Are quantile risk measures suitable for risk-transfer decisions?*

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Abstract: Although controversial from the theoretical point of view, quantile risk measures are widely used by institutions and regulators.

In this paper we show that the use of measures like Value at Risk or Conditional Tail Expectation as optimization criteria for insurance or reinsurance leads to treaties that are not enforceable and/or are clearly bad for the cedent. We argue that this is one further argument against the use of quantile risk measures, at least for the purpose of risk-transfer decisions.

Key words: Coherent risk measures, Conditional Tail Expectation, Risk, Risk measures, Optimal reinsurance, Quantile risk measures, Truncated stop-loss, Value at Risk.

1 Introduction

Risk measures based on quantiles became popular since 1988, when U.S. commercial banks started to determine their regulatory capital requirements for financial market risk exposure using Value at Risk (VaR) models. Value at Risk became widely used with the Basel II accord, which came into force in 2006. Although controversial, the same risk measure was adopted by the European Union within the solvency assessment of insurance companies for the calibration of the Solvency Capital Requirement, in the Solvency II accord, which is scheduled to come into effect in 2012.

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The controversy around the VaR is based on several of its shortcomings: *(i)* It is only concerned about the frequency of default, but not with the size of default, thus it encourages agents to take excessive risk under a false sense of security; *(ii)* VaR estimates are unstable when the losses are not "normally distributed", which is of upmost importance in insurance, where fat tails are very common and mass points may occur; *(iii)* VaR lacks theoretically appealing properties, namely, it fails to be coherent in the sense of Artzner et al. (1999).

This paper provides further evidence against VaR following another line of argument: In our opinion, if decisions based on a risk measure are sound, then the use of that risk measure as an optimization criterion should lead to sound strategies. We argue that VaR clearly fails this test.

We will use the following definition for the VaR.

Definition 1 *For any real random variable X and any constant $\alpha \in (0,1)$, the Value at Risk at probability α is*

$$\mathrm{VaR}_X(\alpha) = \min\{v : \Pr\{X > v\} \leq \alpha\}. \ \square \tag{1}$$

Trying to avoid the shortcomings of VaR, several alternative measures based on quantiles have been proposed in the literature. Rockafellar & Uryasev (2002) derive fundamental properties for the Conditional Tail Expectation (CTE) – in their article designated Conditional Value at Risk (CVaR) – for distributions that can involve discreetness and show that it is a coherent risk measure. We use their definition.

Definition 2 *For any real random variable X and any constant $\alpha \in (0,1)$, the Conditional Tail Expectation at probability α ($\mathrm{CTE}_X(\alpha)$) is the mean of the α tail distribution of X, where the distribution in question is the one with distribution function*

$$F(\zeta) = \begin{cases} 0 & \textit{for } \zeta < \mathrm{VaR}_X(\alpha) \\ \frac{\alpha - \Pr\{X>\zeta\}}{\alpha} & \textit{for } \zeta \geq \mathrm{VaR}_X(\alpha). \ \square \end{cases} \tag{2}$$

The subtlety of the definition resides in the case where the distribution function of X has an atom at $\mathrm{VaR}_X(\alpha)$.

There are two common variants of the Conditional Tail Expectation that appear in the literature, here designated by Upper and Lower Conditional Tail Expectation (called Upper and Lower Conditional Value at Risk in Rockafellar & Uryasev (2002)).

Definition 3 *For any real random variable X and any constant $\alpha \in (0,1)$, the Lower Conditional Tail Expectation at probability α is*

$$\text{CTE}^-_X(\alpha) = E\left[X | X \geq \text{VaR}_X(\alpha)\right] \tag{3}$$

and the Upper Conditional Tail Expectation at probability α is

$$\text{CTE}^+_X(\alpha) = E\left[X | X > \text{VaR}_X(\alpha)\right]. \ \square \tag{4}$$

Both CTE^- and CTE^+ fail to be coherent. Also, $\text{CTE}^+_X(\alpha)$ is not well defined for every random variable because $\Pr\{X > \text{VaR}_X(\alpha)\}$ may be zero. One important example is the case when an insurer detains a risk X and buys stop-loss reinsurance $(X - M)^+$ with $M \leq \text{VaR}_X(\alpha)$. Then, he retains a risk $\min\{Y, M\}$, with Value at Risk equal to M and $\text{CTE}^+_{\min\{Y,M\}}(\alpha)$ not well defined. For this reason, the measure CTE^+ is not considered in this paper.

