Chaos in the square billiard with a modified reflection law

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We study a square billiard with a reflection law such that the reflection angle is a linear contraction of the angle of incidence. Numerical and analytical evidences of the coexistence of one parabolic attractor, one chaotic attractor and several horseshoes are presented. This scenario implies the positivity of the topological entropy, a property in sharp contrast to the integrability of the square billiard with the usual reflection law.

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A billiard is a mechanical system consisting of a point-particle moving freely inside a planar region and being reflected off the perimeter of the region according to some reflection law. The specular reflection law is the familiar rule that prescribes the equality of the angles of incidence and reflection. Billiards with this reflection law are conservative systems, and as such are models for physical systems with elastic collisions. For this reason and their intrinsic mathematical interest, conservative billiards have been extensively studied. Much less studied are dissipative billiards, which originate from reflection laws requiring that the angle of reflection is a contraction of the angle of incidence. These billiards do not preserve the Liouville measure, and therefore can model physical systems with non-elastic collisions. In this paper, we investigate numerically and analytically a dissipative billiard in a square. We find that its dynamics differs strikingly from the one of its conservative counterpart, which is well known to be integrable. Indeed, our results show that a dissipative billiard in a square has a rich dynamics with horseshoes and attractors of parabolic and hyperbolic type coexisting simultaneously.

I. INTRODUCTION

Billiards are among the most studied dynamical systems for two main reasons. Firstly, they serve as models for physical systems of great importance (see e.g. the book[7] and references therein), and secondly, they are simple examples of dynamical systems displaying a rich variety of dynamics ranging from integrability to complete chaoticity. Most of the existing literature on this subject is devoted to the study of conservative billiards (cf.[3,8]). In this case, the point-particle obeys the standard reflection law: the equality between the angles of incidence and reflection.

Another class of systems, which has not received as much attention as conservative billiards, consists of billiards with reflection law stating that the angle of reflection is a contraction of the angle of incidence. This should have the effect of contracting the volume in phase space, as observed computationally.

In this paper, we consider a specific dissipative law of reflection. We assume that the contraction is uniform, i.e. the angle of reflection equals the angle of incidence times a constant factor $0 < \lambda < 1$.

Recently, Markarian, Pujals and Sambarino[6] proved that dissipative planar billiards (called *pinball billiards* in their paper) have two invariant directions such that the growth rate along one direction dominates uniformly the growth rate along the other direction. This property is called *dominated splitting*, and is weaker than hyperbolicity, which requires one growth rate to be greater than one, and the other growth rate to be smaller than one. The result of Markarian, Pujals and Sambarino applies to billiards in regions of different shapes. In particular, it applies to billiards in polygons. This is an interesting fact because the dominated splitting property enjoyed by the dissipative polygonal billiards contrasts the parabolic dynamics observed in the conservative case[6,8].

We take here a further step towards the study of dissipative polygonal billiards considering the square billiard table. Our main finding is that this system does not just have the dominated splitting property, but it is hyperbolic on a subset of the phase space. More precisely, we provide theoretical arguments and numerical evidence that the non-wandering set in the phase space decomposes into two invariant sets, one parabolic and one hyperbolic. This dynamical picture is clearly richer than the one of the conservative square billiard, which is fully integrable.

We also conduct a rather detailed study of the changes in the properties of the invariant sets as the parameter λ varies. We found that there are no hyperbolic attractors for small values of λ but only one horseshoe and one parabolic attractor. For intermediate values of λ, we observe the coexistence of a parabolic attractor, a hyperbolic attractor and a horseshoe, whereas for large values of λ, we only have a parabolic and a hyperbolic attractors.

We should mention that results somewhat similar to ours were obtained for non polygonal billiards (see ref.[1,2]).

The paper is organized as follows. In Section II, we present a detailed description of the map for the dissipative square billiard. Some results concerning the invariant sets of this map are formulated in Section III. To study our map, it is convenient to quotient it by the symmetries of the square. This procedure is described in Section IV, and produces the so-called reduced billiard map. Section V is devoted to the construction of the stable and unstable manifolds of a fixed point of the reduced billiard map (corresponding to a special periodic orbit of the billiard map). In Section VI, we explain that these invariant

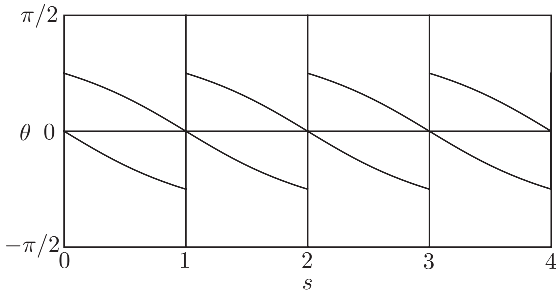


FIG. 1: Phase space $\mathcal{M} \setminus \mathcal{S}^+$.

manifolds have transversal homoclinic intersections, and use this fact to argue that the dissipative square billiard has positive topological entropy. The same section contains also a bifurcation analysis of the limit sets of reduced billiard map.

