CREATION OF HOMOCLINIC TANGENCIES IN HAMILTONIANS BY THE SUSPENSION OF POINCARÉ SECTIONS

MÁRIO BESSA AND JOÃO LOPES DIAS

Abstract. In this note we show that for any Hamiltonian defined on a symplectic 4-manifold M and any point $p \in M$, there exists a C^2-close Hamiltonian whose regular energy surface through p is either Anosov or it contains a homoclinic tangency. Our result is based on a general construction of Hamiltonian suspensions for given symplectomorphisms on Poincaré sections already known to yield similar properties.

MSC 2000: Primary: 37J45, 37D05 ; Secondary: 37D20.
keywords: Hamiltonian vector field, Anosov flow, elliptic point, homoclinic tangency.

1. Introduction and statement of the results

A few years ago Palis conjectured that any dynamical system can be approximated in a certain topology by a hyperbolic system without cycles, or by a system exhibiting either a homoclinic tangency or a heterodimensional cycle (cf. [14, 15]). Later, Pujals and Sambarino [16] proved this conjecture for the C^1 topology in the context of diffeomorphisms on compact surfaces. Notice that there are no heterodimensional cycles for surface diffeomorphisms.

A version for flows appeared in [1] stating that on a 3-dimensional compact manifold, a vector field can be C^1-approximated by another satisfying only one of the following phenomena:

- uniform hyperbolicity with no cycles,
- a homoclinic tangency,
- a singular cycle.

It has been further conjectured ([15, Conjecture 4]) that the last situation above can be replaced by a singular hyperbolic set (see [12] for the definition).

Related results can be obtained when restricting to conservative systems. In fact, any divergence-free vector field defined on a 3-dimensional closed manifold can be C^1-approximated in the same class by a vector field either Anosov or with a homoclinic tangency associated to a hyperbolic closed orbit [4]. This was recently generalized in [9] for a d-dimensional closed manifold, $d \geq 4$: any divergence-free vector field can be C^1-approximated by another one satisfying either one of the properties of the 3-dimensional

Date: July 21, 2011.

case, or with a heterodimensional cycle. In this note we address the problem of obtaining a version of [4] in the Hamiltonian context.

Let (M, ω) be a compact symplectic C^∞ $2d$-manifold, $d \geq 2$, with a smooth boundary ∂M. Let $C^s(M)$, $2 \leq s \leq \infty$, stand for the set of C^s real-valued functions on M constant on each connected component of ∂M, which we call C^s-Hamiltonians. We endow $C^s(M)$ with the C^r-Whitney topology. For each $H \in C^s(M)$ one has the Hamiltonian vector field X_H and the Hamiltonian flow φ_H^t. Consider an *energy* $e \in H(M) \subset \mathbb{R}$ and the associated φ_H^t-invariant *energy level set* $H^{-1}(e)$. An *energy surface* is a connected component of $H^{-1}(e)$. We say that it is regular if it does not contain critical points.

A regular energy surface is *Anosov* if it is uniformly hyperbolic (cf. [5]). It is *far from Anosov* if it is not in the closure of Anosov regular energy surfaces. Moreover, Anosov regular energy surfaces do not contain singularities or elliptic closed orbits.

Let us state the main result in this note.

Theorem 1. *Let $d = 2$, $H \in C^2(M)$ and $p \in M$. There exists a Hamiltonian C^2-close to H whose regular energy surface through p is either Anosov or else it contains a homoclinic tangency associated to some hyperbolic closed orbit.*

Recall that the existence of homoclinic tangencies is a sufficient condition to have elliptic points (see [13, 8]). We see that it is also a necessary condition for, at least, a sufficient C^1-close vector field.

Theorem 2. *Let $d = 2$, $H \in C^2(M)$ and $p \in M$ lies in an elliptic closed orbit of H. Then, there exists a Hamiltonian C^2-close to H whose regular energy surface through p has a homoclinic tangency associated to some hyperbolic closed orbit.*

In the proof of Theorem 2 (section 4) we apply a mechanism introduced in [10] to create homoclinic intersections by perturbations of area-preserving maps with elliptic points (see section 2.5). We use that in our context by finding a Hamiltonian flow (through Theorem 3 below) that yields a Poincaré map with the same properties – see section 3. Theorem 1 is then a direct consequence of Theorem 2 and of the Newhouse dichotomy (Theorem 2.3).

The last result in this note is a Hamiltonian suspension theorem, especially useful for the conversion of perturbative results between symplectomorphisms and Hamiltonian flows in any dimension $2d$. Indeed, if we perturb the Poincaré map of a periodic orbit (cf. section 2.1), there is a nearby Hamiltonian realizing the new map.

Theorem 3 (Hamiltonian suspension)**.** *Let $d \geq 2$ and $H \in C^\infty(M)$ with Poincaré map f at a periodic point p. Then, for any $\epsilon > 0$ there is $\delta > 0$ such that for any symplectomorphism $\widetilde{f}$ being δ-C^3-close to f, there is a Hamiltonian $\widetilde{H}$ ϵ-C^2-close with Poincaré map $\widetilde{f}$.*

The proof of the above theorem is contained in section 3. It is based on the construction using generating functions of an isotopy between f and $\widetilde{f}$, that extends to a Hamiltonian flow with the required properties. This type of suspension of Poincaré maps is already mentioned in [7] when the manifold is the annulus (see also [6]), but without an explicit construction.