Notice that $\text{CTE}^-_X(\alpha) \leq \text{CTE}_X(\alpha)$ holds for any random variable, with strict inequality being possible if X has a probability atom at $\text{VaR}_X(\alpha)$. CTE and CTE^- are related to the Expected Shortfall at probability α,

$$\text{ES}_X(\alpha) = E[(X - \text{VaR}_X(\alpha))^+], \tag{5}$$

by the equalities

$$\text{CTE}_X(\alpha) = \text{VaR}_X(\alpha) + \frac{1}{\alpha}\text{ES}_X(\alpha), \tag{6}$$

$$\text{CTE}^-_X(\alpha) = \text{VaR}_X(\alpha) + \frac{1}{\Pr\{X \geq \text{VaR}_X(\alpha)\}}\text{ES}_X(\alpha). \tag{7}$$

So, we see that the difference between CTE and CTE^- is the greater weight of default sizes in the computation of CTE in the case $\Pr\{X \geq \text{VaR}_X(\alpha)\} > \alpha$.

Under Definition 2, for nonnegative random variables, $\text{CTE}_X(\alpha)$ turns out to be equal to what is usually named $\text{TVaR}_X(\alpha)$ and defined by

$$\text{TVaR}_X(\alpha) = \frac{1}{\alpha}\int_0^\alpha \text{VaR}_X(s)ds, \tag{8}$$

even in the case where X has mass points.

In the real world, insurance arrangements are important means of risk transfer and can take many forms. To simplify, we define insurance as any contract by which the owner of the risk (the cedent) pays a premium which is an amount fixed at the beginning of a given time period and, in exchange for this payment, the counterpart agrees to support part of the total loss generated by the risk during the period, according to some formula specified in the contract. This definition includes both insurance and reinsurance. For convenience, the results in this paper are formulated in the language of reinsurance, but the reader should have in mind that they apply exactly to all forms of insurance as defined above. We believe that, from a qualitative point of view and with necessary adaptations, most of them also apply to more general risk-transfer arrangements.

The popularity of quantile risk measures justifies the significant number of research papers using these criteria for the calculation of optimal reinsurance treaties. Cai and Tan (2007) calculate optimal retention limits for stop loss contracts under the VaR and CTE^- risk measures. Cai et al. (2008) using also the VaR and the CTE^- criteria derive the optimal ceded loss arrangements in a class of increasing convex ceded loss functions, and prove that depending on α and on the safety loading for the reinsurance premium, which is assumed to be calculated according to the expected value principle, the optimal reinsurance can be in the forms of stop loss, quota share or change loss. Tan et al. (2010) show that stop loss minimizes the Conditional Tail Expectation (CTE) of the retained risk, when the reinsurance premium is calculated according to the expected value principle. Bernard and Tian (2009) investigates optimal reinsurance contracts under different tail risk measures subject to regulatory constraints.

In this paper we find that the constraints assumed by Cai et al. (2008) are binding. We solve both problems (VaR and the CTE^-) lifting the constraints on the convexity and monotonicity of the ceded loss functions, and obtain discontinuous functions as optimal solutions. This holds not just when the reinsurance premium is computed by the expected value principle but also when it is either a continuous function of the m first moments or it a risk-adjusted premium calculation principle as defined by Wang (1996). In particular when the optimization criterion is the VaR we are led to the truncated stop loss, which has already appeared in the actuarial literature when the criterion chosen is the survival probability in one period of time, namely in Gajek and Zagrodny (2004), Kaluszka (2005) and Kaluszka and Okolewski (2008).

It is our opinion that discontinuous reinsurance arrangements are not acceptable. Any claim value on the neighborhood of a discontinuity point of the reinsurance arrangement would lead to a conflict between insurer and reinsurer.