II. THE SQUARE BILLIARD

Consider the square $D = [0,1] \times [0,1] \subset \mathbb{R}^2$. For our purposes, D is called the square billiard table. To study the dynamics of the billiard inside this table, it is sufficient to know the angle of incidence at the impact points and the reflection law. For the usual reflection law (the angle of reflection is equal to the angle of incidence) we find the next impact point s' and angle of reflection θ' by the billiard map $(s', \theta') = \mathcal{B}(s, \theta)$ acting on the previous impact (s, θ). This map admits an explicit analytic description. Its domain coincides with the rectangle

$$\mathcal{M} = [0,4] \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

from which the set

$$\mathcal{S}^+ = \{\, (s,\theta) \in \mathcal{M} \colon \{s\} = 0 \text{ or } \{s\} + \tan\theta \in \{0,1\} \,\}$$

is removed. The symbols $[s]$ and $\{s\} = s - [s]$ stand for the integer part and the fractional part of s, respectively. An element of $\mathcal{S}^+$ corresponds to an orbit leaving or reaching a corner of D (see Fig. 1). By reversing the role of time in this description of $\mathcal{S}^+$, one obtains the set

$$\mathcal{S}^- = \left\{\, (s,\theta) \in \mathcal{M} \colon \{s\} = 0 \text{ or } (s, -\lambda^{-1}\theta) \in \mathcal{S}^+ \,\right\}.$$

Both sets $\mathcal{S}^+$ and $\mathcal{S}^-$ consist of finitely many analytic curves. Next, let

$$\begin{aligned}\mathcal{M}_1 &= \{\, (s,\theta) \in \mathcal{M} \colon \{s\} > 0 \text{ and } \{s\} + \tan\theta > 1 \,\}\,, \\ \mathcal{M}_2 &= \{\, (s,\theta) \in \mathcal{M} \colon \{s\} > 0 \text{ and } 0 < \{s\} + \tan\theta < 1 \,\}\,, \\ \mathcal{M}_3 &= \{\, (s,\theta) \in \mathcal{M} \colon \{s\} > 0 \text{ and } \{s\} + \tan\theta < 0 \,\}\,.\end{aligned}$$

The billiard map $\mathcal{B} \colon \mathcal{M} \setminus \mathcal{S}^+ \to \mathcal{M} \setminus \mathcal{S}^-$ is defined by

$$\mathcal{B}(s,\theta) = \begin{cases} \left([s] + 1 + \dfrac{1 - \{s\}}{\tan\theta} \pmod 4, \dfrac{\pi}{2} - \theta \right) & \text{on } \mathcal{M}_1, \\ ([s] - 1 - \{s\} - \tan\theta \pmod 4, -\theta) & \text{on } \mathcal{M}_2, \\ \left([s] + \dfrac{\{s\}}{\tan\theta} \pmod 4, -\dfrac{\pi}{2} - \theta \right) & \text{on } \mathcal{M}_3. \end{cases}$$

This map is clearly an analytic diffeomorphism in its domain. The inverse of $\mathcal{B}$ is easily obtained by noticing that the billiard map is time-reversible. That is, given the map $\mathcal{T}(s,\theta) = (s,-\theta)$, we have

$$\mathcal{B}^{-1} = \mathcal{T} \circ \mathcal{B} \circ \mathcal{T}^{-1}.$$

To modify the reflection law, we compose $\mathcal{B}$ with another map $\mathcal{R} \colon \mathcal{M} \to \mathcal{M}$. The resulting map $\Phi = \mathcal{R} \circ \mathcal{B}$ is called a *billiard map with a modified reflection law.*

Several reflections laws have been considered[1,6]. In this paper, we consider the following "dissipative" law. Given $0 < \lambda < 1$, we set

$$\mathcal{R}_\lambda(s,\theta) = (s, \lambda\theta)\,.$$

According to this law, the direction of motion of the particle after a reflection gets closer to the normal of the perimeter of the square (see Fig. 2). To emphasize the dependence of the billiard map on the parameter λ, we write

$$\Phi_\lambda = \mathcal{R}_\lambda \circ \mathcal{B}.$$

As a side remark, one can also define the map Φ_λ for $\lambda > 1$. In this case, the map $\mathcal{R}_\lambda$ expands uniformly the angle θ, and Φ_λ becomes a map with holes in the phase space. It is interesting to observe that the maps $\Phi_{\lambda^{-1}}$ and Φ_λ^{-1} are conjugated for $0 < \lambda < 1$. Indeed, it is not difficult to check that

$$\Phi_{\lambda^{-1}} = (\mathcal{R}_\lambda \circ \mathcal{T})^{-1} \circ \Phi_\lambda^{-1} \circ (\mathcal{R}_\lambda \circ \mathcal{T}),$$

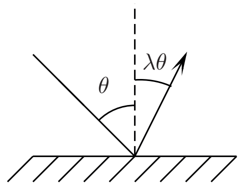


FIG. 2: Dissipative reflection law.

by using the fact that $\mathcal{T}$ and $\mathcal{R}_\lambda$ commute and that $\mathcal{R}_\lambda^{-1} = \mathcal{R}_{\lambda^{-1}}$. Therefore, all the results presented in this paper hold for $\lambda > 1$ as well, provided that we replace the word "attractor" with the word "repeller", and switch the words "stable" and "unstable".

