Optimal *per claim* reinsurance for dependent risks*

Manuel Guerra
CEOC and ISEG -T.U.Lisbon
R. Quelhas 6, 1200-781 Lisboa
mguerra@iseg.utl.pt

Maria de Lourdes Centeno
CEMAPRE, ISEG -T.U.Lisbon
R. Quelhas 6, 1200-781 Lisboa
lcenteno@iseg.utl.pt

Abstract: This paper generalizes the results on optimal reinsurance presented in Centeno and Guerra (2008) to the case of an insurer holding a portfolio of k dependent risks. It is assumed that the number of claims of a risk may depend on the number of claims of the other risks of the portfolio. Our aim is to determine the optimal form of reinsurance for each risk when the cedent seeks to maximize the adjustment coefficient of the retained portfolio - which is equivalent to maximizing the expected utility of wealth, with respect to an exponential utility with a certain coefficient of risk aversion - and restricts the reinsurance strategies to functions of the individual claims.

Assuming that the premium calculation principle is a convex functional we prove existence and uniqueness of solutions and provide a necessary optimality condition. These results are used to find the optimal reinsurance policy for a

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given risk when the reinsurance loading is either proportional to the expected value or increasing with the variance of the ceded claims. The type of the optimal arrangement for a given risk only depends on the premium of that particular risk.

KEY WORDS: optimal reinsurance, dependent risks, adjustment coefficient, expected utility, exponential utility function, convex premium principles.

1 Introduction

Although the topic of dependency among risks has been quite popular in the recent actuarial literature, only very few articles deal with the problem in relation to traditional reinsurance. An exception is Centeno (2005) where it was calculated, from the insurance point of view, the optimal excess of loss retention limits for two dependent risks, when the optimization criteria used were the expected utility of wealth with respect to the exponential utility function and the adjustment coefficient of the retained aggregate claims amount. In that paper we considered that the number of claims is generated by a bivariate Poisson distribution and that the premium calculation principle used for the excess of loss treaties is the expected value principle. So the key point was only on the retention limits chosen and not on the type of reinsurance, which was limited to excess of loss reinsurance, which at least for the independent case we knew that was the optimal form of reinsurance, when the reinsurer priced their treaties according to the expected value principle (see Gerber (1979)). That model was, among others, considered by Wang (2008) on the description of correlated risk portfolios. The dependence between two lines of business arises generally by a common effect, which affects both the number and the claim size of both lines. However in some situations the severities have little correlation, in which case the model can be applied. The classical case of dependence is natural hazards where usually at least two lines of business are affected - homeowners and motor hull.

In Centeno and Guerra (2008) we proved that excess of loss reinsurance is no longer the optimal form of reinsurance if the reinsurer's premium loadings are not proportional to the expected ceded claims. If the loading is an increasing function with the variance of the ceded claims and the number of claims belongs to the Katz family, i.e. if it is Poisson, Negative Binomial or Binomial, it was proved, under some fair assumptions, that the optimal form of reinsurance, when reinsurance is placed on the individual claims satisfies

$$y = Z(y) + \frac{1}{R} \ln \frac{Z(y) + \alpha}{\alpha}, \tag{1}$$

where y is the amount of an individual claim, $Z(y)$ is the ceded claim, R is either the adjustment coefficient of the retained claims or the coefficient of risk aversion, depending on the optimization criterion chosen and α is a positive constant (whose value can easily be calculated, see Centeno and Guerra (2008) and Guerra and Centeno (2008b)). Note that the same form was found for the aggregate case in Guerra and Centeno (2008a).

In the present article we generalize the results of Centeno and Guerra (2008) to k dependent aggregate risks, although the individual claim amounts are supposed to be independent among lines.

2 Assumptions and preliminaries

Let us assume that a insurer seeks reinsurance for $k \geq 1$ risks. Let $\mathbf{N}$ be the random vector $(N_1, N_2, \ldots, N_k)$, with joint probability function

$$p(\mathbf{n}) = \Pr\{N_1 = n_1,\ N_2 = n_2, ..., N_k = n_k\},$$

where $N_i,\ i = 1, 2, \ldots, k$, is the number of claims of risk i in a given period of time. Let $Y_{i,j},\ i = 1, 2, \ldots, k,\ \ j = 1, 2, \ldots, N_i$ be the value of the j^{th} claim of risk i and $\widehat{Y}_i = \sum_{1\le j\le N_i} Y_{i,j}$ the aggregate claim amount of risk i in the same period.

Let $Z_i\left(y\right)$ be the value ceded under the reinsurance policy in force for risk i and $P_i(Z_i)$ the respective reinsurance premium. The set of all possible reinsurance policies for risk i is denoted $\mathcal{Z}_i$ and defined as

$$\mathcal{Z}_i = \left\{\zeta : [0,+\infty[\mapsto I\!\!R |\ \zeta \ \text{ is measurable and } 0\le \zeta\left(y\right) \le y,\ \forall y\ge 0\right\}.$$

Let $\mathcal{Z} = \bigotimes_{i=1}^{k} \mathcal{Z}_i$, be the cartesian product of all $\mathcal{Z}_i$ i.e. the set of the vector functions $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_k)$.

The aggregate ceded claims of risk i is

$$\widehat{Z}_i = \sum_{j=1}^{N_i} Z_i\left(Y_{i,j}\right).$$

Hence, denoting by c the insurer's premium (before reinsurance) of the portfolio of the k risks for each period of time, the insurer's net profit, after reinsuring the k risks, is

$$L_{\mathbf{Z}} = c - \sum_{i=1}^{k}\left[P_i\left(Z_i\right) - \left(\widehat{Y}_i - \widehat{Z}_i\right)\right] =$$

$$= \quad c - \sum_{i=1}^{k}\left[P_i\left(Z_i\right) - \sum_{j=1}^{N_i}\left(Y_{i,j} - Z_i\left(Y_{i,j}\right)\right)\right].$$

Our objective in this article is to determine the vector of reinsurance arrangements $\mathbf{Z}$ that maximizes the adjustment coefficient of the aggregate retained claims of the portfolio. That problem was already considered in Centeno and Guerra (2008) for the case $k = 1$, hence we do not go into many details whenever the generalization is straightforward.

In what follows we consider the assumptions:

Assumption 1 $\Pr\left\{L_{\mathbf{Z}} < 0\right\} > 0$ *holds for every* $\mathbf{Z}\in\mathcal{Z}$.

Assumption 2 *For each* $i \in \{1, 2, \ldots, k\}$*, all* $Y_{i,j}$*,* $j \in \mathbb{N}$*, are i.i.d. nonnegative continuous random variables with common density function* f_i*. We denote by* Y_i *a generic r.v. with density* f_i *and assume that* $E[Y_i^2] < +\infty$.

Assumption 3 *The moment-generating function of* $\mathbf{N}$ *is finite in some neighbourhood of zero, which is equivalent to saying that the probability generating function of* $\mathbf{N}$,

$$\pi(\mathbf{x}) = E\left[\prod_{i=1}^{k} x_i^{N_i}\right] = \sum_{\mathbf{n}\in\mathbb{N}_0^k} \left(\prod_{i=1}^{k} x_i^{n_i}\right) p(\mathbf{n}), \tag{2}$$

is finite in some neighbourhood of the point $\mathbf{x} = (1, 1, ..., 1)$.

Assumption 4 *The random variables* Y_i, $i = 1, 2, \ldots, k$ *are mutually independent and independent of the random vector* $\mathbf{N}$*;*

Assumption 5 *All functionals* $P_i : \mathcal{Z}_i \mapsto [0, +\infty]$ *are convex,* $P_i(0) = 0$*, and are continuous in the mean-squared sense, i.e.*

$$\lim_{m\to\infty} \int_0^{+\infty} \left(Z_{i,m}(y) - Z_i(y)\right)^2 f_i(y)\, dy = 0 \text{ implies that}$$
$$\lim_{m\to\infty} P_i(Z_{i,m}) = P_i(Z_i).$$

We do not distinguish between functions $Z_i, Z_i' \in \mathcal{Z}_i$ which differ only on a set of zero probability with respect to the density f_i.