We remark that the result in [10] holds also for real-analytic Hamiltonians. However, the problem of suspending a real-analytic Poincaré map into a Hamiltonian flow is of a very different sort because of the lack of real-analytic bump functions, and remains an open problem. So, in this case, it is required to find versions of the pertubation results directly for flows.

2. Preliminaries

In this section we assume (M, ω) to be a symplectic $2d$-manifold, with $d \geq 2$.

2.1. **Poincaré maps.**

Consider $H \in C^2(M)$ and a closed orbit $\mathcal{O}$ with least period $T > 0$ for φ_H^t. At a point $p \in \mathcal{O}$ consider a transversal $\Sigma \subset M$ to the flow, i.e. a local $(2d-1)$-submanifold for which X_H is nowhere tangencial. By choosing $e = H(p)$, define the dimension $2d-2$ symplectic submanifold

$$\Sigma_e = \Sigma \cap H^{-1}(\{e\}).$$

Thus, for any $x \in \Sigma_e$,

$$T_x H^{-1}(\{e\}) = T_x \Sigma_e \oplus \mathbb{R} X_H(x),$$

where $\mathbb{R} X_H(x)$ stands for the flow direction.

Let $U \subset M$ be some open neighbourhood of p and $V = U \cap \Sigma_e$. The *Poincaré (section) map* $f \colon V \to \Sigma_e$ is the return map of φ_H^t to Σ_e. It is given by

$$f(x) = \varphi_H^{\tau(x)}(x), \qquad x \in V,$$

where τ is the return time to Σ_e defined implicitely by the relation $\varphi_H^{\tau(x)}(x) \in \Sigma_e$ and satisfying $\tau(p) = T$. In addition, p is a fixed point of f. Notice that one needs to assume that U is a small enough neighbourhood of p. Thus, f is a C^1-symplectomorphism between V and its image. Moreover, any two Poincaré section maps of the same closed orbit are conjugate by a symplectomorphism.

2.2. **Homoclinic tangencies.**

Take $H \in C^2(M)$, a non-constant hyperbolic closed orbit $\mathcal{O}$ and a transversal section at a point $p \in \mathcal{O}$. Let W_p^s be the stable manifold at p of the Poincaré map, and W_p^u the unstable manifold. We say that $\mathcal{O}$ has a *homoclinic tangency* at $q \neq p$ if the invariant manifolds W_p^s and W_p^u have a non transversal intersection, i.e.:

- $T_q W_p^s \cap T_q W_p^u$ contains a nonzero vector,
- $T_q W_p^s \oplus T_q W_p^u \oplus \mathbb{R} X(q) \neq T_q H^{-1}(p)$.

2.3. **Hamiltonian flowtube coordinates.**

Denote the coordinates in $\mathbb{R}^{2d}$ as $(x_1, \ldots, x_d, y_1, \ldots, y_d)$. The canonical symplectic form is given by

$$\omega_0 = \sum_{i=1}^{d} dx_i \wedge dy_i.$$

The Hamiltonian vector field of any smooth Hamiltonian H on $(\mathbb{R}^{2d}, \omega_0)$ is then

$$X_H = \mathbb{J} \nabla H,$$

where $\mathbb{J} = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ and I is the $d \times d$ identity matrix.

Consider $H_0 \colon \mathbb{R}^{2d} \to \mathbb{R}$ given by $H_0 = y_d$, so that

$$X_{H_0} = \frac{\partial}{\partial x_d}.$$

Hence, the flow is $\varphi^t_{H_0} = \mathrm{id} + (0, \ldots, t, 0, \ldots, 0)$.

The following results provide us with the above coordinates, useful to perform local perturbations of a Hamiltonian defined on any symplectic manifold (M, ω).

Theorem 2.1 (Hamiltonian flowbox, cf. e.g. [3])**.** *Let $H \in C^s(M)$, $s \geq 2$ or $s = \infty$, and $p \in M$. If $dH(p) \neq 0$, there exists a neighborhood $U \subset M$ of p and a local C^{s-1}-symplectomorphism $g \colon (U, \omega) \to (\mathbb{R}^{2d}, \omega_0)$ such that $H = H_0 \circ g$ on U.*

By considering neighbourhoods as above taken along a piece of a trajectory, we can find a small tubular neighborhood where the flow is again straightened. This is the content of the next result.

Theorem 2.2 (Hamiltonian flowtube)**.** *Let $H \in C^s(M)$, $s \geq 2$ or $s = \infty$, and a non-closed compact self-avoiding arc of trajectory $\Gamma \subset M$. There exists a neighborhood $W \subset M$ of Γ and a local C^{s-1}-symplectomorphism $\phi \colon (W, \omega) \to (\mathbb{R}^{2d}, \omega_0)$ such that $H = H_0 \circ \phi$ on W.*

2.4. **Density of elliptic closed orbits.** The next result is the Hamiltonian version of the Newhouse dichotomy [13] for 4-dimensional Hamiltonians. As previously mentioned, it will be used in the proof of Theorem 1 (see section 4).