Further, the optimal treaties provide effective protection against small or moderate losses (which could possibly be well provided by adequate reserves) and little protection, if any, against large or very large losses where reinsurance is crucial.

The insurer is expected to make a profit in the event of small claims. Hence by ceding the small claims and keeping the large ones, the insurer is in fact renouncing to the source of profits without gaining any protection against catastrophic events. Indeed, in many situations the optimal treaty exchanges a profitable situation against one where the insurer is certain to loose money and still have the same unbounded tail on the amount of the losses.

The reader may argue that at least part of these shortcomings could be eliminated by including constraints to the admissible functions, like monotonicity and convexity. Our opinion is that the optimization criterion should be worth by itself. If without further constraints it provides optimal solutions that have clear and systematic shortcomings, then it should be seen at least with reserve.

If instead of VaR we consider the coherent CTE, then the optimal treaties look more acceptable. At least for "good" premium principles the optimal solution is now continuous and monotonically increasing, being generically a stop-loss with a ceiling. This type of contract is common in the industry, mainly because stop-loss reinsurance is too expensive to be acceptable in most practical cases. However, we give strong evidence that CTE provides solutions with reinsurance excessively concentrated on small/moderate claim sizes. Optimal treaties by which the insurer is certain to loose money and retains an unbounded tail of the losses are still possible.

The results for CTE^- are intermediate between VaR and CTE: some more reinsurance of large claims is provided than in the VaR case, but the optimal treaties are again discontinuous and non-monotonic. This is not surprising, considering that CTE^- takes the size of defaults into account but with a smaller weight than CTE.

Quantile risk measures are only concerned with a single period, take into account only the claims, ignoring important parameters of the overall situation of the insurer, like reserves and premiums revenue, that presumably should have some bearing in the choice of the reinsurance strategy. So, maybe other criteria that take into account the long-term fitness of the insurer are more adequate to select good reinsurance policies.

This paper is organized as follows: In Section 2 we formalize the optimal reinsurance problem and state the general assumptions used in the paper. In Section 3, we introduce random treaties, a mathematical tool that allows a unified analysis of the optimal reinsurance problem under the different criteria used in this paper. Full proofs of the main properties of random treaties are provided in Appendix. Sections 4, 5 and 6 deal with the VaR, CTE and CTE^{-} criteria, respectively. Implications of these results concerning suitability of the risk measures are discussed in Section 7.

2 Problem setting

We consider the case when the direct insurer is concerned with a single risk. This can be a single policy or a portfolio of policies. The aggregate value of claims on this risk during a given interval of time is a non-negative random variable $Y : \Omega \mapsto [0, +\infty)$, defined in a probability space $(\Omega, \mathcal{F}, \mu)$. A reinsurance policy is a function $Z : [0, +\infty) \mapsto [0, +\infty)$, mapping each possible aggregate value of the claims into the corresponding value refunded under the reinsurance contract. The set of all possible reinsurance policies is:

$$\mathcal{Z} = \{Z : [0, +\infty) \mapsto \mathbb{R} | Z \text{ is measurable and } 0 \leq Z(y) \leq y, \forall y \geq 0\}.$$

We do not distinguish between functions which differ only on a set of zero probability. i.e., two measurable functions, ϕ and ϕ' are considered to be the same whenever $\Pr\{\phi(Y) = \phi'(Y)\} = 1$. Similarly, a measurable function Z is an element of $\mathcal{Z}$ whenever $\Pr\{0 \leq Z(Y) \leq Y\} = 1$.