III. HYPERBOLICITY

Let (s_0, θ_0) be an element of $\mathcal{M} \setminus \mathcal{S}^+$. Set $(s_1, \theta_1) = \Phi_\lambda(s_0, \theta_0)$, and denote by $t(s_0, \theta_0)$ the length of the segment connecting s_0 and s_1. Using elementary trigonometry, one can show in a straightforward manner that the derivative of Φ_λ takes the following form:

$$D\Phi_\lambda(s_0, \theta_0) = -\begin{pmatrix} \dfrac{\cos\theta_0}{\cos\lambda^{-1}\theta_1} & \dfrac{t(s_0,\theta_0)}{\cos\lambda^{-1}\theta_1} \\ 0 & \lambda \end{pmatrix}.$$

In fact, the previous formula holds for every polygon, and not just for the square (see Formula 2.26 in ref.[3]).

Now, suppose that $\{(s_i, \theta_i)\}_{i=0}^n$ are $n+1$ consecutive iterates of Φ_λ. Then, we see that

$$D\Phi_\lambda^n(s_0, \theta_0) = (-1)^n \begin{pmatrix} \alpha_n(s_0,\theta_0) & \gamma_n(s_0,\theta_0) \\ 0 & \beta_n(s_0,\theta_0) \end{pmatrix},$$

where

$$\alpha_n(s_0,\theta_0) = \frac{\cos\theta_0}{\cos\lambda^{-1}\theta_n} \prod_{i=1}^{n-1} \frac{\cos\theta_i}{\cos\lambda^{-1}\theta_i}, \quad \beta_n(s_0,\theta_0) = \lambda^n,$$

and

$$\gamma_n(s_0,\theta_0) = \frac{1}{\cos\lambda^{-1}\theta_n} \sum_{i=0}^{n-1} \lambda^i t(s_i,\theta_i) \prod_{k=i+1}^{n-1} \frac{\cos\theta_k}{\cos\lambda^{-1}\theta_k}.$$

We now prove a simple lemma concerning the stability of the periodic points of Φ_λ. It is not difficult to see that this lemma remains valid for every polygon and for other reflection laws (e.g. $\mathcal{R}_\lambda(s,\theta) = (s, \theta - c\sin 2\theta)$ with $0 < c < 1/2$ as in[6]).

Lemma III.1. *For every $\lambda \in (0,1)$, the periodic points of Φ_λ of period 2 and period greater than 2 are parabolic and hyperbolic, respectively.*

Proof. Suppose that (s_0, θ_0) is a periodic point of Φ_λ with period n. Since $(s_n, \theta_n) = (s_0, \theta_0)$, it turns out that

$$\alpha_n(s_0, \theta_0) = \prod_{i=0}^{n-1} \frac{\cos \theta_i}{\cos \lambda^{-1} \theta_i}.$$

Before continuing with the proof, we make two simple remarks. The first remark is that each term $\cos \theta_i / \cos \lambda^{-1} \theta_i$ of the expression of $\alpha_n(s_0, \theta_0)$ is equal or greater than 1 with equality if and only if $\theta_i = 0$. The second remark is that $D\Phi_\lambda^n(s_0, \theta_0)$ is a triangular matrix, and so the moduli of its eigenvalues are $\alpha_n(s_0, \theta_0)$ and $\lambda^n < 1$. To determine the stability of (s_0, θ_0) is therefore enough to find out whether or not $\alpha_n(s_0, \theta_0)$ is greater than 1.

If $n = 2$, it is easy to see that the trajectory of (s_0, θ_0) must always hit the boundary of D perpendicularly. In other words, we have $\theta_0 = \theta_1 = \theta_2 = 0$, and so $\alpha_2(s_0, \theta_0) = 1$. Periodic points of period 2 are therefore parabolic. Clearly, a necessary condition for a polygon to admit periodic points of period 2 is that the polygon must have at least 2 parallel sides (not a sufficient condition though).

Now, suppose that $n > 2$. In this case, we claim that (s_0, θ_0) is hyperbolic. Indeed, when $n > 2$, the billiard trajectory of (s_0, θ_0) must have at least two non-perpendicular collisions with the boundary of D, and since $\cos \theta_i / \cos \lambda^{-1} \theta_i > 1$ for such collisions, we can immediately conclude that $\alpha_n(s_0, \theta_0) > 1$. □

A more elaborated analysis along the lines of the proof of Lemma III.1 yields some general conclusions on the chaotic behavior of dissipative polygonal billiards. We state below two interesting results of this type. Their proofs are behind the scope of this paper, and will appear elsewhere[4]. Unless specified otherwise, Φ_λ denotes the map of a dissipative billiard in a general polygon D throughout the rest of this section.

