}\left(e^{-\delta T_u} | T_u < \tau_u\right) = \frac{\lambda}{c} \frac{e^{-r_2 u} - e^{-r_1 u}}{(r_1+\alpha)e^{-r_2 b} - (r_2+\alpha)e^{-r_1 b}} \chi(u,b)^{-1}. \tag{7.3}$$

Result (7.2) is not surprising, as the conditional variables, given that $T_u < \tau_u$, of the time to dividend and respective dividend amount are independent, respectively $\tilde{T}_u$ and $\tilde{D}_u$. This case is the analogue to the classical risk model, between the conditional severity of ruin and time to ruin, given that ruin occurs [see Gerber (1979), for instance]. Hence, Expression (7.3) gives the Laplace transform of the time to dividend.

There are other features for the future discounted expected dividends that are worth mentioning when $u = b$. Let's retrieve Formula (7.1), for $u = b$ we have

$$\mathbb{E}\left(e^{-\delta T_b}\right) = \frac{\lambda}{c} \frac{e^{-r_2 b} - e^{-r_1 b}}{(r_1+\alpha)e^{-r_2 b} - (r_2+\alpha)e^{-r_1 b}} = \frac{\lambda}{c} \frac{e^{(r_1-r_2)b} - 1}{(r_1+\alpha)e^{(r_1-r_2)b} - (r_2+\alpha)}.$$

Taking the limit as $b \to \infty$, simplifying and writing $r_2 = R_\delta$ we get

$$\lim_{b\to\infty} \mathbb{E}\left(e^{-\delta T_b}\right) = \frac{\lambda}{c} \frac{1}{\alpha + R_\delta},$$

where R_δ is as defined in (2.4). Again from (7.1), we have that

$$\begin{aligned} \lim_{b\to\infty} \mathbb{E}\left(e^{-\delta T_b} D_b\right) &= \frac{\lambda}{\alpha c} \frac{1}{\alpha + R_\delta}, \\ \lim_{b\to\infty} V(b; b, \delta) &= \frac{\lambda}{\alpha c\,(\alpha + R_\delta) - \lambda}. \end{aligned}$$

It it interesting to see that if the initial investment equals the barrier level the resulting expected dividends have a limiting value. We can find similar limiting formulae for higher moments, likewise.

7.2 Combination of exponentials

We use an illustration example also worked by Avanzi *et al.* (2007), where

$$p(x) = 3e^{-3x/2} - 3e^{-3x},\ x > 0.$$

In this case we don't have analytical expressions and we worked the example numerically. We set a situation where on average we have one jump per unit time with an average amount of one again, i.e., $\lambda = 1$. In addition we consider $c = 0.75$ and $\delta = 0.02$. First we produce numerial values for the case $u = b$ and after we consider other values of u. The idea is to evaluate the contribution of $V(b; b, \delta)$ in $V(u; b, \delta)$, see Formula (4.4). Table 7.1 shows figures for $E[e^{-\delta T_b}]$, $E[e^{-\delta T_b} D_b]$, $V(b; b, 0.02)$ and $\chi(b, b)$ for some values of b, we inserted the optimal value $b^* = 6.48298$. Like in the previous example, we see that $V(b; b, 0.02)$ has a limiting value, due to the quantities $E[e^{-\delta T_b}]$, $E[e^{-\delta T_b} D_b]$, as $b \to \infty$. This sounds natural as the probability $\chi(b, b)$ gets closer to one. This may not be true for any single gain distribution.