Theorem 2.3 ([2])**.** *Let $d = 2$. Given an open set $U \subset M$ intersecting a far from Anosov regular energy surface of $H \in C^2(M)$, there is a C^2-nearby Hamiltonian having an elliptic closed orbit through U. Moreover, this implies that, for far from Anosov regular energy surfaces of a C^2-generic Hamiltonian, the elliptic closed orbits are dense.*

2.5. **Creation of homoclinic tangencies.** The next result is central to the proof of Theorem 1. It deals with symplectomorphisms on a symplectic 2-manifold, i.e. area-preserving maps.

Theorem 2.4 (Gelfreich and Turaev [10])**.** *Let $r \in \mathbb{N} \cup \{\infty, \omega\}$. Any C^r-area-preserving map with an elliptic point can be C^r-approximated by another area-preserving map with a homoclinic tangency.*

3. Hamiltonian realization of a perturbed Poincaré map

Consider a Hamiltonian flow with a closed orbit and an associated Poincaré section map in an energy surface. Our goal in this section is to find a nearby Hamiltonian exhibiting a perturbed Poincaré map (Theorem 3). In order to prove Theorem 2, we will only make use of the case $d = 2$. Nevertheless, we study here the general situation for future use.

3.1. **Suspension of Poincaré maps.** Let $H \in C^\infty(M)$. Consider a closed orbit $\mathcal{O}$ with least period $T>0$, $p \in \mathcal{O}$ and $e = H(p)$. The Poincaré map is given by $f\colon V \to \Sigma_e$ as in section 2.1, having a fixed point at p.

The return time $\tau\colon V \to \mathbb{R}^+$ is close to T. So, choose $T_0, T_1 > 0$ such that $T_0 + T_1 \leq \frac{1}{2}\min\{\tau(x)\colon x \in V\}$. Take the arc of trajectory

$$\Gamma = \{\varphi_H^t(p)\colon T_0 \leq t \leq T - T_1\} \subset \mathcal{O}.$$

By Theorem 2.2, in a tubular neighbourhood $W \subset M$ of Γ we have $H = H_0 \circ \phi$. One can always compose ϕ with some symplectomorphism ψ so that $S_0, S_1 \subset \psi \circ \phi(W)$, where

$$S_0 = \{(x_1, \dots, x_d, y_1, \dots, y_d) \in \mathbb{R}^{2d}\colon x_d = y_d = 0\}$$

and $S_1 = \varphi^1_{H_0}(S_0)$. We assume that ϕ is in fact $\psi \circ \phi$ in order to simplify notations. Furthermore,

$$\varphi^1_{H_0}|S_0 = \phi \circ \varphi_H^{-T_1} \circ f \circ \varphi_H^{-T_0} \circ \phi^{-1},$$

which is simply given by $\varphi^1_{H_0}(x,0,y,0) = (x,1,y,0)$ with

$$(x,y) = (x_1, \dots, x_{d-1}, y_1, \dots, y_{d-1}) \in \mathbb{R}^{2d-2}.$$

This means that $\Pi \circ \varphi^1_{H_0}|S_0 = \mathrm{id}$ by using the projection $\Pi\colon \mathbb{R}^{2d} \to \mathbb{R}^{2d-2}$, $(x, x_d, y, y_d) \mapsto (x,y)$.

Given a C^∞-symplectomorphism $\widetilde{f}$ on V that is C^1-close to f, we want to find a Hamiltonian $\widetilde{H}$ having $\widetilde{f}$ as Poincaré map. The perturbation is constructed inside W, hence being enough to find $\widetilde{H}_0 = \widetilde{H} \circ \phi^{-1}$ such that

$$\varphi^1_{\widetilde{H}_0}|S_0 = \phi \circ \varphi_H^{-T_1} \circ \widetilde{f} \circ \varphi_H^{-T_0} \circ \phi^{-1}.$$

Then, $g = \Pi \circ \varphi^1_{\widetilde{H}_0}|S_0$ is a C^∞-symplectomorphism on $\mathbb{R}^{2d-2}$. From the above considerations we know that for any $r \geq 0$,

$$\|g - \mathrm{id}\|_{C^r} \leq c\|\widetilde{f} - f\|_{C^r}$$

for some $c_r > 0$ depending on H.

Let $\rho > 0$ and the euclidean open ball

$$B_\rho = \{(x,y) \in \mathbb{R}^{2d-2}\colon \|(x,y)\| < \rho\}.$$

The radius ρ is chosen small enough so that $B_\rho \times \{0 \leq x_d \leq 1, |y_d| < \rho\} \subset \phi(W)$.

Proposition 3.1. *There is $\delta, c > 0$ such that for any C^∞-symplectomorphism g compactly supported in B_ρ, δ-C^1-close to the identity, we can find $\widetilde{H}_0 \in C^\infty(\mathbb{R}^{2d})$ compactly supported in B_ρ verifying*

$$\Pi \circ \varphi^1_{\widetilde{H}_0}|S_0 = g$$

and

$$\left\|\widetilde{H}_0 - H_0\right\|_{C^2} \leq c(1 + \rho + \rho^{-1} + \rho\|g - \mathrm{id}\,\|^2_{C^3})\ \|g - \mathrm{id}\|_{C^1}. \tag{1}$$

Moreover, if g fixes the origin, then $\varphi^1_{\widetilde{H}_0}(0) = (0,1,0,0)$.