The premium charged for each admissible reinsurance policy is computed by a real functional $P : \mathcal{Z} \mapsto [0, +\infty]$. Thus, if the direct insurer buys a particular reinsurance policy $Z \in \mathcal{Z}$, the retained risk (net of premium) is

$$R_Z = Y - Z + P(Z).$$

We assume that the insurer rates risks using a given risk measure ρ and has full knowledge of the functional P. Thus, he wishes to select a reinsurance policy that solves the following problem:

Problem 1 *Find $\hat{Z} \in \mathcal{Z}$ such that*

$$\rho(R_{\hat{Z}}) = \min \{\rho(R_Z) : Z \in \mathcal{Z}\}. \square$$

In this paper we consider that ρ is VaR, CTE or CTE^{-}. All three risk measures are cash-invariant, which implies

$$\rho(R_Z) = P(Z) + \rho(Y - Z), \qquad \forall Z \in \mathcal{Z}. \tag{9}$$

As already mentioned, $\mathrm{CTE}_X(\alpha)$ is coherent in the sense of Artzner et al. (1999) while $X \mapsto \mathrm{VaR}_X(\alpha)$ and $X \mapsto \mathrm{CTE}_X^-(\alpha)$ are nonconvex functionals and hence are incoherent.

All the results presented in this paper are obtained under the following blanket assumptions:

Assumption 1 *The random variable Y is integrable, i.e., $E[Y] < +\infty$.* □

This assumption is sufficient to guarantee existence of relaxed solutions (see Section 3 below). Notice that nonintegrable risks are generally considered to be non-insurable.

Assumption 2 *The random variable Y is a mixture of a continuous random variable with the degenerate random variable $Y_0 \equiv 0$.* □

This assumption is included only for convenience. A brief discussion of the consequences of its relaxation is included as a footnote to the proof of Proposition 2.

Concerning the premium calculation principle, we take one of the following alternative assumptions:

Assumption 3 *The premium calculation principle is a functional*

$$P(Z) = \gamma\left(E[Z], E[Z^2], \ldots, E[Z^m]\right), \qquad \forall Z \in \mathcal{Z}, \tag{10}$$

where $\gamma : [0, +\infty]^m \mapsto [0, +\infty]$ is a continuous function, monotonically increasing with respect to x_m (the m-th coordinate of its argument), such that $\lim_{x_m \to +\infty} \gamma(x_1, x_2, \ldots, x_m) = +\infty$ uniformly with respect to $(x_1, x_2, \ldots, x_{m-1})$. □

Assumption 4 *The premium calculation principle is a functional*

$$P(Z) = \int_0^{+\infty} w(\Pr\{Z > t\})dt, \qquad \forall Z \in \mathcal{Z}, \tag{11}$$

where $w : [0, 1] \mapsto [0, 1]$ is continuous, concave, monotonic increasing function with $w(0) = 0$, $w(1) = 1$. □

Examples of premium calculation principles of the type (10) include the expected value principle ($m = 1$, $\gamma(x) = (1 + \beta)x$), the variance principle ($m = 2$, $\gamma(x_1, x_2) = x_1 + \beta(x_2 - x_1^2)^+$), and the standard deviation principle ($\gamma(x_1, x_2) = x_1 + \beta\sqrt{(x_2 - x_1^2)^+}$).

Since the premium is the minimal monetary compensation the reinsurer will take to accept a given risk, it is a risk measure in the sense of Artzner et al. (1999). If we accept that risk measures in general should be coherent, then premium calculation principles should be coherent. Functionals of type (10) include both coherent and incoherent principles, while all principles of type (11) are coherent (see Kusuoka (2001)).

3 Random treaties

The risk measures VaR and CTE^- and many of the premium principles (10) fail to be convex. Optimization of nonconvex functionals poses special difficulties. The main one being that existence of any solution is difficult to guarantee. This is inconvenient not only because any attempt to solve a problem without solution is obviously pointless, but also because approaches based on necessary optimality conditions are much simplified when the solution is a priori known to exist. We overcome this difficulty by considering random treaties.

A random treaty is a contract that defines the amount to be refunded as a function of the claim amount and some other random variable. Thus, the value refunded under a random treaty given a particular claim amount $Y = y$ is a random variable $Z(y, X)$ rather than a constant value given by the function $Z(y)$. One simple example is a contract stating that the insurer will be refunded for all his losses if the winning number in the lottery is even and will receive nothing if it is odd. On average the direct insurer expects to be refunded for half his losses, but this contract is obviously not the same as the quota-share $Z(y) = \frac{1}{2}y$. Such contracts are enforceable in practice as long as the auxiliary random variable is public and cannot be tampered by any of the contracting parts (as it is presumably the case in the example above). The results in Sections 4, 5 and 6 show that, under Assumptions 1, 2, a non-random optimal treaty always exists. Hence the reader who is not comfortable with the idea of random treaties can view them as mere mathematical tool that allows us to deal with the problems outlined in the previous section in a unified approach.