A set $\Sigma \subset \mathcal{M}$ is called *invariant* if $\Phi_\lambda^{-1}(\Sigma) = \Sigma$. An invariant set Σ is called *hyperbolic* if there exist a norm $\|\cdot\|$ on $\mathcal{M}$, a non-trivial invariant measurable splitting $T_\Sigma \mathcal{M} = E^s \oplus E^u$ and two measurable functions $0 < \mu < 1$ and $K > 0$ on Σ such that for every $(s, \theta) \in \Sigma$ and every $n \geq 1$, we have

$$\|D\Phi_\lambda^n|_{E^s(s,\theta)}\| \leq K(s,\theta)\mu(s,\theta)^n,$$

$$\|D\Phi_\lambda^{-n}|_{E^u(\Phi_\lambda^n(s,\theta))}\| \leq K(s,\theta)\mu(s,\theta)^n.$$

If the functions μ and K can be replaced by constants, then Σ is called *uniformly hyperbolic*, otherwise it is called *non-uniformly hyperbolic*.

The first result we present concerns billiards in regular polygons without parallel sides. For such polygons, the map Φ_λ does not have periodic points of period 2.

Proposition III.2. *Let D be a polygon without parallel sides, and suppose that Σ is an invariant set of Φ_λ. Then Σ is uniformly hyperbolic for every $\lambda \in (0,1)$.*

The second result concerns billiards in rectangles. Now, the map Φ_λ admits periodic points of period 2, and it is not difficult to check that the set of all these points is a parabolic attractor P for every $\lambda \in (0,1)$.

Proposition III.3. *Let D be a rectangle, and suppose that Σ is an invariant set of Φ_λ not intersecting the basin of attraction of P. Then there exists $\lambda_* \in (0,1)$ such that Σ is hyperbolic for every $\lambda \in (0,\lambda_*)$, and is uniformly hyperbolic for every $\lambda \in (\lambda_*,1)$.*

IV. THE REDUCED BILLIARD MAP

The analysis of the billiard dynamics can be simplified if we reduce the phase space. First, we identify all sides of the square by taking the quotient with the translations by integers of the s-component. Then, due to the symmetry along the vertical axis at the midpoint of the square, we can also identify each point (s,θ) with $(1-s,-\theta)$. To formulate the reducing procedure more precisely, we define an equivalence relation $\sim$ on $\mathcal{M}$ by $(s_1,\theta_1) \sim (s_2,\theta_2)$ if and only if $\pi(s_1,\theta_1) = \pi(s_2,\theta_2)$, where $\pi\colon \mathcal{M} \to \mathcal{M}$ is the function defined by

$$\pi(s,\theta) = \begin{cases} (\{s\},\theta) & \text{if } \theta \in \left[0,\frac{\pi}{2}\right), \\ (1-\{s\},-\theta) & \text{if } \theta \in \left(-\frac{\pi}{2},0\right). \end{cases}$$

Let M denote the image of π. Clearly, we have

$$M = (0,1) \times \left[0,\frac{\pi}{2}\right).$$

Note that it is possible to identify the set M with the quotient space $\mathcal{M}/\sim$. We call M the *reduced phase space*. The induced billiard map on M is the *reduced map*, which we will denote by ϕ_λ.

It is clear from the definition of π that $\pi^{-1}(s, \theta)$ consists of 8 elements for every $(s, \theta) \in M$, and so $(\mathcal{M}, \pi)$ is an 8-fold covering of M. It is then easy to see that the reduced billiard map ϕ_λ is a factor of the original billiard map Φ_λ by noting that the quotient map π is indeed a semiconjugacy between ϕ_λ and Φ_λ, i.e., we have that $\pi \circ \Phi_\lambda = \phi_\lambda \circ \pi$.

In what concerns the relation between the dynamical systems defined by Φ_λ and ϕ_λ, there are several key points that are worth remarking. First, we note that periodic points of ϕ_λ lift to periodic points of Φ_λ. To be more precise, an orbit of period n under ϕ_λ is lifted to eight orbits of period n, four orbits of period $2n$, two orbits of period $4n$ and one orbit of period $8n$ for Φ_λ. Analogous statements can be produced for the lifts of transitive sets and the existence of invariant measures. Namely, transitive sets for ϕ_λ are lifted to a finite number of transitive sets for Φ_λ, and any invariant measure under the dynamics of ϕ_λ corresponds to a finite number of invariant measures under Φ_λ. Finally, we remark that the reduced map ϕ_λ has positive topological entropy if and only if this is the case for the billiard map Φ_λ.

By studying the trajectories of the billiard map we have basically two cases: either the billiard orbit hits a neighboring side of the square or the opposite side. Separating these cases there is a corner which is reachable only if the initial position $(s, \theta) \in M$ is in the singular curve

$$S^+ = \{(s, \theta) \in M \colon s + \tan\theta = 1\}.$$

This curve separates the reduced phase space in two connected open sets: M_1 below S^+ and M_2 above S^+.