We now use the above proposition (to be proved in section 3.2 below) to complete the proof of Theorem 3. Consider

$$\widetilde{H} = \begin{cases} H, & \text{on } M \setminus W \\ H + (\widetilde{H}_0 - H_0) \circ \phi, & \text{otherwise.} \end{cases}$$

Therefore, combining the estimates above and assuming that $\widetilde{f}$ is C^3-close to f, one gets

$$\|\widetilde{H} - H\|_{C^2} \leq c\|\widetilde{f} - f\|_{C^1}$$

for some $c > 0$.

3.2. **Proof of Proposition 3.1.** Since the group of smooth symplectomorphisms isotopic to the identity is path-connected, we can always find an isotopy g_α, $\alpha \in [0,1]$, of symplectomorphisms from the identity to g. The corresponding non-autonomous vector field $X_\alpha = \dot{g}_\alpha \circ g_\alpha^{-1}$ is symplectic (for each α), and in fact Hamiltonian since we are in a simply connected space. The proof of Proposition 3.1 relies on this well-known fact, but it also requires a control on the size of the derivatives of $(x, y, \alpha) \mapsto g_\alpha(x, y)$. For this reason we need to construct g_α through a simple isotopy of generating functions, whose norms are easily estimated. Later, by adding a flow direction coordinate ($\alpha = x_d$) and its symplectic conjugate (the "energy" y_d), we will extend our Hamiltonian to $\mathbb{R}^{2d}$.

For functions $F \colon D \to \mathbb{R}^m$, $D \subset \mathbb{R}^{2d}$, consider the C^s-norm, with $s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$$\|F\|_{C^s} = \max_{i=1,\dots,m} \max_{|\sigma| \leq s} \sup_D \left| \frac{\partial^{|\sigma|} F_i}{\partial^{\sigma_1} x_1 \dots \partial^{\sigma_{2d}} y_d} \right|$$

where $\sigma = (\sigma_1, \dots, \sigma_{2d}) \in \mathbb{N}_0^{2d}$ and $|\sigma| = \sum_i \sigma_i$. Moreover, $\langle \cdot, \cdot \rangle$ denotes the usual euclidean scalar product and we introduce the projections $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$.

Let $V \in C^\infty(\mathbb{R}^{2d-2})$ such that

$$W(x', y) = \langle x', y \rangle + V(x', y)$$

is a generating function of g. More specifically, writing $(x', y') = g(x, y)$, since $\det D_1 x' \neq 0$,

$$x = \frac{\partial W}{\partial y}(x', y) \quad \text{and} \quad y' = \frac{\partial W}{\partial x'}(x', y).$$

Therefore,

$$g(x, y) = (x, y) - \mathbb{J}\nabla V \circ G(x, y).$$

where $G(x, y) = (\pi_1 g(x, y), y)$ and $\|\nabla V\|_{C^0} = \|g - \mathrm{id}\|_{C^0}$. We assume that g is sufficiently C^1-close to the identity, thus G is a diffeomorphism.

Lemma 3.2. *For $r \geq 1$, there is $c_r > 0$ such that*

$$\|\nabla V\|_{C^r} \leq c_r \max\{1, \|G^{-1}\|_{C^r}^r\} \, \|g - \mathrm{id}\|_{C^r}.$$

Proof. Write $\phi = g - \mathrm{id}$ and $\beta = G^{-1}$ so that $\phi \circ \beta = -\mathbb{J}\nabla V$. Recall the Faà di Bruno formula for the higher derivative chain rule:

$$D^r(\phi\circ\beta) = \sum \frac{r!}{k_1!\dots k_r!1!^{k_1}\dots r!^{k_r}} D^{|k|}\phi(\beta)\,(\underbrace{D\beta,\dots,D\beta}_{k_1},\dots,\underbrace{D^r\beta,\dots,D^r\beta}_{k_r}) \tag{2}$$

where the sum is over every $k = (k_1,\dots,k_r) \in \mathbb{N}_0^r$ such that

$$\langle k, (1,2,\dots,r)\rangle = r.$$

Therefore, there is a constant $c_r > 0$ depending on r, satisfying

$$\|\nabla V\|_{C^r} \le c_r \max\{1, \|\beta\|_{C^r}^r\}\,\|\phi\|_{C^r},$$

where we have used that $\|\beta\|_{C^{k_i}}^{k_i} \le \|\beta\|_{C^r}^{k_i} \le \max\{1, \|\beta\|_{C^r}^r\}$. □

Let $\ell \in C^\infty(\mathbb{R})$ be a bump function verifying

$$\ell(\alpha) = \begin{cases} 1, & \alpha \ge \xi \\ 0, & \alpha \le 0 \end{cases}$$

for some choice of $0 < \xi < 1$ such that $\ell' > 0$ in $(0,\xi)$. We can now construct the following smooth 1-family of generating functions:

$$W_\alpha(x', y) = \langle x', y\rangle + \ell(\alpha)\,V(x', y).$$

For each $\alpha \in \mathbb{R}$ we obtain a C^∞-symplectomorphism g_α generated by W_α. Clearly, $g_0 = \mathrm{id}$ and $g_1 = g$. Hence, g_α is a C^∞-isotopy between id and g implicitly given by

$$g_\alpha = \mathrm{id} - \ell(\alpha)\,\mathbb{J}\nabla V \circ G_\alpha,$$

where $G_\alpha = (\pi_1 g_\alpha, \pi_2)$ and $\|g_\alpha - \mathrm{id}\|_{C^0} \le \|\nabla V\|_{C^0} = \|g - \mathrm{id}\|_{C^0}$.