Table 7.2 shows values for $\mathbb{E}[e^{-\delta T_u}]$, $\mathbb{E}[e^{-\delta T_u} D_u]$, $V(u; b, 0.02)$, $V(b; b, 0.02)$, $\mathbb{E}[D_u]$, $\mathbb{E}[{D_u}^2]$, $\mathbb{E}[{D_u}^3]$, standard deviation and skewness of D_u (denoted as $Sd[D_u]$ and $Sk[D_u]$, respectively), for a choice of values of $(u; b)$. We can see from this table that the first discounted dividend is on average quite small when compared to the discounted remainder future dividends, $\mathbb{E}[e^{-\delta T_u}] \times V(b; b, 0.02)$, for our choices of (u, b). In view of Formula (4.4) we can evaluate their contribution to $V(u; b, 0.02)$. This is shown in rows (1) and (4). Figures in row (2) show $E[e^{-\delta T_u}]$, it means the average discounted amount of a first dividend of one unit amount, and it gives the idea of how much a dividend of one unit is discounted, on average. We can have other readings from the figures in the table. For instance, an investment of three capital units, three times greater than the average jump, with a dividend barrier of six (the double of the investment) is expected to give back a present value of capital return of more than eight units. It is almost three times of the invested capital. However, for some higher dividend levels the returned capital is lower. Note that the barrier $b = 6$ is close to the optimal value, for a fixed u (for more details on optimal values please see Avanzi *et al.* (2007)). Note that for $u = 1$ and $b = 2$ for instance, we get a discounted return of more 36% of the invested capital on the first dividend. This gives a first insight how quick can be the capital recovery of the investment. Table 7.2 also shows figures for the probability of getting a dividend payment so that we can have some idea of how probable can be this dividend payment as function of the initial investment (u) and of the dividend barrier (b).

We further show the first three moments of the amount of a single dividend (undiscounted), as well as standard deviation and skewness so that we have some understanding of its distribution. These quantities can be evaluated either through the function $\phi_n(b - u; b, 0)$ or computed numerically using the density function $g(u, b; x)$. Knowing that a single dividend is part of a particular gain (with mean size one) we see that the average dividend is much smaller. The standard deviation values are quite similar and the skewness coefficient ones are also not too different. The distribution is lightly positively skewed. We further show a graph with different plots for the probability density function of the first dividend amounts $g(u, 10; x)$, for some given values of the initial surplus (see Figure 2).

Table 7.3 shows values for the expected discounted finite future dividends taken from Formula (4.9), for different values of n and the same choice of (u, b) so that we can have some insight of the contribution of the early dividends to the global $V(u, b, 0.02)$. We can see that the first twenty dividends get the most of it.

Table 7.4 show some figures and parameters of the distribution of the number of dividends. Figures for the moments and parameters in this table can somehow be related with those from Table 7.3, although the connection does not seem to be direct, except perhaps the limiting behaviour. We can see that for some choices of (u, b) the expected number of dividends is quite small and in others it is the opposite. For instance, with $(u, b) = (1, 2)$ a few dividends are expected, this should be due to the fact that both the initial surplus and the barrier levels are quite small. On the opposite side with $(15, 40)$ the surplus process is expected to *to be travelling* around the barrier level crossing it often. Relating the cases $(1, 10)$ and $(5, 10)$ we see that the expected number are quite different, this should be due to the fact that in the first case the ruin probability is higher, in one hand, and in the other the time expected to reach for the first time the barrier should also be quite different (see the corresponding values of the Laplace transform $E[e^{-\delta T_u}]$ in row (2) of Table 7.2).

Tables 7.5 and 7.6 show figures for the second and third moments for the discounted future dividends and related figures [see Formulae (4.7) and (4.8)]. For both cases we can evaluate the contribution of each component in the table for the final values $V_2(u; b, 0.02)$ and $V_3(u; b, 0.02)$, respectively. We can see that the heavier contribution comes from the expectations $V_2(b; b, 0.02)$ and $V_3(b; b, 0.02)$, respectively.

7.3 Damped sine distribution

We consider now Example 3 worked by Cheung and Drekic (2008), with

$$p(x) = 2e^{-x}(1 - \sin x),\ x > 0.$$

We keep the parameter values $\lambda = 1$, $c = 0.75$ and $\delta = 0.02$. Figures for this example are shown in Tables 7.7-7.12. Like in Table 7.1 we inserted the optimal $b^* = 7.92010$, and $V(b^*; b^*, 0.02)$, in Table 7.7. In this example we observe similar features to those found in the previous example. The main differences are of course due to the shape of the single gain distribution, although it has the same mean other characteristics are different. For instance, its standard deviation is the now $\sqrt{2}$ (that is almost the double of the previous case). We also observe the existence of limiting values for $\mathbb{E}[e^{-\delta T_b}]$, $\mathbb{E}[e^{-\delta T_b} D_b]$ and $V(b; b, 0.02)$.