Let $f_1 \colon M_1 \to M$ and $f_2 \colon M_2 \to M$ be the transformations defined by

$$\begin{aligned} f_1(s, \theta) &= (s + \tan\theta, \lambda\theta) && \text{for } (s, \theta) \in M_1, \\ f_2(s, \theta) &= \Big((1 - s)\cot\theta, \lambda\left(\frac{\pi}{2} - \theta\right)\Big) && \text{for } (s, \theta) \in M_2. \end{aligned}$$

The reduced billiard map is then given by

$$\phi_\lambda = \begin{cases} f_1 & \text{on } M_1, \\ f_2 & \text{on } M_2. \end{cases}$$

Its domain and range are $M \setminus S^+$ and $M \setminus S^-$, respectively, where

$$S^- = \left\{(s, \theta) \in M \colon s - \tan(\lambda^{-1}\theta) = 0\right\}.$$

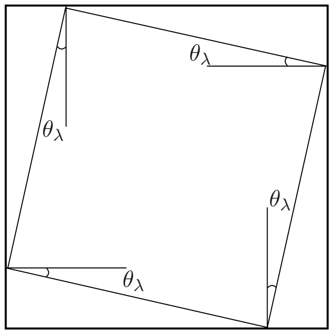


FIG. 3: Periodic orbit.

Like the billiard map Φ_λ, the reduced billiard map ϕ_λ is an analytic diffeomorphism. Notice that ϕ_λ maps horizontal lines into horizontal lines, a consequence of the fact that its second component is independent of s.

Finally, we observe that the subsets of M where the maps ϕ_λ^n and ϕ_λ^{-n} are defined for every $n \geq 0$ are, respectively,

$$M^+ = M \setminus \bigcup_{n\geq 0} \phi_\lambda^{-n}(S^+) \quad \text{and} \quad M^- = M \setminus \bigcup_{n\geq 0} \phi_\lambda^{n}(S^-).$$

V. ATTRACTORS AND HORSESHOES

We start this section by collecting several definitions, which will be used later. Let $\|\cdot\|$ be the Euclidean norm on M. The *stable set* of an element $q \in M$ is defined by

$$W^s(q) = \left\{ x \in M^+ \colon \lim_{n\to+\infty} \|\phi_\lambda^n(x) - \phi_\lambda^n(q)\| = 0 \right\}.$$

In the case of an invariant set $\Lambda = \phi_\lambda(\Lambda)$, we define its stable set to be

$$W^s(\Lambda) = \bigcup_{q\in\Lambda} W^s(q).$$

The *unstable sets* $W^u(q)$ and $W^u(\Lambda)$ are defined analogously by replacing ϕ_λ with ϕ_λ^{-1} and M^+ with M^-.

We say that an invariant set Λ is an *attractor* if and only if $\Lambda = W^u(\Lambda)$ and $W^s(\Lambda)$ is open in M^+.

We say that an invariant set Λ is a *horseshoe* if and only if neither $W^s(\Lambda)$ is an open set in M^+ nor is $W^u(\Lambda)$ an open set in M^-. Note that a saddle periodic orbit is a horseshoe according to this definition.

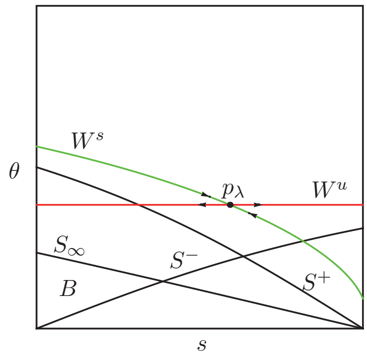


FIG. 4: Invariant manifolds of p_λ and singular curves for the reduced billiard map.

Finally, we say that a finite union of hyperbolic invariant sets $A_1, \ldots, A_m$ is a *hyperbolic chain* if

$$W^u(A_i) \cap W^s(A_{i+1}) \neq \emptyset \qquad \text{for } i = 1, \ldots, m-1.$$

A. Parabolic attractor

It is simple to check that the set

$$P = \{(s, \theta) \in M : \theta = 0\}$$

consists of parabolic fixed points coming from period 2 orbits of the original billiard (orbits that bounce between parallel sides of the square). It is an attractor and $W^s(P)$ includes the set of points B that are below the forward invariant curve

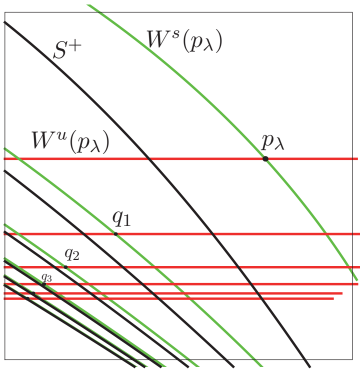
$$S_\infty = \left\{ (s, \theta) \in M : s + \sum_{i=0}^{+\infty} \tan(\lambda^i \theta) = 1 \right\}.$$

The sequence $\phi_\lambda^n(S_\infty)$ converges to the point $(1, 0)$. The pre-image of B is at the top of phase space. Moreover, its basin of attraction is

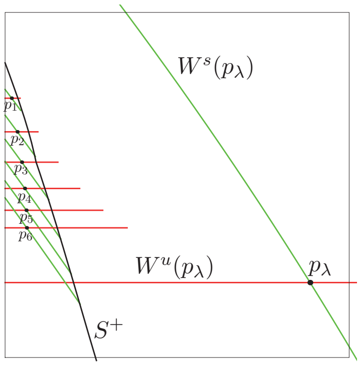
$$W^s(P) = \bigcup_{n \geq 0} \phi_\lambda^{-n}(B).$$

B. Fixed point and its invariant manifolds

The map Φ_λ has many periodic orbits. Two special periodic orbits of period 4 can be found by using the following simple argument. A simple computation shows that if an orbit



(a)



(b)

FIG. 5: (a) Points q_n together with their local stable (red) and unstable (green) manifolds for $\lambda = 0.6$.