Lemma 3.3. *For $r \ge 1$, there is $c_r > 0$ such that for any $\alpha \in \mathbb{R}$, if $\|g - \mathrm{id}\|_{C^1}$ is sufficiently small, then*

$$\|g_\alpha - \mathrm{id}\|_{C^r} \le \frac{c_r}{1 - \|\nabla V\|_{C^1}}\,\|g - \mathrm{id}\|_{C^{r-1}}^r \|\nabla V\|_{C^r}.$$

Proof. Write $v_\alpha = -\ell(\alpha)\,\mathbb{J}\nabla V$ so that $\|v_\alpha\|_{C^r} \le \|\nabla V\|_{C^r}$. Using again the Faà di Bruno formula,

$$D^r(g_\alpha - \mathrm{id}) = \sum_{k_r=0} c_{k,r} D^{|k|} v_\alpha(G_\alpha)\,(\underbrace{DG_\alpha,\dots,DG_\alpha}_{k_1},\dots,\underbrace{D^{r-1}G_\alpha,\dots,D^{r-1}G_\alpha}_{k_{r-1}})$$
$$+ Dv_\alpha(G_\alpha)\,D^r G_\alpha,$$

where $c_{k,r}$ are the coefficients as in (2) and we have split the sum in the terms corresponding to the vectors $k = (k_1,\dots,k_{r-1},0)$ and $k = (0,\dots,0,1)$. Taking the norms, with $c_r > 0$ depending on r,

$$\|g_\alpha - \mathrm{id}\|_{C^r} \le c_r \|v_\alpha\|_{C^r} \|g_\alpha - \mathrm{id}\|_{C^{r-1}}^r + \|v_\alpha\|_{C^1}\,\|g_\alpha - \mathrm{id}\|_{C^r}.$$

Therefore,

$$\|g_\alpha - \mathrm{id}\|_{C^r} \le \frac{c_r}{1 - \|v_\alpha\|_{C^1}} \|g_\alpha - \mathrm{id}\|_{C^{r-1}}^r \|v_\alpha\|_{C^r}.$$

The claim follows from applying Lemma 3.2. □

Consider now the C^∞-vector field $\dot g_\alpha = \frac{d}{d\alpha} g_\alpha$ on $\mathbb{R}^{2d-2}$ that generates the isotopy g_α. The non-autonomous vector field

$$X_\alpha = \dot g_\alpha \circ g_\alpha^{-1}$$

is symplectic, i.e. $\iota_{X_\alpha}\omega_0$ is a closed 1-form. By the Poincaré lemma, since our space is simply-connected, it is also exact. Therefore, for each α there exists a C^∞-function $K_\alpha\colon \mathbb{R}^{2d-2} \to \mathbb{R}$ with compact support such that $\iota_{X_\alpha}\omega_0 = dK_\alpha$, i.e. $\nabla K_\alpha = -\mathbb{J}X_\alpha$ and using the notation of a Hamiltonian vector field

$$X_{K_\alpha} = X_\alpha.$$

Up to a constant (chosen so that K_α has compact support), it is given by

$$K_\alpha(x,y) = \int_{[0,(x,y)]} \iota_{X_\alpha}\omega_0 = \int_0^1 \langle X_{K_\alpha}(s(x,y)), (y,-x)\rangle \, ds, \tag{3}$$

where the integration is along the straight path $[0,(x,y)]$ that connects (x,y) to the origin. Notice that the vector field that determines g as the time-1 map is non-autonomous, not preserving the "energy" K. Also, $K_\alpha = 0$ for any $\alpha \notin (0,1)$.

We can extend the dimension of the space to $\mathbb{R}^{2d}$ by considering the variables $x_d = \alpha$ (seen as the time direction) and y_d (the "energy" K).

Let $\widetilde\ell \in C^\infty(\mathbb{R})$ be another bump function satisfying

$$\widetilde\ell(y_d) = \begin{cases} 1, & |y_d| \le \nu\rho \\ 0, & |y_d| \ge \rho \end{cases}$$

for any choice of $0<\nu<1$, such that $\|\widetilde\ell\|_{C^0} \le 1$,

$$\|\widetilde\ell'\|_{C^0} \le \frac{2}{(1-\nu)\rho} \quad \text{and} \quad \|\widetilde\ell''\|_{C^0} \le \frac{4}{(1-\nu)\rho^2}.$$

We define the (autonomous) C^∞-Hamiltonian $\widetilde H_0\colon \mathbb{R}^{2d} \to \mathbb{R}$ as

$$\widetilde H_0(x, x_d, y, y_d) = H_0(y_d) + K_{x_d}(x,y)\, \widetilde\ell(y_d)$$

with $H_0(y_d) = y_d$. Hence,

$$\nabla(\widetilde H_0 - H_0) = \left(\widetilde\ell\,\frac{\partial K}{\partial x}, \widetilde\ell\,\frac{\partial K}{\partial x_d}, \widetilde\ell\,\frac{\partial K}{\partial y}, \widetilde\ell'\,K\right). \tag{4}$$