Comparing figures with Example 2, we see that values of $V(b; b, 0.02)$ are smaller for lower values of b and higher in for bigger values. We can also observe that a limiting value exists, but convergence is slower. The behaviour of $V(u; b, 0.02)$ depends on the parameter choice. The probability $\xi(u, b)$ is now higher for the same choice of (u, b). This implies that the expected values of M are lower, also is the standard deviation. The mean value of the (undiscounted) single dividend is higher. In terms of discounted values the differences are not similar, for instance the Laplace transform $E[e^{-\delta T_u}]$ are smaller in some cases and the discounted expected single dividend $E[e^{-\delta T_u} D_u]$ are higher. Somehow, the weight of the quantity D_u is greater. However, conclusions on the similarities or differences with the previous example are not definite.

In Figure 3 we show a graph for the probability density function of the first dividend amount $g(u, 10; x)$, with the same choice of plots as the previous example. The shape of these densities are quite different from those of the combination of exponentials example, this is in first place due to the features of the density of the single gains amount.

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Lourdes B. Afonso
Depart. de Matemática
and CMA
Faculdade Ciências e Tecnologia
Universidade Nova de Lisboa
2829-516 Caparica
Portugal

lbafonso@fct.unl.pt

Rui M.R. Cardoso
Depart. de Matemática
and CMA
Faculdade Ciências e Tecnologia
Universidade Nova de Lisboa
2829-516 Caparica
Portugal

rrc@fct.unl.pt

Alfredo D. Egídio dos Reis
Department of Management
ISEG and CEMAPRE
Technical University of Lisbon
Rua do Quelhas 6
1200-781 Lisboa
Portugal

alfredo@iseg.utl.pt

Table 7.1: $\mathbb{E}[e^{-\delta T_b}]$, $\mathbb{E}[e^{-\delta T_b} D_b]$, $V(b; b, 0.02)$ and $\chi(b, b)$, combination of exponentials.

b	$E[e^{-\delta T_b}]$	$E[e^{-\delta T_b} D_b]$	$V(b; b, 0.02)$	$\chi(b, b)$
2	0.81844	0.66529	3.66439	0.83443
3	0.88286	0.71173	6.07590	0.90686
5	0.92887	0.74490	10.47248	0.96518
6	0.93723	0.75093	11.96304	0.97787
b^*	0.93978	0.75277	12.50000	0.98214
7	0.94181	0.75423	12.96088	0.98576
10	0.94656	0.75765	14.17653	0.99606
15	0.94752	0.75835	14.44933	0.99952
20	0.94757	0.75838	14.46502	0.99994
30	0.94757	0.75839	14.46596	1.00000
40	0.94757	0.75839	14.46596	1.00000

Table 7.2: $\mathbb{E}[e^{-\delta T_u}]$, $\mathbb{E}[e^{-\delta T_u} D_u]$, $V(u; b, 0.02)$, $\mathbb{E}[D_u]$, $\mathbb{E}[{D_u}^2]$, $Sd[D_u]$, $\mathbb{E}[{D_u}^3]$ and $Sk[D_u]$, combination of exponentials.