(b) Points p_n together with their local stable (red) and unstable (green) manifolds for $\lambda = 0.85$.

hits two adjacent sides of the square with the same reflection angle θ_λ, then

$$\theta_\lambda = \frac{\pi\lambda}{2(1+\lambda)}.$$

If we further impose the condition that the orbit hits the two sides at s_1 and s_2 in such a way that $\{s_1\} = \{s_2\} = s_\lambda$, then we obtain

$$s_\lambda = \frac{1}{1+\tan\theta_\lambda}.$$

By symmetry, we conclude that $\Phi_\lambda^4(s_\lambda, \theta_\lambda) = (s_\lambda, \theta_\lambda)$. By the symmetry of the square, we also have $\Phi_\lambda^4(1-s_\lambda, -\theta_\lambda) = (1-s_\lambda, -\theta_\lambda)$. One of these orbits is depicted in Fig. 3.

Due to the phase space reduction, the periodic orbits just described correspond to the fixed point

$$p_\lambda = (s_\lambda, \theta_\lambda)$$

of ϕ_λ. This is actually the unique fixed point of ϕ_λ outside P and it lies in M_2. By Lemma III.1, p_λ is hyperbolic thus it has local stable and unstable manifolds $W^{s,u}_{\text{loc}}(p_\lambda)$ for every $\lambda \in (0,1)$. Since ϕ_λ maps horizontal lines into horizontal lines, and the set S^- does not intersect the horizontal line through p_λ, we see that the local unstable manifold of p_λ is given by

$$W^u_{\text{loc}}(p_\lambda) = \{\, (s,\theta) \in M \colon \theta = \theta_\lambda \,\}.$$

In fact, the global unstable manifold consists of a collection of horizontal lines cut by the images of S^-.

The geometry of the stable manifold is more complicated. Since by definition points on the stable manifold converge to the fixed point, $W^s_{\text{loc}}(p_\lambda)$ can not cross S^+, so it must remain in M_2. The graph transform associated with the corresponding branch of ϕ_λ is the transformation

$$\Gamma(h)(\theta) = 1 - h(g_\lambda(\theta))\, \tan\theta,$$

where $g_\lambda\colon [0,\pi/2) \to [0,\pi/2)$ denotes the affine contraction

$$g_\lambda(\theta) = \lambda \left(\frac{\pi}{2} - \theta\right).$$

Iterating k times the zero function by Γ we obtain

$$\Gamma^k(0)(\theta) = \sum_{n=0}^{k-1} (-1)^n \prod_{i=0}^{n-1} \tan(g^i_\lambda(\theta)).$$

Hence, the local stable manifold of p_λ is the curve

$$W^s_{\text{loc}}(p_\lambda) = \left\{ (h_\lambda(\theta), \theta)\colon 0 \le \theta < \frac{\pi}{2} \text{ and } 0 < h(\theta) < 1 \right\},$$

where

$$h_\lambda(\theta) = \sum_{n=0}^{\infty} (-1)^n \prod_{i=0}^{n-1} \tan(g^i_\lambda(\theta)). \tag{1}$$

This series converges uniformly and absolutely because $\tan(g^n_\lambda(\theta))$ converges to $\tan\theta_\lambda$ as $n \to \infty$, and $0 < \tan\theta_\lambda < 1$. The same is true for the series of the derivatives of h_λ, and so h_λ is analytic.

The invariant manifolds of p_λ, the singular curves of the reduced billiard map and the upper boundary S_∞ of B are depicted in Fig. 4.

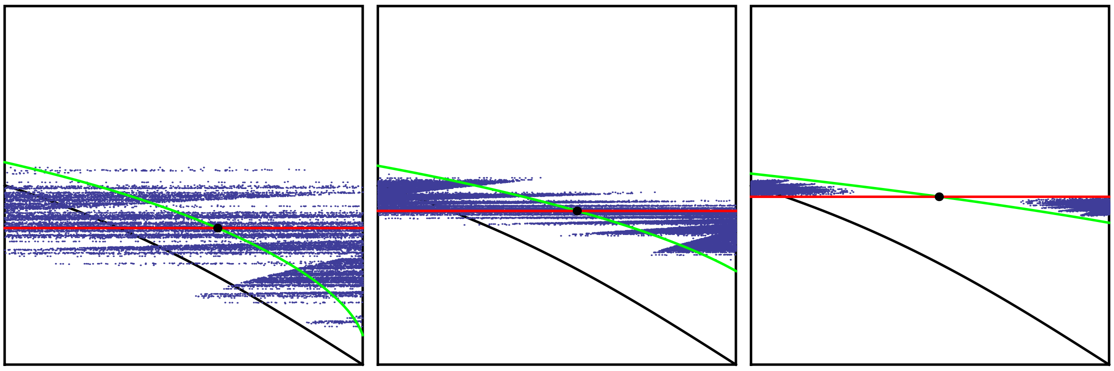
C. Two Families of Periodic Orbits

A straightforward computation shows that for each $n \ge 1$, there is a single periodic point q_n of period $n+2$ such that

$$\phi^{n+2}_\lambda(q_n) = f^2_2 \circ f^n_1(q_n) = q_n,$$

and a single periodic point p_n of period $2n$ such that

$$\phi^{2n}_\lambda(p_n) = f^{2n-1}_2 \circ f_1(p_n) = p_n.$$



(a) $\lambda = 0.615 \in (\lambda_0, \lambda_1)$ (b) $\lambda = 0.75 \in (\lambda_1, \lambda_2)$ (c) $\lambda = 0.88 \in (\lambda_2, 1)$