The space of random treaties is defined as follows

Definition 4 *Any Borel probability measure η over $\mathbb{R}^2$ satisfying*

(i) $\eta\{(y, z) : 0 \le z \le y\} = 1$*;*

(ii) $\eta(A \times \mathbb{R}) = \Pr\{Y \in A\}$ *for every Borel set* $A \subset [0, +\infty)$*,*

is called a random treaty. The space of all random treaties is denoted by $\mathcal{M}$. $\square$

The space of (deterministic) treaties, $\mathcal{Z}$, can be embedded in $\mathcal{M}$ by identifying each function $Z \in \mathcal{Z}$ with the corresponding measure

$$\eta_Z(A) = \int_{\mathbb{R}^2} \chi_A(y, z)\delta_{Z(y)}(dz)\mu(dy), \qquad A \in \mathcal{B}_{\mathbb{R}^2}, \tag{12}$$

where δ_a denotes the Dirac measure concentrated at the point a and

$$\chi_A(y,z) = \begin{cases} 1, & \text{if } (y,z) \in A, \\ 0, & \text{if } (y,z) \notin A \end{cases}$$

is the characteristic function of the set A. A functional $\hat{\Phi} : \mathcal{M} \mapsto [-\infty, +\infty]$ is said to be an extension of a functional $\Phi : \mathcal{Z} \mapsto [-\infty, +\infty]$ into $\mathcal{M}$ if

$$\hat{\Phi}(\eta_Z) = \Phi(Z), \qquad \forall Z \in \mathcal{Z}.$$

Using (12), it is easy to check that the following are "natural" extensions for the functionals $Z \mapsto \mathrm{VaR}_{Y-Z}(\alpha)$, $Z \mapsto \mathrm{CTE}_{Y-Z}(\alpha)$, $Z \mapsto \mathrm{CTE}^-_{Y-Z}(\alpha)$, (10) and (11), respectively:

$$\hat{\mathrm{VaR}}_\eta(\alpha) = \inf\{v : \eta\{(y,z) : y - z > v\} \le \alpha\}; \tag{13}$$

$$\hat{\mathrm{CTE}}_\eta(\alpha) = \frac{\alpha - \eta\{(y,z) : y - z > \mathrm{VaR}_\eta(\alpha)\}}{\alpha}\mathrm{VaR}_\eta(\alpha) + \\ + \frac{1}{\alpha}\int_{\{(y,z): y-z > \mathrm{VaR}_\eta(\alpha)\}} (y-z)\,d\eta; \tag{14}$$

$$\hat{\mathrm{CTE}}^-_\eta(\alpha) = \frac{\int_{\{(y,z): y-z \ge \mathrm{VaR}_\eta(\alpha)\}} (y-z)\,d\eta}{\eta\{(y,z) : y - z \ge \mathrm{VaR}_\eta(\alpha)\}}; \tag{15}$$

$$\hat{P}(\eta) = \gamma\left(\int_{R^2} (z, z^2, \ldots, z^m)\,d\eta\right); \tag{16}$$

$$\hat{P}(\eta) = \int_0^{+\infty} w(\eta\{(y,z) : z > t\})\,dt. \tag{17}$$

So, we introduce the relaxed problem:

Problem 2 *Find $\hat{\eta} \in \mathcal{M}$ such that*

$$\hat{P}(\hat{\eta}) + \hat{\rho}(\hat{\eta}) = \min\{\hat{P}(\eta) + \hat{\rho}(\eta) : \eta \in \mathcal{M}\},$$

where $\hat{\rho}$ is (13)*,* (14) *or* (15) *and $\hat{P}$ is* (16) *or* (17)*.* □

In the remaining of this paper $\hat{\rho}(\eta)$ always denotes one of the extensions (13), (14) or (15). $\hat{P}(\eta)$ denotes (16) or (17).