	(u, b)	$(1, 2)$	$(1, 10)$	$(3, 6)$	$(5, 10)$	$(10, 30)$	$(15, 40)$
(1)	$\mathbb{E}[e^{-\delta T_u} D_u]$	0.36207	0.16630	0.47354	0.46718	0.18343	0.13237
(2)	$\mathbb{E}[e^{-\delta T_u}]$	0.49939	0.23068	0.65688	0.64807	0.25445	0.18362
(3)	$V(b; b, 0.02)$	3.66439	14.17653	11.96304	14.17653	14.46596	14.46596
(4)	(2) $\times$ (3)	1.82994	3.27027	7.85825	9.18735	3.68080	2.65627
	$V(u; b, 0.02)$	2.19201	3.43657	8.33179	9.65453	3.86423	2.78864
	$\mathbb{E}[D_u]$	0.37078	0.24945	0.54977	0.63952	0.71008	0.71971
	$\mathbb{E}[{D_u}^2]$	0.51430	0.34514	0.76068	0.88486	0.98249	0.99581
	$Sd[D_u]$	0.61386	0.53190	0.67708	0.68983	0.69157	0.69125
	$\mathbb{E}[{D_u}^3]$	1.04852	0.70283	1.54902	1.80189	2.00069	2.02781
	$Sk[D_u]$	2,50047	3,16039	2,01920	1,91102	1,88601	1,88713
	$\chi(u.b)$	0.51135	0.34594	0.76244	0.88692	0.98477	0.99812

Table 7.3: $V(u;b,0.02,n)$, $n=1,5,10,20,50,100,300$, combination of exponentials.

(u,b)	$(1,2)$	$(1,10)$	$(3,6)$	$(5,10)$	$(10,30)$	$(15,40)$
$V(u;b,0.02,1)$	0.36207	0.16630	0.47354	0.46718	0.18343	0.13237
$V(u;b,0.02,5)$	1.37091	0.81133	2.26849	2.27931	0.89670	0.64710
$V(u;b,0.02,10)$	1.89047	1.44177	3.94711	4.05043	1.59717	1.15261
$V(u;b,0.02,20)$	2.15134	2.28481	6.03883	6.41883	2.54112	1.83381
$V(u;b,0.02,50)$	2.19191	3.21488	8.00387	9.03172	3.60121	2.59883
$V(u;b,0.02,100)$	2.19201	3.42234	8.31896	9.61457	3.84642	2.77579
$V(u;b,0.02,300)$	2.19201	3.43657	8.33179	9.65453	3.86423	2.78864

Table 7.4: P.f. of the number of dividends and parameters, combination of exponentials.

(u,b)	$(1,2)$	$(1,10)$	$(3,6)$	$(5,10)$	$(10,30)$	$(15,40)$
$Pr[M=0]$	0.48865	0.65406	0.23756	0.11308	0.01523	0.00188
$Pr[M=1]$	0.08466	0.00136	0.01687	0.00349	0.00000	0.00000
$Pr[M=2]$	0.07065	0.00136	0.01650	0.00348	0.00000	0.00000
$Pr[M=3]$	0.05895	0.00135	0.01613	0.00347	0.00000	0.00000
$\mathbb{E}[M]$	3.08839	87.8479	34.4576	225.222	1089824	72327477
$Sd[M]$	4.96784	191.861	43.5057	251.863	1106555	72463639
$Sk[M]$	2.52037	3.32402	2.14209	2.03495	2.00069	2.00001

Table 7.5: Figures for $V_2(u,b,0.02)$, combination of exponentials.

(u,b)	$(1,2)$	$(1,10)$	$(3,6)$	$(5,10)$	$(10,30)$	$(15,40)$
$V_2(b,b,0.02)$	29.1671	236.480	189.685	236.480	242.033	242.033
$\mathbb{E}[e^{-2\delta T_b}D_b^2]$	0.95063	1.05528	1.05014	1.05528	1.05561	1.05561
$\mathbb{E}[e^{-2\delta T_b}D_b]$	0.65379	0.72943	0.72572	0.72943	0.72968	0.72968
$\mathbb{E}[e^{-2\delta T_b}]$	0.80313	0.90808	0.90292	0.90808	0.90842	0.90842
$V_2(u,b,0.02)$	17.3152	42.1881	119.549	129.070	24.1971	13.6212
$\mathbb{E}[e^{-2\delta T_u}D_u^2]$	0.49060	0.16308	0.57323	0.49894	0.09142	0.05146
$\mathbb{E}[e^{-2\delta T_u}D_u]$	0.35374	0.11789	0.41438	0.36068	0.06609	0.03720
$\mathbb{E}[e^{-2\delta T_u}]$	0.48795	0.16358	0.57496	0.50044	0.09170	0.05162

Table 7.6: Figures for $V_3(u, b, 0.02)$, combination of exponentials.