FIG. 6: Local stable (green curve) and unstable (red curve) manifolds of p_λ, and attractor A_λ (blue region).

By Lemma III.1, these periodic points are hyperbolic. It is easy to show that q_n exists for all $\lambda \in (0, c_n]$, and that c_n is a decreasing sequence in n. Numerical computations show that $p_1, \dots, p_{16}$ exist for all $\lambda \in (0, 1)$, whereas for every $n \geq 17$, the point p_n exists for all $\lambda \in (0, a_n] \cup [b_n, 1)$ with a_n and b_n being decreasing and increasing sequences, respectively. Our numerical computations also suggest that all the q_n's are homoclinically related to each other, and that the same property seems to hold also for most of the p_n's (see Fig. 5).

VI. BIFURCATION OF THE LIMIT SET

Let Ω be the non-wandering set of the map ϕ_λ. We now formulate a conjecture on the bifurcation of the set Ω as λ varies.

Conjecture VI.1. *For any $0 < \lambda < 1$, the non-wandering set Ω is a union of three sets:*

$$\Omega = P \cup H_\lambda \cup A_\lambda,$$

where P is the parabolic attractor (see section V A), A_λ is a hyperbolic transitive attractor, and H_λ is a horseshoe. Moreover, H_λ is either transitive or else a (possibly empty) hyperbolic chain of transitive horseshoes. In particular,

$$M^+ = W^s(P) \cup W^s(H_\lambda) \cup W^s(A_\lambda).$$

To support our conjecture, we present some facts and other more specific conjectures. Numerically, we found three constants

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < 1$$

such that

(F1) If $0 < \lambda < \lambda_0$, then $A_\lambda = \emptyset$ and H_λ is the maximal invariant set outside $W^s(P)$, the basin of attraction of P.

(F2) If $\lambda_0 \leq \lambda < \lambda_2$, then H_λ contains infinitely many q_n's, while the attractor A_λ contains p_λ, all p_n's and the remaining q_n's. As λ approaches λ_2, the horseshoe H_λ first shrinks due to the loss of periodic points q_n, then it becomes a short chain of periodic saddles, and eventually it disappears when all points q_n vanish.

(F3) If $\lambda_1 \leq \lambda < 1$, then $W^s(P) = B \cup \phi_\lambda^{-1}(B)$. In particular, $W^s(A_\lambda) = M^+ \setminus (B \cup \phi_\lambda^{-1}(B) \cup H_\lambda)$.

(F4) If $\lambda_2 \leq \lambda < 1$, then all q_n's vanish, the attractor A_λ contains some of the periodic points p_n, and the horseshoe H_λ contains p_λ as well as the remaining surviving p_n's.

In Fig. 6, the attractor A_λ is depicted for three different values of λ.

A. Conjecture for the value of λ_0

According to our numerical experiments, we see that $W^u(p_\lambda)$ is contained in $W^s(A_\lambda)$ for $\lambda < \lambda_2$ (see Fig. 6). Iterating numerically the unstable manifold of p_λ until it enters the basin of attraction of P, we obtain the following lower bound for λ_0:

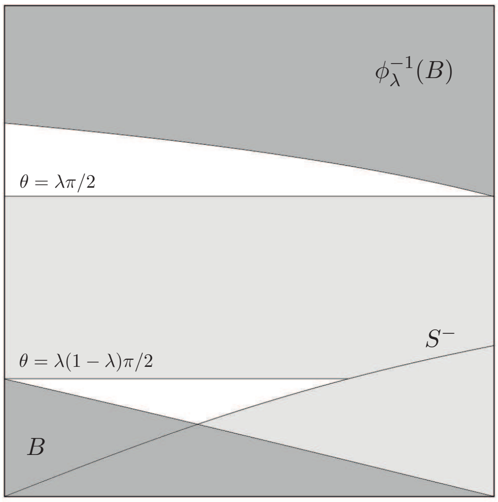
$$\lambda_0 \geq 0.607.$$

Set $q_0 = p_\lambda$ (this is consistent with the definition of q_n), and let $\bar{\lambda}_n$ be the maximum of all $0 < \lambda < 1$ such that

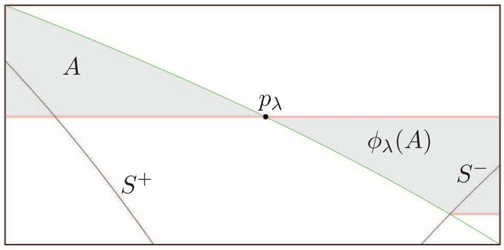
$$W^u_{\text{loc}}(q_n) \cap W^s_{\text{loc}}(q_{n+1}) \neq \emptyset.$$

We conjecture that $\lambda_0 = \min_n \bar{\lambda}_n$. Numerically, we see that

$$\min_n \bar{\lambda}_n = 0.6143916...$$



(a) $\lambda = \lambda_1$. Since the image of the map ϕ_λ is always below the line $\theta = \lambda\pi/2$, the basin of attraction of B is only $\phi_\lambda^{-1}(B)$. Moreover, the region in light gray is invariant.