The adjustment coefficient of the aggregate retained portfolio of risks, for a particular combination of reinsurance policies $\mathbf{Z}$, is defined as the unique positive solution, when such a root exists, of the equation

$$G(R, \mathbf{Z}) = 1, \tag{3}$$

where $G : [0, +\infty[\times \mathcal{Z} \mapsto [0, +\infty]$ is the mapping

$$G(R, \mathbf{Z}) = E\left[e^{-RL_{\mathbf{Z}}}\right], \tag{4}$$

which under assumptions 1-5 is

$$G(R, Z) = \\ = e^{R\left(\sum_{i=1}^{k} P_i(Z_i) - c\right)} \pi\left(E\left[e^{R(Y_1 - Z_1)}\right], E\left[e^{R(Y_2 - Z_2)}\right], ..., E\left[e^{R(Y_k - Z_k)}\right]\right). \tag{5}$$

Let $\mathcal{Z}^{+} = \{\mathbf{Z} \in \mathcal{Z} : (3) \text{ admits a positive solution}\}$. For each $\mathbf{Z} \in \mathcal{Z}^{+}$, we denote by $R_{\mathbf{Z}}$ the corresponding positive solution of (3). Our main objective is to solve the problem

Problem 1 *Find* $(R^*, \mathbf{Z}^*) \in (0, +\infty) \times \mathcal{Z}^{+}$ *such that*

$$R^* = R_{\mathbf{Z}^*} = \max \left\{ R_{\mathbf{Z}} : \mathbf{Z} \in \mathcal{Z}^{+} \right\}.$$

A policy $\widehat{\mathbf{Z}} \in \mathcal{Z}^{+}$ is said to be **optimal for the adjustment coefficient criterion** if $\left(R_{\hat{\mathbf{Z}}}, \hat{\mathbf{Z}} \right)$ solves this problem.

Consider the exponential utility function with coefficient of risk aversion $R > 0$:

$$U_R(w) = -e^{-Rw}.$$

As for exponential utility functions the utility of wealth, after a period of time, is proportional to the utility of the profit after the same period of time, maximizing the expected utility of the profit is equivalent to maximizing the expected utility of wealth. For any given coefficient of risk aversion, $R > 0$, the expected utility of the profit obtained by the insurance company in a given period of time is

$$E\left[U_R(L_{\mathbf{Z}})\right] = -G(R, \mathbf{Z}). \tag{6}$$

Consider the maximization problem:

Problem 2 *Find* $\mathbf{Z}^* \in \mathcal{Z}$*, such that*

$$E\left[U_R(L_{\mathbf{Z}^*})\right] = \max\left\{E\left[U_R(L_{\mathbf{Z}})\right] : \mathbf{Z} \in \mathcal{Z}\right\},$$

where $R > 0$ *is a given constant (fixed).*$\square$

A policy $\mathbf{Z} \in \mathcal{Z}$ is said to be **optimal for the expected utility criterion** with coefficient of risk aversion R if it solves Problem 2 for that particular R.

The adjustment coefficient problem is related to the expected utility problem, for an exponential utility function, in the sense of the following proposition (see the proof in Centeno and Guerra (2008) for the case $k = 1$).

Proposition 1 *A pair* $(R^*, \mathbf{Z}^*) \in (0, +\infty) \times \mathcal{Z}$ *solves the adjustment coefficient problem if and only if it satisfies the following conditions:*

1. $\mathbf{Z}^*$ *is optimal for the expected utility criterion with coefficient of risk aversion* $R = R^*$,

2. $G(R^*, \mathbf{Z}^*) = 1$.

Proposition 1 shows that, Problem 1 can be solved in two steps:

1. For each $R \in]0, +\infty[$ find $\mathbf{Z}_R$, the respective optimal policy for the expected utility criterion. Equivalently, find

$$\mathbf{Z}_R = \arg\min \{G(R, \mathbf{Z}) : \mathbf{Z} \in \mathcal{Z}\};$$

2. Solve the equation with one single real variable

$$G(R, \mathbf{Z}_R) = 1.$$

3 Existence and uniqueness of optimal policy

This relation between Problems 1 and 2 and the assumptions made, namely on the convexity of the reinsurance premiums, which imply the convexity of the functional G for each R, allow us to state the following theorem:

Theorem 1 *For each* $R \in (0, +\infty)$ *there exists an optimal policy for the expected utility criterion.*

There exists an optimal policy for the adjustment coefficient criterion.

Given an optimal combination of policies $\mathbf{Z}^* = (Z_1^*, Z_2^*, \ldots, Z_k^*) \in \mathcal{Z}$, *a combination of policies* $\mathbf{Z}' = (Z_1', Z_2', \ldots, Z_k') \in \mathcal{Z}$ *is optimal for the same* $R \in (0, +\infty)$ *if and only if*

(a) $\Pr\{Z_i'(Y_i) = Z_i^*(Y_i)\} = 1$ *holds for every* $i \in \{1, 2, \ldots, k\}$ *such that* $\Pr\{N_i = m\} < 1 \ \forall m \in \mathbb{N}$;

(b) $\Pr\left\{Z_i'(Y_i) - Z_i^*(Y_i) = \frac{P_i(Z_i') - P_i(Z_i^*)}{m}\right\} = 1$ *holds for every* i, m *such that* $\Pr\{N_i = m\} = 1$.

Therefore, the optimal combination of policies is unique whenever $\Pr\{N_i = m\} < 1,\ \forall i \in \{1, 2, \ldots, k\}, m \in \mathbb{N}$.

Notice that condition *(b)* in Theorem 1 means that the policies Z_i^*, Z_i' differ by a constant equal to the difference between the correspondent premiums divided by the number of claims. Thus, multiple solutions occur only in cases where the number of claims is a degenerate random variable. Further, since our model does not take into account the different times of premium payment and refund of claims, multiple solutions are always equivalent from the economical point of view.

4 Necessary condition for optimality

Fix $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_k) \in \mathcal{Z}$. We consider needle-like perturbations of $\mathbf{Z}$, i.e., we consider reinsurance policies of type

$$Z_i^{\alpha,\upsilon,\varepsilon}(y) = \begin{cases} Z_i(y), & \text{if } y \notin [\upsilon, \upsilon + \varepsilon]; \\ \alpha, & \text{if } y \in [\upsilon, \upsilon + \varepsilon]. \end{cases}$$

$$\mathbf{Z}^{i,\alpha,\upsilon,\varepsilon} = \left(Z_1, ..., Z_{i-1}, Z_i^{\alpha,\upsilon,\varepsilon}, Z_{i+1}, ..., Z_k\right).$$

We use the short-hand notation

$$\delta = (\alpha, \upsilon, \varepsilon), \qquad Z_i^\delta = Z_i^{\alpha,\upsilon,\varepsilon}, \qquad \mathbf{Z}^{i,\delta} = \mathbf{Z}^{i,\alpha,\upsilon,\varepsilon},$$

and in what follows we assume that if $\mathbf{Z}$ is optimal for the expected utility criterion, then the limits

$$\Delta P_{i,Z_i}(\upsilon) = \lim_{\substack{\alpha \to Z_i(\upsilon) \\ 0 \le \alpha \le \upsilon}} \lim_{\varepsilon \to 0^+} \frac{P_i\left(Z_i^\delta\right) - P_i\left(Z_i\right)}{\varepsilon\left(\alpha - Z_i\left(\upsilon\right)\right)}, \qquad i = 1, 2, ..., k$$

are well defined functions, except possibly on some null (zero probability) subsets of $[0, +\infty)$.