(u,b)	$(1,2)$	$(1,10)$	$(3,6)$	$(5,10)$	$(10,30)$	$(15,40)$
$V_3(b;b,0.02)$	323.650	4416.26	3465.34	4416.26	4523.66	4523.66
$\mathbb{E}[e^{-3\delta T_b} D_b^3]$	1.94823	2.12644	2.12033	2.12644	2.12669	2.12669
$\mathbb{E}[e^{-3\delta T_b} D_b^2]$	0.93507	1.02260	1.01960	1.02260	1.02273	1.02273
$\mathbb{E}[e^{-3\delta T_b} D_b]$	0.64275	0.70602	0.70386	0.70602	0.70611	0.70611
$\mathbb{E}[e^{-3\delta T_b}]$	0.78845	0.87625	0.87325	0.87625	0.87638	0.87639
$V_3(u;b,0.02)$	190.889	601.776	1994.37	1994.18	202.075	97.7136
$\mathbb{E}[e^{-3\delta T_u} D_u^3]$	0.97756	0.24561	1.03420	0.81389	0.08045	0.03699
$\mathbb{E}[e^{-3\delta T_u} D_u^2]$	0.47953	0.12063	0.50794	0.39974	0.03939	0.01496
$\mathbb{E}[e^{-3\delta T_u} D_u]$	0.34576	0.08721	0.36723	0.28900	0.02884	0.02006
$\mathbb{E}[e^{-3\delta T_u}]$	0.47701	0.12104	0.50966	0.40109	0.03965	0.01823