Notice that outside $\{x_d \in (0,1), |y_d| < \rho\} \subset \mathbb{R}^{2d}$ we have $\widetilde H_0 = H_0$. By contrast, the Hamiltonian vector field for $x_d \in [0,1]$ and $|y_d| \le \nu\rho$ is

$$X_{\widetilde H_0} = \left(\pi_1 X_K, 1, \pi_2 X_K, -\frac{\partial K}{\partial x_d}\right).$$

Lemma 3.4. *There is $\delta > 0$ and $c > 0$ such that, if $\|g - \mathrm{id}\,\|_{C^1} \le \delta$, then* (1) *holds.*

Proof. We write a dot to represent the derivative with respect to x_d and D for the derivative with respect to (x,y). Recall that $X_K(x,x_d,y,y_d) = \dot g_{x_d} \circ g_{x_d}^{-1}(x,y)$. We will use Lemmas 3.2 and 3.3 without explicit mention.

From (4) we have

$$\left\|\widetilde H_0 - H_0\right\|_{C^1} \le \max\left\{\|K\|_{C^0}, \|X_K\|_{C^0}, \|\dot K\|_{C^0}, \|\widetilde\ell'\|_{C^0}\,\|K\|_{C^0}\right\}.$$

Now, the second order derivatives of $\widetilde{H}_0$ are

$$\frac{\partial^2 \widetilde{H}_0}{\partial z_i \partial z_j} = \widetilde{\ell}\,\frac{\partial^2 K}{\partial z_i \partial z_j}$$
$$\frac{\partial^2 \widetilde{H}_0}{\partial z_i \partial x_d} = \widetilde{\ell}\,\frac{\partial \dot{K}}{\partial z_i}$$
$$\frac{\partial^2 \widetilde{H}_0}{\partial^2 x_d} = \widetilde{\ell}\,\ddot{K}$$
$$\frac{\partial^2 \widetilde{H}_0}{\partial z_i \partial y_d} = \widetilde{\ell}'\,\frac{\partial K}{\partial z_i}$$
$$\frac{\partial^2 \widetilde{H}_0}{\partial x_d \partial y_d} = \widetilde{\ell}'\,\dot{K}$$
$$\frac{\partial^2 \widetilde{H}_0}{\partial^2 y_d} = \widetilde{\ell}''\,K$$

where $z = (x, y)$ and $i, j = 1, \dots, 2d-2$. So,

$$\left\| \widetilde{H}_0 - H_0 \right\|_{C^2} \leq \max\Big\{\, \|X_K\|_{C^1}\,, \|\widetilde{\ell}'\|_{C^0}\,\|X_K\|_{C^0}\,, \|\ddot{K}\|_{C^0}, \max\{1, \|\widetilde{\ell}'\|_{C^0}\}\|\dot{K}\|_{C^0}, \max\{1, \|\widetilde{\ell}'\|_{C^0}, \|\widetilde{\ell}''\|_{C^0}\}\,\|K\|_{C^0} \Big\}\,.$$

By writing $v = -\mathbb{J}\nabla V$, we have that

$$\|\dot{g}\|_{C^0} \leq \|\ell\|_{C^1}\,\|v\|_{C^0} + \|v\|_{C^1}\|\dot{g}\|_{C^0}.$$

Therefore,

$$\|\dot{g}\|_{C^0} \leq \frac{\|\ell\|_{C^1}\,\|g - \mathrm{id}\,\|_{C^0}}{1 - \|v\|_{C^1}} \leq c\,\|g - \mathrm{id}\|_{C^0}$$

for some $c > 0$. Similarly,

$$\begin{aligned}\|\ddot{g}\|_{C^0} &\leq \frac{\|\ell\|_{C^2}\,\|v\|_{C^0} + 2\,\|\ell\|_{C^1}\,\|v\|_{C^1}\,\|\dot{g}\|_{C^0} + \|v\|_{C^2}\,\|\dot{g}\|^2_{C^0}}{1 - \|v\|_{C^1}} \\ &\leq c\,\|g - \mathrm{id}\|_{C^0}\end{aligned}$$

for some $c > 0$. Moreover,

$$\|D\dot{g}\|_{C^0} \leq \|\ell\|_{C^1}\,\|v\|_{C^1}\,\|g\|_{C^1} + \|v\|_{C^2}\,\|g\|_{C^1}\,\|\dot{g}\|_{C^0} + \|v\|_{C^1}\,\|D\dot{g}\|_{C^0},$$

thus

$$\begin{aligned}\|D\dot{g}\|_{C^0} &\leq \frac{\|\ell\|_{C^1}\,\|v\|_{C^1}\,\|g\|_{C^1} + \|v\|_{C^2}\|\dot{g}\|_{C^0}\,\|g\|_{C^1}}{1 - \|v\|_{C^1}} \\ &\leq c\,\|g - \mathrm{id}\|_{C^1}\end{aligned}$$

for some $c > 0$.

From $\dot{X}_K = \ddot{g} \circ g^{-1} + D\dot{g} \circ g^{-1}\,\dot{g}^{-1}$ and $DX_K = D\dot{g} \circ g^{-1}\,Dg^{-1}$,

$$\|X_K\|_{C^1} \leq c\,\|g - \mathrm{id}\|_{C^1}\,.$$

From (3), $\|K\|_{C^0} \leq \rho\,\|X_K\|_{C^0}$, $\|\dot{K}\|_{C^0} \leq \rho\|X_K\|_{C^1}$ and also $\|\ddot{K}\|_{C^0} \leq \rho\|\dot{X}_K\|_{C^0}$. Thus, it remains to bound $\|\dot{X}_K\|_{C^0}$.