In order to obtain some of the following results we also need to consider compositions of needle-like perturbations. These are treaties of the type

$$Z_i^{\delta_1\delta_2}(y) = \begin{cases} Z_i(y), & \text{if } y \notin [\upsilon_1, \upsilon_1+\varepsilon_1] \cup [\upsilon_2, \upsilon_2+\varepsilon_2]; \\ \alpha_1, & \text{if } y \in [\upsilon_1, \upsilon_1+\varepsilon_1]; \\ \alpha_2, & \text{if } y \in [\upsilon_2, \upsilon_2+\varepsilon_2], \end{cases}$$

$$\mathbf{Z}^{i,\delta_1\delta_2} = \left(Z_1, \ldots, Z_{i-1}, Z_i^{\delta_1\delta_2}, Z_{i+1}, \ldots, Z_k\right),$$

for arbitrary $\upsilon_1 \neq \upsilon_2$ and sufficiently small $\varepsilon_1, \varepsilon_2 > 0$. We assume that the functions $\Delta P_{i,Z_i}$ suitably approximate the effect of double needle-like perturbations, in the sense that the approximation

$$\begin{aligned} P_i\left(Z_i^{\delta_1\delta_2}\right) =& P_i(Z_i) + \varepsilon_1(\alpha_1 - Z_i(\upsilon_1))\Delta P_{i,Zi}(\upsilon_1) + \varepsilon_2(\alpha_2 - Z_i(\upsilon_2))\Delta P_{i,Zi}(\upsilon_2) + \\ &+ \varepsilon_1 o(\alpha_1 - Z_i(\upsilon_1)) + \varepsilon_2 o(\alpha_2 - Z_i(\upsilon_2)) + o(\varepsilon_1 + \varepsilon_2) \end{aligned} \tag{7}$$

holds for every $\upsilon_1 \neq \upsilon_2$ chosen in a set of probability equal to one with respect to the density f_i.

Theorem 2 *Let $\mathbf{Z} \in \mathcal{Z}$ be optimal for the expected utility criterion. Then, there exist constants $C_i \in (0, +\infty)$, $i = 1, 2, \ldots, k$ such that the following conditions hold with probability equal to one:*

$$\begin{cases} \Delta P_{i,Z_i}(y) \geq C_i e^{Ry} f_i(y), & \text{whenever } Z_i(y) = 0; \\ \Delta P_{i,Z_i}(y) = C_i e^{R(y - Z_i(y))} f_i(y), & \text{whenever } 0 < Z_i(y) < y; \\ \Delta P_{i,Z_i}(y) \leq C_i f_i(y), & \text{whenever } Z_i(y) = y. \end{cases} \tag{8}$$

The constants C_i satisfy

$$C_i \geq \frac{\frac{\partial}{\partial x_i}\pi\left(E\left[e^{R(Y_1-Z_1)}\right], E\left[e^{R(Y_2-Z_2)}\right], \ldots, E\left[e^{R(Y_k-Z_k)}\right]\right)}{\pi\left(E\left[e^{R(Y_1-Z_1)}\right], E\left[e^{R(Y_2-Z_2)}\right], \ldots, E\left[e^{R(Y_k-Z_k)}\right]\right)}, \tag{9}$$

with equality holding in (9) whenever there exists some $x_i > E\left[e^{R(Y_i-Z_i)}\right]$ such that

$$\pi\left(E\left[e^{R(Y_1-Z_1)}\right], \ldots, x_i, \ldots, E\left[e^{R(Y_k-Z_k)}\right]\right) < +\infty. \ \square \tag{10}$$

Proof. We introduce the short notation

$$\begin{aligned}
&x_i = E\left[e^{R(Y_i-Z_i)}\right], \qquad \mathbf{x} = (x_1, x_2, \ldots, x_k),\\
&x_i^\delta = E\left[e^{R(Y_i-Z_i^\delta)}\right], \qquad \mathbf{x}^{i,\delta} = (x_1, \ldots, x_{i-1}, x_i^\delta, x_{i+1}, \ldots, x_k),\\
&P_i = P_i(Z_i), \qquad P_i^\delta = P_i\left(Z_i^\delta\right).
\end{aligned}$$

Fix $i \in \{1, 2, ..., k\}$. Optimality of $\mathbf{Z}$ means that for any perturbation $\mathbf{Z}^{i,\delta}$, we must have

$$G\left(R, \mathbf{Z}^{i,\delta}\right) - G\left(R, \mathbf{Z}\right) \geq 0. \tag{11}$$

But we have the first order Taylor expansion:

$$\begin{aligned}
&G\left(R, \mathbf{Z}^{i,\delta}\right) - G\left(R, \mathbf{Z}\right) = e^{R\left(\sum\limits_{j=1}^{k} P_j - c + P_i^\delta - P_i\right)} \pi\left(\mathbf{x}^{i,\delta}\right) - e^{R\left(\sum\limits_{j=1}^{k} P_j - c\right)} \pi\left(\mathbf{x}\right) =\\
&= e^{R\left(\sum\limits_{j=1}^{k} P_j - c\right)} \pi(\mathbf{x}) R\left(P_i^\delta - P_i\right) + e^{R\left(\sum\limits_{j=1}^{k} P_j - c\right)} \frac{\partial \pi(\mathbf{x})}{\partial x_i}\left(x_i^\delta - x_i\right) +\\
&\qquad + o\left(\left|P_i^\delta - P_i\right| + \left|x_i^\delta - x_i\right|\right) =\\
&= e^{R\left(\sum\limits_{j=1}^{k} P_j - c\right)} \left[\pi(\mathbf{x}) R\left(P_i^\delta - P_i\right) + \frac{\partial \pi(\mathbf{x})}{\partial x_i}\left(x_i^\delta - x_i\right)\right] +\\
&\qquad + o\left(\left|P_i^\delta - P_i\right| + \left|x_i^\delta - x_i\right|\right).
\end{aligned} \tag{12}$$

It can easily be checked that

$$x_i^\delta - x_i = \int_\upsilon^{\upsilon+\varepsilon} \left(e^{R(y-\alpha)} - e^{R(y-Z_i(y))}\right) f_i(y) dy.$$

Provided υ is a Lebesgue point of the functions $y \mapsto f_i(y)$ and $y \mapsto e^{R(y-Z_i(y))} f_i(y)$, this further reduces to

$$x_i^\delta - x_i = -\varepsilon R e^{R(\upsilon - Z_i(\upsilon))}(\alpha - Z_i(\upsilon)) f_i(\upsilon) + \varepsilon o(\alpha - Z_i(\upsilon)) + o(\varepsilon).$$

In this case, equality (12) reduces to

$$G\left(R,\mathbf{Z}^{i,\delta}\right)-G\left(R,\mathbf{Z}\right)=$$
$$=Re^{R\left(\sum\limits_{j=1}^{k}P_j-c\right)}\left(\pi(\mathbf{x})\left(P_i^{\delta}-P_i\right)-\varepsilon\frac{\partial\pi(\mathbf{x})}{\partial x_i}e^{R(\upsilon-Z_i(\upsilon))}(\alpha-Z_i(\upsilon))f_i(\upsilon)\right)+$$
$$+\varepsilon o(\alpha-Z_i(\upsilon))+o\left(\varepsilon\right).$$

Using (11) and dividing by ε, this implies

$$\pi(\mathbf{x})\lim_{\varepsilon\to0^+}\frac{P_i^{\delta}-P_i}{\varepsilon}-\frac{\partial\pi(\mathbf{x})}{\partial x_i}e^{R(\upsilon-Z_i(\upsilon))}(\alpha-Z_i(\upsilon))f_i(\upsilon)+o(\alpha-Z_i(\upsilon))\geq0.$$

Provided $Z_i(\upsilon)<\upsilon$, we can make $\alpha\to Z_i(\upsilon)^+$ to obtain

$$\pi(\mathbf{x})\Delta P_{i,Z_i}(\upsilon)-\frac{\partial\pi(\mathbf{x})}{\partial x_i}e^{R(\upsilon-Z_i(\upsilon))}f_i(\upsilon)\geq0. \tag{13}$$

If (10) holds and $Z_i(\upsilon)>0$, we can also make $\alpha\to Z_i(\upsilon)^-$ and obtain the converse inequality

$$\pi(\mathbf{x})\Delta P_{i,Z_i}(\upsilon)-\frac{\partial\pi(\mathbf{x})}{\partial x_i}e^{R(\upsilon-Z_i(\upsilon))}f_i(\upsilon)\leq0. \tag{14}$$

This proves the Theorem in the case when (10) holds.