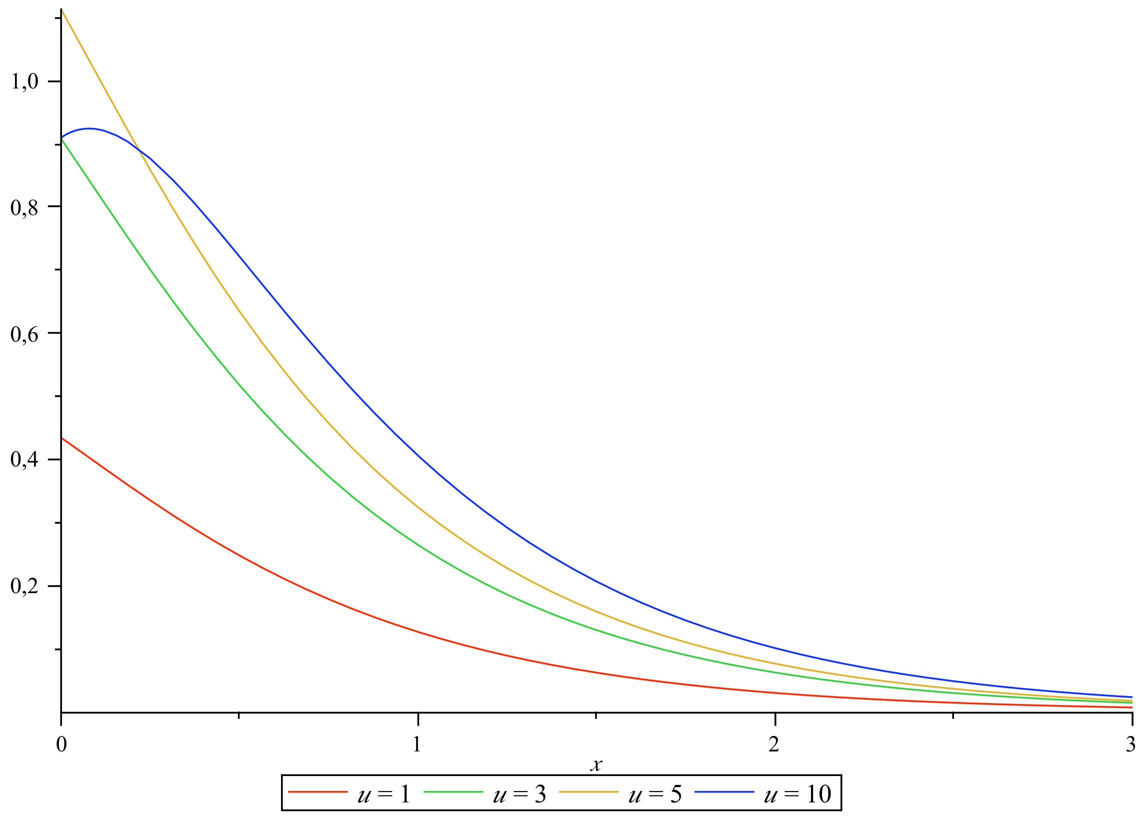

Figure 2: $g(u, 10; x)$, p.d.f. of the first dividend amount, combination of exponentials.

Table 7.7: Values for $E[e^{-\delta T_b}]$, $E[e^{-\delta T_b} D_b]$, $V(b; b, 0.02)$ and $\chi(b, b)$, damped sine distr.

b	$E[e^{-\delta T_b}]$	$E[e^{-\delta T_b} D_b]$	$V(b; b, 0.02)$	$\chi(b, b)$
2	0.66245	1.06384	3.15169	0.67593
3	0.75713	1.20045	4.94285	0.77953
5	0.84581	1.31577	8.53329	0.88456
6	0.86703	1.34562	10.11996	0.91291
7	0.88104	1.36509	11.47503	0.93328
b^*	0.88982	1.37723	12.50000	0.94725
8	0.89044	1.37809	12.57913	0.94830
10	0.90122	1.39301	14.10296	0.96822
15	0.90951	1.40450	15.52190	0.98989
20	0.91087	1.40638	15.77966	0.99665
30	0.91114	1.40674	15.83059	0.99962
40	0.91114	1.40675	15.83201	0.99996

Table 7.8: $\mathbb{E}[e^{-\delta T_u}]$, $\mathbb{E}[e^{-\delta T_u} D_u]$, $V(u; b, 0.02)$, $\mathbb{E}[D_u]$, $\mathbb{E}[{D_u}^2]$, $Sd[D_u]$, $\mathbb{E}[{D_u}^3]$ and $Sk[D_u]$, damped sine distr.

	(u, b)	$(1, 2)$	$(1, 10)$	$(3, 6)$	$(5, 10)$	$(10, 30)$	$(15, 40)$
(1)	$\mathbb{E}[e^{-\delta T_u} D_u]$	0.69180	0.23178	0.73100	0.81371	0.38795	0.29708
(2)	$\mathbb{E}[e^{-\delta T_u}]$	0.33229	0.16731	0.55340	0.58381	0.28013	0.21452
(3)	$V(b; b, 0.02)$	3.15169	14.10296	10.11996	14.10296	15.83059	15.83201
	$V(u; b, 0.02)$	1.73909	2.59135	6.33141	9.04720	4.82260	3.69335
(5)	(1) $\times$ (3)	2.18034	3.26881	7.39770	11.47579	6.14145	4.70342
	$\mathbb{E}[D_u]$	0.70505	0.29630	0.80365	1.01086	1.23016	1.33398
	$\mathbb{E}[{D_u}^2]$	1.90169	0.68361	1.81506	2.33841	2.83747	3.07693
	$Sd[D_u]$	1.18515	0.77190	1.08130	1.14742	1.15073	1.13905
	$\mathbb{E}[{D_u}^3]$	6.08147	2.07549	5.47248	7.10697	8.61320	9.34009
	$Sk[D_u]$	1.65804	3.30467	1.68836	1.37782	1.22379	1.20043
	$\chi(u.b)$	0.33894	0.21349	0.60498	0.72475	0.88661	0.96143

Table 7.9: $V(u; b, 0.02, n)$, $n = 1, 5, 10, 20, 50, 100, 300$, damped sine distr.