As before, we obtain the following bounds:

$$\|\dddot{g}\|_{C^0} \leq \frac{1}{1-\|v\|_{C^1}}\Big(\|\ell\|_{C^3}\|v\|_{C^0} + 3\|\ell\|_{C^2}\|v\|_{C^1}\|\dot{g}\|_{C^0} + 3\|\ell\|_{C^1}\|v\|_{C^2}\|\dot{g}\|^2_{C^0} + 3\|\ell\|_{C^1}\|v\|_{C^1}\|\ddot{g}\|_{C^0} + \|v\|_{C^3}\|\dot{g}\|^3_{C^0}\Big)$$

$$\|D^2\dot{g}\|_{C^0} \leq \frac{1}{1-\|v\|_{C^1}}\Big(\|\ell\|_{C^1}\|v\|_{C^1}\|g\|^2_{C^1} + \|\ell\|_{C^1}\|v\|_{C^1}\|D^2 g\|_{C^0} + \|v\|_{C^3}\|g\|^2_{C^1}\|\dot{g}\|_{C^0} + \|v\|_{C^2}\|D^2 g\|_{C^0}\|\dot{g}\|_{C^0} + 2\|v\|_{C^2}\|g\|_{C^1}\|\dot{g}\|_{C^1}\Big)$$

$$\|D\ddot{g}\|_{C^0} \leq \frac{1}{1-\|v\|_{C^1}}\Big(\|\ell\|_{C^2}\|v\|_{C^1}\|g\|_{C^1} + 2\|\ell\|_{C^1}\|v\|_{C^2}\|g\|_{C^1}\|\dot{g}\|_{C^0} + 2\|\ell\|_{C^1}\|v\|_{C^1}\|\dot{g}\|_{C^1} + \|v\|_{C^3}\|g\|_{C^1}\|\dot{g}\|^2_{C^0} + 2\|v\|_{C^2}\|\dot{g}\|_{C^1}\|\dot{g}\|_{C^0} + \|v\|_{C^2}\|g\|_{C^1}\|\ddot{g}\|_{C^0}\Big)$$

Finally, we use the fact that $\ddot{X}_K = \dddot{g}\circ g^{-1} + 2D\ddot{g}\circ g^{-1}\,\dot{g}^{-1} + D^2\dot{g}\circ g^{-1}\,(\dot{g}^{-1},\dot{g}^{-1}) + D\dot{g}\circ g^{-1}\,\ddot{g}^{-1}$. So,

$$\|\ddot{X}_K\|_{C^0} \leq c\left(1+\|g-\mathrm{id}\|^2_{C^3}\right)\|g-\mathrm{id}\|_{C^1}$$

for some constant $c>0$. Evaluating all the above estimates together, one gets

$$\left\|\widetilde{H}_0 - H_0\right\|_{C^2} \leq c\left(1+\rho+\rho^{-1}+\rho\|g-\mathrm{id}\|^2_{C^3}\right)\|g-\mathrm{id}\|_{C^1}$$

for some universal constant $c>0$ that only depends on the norms of the bump functions. □

Remark 3.1. *In the above lemma there is the need to bound the size of higher derivatives of g. This loss of differentiability is caused by our specific construction of the isotopy g_α. It should be possible to use a different isotopy that avoids this phenomenon. Our choice was done for the sake of simplicity, since it does not restrict our main results.*

The Hamiltonian flow for $x_d\in[0,1]$ and $|y_d|\leq\nu\rho$ is given by

$$\varphi^t_{\widetilde{H}_0}(x,x_d,y,y_d) = \Big(\pi_1 g_{x_d+t}\circ g^{-1}_{x_d}(x,y),\; x_d+t,\; \pi_2 g_{x_d+t}\circ g^{-1}_{x_d}(x,y),\; y_d - \int_0^t \frac{\partial K_{x_d+s}}{\partial x_d}\circ g_{x_d+t}\circ g^{-1}_{x_d}(x,y)\,ds\Big).$$

Using estimates in the proof of Lemma 3.4, one gets that the increment in the last coordinate for $t\in[0,1]$ is bounded from above by

$$\left\|\frac{\partial K}{\partial x_d}\right\|_{C^0} \leq \rho\|X_K\|_{C^0} \leq \nu\rho$$

as long as $\|g-\mathrm{id}\|_{C^1}$ is small. Finally, the time-1 flow acts on the transversal $\{(x,0,y,0)\}$ by

$$\varphi^1_{\widetilde{H}_0}(x,0,y,0) = \left(\pi_1 g(x,y), 1, \pi_2 g(x,y), -\int_0^1 \frac{\partial K_s}{\partial x_d}\circ g(x,y)\,ds\right).$$

In particular, if $g(0) = (0)$, $\varphi^1_{\widetilde{H}_0}(0) = (0, 1, 0, 0)$ because $\frac{\partial}{\partial x_d} K(0, 0) = 0$.