If (10) fails, then $G\left(R,\mathbf{Z}^{i,\delta}\right)=+\infty$ holds whenever $\alpha<Z_i(\upsilon)$ and υ is a Lebesgue point such that $f_i(\upsilon)>0$. Therefore, the argument above can not be used to deduce (14). In this case, we require double perturbations to prove the Theorem.

Notice that

$$x_i^{\delta_1\delta_2}-x_i=\int_{\upsilon_1}^{\upsilon_1+\varepsilon_1}\left(e^{R(y-\alpha_1)}-e^{R(y-Z_i(y))}\right)f_i(y)dy+$$
$$+\int_{\upsilon_2}^{\upsilon_2+\varepsilon_2}\left(e^{R(y-\alpha_2)}-e^{R(y-Z_i(y))}\right)f_i(y)dy=$$
$$=\varepsilon_1Re^{R(\upsilon_1-Z_i(\upsilon_1))}(\alpha_1-Z_i(\upsilon_1))f_i(\upsilon_1)+\varepsilon_2Re^{R(\upsilon_2-Z_i(\upsilon_2))}(\alpha_2-Z_i(\upsilon_2))f_i(\upsilon_2)+$$
$$+\varepsilon_1o(\alpha_1-Z_i(\upsilon_1))+\varepsilon_2o(\alpha_2-Z_i(\upsilon_2))+o(\varepsilon_1+\varepsilon_2)$$

holds provided $\upsilon_1,\ \upsilon_2$ are Lebesgue points of the functions $y\mapsto f_i(y)$ and $y\mapsto e^{R(y-Z_i(y))}f_i(y)$.

Using an implicit-function type argument it can be shown that for each $\alpha_1 - Z_i(\upsilon_1) > 0$, $\alpha_2 - Z_i(\upsilon_2) < 0$ and every sufficiently small $\varepsilon > 0$ there exist $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\varepsilon_1 = -\varepsilon \left(e^{R(\upsilon_2 - Z_i(\upsilon_2))}(\alpha_2 - Z_i(\upsilon_2)) + o(\alpha_2 - Z_i(\upsilon_2)) \right) + o(\varepsilon); \tag{15}$$

$$\varepsilon_2 = \varepsilon \left(e^{R(\upsilon_1 - Z_i(\upsilon_1))}(\alpha_1 - Z_i(\upsilon_1)) + o(\alpha_1 - Z_i(\upsilon_1)) \right) + o(\varepsilon); \tag{16}$$

$$x_i^{\delta_1 \delta_2} = x_i. \tag{17}$$

$>$From (17) it follows that $G(R, \mathbf{Z}^{i,\delta_1\delta_2}) \geq G(R, \mathbf{Z})$ holds if and only if $P_i^{\delta_1\delta_2} - P_i \geq 0$ holds. Due to (7), this is

$$\varepsilon_1 \left((\alpha_1 - Z_i(\upsilon_1)) \Delta P_{i,Z_i}(\upsilon_1) + o(\alpha_1 - Z_i(\upsilon_1)) \right) + \\ \varepsilon_2 \left((\alpha_2 - Z_i(\upsilon_2)) \Delta P_{i,Z_i}(\upsilon_2) + o(\alpha_2 - Z_i(\upsilon_2)) \right) + o(\varepsilon_1 + \varepsilon_2) \geq 0.$$

Setting $\alpha_2 - Z(\upsilon_2) = -\left(\alpha_1 - Z(\upsilon_1)\right)$ and using (15),(16), we obtain

$$e^{R(\upsilon_2 - Z_i(\upsilon_2))}(\alpha_1 - Z_i(\upsilon_1))^2 \Delta P_{i,Z_i}(\upsilon_1) f_i(\upsilon_2) - \\ - e^{R(\upsilon_1 - Z_i(\upsilon_1))}(\alpha_1 - Z_i(\upsilon_1))^2 \Delta P_{i,Z_i}(\upsilon_2) f_i(\upsilon_1) + o(\alpha_1 - Z_i(\upsilon_1))^2 \geq 0.$$

Dividing by $(\alpha_1 - Z_i(\upsilon_1))^2$ and making $\alpha_1 \to Z_i(\upsilon_1)^+$, we show that the inequality

$$\frac{\Delta P_{i,Z_i}(\upsilon_1)}{e^{R(\upsilon_1 - Z_i(\upsilon_1))} f_i(\upsilon_1)} \geq \frac{\Delta P_{i,Z_i}(\upsilon_2)}{e^{R(\upsilon_2 - Z_i(\upsilon_2))} f_i(\upsilon_2)}$$

holds for every pair of Lebesgue points of $y \mapsto f_i(y)$, $y \mapsto e^{R(y - Z_i(y))} f_i(y)$ such that $f_i(\upsilon_1) > 0$, $f_i(\upsilon_2) > 0$, $Z_i(\upsilon_1) < \upsilon_1$, $Z_i(\upsilon_2) > 0$.

This proves that (8) must hold for some constant $C_i \in \mathbb{R}$. Inequality (13) shows that C_i must satisfy (9). ■

Theorem 2 presents an optimality condition which is suitable for computations. However, it is useful to make some remarks concerning its economic meaning.

It can be proved (see Appendix) that under the Assumptions 1–4, the following equalities hold:

$$E\left[e^{R\sum_{j=1}^{k}(\hat{Y}_j-\hat{Z}_j)}\right] = \pi\left(E\left[e^{R(Y_1-Z_1)}\right],\ldots,E\left[e^{R(Y_k-Z_k)}\right]\right);$$

$$\lim_{\varepsilon\to 0^+}\frac{E\left[e^{R\sum_{j=1}^{k}(\hat{Y}_j-\hat{Z}_j)}\chi_{\{\exists m\in\{1,2,\ldots,N_i\}:Y_{i,m}\in[y-\varepsilon,y+\varepsilon]\}}\right]}{\Pr\{\exists m\in\{1,2,\ldots,N_i\}:Y_{i,m}\in[y-\varepsilon,y+\varepsilon]\}} =$$
$$= \frac{\partial\pi}{\partial x_i}\left(E\left[e^{R(Y_1-Z_1)}\right],\ldots,E\left[e^{R(Y_k-Z_k)}\right]\right)\frac{e^{R(y-Z_i(y))}}{E[N_i]}, \qquad (18)$$

the last equality holding with probability equal to one with respect to the density f_i. Recalling that

$$E[U_R(L_{\mathbf{Z}})] = e^{R(\sum_{j=1}^{k}P_j-c)}E\left[e^{R\sum_{j=1}^{k}(\hat{Y}_j-\hat{Z}_j)}\right],$$

equality (18) can be interpreted as

$$E\left[U_R(L_{\mathbf{Z}})|\,\exists m\in\{1,2,\ldots,N_i\}:Y_{i,m}=y\right] =$$
$$= e^{R(\sum_{j=1}^{k}P_j-c)}\frac{\partial\pi}{\partial x_i}\left(E\left[e^{R(Y_1-Z_1)}\right],\ldots,E\left[e^{R(Y_k-Z_k)}\right]\right)\frac{e^{R(y-Z_i(y))}}{E[N_i]}.$$

Therefore,

$$\frac{\frac{\partial}{\partial x_i}\pi\left(E\left[e^{R(Y_1-Z_1)}\right],\ldots,E\left[e^{R(Y_k-Z_k)}\right]\right)}{\pi\left(E\left[e^{R(Y_1-Z_1)}\right],\ldots,E\left[e^{R(Y_k-Z_k)}\right]\right)}e^{R(y-Z_i(y))} =$$
$$= E[N_i]\frac{E\left[U_R(L_Z)|\,\exists m\in\{1,2,\ldots,N_i\}:Y_{i,m}=y\right]}{E\left[U_R(L_Z)\right]}$$

and condition (10) means that it is possible to decrease slightly the reinsurance cover of risk i without causing the expected utility of the portfolio's wealth to become $-\infty$. Thus, Theorem 2 implies the following more economically meaningful corollary:

Corollary 1 *Let $Z\in\mathcal{Z}$ be optimal for the expected utility criterion, and sup-*

pose there exists $\tilde{Z}_i \in \mathcal{Z}_i$ *such that:*

$$\Pr\left\{\tilde{Z}_i \leq Z_i\right\} = 1, \qquad \Pr\left\{\tilde{Z}_i < Z_i\right\} > 0,$$
$$E\left[U_R(L_{(Z_1,\ldots,Z_{i-1},\tilde{Z}_i,Z_{i+1},\ldots,Z_k)})\right] > -\infty.$$

Then, the following holds with probability equal to one:

$$\begin{cases} \frac{\Delta P_{i,Z_i}(y)}{E[N_i]} \geq \frac{E[U_R(L_Z)|\ \exists m\in\{1,\ldots,N_i\}:Y_{i,m}=y]}{E[U_R(L_Z)]} f_i\left(y\right), & \text{whenever } Z_i(y)=0; \\ \frac{\Delta P_{i,Z_i}(y)}{E[N_i]} = \frac{E[U_R(L_Z)|\ \exists m\in\{1,\ldots,N_i\}:Y_{i,m}=y]}{E[U_R(L_Z)]} f_i\left(y\right), & \text{whenever } 0<Z_i(y)<y; \\ \frac{\Delta P_{i,Z_i}(y)}{E[N_i]} \leq \frac{E[U_R(L_Z)|\ \exists m\in\{1,\ldots,N_i\}:Y_{i,m}=y]}{E[U_R(L_Z)]} f_i\left(y\right), & \text{whenever } Z_i(y)=y. \ \square \end{cases}$$

Thus, the marginal cost of reinsurance of a claim of amount y on risk i is balanced against the ratio between the expected utility of the portfolio given that a claim o value y is placed on risk i and the a priori expected utility of the portfolio.

5 The expected value principle

In this section we consider the case when the reinsurance premium for one of the risks in the portfolio is computed by the expected value principle:

$$P_i\left(Z_i\right) = (1+\beta)\, E[\hat{Z}_i] = (1+\beta)\, E\left[N_i\right] E\left[Z_i\right],$$

where β is a positive constant.

For this case the necessary conditions of Theorem 2 reduce to the following

Corollary 2 *Suppose the reinsurance premium for the risk* i *is computed by an expected value principle. The optimal treaty for the risk* i *is a stop-loss contract:*

$$Z_i(y) = \begin{cases} 0, & \text{if } y \leq M; \\ y-M, & \text{if } y \geq M. \end{cases}$$

If (10) holds, then

$$M = \frac{1}{R}\ln\frac{(1+\beta)E[N_i]\pi\left(E\left[e^{R(Y_1-Z_1)}\right],\ldots,E\left[e^{R(Y_k-Z_k)}\right]\right)}{\frac{\partial\pi}{\partial x_i}\left(E\left[e^{R(Y_1-Z_1)}\right],\ldots,E\left[e^{R(Y_k-Z_k)}\right]\right)}.$$

If (10) fails, then

$$M \leq \frac{1}{R} \ln \frac{(1+\beta)E[N_i]\pi\left(E\left[e^{R(Y_1-Z_1)}\right], \ldots, E\left[e^{R(Y_k-Z_k)}\right]\right)}{\frac{\partial \pi}{\partial x_i}\left(E\left[e^{R(Y_1-Z_1)}\right], \ldots, E\left[e^{R(Y_k-Z_k)}\right]\right)}. \ \square$$

Proof. A simple computation shows that

$$\Delta P_{i,Z_i}(\upsilon) = (1+\beta)\, E[N_i] f_i(\upsilon).$$

Substituting in (8), one obtains

$$\begin{cases} (1+\beta)\, E[N_i] \geq C_i e^{Ry}, & \text{whenever } Z_i(y) = 0; \\ (1+\beta)\, E[N_i] = C_i e^{R(y - Z_i(y))}, & \text{whenever } 0 < Z_i(y) < y; \\ (1+\beta)\, E[N_i] \leq C_i, & \text{whenever } Z_i(y) = y, \end{cases}$$

i.e.,

$$\begin{cases} y \leq \frac{1}{R} \ln \frac{(1+\beta)E[N_i]}{C_i}, & \text{whenever } Z_i(y) = 0; \\ Z_i(y) = y - \frac{1}{R} \ln \frac{(1+\beta)E[N_i]}{C_i}, & \text{whenever } 0 < Z_i(y) < y; \\ \frac{(1+\beta)E[N_i]}{C_i} \leq 1, & \text{whenever } Z_i(y) = y. \end{cases}$$

The proof is complete by taking (9) into account. ■

Notice that Corollary 2 does not rule out the possibility that $Z_i \equiv Y_i$, i.e., the totality of risk i should be ceded.

6 Variance-related principles

In this section we consider the case when the reinsurance premium for one of the risks in the portfolio is computed by a variance-related principle:

$$\begin{aligned} P_i(Z_i) =& E[\hat{Z}_i] + g\left(Var[\hat{Z}_i]\right) = \\ =& E[N_i]E[Z_i] + g\left(E[N_i]Var[Z_i] + Var[N_i]E[Z_i]^2\right), \end{aligned} \tag{19}$$

where $g:[0,+\infty)\mapsto[0,+\infty)$ is a continuous function, smooth in $(0,+\infty)$ such that

$$g(0)=0, \qquad g'(t)>0, \quad \forall t>0, \tag{20}$$

and

$$\frac{g''(t)}{g'(t)}\geq -\frac{1}{2t}, \qquad \forall t\in(0,B), \tag{21}$$

with $B=\sup\left\{Var[\hat{Z}_i]:Z_i\in\mathcal{Z}_i\right\}$. Following Guerra and Centeno (2008b), we can say that if g satisfies (20) then the premium (19) is convex if and only if (21) is fulfilled.

For this case the necessary conditions of Theorem 2 reduce to the following

Corollary 3 *Suppose that the reinsurance premium for the risk i is computed by a variance-related principle (19). If g' is bounded in a neighbourhood of zero, then the following set of conditions holds with probability equal to one with respect to the density f_i.*

$$y\leq\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}, \qquad \text{when } Z_i(y)=0; \tag{22}$$

$$y=Z_i(y)+\frac{1}{R}\ln\frac{Z_i(y)-\alpha_2}{\alpha_1}, \qquad \text{when } 0<Z_i(y)<y; \tag{23}$$

$$y\leq\alpha_1+\alpha_2, \qquad \text{when } Z_i(y)=y. \tag{24}$$

α_1, α_2 are constants satisfying

$$\alpha_1=\frac{\frac{\partial\pi}{\partial x_i}\left(E\left[e^{R(Y_1-Z_1)}\right],\ldots,E\left[e^{R(Y_k-Z_k)}\right]\right)}{E[N_i]\pi\left(E\left[e^{R(Y_1-Z_1)}\right],\ldots,E\left[e^{R(Y_k-Z_k)}\right]\right)2g'\left(Var[\hat{Z}_i]\right)} \quad \text{if (10) holds;}$$

$$\alpha_1\geq\frac{\frac{\partial\pi}{\partial x_i}\left(E\left[e^{R(Y_1-Z_1)}\right],\ldots,E\left[e^{R(Y_k-Z_k)}\right]\right)}{E[N_i]\pi\left(E\left[e^{R(Y_1-Z_1)}\right],\ldots,E\left[e^{R(Y_k-Z_k)}\right]\right)2g'\left(Var[\hat{Z}_i]\right)} \quad \text{if (10) fails;}$$

$$\alpha_2=\frac{E[N_i]-Var[Z_i]}{E[N_i]}E[Z_i]-\frac{1}{2g'\left(Var[\hat{Z}_i]\right)}.$$