(u, b)	$(1, 2)$	$(1, 10)$	$(3, 6)$	$(5, 10)$	$(10, 30)$	$(15, 40)$
$V(u; b, 0.02, 1)$	0.69180	0.23178	0.73100	0.81371	0.38795	0.29708
$V(u; b, 0.02, 5)$	1.53740	1.03479	3.16649	3.61573	1.76630	1.35262
$V(u; b, 0.02, 10)$	1.71336	1.66594	4.78066	5.81808	2.90342	2.22345
$V(u; b, 0.02, 20)$	1.73867	2.26426	5.95910	7.90585	4.06584	3.11370
$V(u; b, 0.02, 50)$	1.73909	2.57690	6.32625	8.99680	4.77620	3.65780
$V(u; b, 0.02, 100)$	1.73909	2.59127	6.33140	9.04692	4.82216	3.69301
$V(u; b, 0.02, 300)$	1.73909	2.59135	6.33141	9.04720	4.82260	3.69335

Table 7.10: P.f. of the number of dividends and parameters, damped sine distr.

(u,b)	$(1,2)$	$(1,10)$	$(3,6)$	$(5,10)$	$(10,30)$	$(15,40)$
$Pr[M=0]$	0.66106	0.78651	0.39502	0.27525	0.11339	0.03857
$Pr[M=1]$	0.10984	0.00678	0.05269	0.02303	0.00034	0.00004
$Pr[M=2]$	0.07424	0.00657	0.04810	0.02230	0.00034	0.00004
$Pr[M=3]$	0.05018	0.00636	0.04391	0.02159	0.00034	0.00004
$\mathbb{E}[M]$	1.04590	6.71874	6.94676	22.8086	2332.42	22130.5
$Sd[M]$	2.07727	19.2622	10.2142	29.8762	2613.32	23000.7
$Sk[M]$	2.98465	4.32115	2.35956	2.18685	2.03613	2.00435

Table 7.11: Figures for $V_2(u,b)$, damped sine distr.

(u,b)	$(1,2)$	$(1,10)$	$(3,6)$	$(5,10)$	$(10,30)$	$(15,40)$
$V_2(b;b,0.02)$	27.5848	270.805	171.691	270.805	310.445	310.471
$\mathbb{E}[e^{-2\delta T_b}D_b^2]$	3.11967	3.72854	3.65843	3.72854	3.74129	3.74129
$\mathbb{E}[e^{-2\delta T_b}D_b]$	1.03884	1.31408	1.28364	1.31408	1.31961	1.31961
$\mathbb{E}[e^{-2\delta T_b}]$	0.64952	0.84936	0.82737	0.84936	0.85337	0.85337
$V_2(u;b,0.02)$	15.1021	42.4331	102.591	152.208	44.8324	27.9520
$\mathbb{E}[e^{-2\delta T_u}D_u^2]$	1.83325	0.43227	1.50436	1.56643	0.39966	0.24916
$\mathbb{E}[e^{-2\delta T_u}D_u]$	0.67897	0.18747	0.66970	0.67663	0.17340	0.10810
$\mathbb{E}[e^{-2\delta T_u}]$	0.32587	0.13557	0.50982	0.48580	0.12544	0.07820

Table 7.12: Figures for $V_3(u,b)$, damped sine distr.