4. Proof of Theorems 1 and 2

The proof of Theorem 2 follows from the following steps:

(1) Since elliptic closed orbits are stable, we can find a C^∞ approximation $\widetilde{H}$ keeping the same (i.e. its analytic continuation) elliptic closed orbit.
(2) Consider the C^∞ Poincaré map f of $\varphi^t_{\widetilde{H}}$ on a transversal to the elliptic closed orbit restricted to an energy surface.
(3) Use Theorem 2.4 to obtain a C^∞-symplectomorphim $\widetilde{f}$ close to f with a homoclinic tangency.
(4) Finally, Theorem 3 allows us to construct a Hamiltonian C^2-close to $\widetilde{H}$, which realizes the Poincaré map $\widetilde{f}$ on the energy surface.

Assume that the energy level $H^{-1}(\{H(p)\})$ is far from Anosov. The proof of Theorem 1 follows from Theorem 2 after applying Theorem 2.3 that gives elliptic closed orbits for some Hamiltonian C^2-close.

Finally, we would like to mention a possible alternative strategy to prove Theorem 2 in the absence of Theorem 2.4. We first observe that an area-preserving diffeomorphism yielding an irrational invariant curve can be perturbed in order to create homoclinic tangencies, as proved in [11]. So, starting from a Hamiltonian with an elliptic closed orbit, one can perturb its tangent map and get a new Hamiltonian (using a version of Franks Lemma[17]) whose Poincaré map is an area-preserving map satisfying a twist condition along a diophantine invariant curve. KAM theory then assures us the stability of this structure, and a suspension of the result in [11] holds homoclinic tangencies for a nearby Hamiltonian.

Acknowledgements

MB was partially supported by Fundação para a Ciência e a Tecnologia through SFRH/BPD/20890/2004 and the project PTDC/MAT/099493/2008. JLD was partially supported by Fundação para a Ciência e a Tecnologia through the project “Randomness in Deterministic Dynamical Systems and Applications” PTDC/MAT/105448/2008.

References

[1] A. Arroyo and F. Rodriguez-Hertz, *Homoclinic bifurcations and uniform hyperbolicity for three-dimensional flows*, Ann. Inst. H. Poincaré Anal. Non Linéaire 20, 805–841, 2003.
[2] M. Bessa and J. Lopes Dias, *Hamiltonian elliptic dynamics on symplectic 4-manifolds*, Proc. Amer. Math. Soc. 137, 585–592, 2009.
[3] M. Bessa and J. Lopes Dias, *Generic dynamics of 4-dimensional C^2 Hamiltonian systems*, Comm. Math. Phys. 281, 597–619, 2008.
[4] M. Bessa and J. Rocha, *Homoclinic tangencies versus uniform hyperbolicity for conservative 3-flows*, Jr. Diff. Eq. 247, 2913-2923, 2009.
[5] M. Bessa, C. Ferreira and J. Rocha, *On the stability of the set of hyperbolic closed orbits of a Hamiltonian*, Math. Proc. Cambridge Phil. Soc. 149, 373–383, 2010.
[6] P. J. Channell, *Hamiltonian suspensions of symplectomorphisms: an alternative approach to design problems*, Physica D 127 (1999), 117–130.

[7] R. Douady, *Une démonstration directe de l'équivalence des théorèmes de tores invariants pour difféomorphismes et champs de vecteurs*, C. R. Acad. Sc. Paris I 295, 201–204, 1982.
[8] P. Duarte, *Abundance of elliptic isles at conservative bifurcations*, Dyn. and Stab. of Syst. 14, 339–356, 1999.
[9] C. Ferreira, *Stability properties of divergence-free vector fields*, Preprint 2010 arXiv:1004.2893.
[10] V. Gelfreich and D. Turaev, *Universal dynamics in a neighborhood of a generic elliptic periodic point*, Regul. Chaotic Dyn. 15, 159-164, 2010.
[11] L. Mora and N. Romero, *Persistence of homoclinic tangencies for area-preserving maps*, Ann. Fac. Sci. Toulouse Math. 6, 711–725, 1997.
[12] C. A. Morales, M. J. Pacifico, and E. R. Pujals, *Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers*, Ann. of Math. 160(2), 375–432, 2004.
[13] S. Newhouse, *Quasi-elliptic periodic points in conservative dynamical systems*, Am. J. Math., **99** (1977), 1061–1087.
[14] J. Palis, *A global view of dynamics and a conjecture on the denseness of finitude of attractors*, Astérisque, **261** (2000), 339–351.
[15] J. Palis, *Open questions leading to a global perspective in dynamics*, Nonlinearity, **21** (2008), T37.
[16] E. Pujals and M. Sambarino, *Homoclinic tangencies and hyperbolicity for surface diffeomorphisms*, Ann. Math., **151**, 3 (2000), 961–1023.
[17] Vivier, T., *Robustly transitive 3-dimensional regular energy surface are Anosov*, Institut de Mathématiques de Bourgogne, Dijon, Preprint 412 (2005). http://math.u-bourgogne.fr/topo/prepub/pre05.html.

Departamento de Matemática, Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal and ESTGOH-Instituto Politécnico de Coimbra, Rua General Santos Costa, 3400-124 Oliveira do Hospital, Portugal
E-mail address: bessa@fc.up.pt

Departamento de Matemática and CEMAPRE, ISEG, Universidade Técnica de Lisboa, Rua do Quelhas 6, 1200-781 Lisboa, Portugal
E-mail address: jldias@iseg.utl.pt