(b) $\lambda = \lambda_2$. The region A between the stable and unstable local manifolds of p_λ is mapped by ϕ_λ into itself.

FIG. 7: Trapping regions for ϕ_λ.

For $\lambda < \lambda_0$, all periodic points p_n and q_n are homoclinically related to p_λ, and the horseshoe H_λ is the homoclinic class of p_λ. This would imply that H_λ is a transitive set. For n large enough, it is easy to check that $W^u(q_n) \cap W^s(P) \neq \emptyset$. Provided that H_λ is transitive, this fact implies that H_λ is a horseshoe.

B. Obtaining λ_1

Define

$$\sigma_\infty(\theta) = 1 - \sum_{i=0}^{\infty} \tan(\lambda^i \theta),$$

and

$$S_\infty = \left\{ (\sigma_\infty(\theta), \theta) : \theta \in \left[0, \frac{\pi}{2}\right) \text{ and } \sigma_\infty(\theta) \geq 0 \right\}.$$

The value

$$\lambda_1 = 0.6218...$$

is the single root of the equation

$$\sigma_\infty\left(\lambda(1-\lambda)\frac{\pi}{2}\right) = 0.$$

For $\lambda \geq \lambda_1$, the light-colored region in Fig. 7(a) is a trapping set containing all the points q_n. This fact may be used to prove (F3), and it also implies that for n sufficiently large, $W^u(q_n) \cap W^s(P) = \emptyset$ if $\lambda > \lambda_1$, and $W^u(q_n) \cap W^s(P) \neq \emptyset$ if $\lambda < \lambda_1$. In particular, this proves that $W^u(H_\lambda) \subset W^s(A_\lambda)$.

C. The value λ_2

Let $h_\lambda(\theta)$ be the function defined in (1), whose graph is the local stable manifold of p_λ. The value

$$\lambda_2 = 0.8736...$$

is defined to be the solution of

$$h_\lambda(\lambda\theta_\lambda) = \tan(\theta_\lambda)$$

(see Fig. 7(b)). This yields that $0 < \lambda < \lambda_2$ if and only if

$$f_1(W^u_{\text{loc}}(p_\lambda) \cap M_1) \cap W^s_{\text{loc}}(p_\lambda) \neq \emptyset.$$

The periodic points q_n persist for $\lambda \in (0, c_n]$ (see Section V C) and we know that $\lambda_1 < c_{n+1} < c_n < \lambda_2$ for every $n \geq 1$, while $\lambda_1 = \lim_{n\to\infty} c_n$. Hence, the periodic points q_n disappear as λ increases from λ_1 to λ_2, with q_1 being the last periodic point to disappear for a value of λ close to λ_2.

As λ approaches λ_2, the set H_λ consists of a hyperbolic chain plus the orbits of q_1 and q_2. As λ increases further and q_2 vanishes, H_λ is just the single orbit of q_1. Finally, when $\lambda < \lambda_2$ is sufficiently large so that the periodic point q_1 disappears, H_λ becomes empty.

Regarding (F4), one proves that the shadowed region in Fig. 7(b) is a trapping set. For $\lambda \geq \lambda_2$ the non-wandering set in this region consists of the fixed point p_λ and an attractor $\widetilde{A}_\lambda$ whose closure is strictly contained in the interior of the region. We do not claim that $\widetilde{A}_\lambda$ is always an attractor. In fact, as λ increases, we see numerically that the periodic points p_n with $n \geq 17$ persist for $\lambda \in (0, a_n] \cup [b_n, 1)$. There is also evidence that $a_n \searrow \lambda_2$ and $b_n \nearrow 1$. We thus believe that the attractor $\widetilde{A}_\lambda$ decomposes into a horseshoe H_λ containing some of the p_n, and an attractor A_λ containing the remaining p_n's. However, we do not know how these periodic points are distributed among H_λ and A_λ.

We have found that the fixed point p_λ has transversal homoclinic intersections for $0 < \lambda < \lambda_2$. This implies that ϕ_λ has positive topological entropy[5] for $0 < \lambda < \lambda_2$. We believe that for $\lambda_2 \leq \lambda < 1$, there exists n such that p_n is homoclinically related to p_{n+1}, implying that ϕ_λ has indeed positive topological entropy for every $0 < \lambda < 1$. This property has an alternative explanation. For $\lambda \geq \lambda_2$, we know that there exists a hyperbolic attractor, and that such an attractor typically admits some physical measures. If this is the case, then our billiard would have positive metric entropy, and so positive topological entropy.

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