(u,b)	$(1,2)$	$(1,10)$	$(3,6)$	$(5,10)$	$(10,30)$	$(15,40)$
$V_3(b;b,0.02)$	341.487	6111.62	3627.96	6111.62	7058.36	7058.96
$\mathbb{E}[e^{-3\delta T_b}D_b^3]$	11.0861	12.5859	12.4453	12.5859	12.6035	12.6035
$\mathbb{E}[e^{-3\delta T_b}D_b^2]$	3.05146	3.56754	3.52117	3.56754	3.57333	3.57333
$\mathbb{E}[e^{-3\delta T_b}D_b]$	1.01488	1.24900	1.22885	1.24900	1.25151	1.25151
$\mathbb{E}[e^{-3\delta T_b}]$	0.63710	0.80721	0.79264	0.80721	0.80904	0.80904
$V_3(u;b,0.02)$	187.105	830.483	2078.45	3058.09	565.840	302.528
$\mathbb{E}[e^{-3\delta T_u}D_u^3]$	5.76386	1.08834	4.15886	4.06211	0.64247	0.34347
$\mathbb{E}[e^{-3\delta T_u}D_u^2]$	1.80065	0.35818	1.38287	1.33419	0.21151	0.11308
$\mathbb{E}[e^{-3\delta T_u}D_u]$	0.66655	0.15538	0.61733	0.57608	0.09180	0.04908
$\mathbb{E}[e^{-3\delta T_u}]$	0.31965	0.11257	0.47254	0.41389	0.06654	0.03557

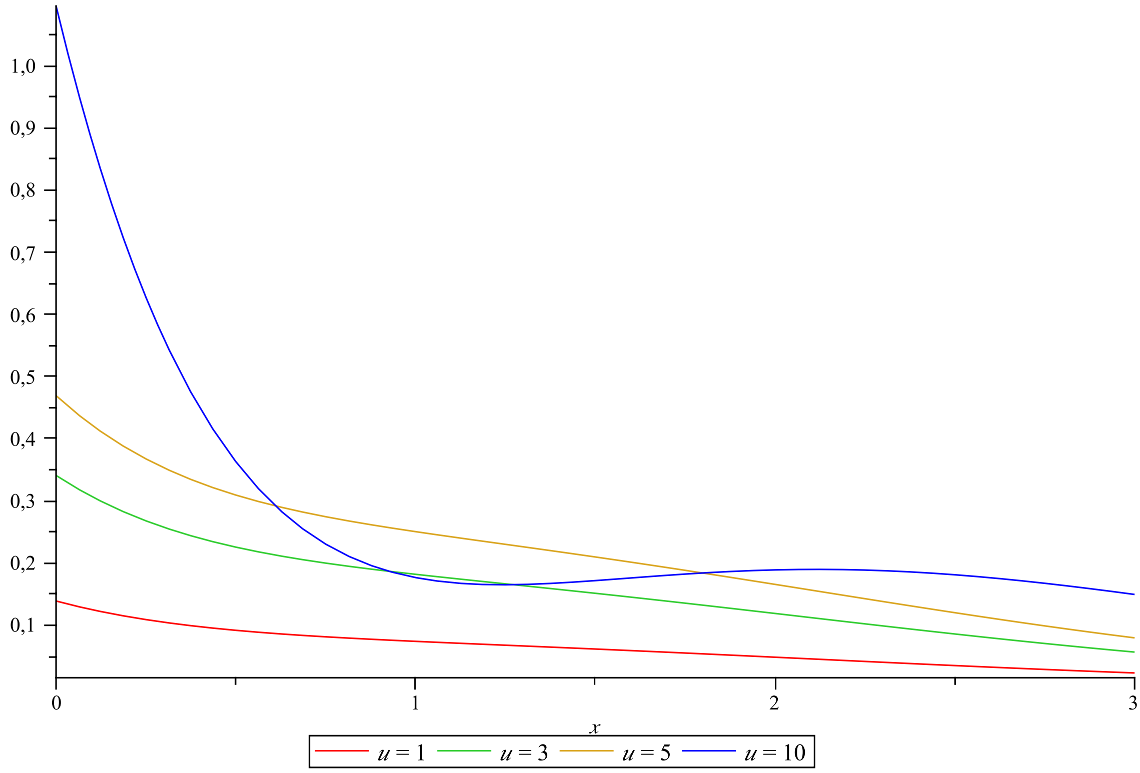

Figure 3: $g(u, 10; x)$, p.d.f. of the first dividend amount, damped sine distr.