If g' is unbounded in any neighbourhood of zero, then the optimal treaty must be either a function of type (22)-(24) or $Z_i\equiv 0$ (no reinsurance at all). $\square$

Proof. Follows the same argument as the proof found in Centeno and Guerra (2008) for the analogous single risk case. ■

7 A numerical example

We consider an insurance company dealing with two risks. The value of individual claims in each risk is a Pareto random variable, i.e., the densities of Y_i are functions

$$f_i(y) = a_i b_i^{a_i} (b_i + y)^{-a_i - 1}, \qquad i = 1, 2,$$

with parameters $a_i > 2$, $b_i > 0$. The claim events are Poisson processes with random intensities driven by a common Gamma random variable:

$$\begin{aligned} \Pr\{N_i = n | \Theta = \theta\} &= e^{-\theta\lambda_i} \frac{(\theta\lambda_i)^n}{n!}, \qquad i = 1, 2; \\ \Pr\{N_1 = n_1, N_2 = n_2\} &= \int_0^{+\infty} e^{-\theta(\lambda_1+\lambda_2)} \frac{(\theta\lambda_1)^{n_1} (\theta\lambda_2)^{n_2}}{n_1! n_2!} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} d\theta, \end{aligned}$$

with parameters $\alpha > 0$, $\beta > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$. The probability generating function for the number of claims in this model is

$$\pi(x_1, x_2) = \left(\frac{\beta}{\beta - \lambda_1(x_1 - 1) - \lambda_2(x_2 - 1)} \right)^\alpha.$$

We chose the parameters of $\alpha, \beta, \lambda_i, a_i, b_i$ in order to model a situation where the risks have different properties. One risk (say, risk 1) has a smaller number of expected claims but its claim size has a larger average and heavier tail compared to risk 2. However, the aggregate claim amount of risk 2 is larger than risk 1[1].

[1] This is the interesting case: If the risk with the heaviest tail is also "large" in aggregate terms, then one can expect that the best reinsurance strategy is to reinsure heavily the first risk, irrespective of dependency.

Thus, we set the following parameter values[2]:

$$a_1 = 3, \qquad a_2 = 4,$$

$$E[N_1] = 1, \qquad E[N_2] = 5, \qquad \mathrm{Corr}(N_1, N_2) = 0.5,$$

$$E[\hat{Y}_1] = \frac{1}{4}, \qquad E[\hat{Y}_2] = \frac{3}{4}$$

$>$From this choice, the remaining parameters of the model follow:

$$b_1 = \frac{1}{2}, \qquad b_2 = \frac{9}{20}, \qquad \lambda_1 = 1, \qquad \lambda_2 = 5, \qquad \alpha = \beta = 1.89898.$$

This gives the statistics

$$Var[N_1] = 1.52660, \qquad Var[N_2] = 18.165,$$

$$E[Y_1] = 0.25, \quad E[Y_2] = 0.15, \quad \mathrm{Var}[Y_1] = 0.1875, \quad \mathrm{Var}[Y_2] = 0.045,$$

$$\mathrm{Var}[\hat{Y}_1] = 0.282912, \qquad \mathrm{Var}[\hat{Y}_2] = 0.633712.$$

We assume reinsurance policies for both risks are priced using the standard deviation principle with loading coefficient equal to 0.3, i.e.,

$$P_i(Z) = E[\hat{Z}] + 0.3\sqrt{\mathrm{Var}[\hat{Z}]}, \qquad i = 1, 2.$$

The direct insurer prices its own policies by the same principle but with a smaller loading equal to 0.15. This means the total income of the insurer by time unit is $c = 1.19919$.

We compare the optimal treaties for the adjustment coefficient criterion in two situations, which we will call the "dependent" and "independent" cases, respectively. the first is the model described above, while the independent case is identical, except that the number of claims N_1, N_2 are independent but have the same marginal distributions as in the dependent case. In each of these models we also compare the optimal treaty with the best excess–of–loss treaty.

The solutions and some key indicators are presented in Table 1.

[2] The total aggregate claim is normalized: $E[\hat{Y}_1] + E[\hat{Y}_2] = 1$. It is clear that this can always be made by a suitable choice of the account unit.

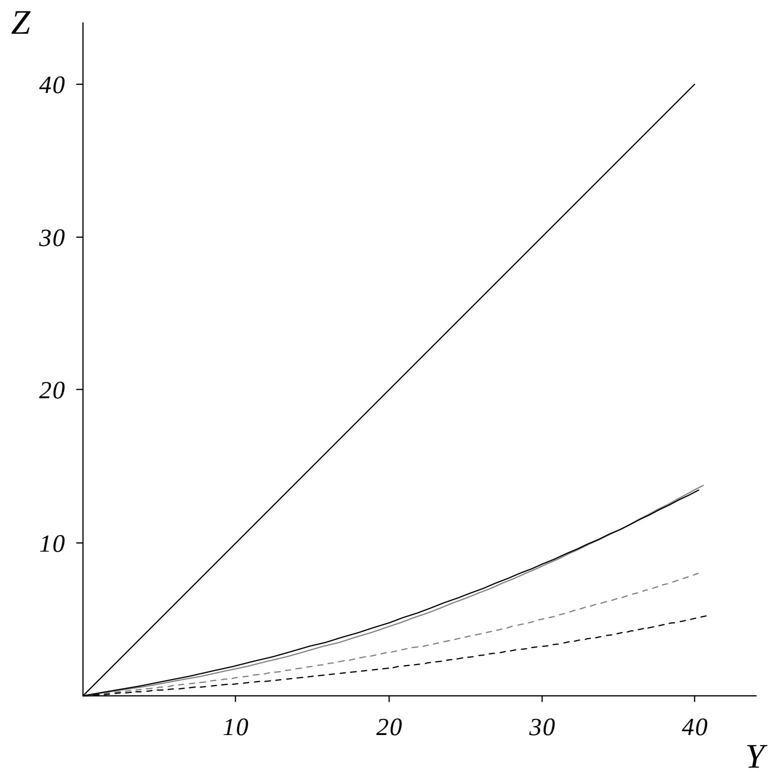

Figure 1: Best treaties. Grey lines – the independent case; black lines – the dependent case; solid lines - treaties for risk 1; dashed lines – treaties for risk 2.

Table 1: **Optimal treaties in the independent case vs the dependent case** (adjustment coefficient criterion)

	Independent N_1, N_2		$\text{Corr}(N_1, N_2) = 0.5$	
	best treaties	best e. loss	best treaties	best e. loss
	$\alpha_1 = 0.487313$ $\alpha_2 = 0.342036$	$M_1 = 8.94428$ $M_2 = 15.8155$	$\alpha_1 = 0.711130$ $\alpha_2 = 0.263398$	$M_1 = 11.7585$ $M_2 = 21.0894$
R	0.311772	0.284421	0.260465	0.238882
$E[\hat{Z}_1]$	0.037230	0.000701	0.042914	0.000416
$E[\hat{Z}_2]$	0.015349	3.176×10^{-6}	0.010168	1.368×10^{-6}
$\frac{E[\hat{Z}_1]}{E[\hat{Y}_1]}$	0.148920	0.002803	0.171655	0.001664
$\frac{E[\hat{Z}_2]}{E[\hat{Y}_2]}$	0.102326	0.000021	0.067786	9.12×10^{-6}
$P_1(Z_1)$	0.079324	0.035215	0.086105	0.030710
$P_2(Z_2)$	0.103890	0.004838	0.069060	0.003648

Similar to what we observed for the single risk aggregate claims case in Guerra and Centeno (2008a) and Guerra and Centeno (2008b), the optimal treaties cede a larger proprotion of risk (measured by $E[Z]$)compared with the best excess of loss treaties (stop-loss in the aggregate case).

In order to compare the risk ceded on each risk, one should have in mind that they have different expected aggregate claims amounts and different expected claim sizes. Therefore, it is more meaningfull to compare amounts ceded normalized by the expected value of the respective risk. Thus, in Table 1 we present the normalized aggregated ceded risks, $E[\hat{Z}_i]/E[\hat{Y}_i]$ and in figures 2 and 4 we present normalized ceded risks $Z_i/E[Y_i]$ as functions of normalized claim size $Y_i/E[Y_i]$.

As expected, the amount of the first risk that should be ceded is larger for risk 1 than for risk 2, which has a comparatively lighter tail thought its aggregate variance is larger. This holds for the dependent and independent case,

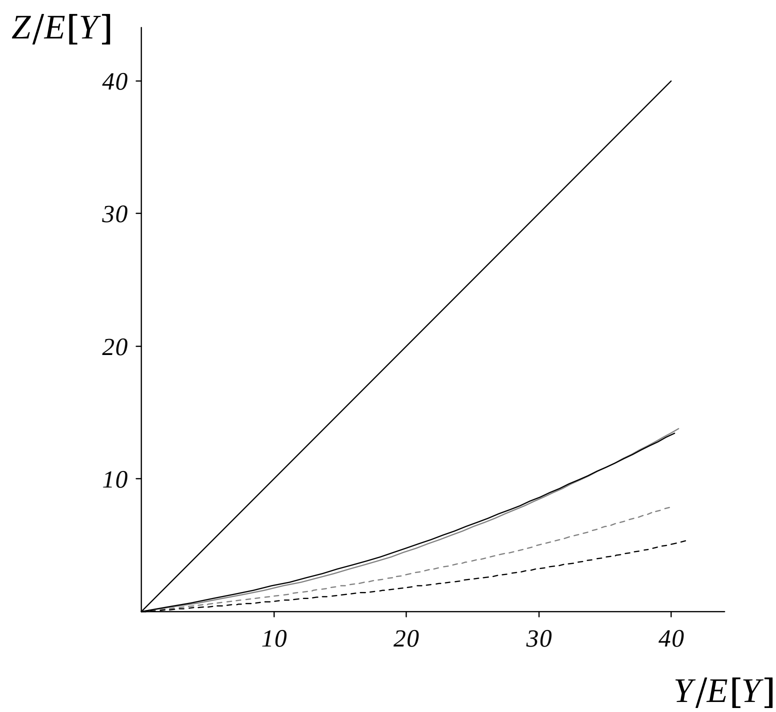

Figure 2: The same as Figure 1, with the values for each treaty scaled by the expected claim size of the respective risk.

for the optimal treaties and for the best excess–of–loss, thought the difference is atenuated when the comparison is made through normalized values.

When we compare the optimal treaties for the independent and the depend cases, the amount of risk 1 ceded increases, compensating a decrease of the amount ceded on risk 2. However, we see that the amount ceded in the tail of the claim size distribution decreases for both risks (see Figures 1,2). This contrasts with the best excess–of–loss, where the amount of risk ceded decreases for both risks when we pass from the independent to the dependent case (see Figures 3,4). Of course, excess–of–loss treaties don't provide for any modulation of retention in tails.

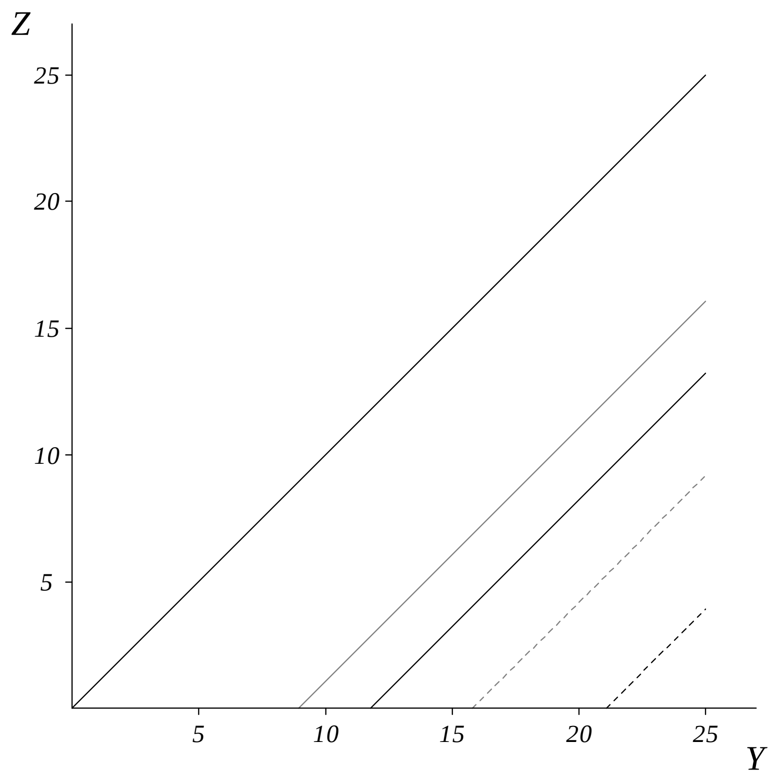

Figure 3: Best expect of loss treaties. Grey lines – the independent case; black lines – the dependent case; solid lines - treaties for risk 1; dashed lines – treaties for risk 2.

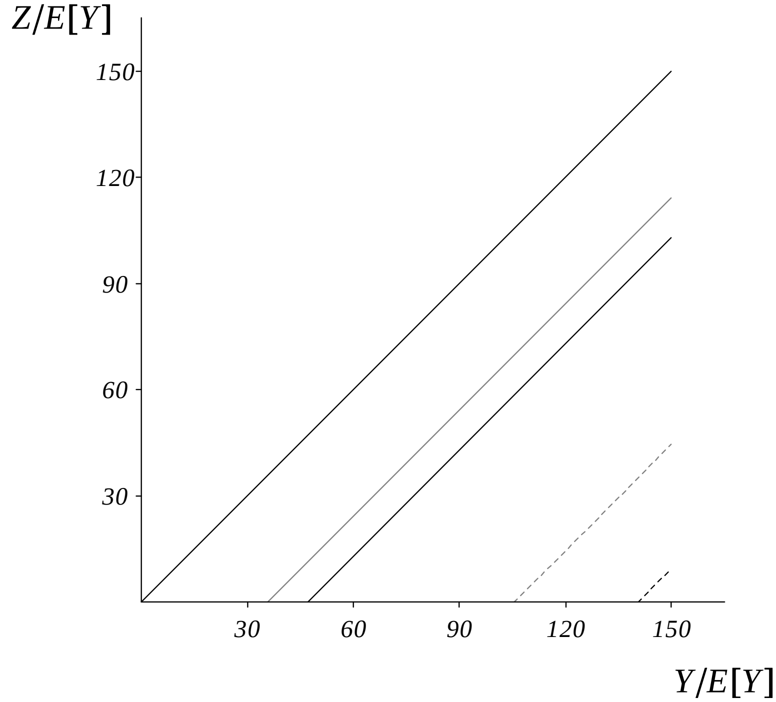

Figure 4: The same as Figure 3, with the values for each treaty scaled by the expected claim size of the respective risk.

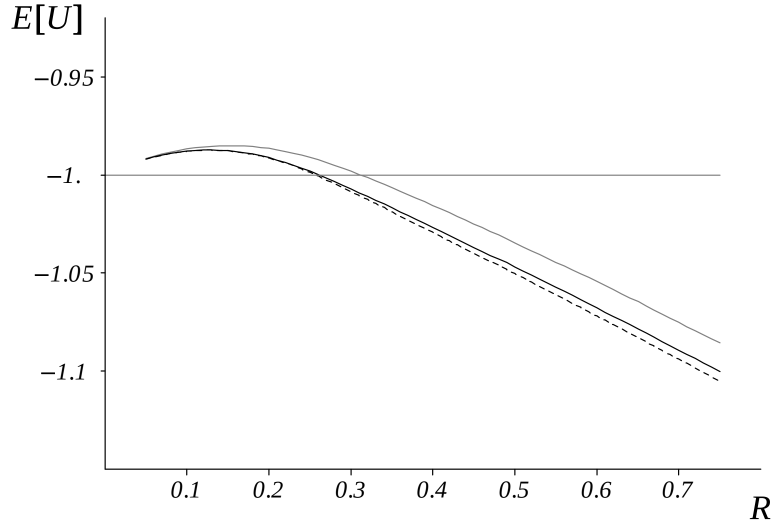

Figure 5: dashed line – Performance of best treaties for independent risks when the risks are dependent.

Now, suppose the direct insurer choses the treaties assuming that the risks are independent but that in fact they are not. One interesting question is what are the consequences of this mistake. To see this, let Z_1^i, Z_2^i be the optimal treaties assuming the independent case and Z_1^d, Z_2^d the optimal treaties in the dependent case. We wish to compute the adjustment coefficient of the retained risk under treaties Z_1^i, Z_2^i when the risks are dependent and compare it to the maximal adjustment coefficient (i.e., the adjustment coefficient of the retained risk under Z_1^d, Z_2^d). It turns out that in the present case the answer is $R_{Z_1^i, Z_2^i} = 0.258863$. This suggests that the most serious consequence of neglecting dependencies (at least of the type considered in this paper) is a substancial underestimated of the risk being undertaken, while the treaties chosen in the assumption of independence do not perform much worse than the best treaty one can chose taking dependence into account. This seems to be a quite robust feature at least for this type of model, persisting under quite different choices of parameters and distributions of claim sizes. However, in the absence of theoretical results caution is fundamental. One possibility is that this phenomenon might be linked to the particular structure of the Gamma-Poisson structure of dependency. We leave this as an open problem.

Finaly, we consider the expected utility for the optimal treaties as a function of the coefficient of risk aversion. The results are shown in Figure 5. Again we see that dependency decreases the expected utility for the best treaties but the difference between $E\left[U_R(L_{Z_1^i,Z_2^i})\right]$ and $E\left[U_R(L_{Z_1^d,Z_2^d})\right]$ (both expected utilities computed in the dependent case) is substantially smaller than the diference with $E\left[U_R(L_{Z_1^i,Z_2^i})\right]$ computed in the independent case, specially for coefficients of risk aversion similar or smaller that the maximal adjustment coefficient.

Appendix: proof of equality (18)

Without loss of generality, we prove the equality for $i = 1$.

For any measurable set $A \subset [0, +\infty)$, we have:

$$\begin{aligned}
&E\left[e^{R\sum_{j=1}^{k}(\hat{Y}_j-\hat{Z}_j)}\chi_{\{\exists m\in\{1,2,\dots N_1\}:Y_{1,m}\in A\}}\right] = \\
=&\sum_{\mathbf{n}\in\mathbb{N}_0^k} E\left[e^{R\sum_{j=1}^{k}(\hat{Y}_j-\hat{Z}_j)}\chi_{\{\exists m\in\{1,2,\dots N_1\}:Y_{1,m}\in A\}}\middle|\, \mathbf{N}=\mathbf{n}\right]p(\mathbf{n}) = \\
=&\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_k=0}^{\infty} E\left[e^{R(\hat{Y}_1-\hat{Z}_1)}\chi_{\{\exists m\in\{1,2,\dots N_1\}:Y_{1,m}\in A\}}\middle|\, N_1=n_1\right]\times \\
&\qquad\times\prod_{j=2}^{k}E[e^{R(Y_j-Z_j)}]^{n_j}p(\mathbf{n}). \qquad (25)
\end{aligned}$$

Now,

$$\begin{aligned}
&E\left[e^{R(\hat{Y}_1-\hat{Z}_1)}\chi_{\{\exists m\in\{1,2,\dots N_1\}:Y_{1,m}\in A\}}\middle|\, N_1=n_1\right] = \\
=&E\left[e^{R(\hat{Y}_1-\hat{Z}_1)}\middle|\, N_1=n_1\right]-E\left[e^{R(\hat{Y}_1-\hat{Z}_1}\chi_{\{\forall m,Y_{1,m}\notin A\}}\middle|\, N_1=n_1\right] = \\
=&\left(\int_{A\cup A^c}e^{R(y-Z_1(y))}dF_1(y)\right)^{n_1}-\left(\int_{A^c}e^{R(y-Z_1(y))}dF_1(y)\right)^{n_1}.
\end{aligned}$$

Thus, the mean value theorem states that there is $\theta_{n_1} \in [0,1]$ such that

$$\begin{aligned} &E\left[e^{R(\hat{Y}_1 - \hat{Z}_1)} \chi_{\{\exists m \in \{1,2,\dots N_1\} : Y_{1,m} \in A\}} \middle| N_1 = n_1 \right] = \\ =&n_1 \left(\int_{A^c} e^{R(y - Z_1(y))} dF_1(y) + \theta_{n_1} \int_A e^{R(y - Z_1(y))} dF_1(y) \right)^{n_1 - 1} \times \\ &\times \int_A e^{R(y - Z_1(y))} dF_1(y) = \\ =&n_1 \left(E[e^{R(Y_1 - Z_1)}] + O(\Pr\{Y_1 \in A\}) \right)^{n_1 - 1} \int_A e^{R(y - Z_1(y))} dF_1(y), \end{aligned}$$

where the error term $O(\Pr\{Y_1 \in A\})$ is always negative. Substituting in (25), one obtains

$$\begin{aligned} &E\left[e^{R \sum_{j=1}^k (\hat{Y}_j - \hat{Z}_j)} \chi_{\{\exists m \in \{1,2,\dots N_1\} : Y_{1,m} \in A\}} \right] = \\ =&\sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} n_1 \left(E\left[e^{R(Y_1 - Z_1)} \right] + \varepsilon_{n_1} \right)^{n_1 - 1} \prod_{j=2}^k E[e^{R(Y_j - Z_j)}]^{n_j} p(\mathbf{n}) \times \\ &\times \int_A e^{R(y - Z_1(y))} dF_1(y), \end{aligned}$$

with $\sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} n_1 \left(E\left[e^{R(Y_1 - Z_1)} \right] + \varepsilon_{n_1} \right)^{n_1 - 1} \prod_{j=2}^k E[e^{R(Y_j - Z_j)}]^{n_j} p(\mathbf{n}) \to \frac{\partial \pi}{\partial x_1} \left(E\left[e^{R(Y_1 - Z_1)} \right], E\left[e^{R(Y_2 - Z_2)} \right], \dots, E\left[e^{R(Y_k - Z_k)} \right] \right)$ when $\Pr\{Y_1 \in A\} \to 0$.

A similar argument shows that

$$\begin{aligned} &\Pr\left\{ \exists m \in \{1,2,\dots,N_1\} : Y_{1,m} \in A \right\} = \\ =&\sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} n_1 \left(1 + \eta_{n_1} \right)^{n_1 - 1} p(\mathbf{n}) \Pr\{Y_1 \in A\}, \end{aligned}$$

and $\sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} n_1 \left(1 + \eta_{n_1} \right)^{n_1 - 1} p(\mathbf{n}) \to E[N_1]$ when $\Pr\{Y_1 \in A\} \to 0$. Fix $v \in [0, +\infty)$, a Lebesgue point of the functions $y \mapsto f_1(y)$, $y \mapsto e^{R(y - Z_1(y))} f_1(y)$ such that $f_1(v) > 0$, and let $A = [v - \varepsilon, v + \varepsilon]$. Then,

$$\begin{aligned} &\int_A e^{R(y - Z_1(y))} dF_1(y) = 2\varepsilon e^{R(v - Z_1(v))} f_1(v) + o(\varepsilon), \\ &\Pr\{Y_1 \in A\} = 2\varepsilon f_1(v) + o(\varepsilon), \end{aligned}$$

and equality (18) follows imediately.

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