The relaxation of Problem 1 into Problem 2 is similar to Kantorovich relaxation in the Monge optimal mass transfer problem (see e.g. Evans (1999)). It is partly justified by the following Proposition:

Proposition 1 *Problem 2 admits a solution in $\mathcal{M}$.* □

Proof. A full proof is provided in the Appendix. Here we just state the key arguments:

We provide the space of random treaties with the topology of weak convergence over the space C_c, of all continuous functions $g : \mathbb{R}^2 \mapsto \mathbb{R}$ with compact support. This means that a sequence $\{\eta_n \in \mathcal{M}\}_{n\in\mathbb{N}}$ is said to converge to $\eta \in \mathcal{M}$ if and only if

$$\lim \int_{\mathbb{R}^2} g d\eta_n = \int_{\mathbb{R}^2} g d\eta$$

holds for every $g \in C_c$. This makes $\mathcal{M}$ into a compact topological space.

All the functionals (13)–(17) are lower semicontinuous with respect to the above topology.

Hence the result follows by the Weierstrass' Theorem. ■

Our approach consists in finding necessary optimality for the relaxed problem and then prove that for each candidate optimal solution there is a nonrandom treaty that has the same rating.

4 The VaR measure

In this section, we solve Problem 1 when the insurer's risk measure is the VaR.

4.1 Premiums depending on moments

Suppose the premium calculation principle is of type (10). For each $\eta \in \mathcal{M}$, we consider the polynomial function $\zeta \mapsto Q_\eta(\zeta)$ defined as

$$Q_\eta(\zeta) = \sum_{i=1}^{m} \frac{\partial \gamma}{\partial x_i} \zeta^i, \qquad \zeta \in \mathbb{R}, \tag{18}$$

where $\frac{\partial \gamma}{\partial x_i}$ denotes the partial derivative of γ with respect to its i-th argument, evaluated at the point $x = \int_{\mathbb{R}^2} \left(z, z^2, \dots z^m\right) d\eta$. To simplify, we denote by Q_Z the polynomial Q_{η_Z}, when $Z \in \mathcal{Z}$ is a nonrandom treaty.

Proposition 2 *(under Assumptions 1, 2 and 3):*

Let $\hat{\eta}$ solve Problem 2. If γ is differentiable at $\hat{x} = \int_{\mathbb{R}^2} \left(z, z^2, \dots z^m\right) d\hat{\eta}$ and $\nabla\gamma(\hat{x}) \neq 0$, then Problem 1 admits an optimal treaty $\hat{Z} \in \mathcal{Z}$ such that, for each claim size $y > 0$ we have:

$$\hat{Z}(y) = \arg \min_{\zeta \in [0,y]} Q_{\hat{Z}}(\zeta) \quad \text{or} \quad \hat{Z}(y) = \arg \min_{\zeta \in [(y-v)^+, y]} Q_{\hat{Z}}(\zeta),$$

with $v = \mathrm{VaR}_{Y-\hat{Z}}(\alpha)$*. Further, there is a constant* $C \in [0, +\infty)$ *such that*

$$\hat{Z}(y) = \arg\min_{\zeta \in [(y-v)^+, y]} Q_{\hat{Z}}(\zeta) \qquad \text{if} \min_{\zeta \in [(y-v)^+, y]} Q_{\hat{Z}}(\zeta) - \min_{\zeta \in [0,y]} Q_{\hat{Z}}(\zeta) < C;$$

$$\hat{Z}(y) = \arg\min_{\zeta \in [0,y]} Q_{\hat{Z}} \qquad \text{if} \min_{\zeta \in [(y-v)^+, y]} Q_{\hat{Z}}(\zeta) - \min_{\zeta \in [0,y]} Q_{\hat{Z}}(\zeta) > C. \ \square$$

Proof. Fix $\hat{\eta} \in \mathcal{M}$, a solution of Problem 2 satisfying the assumptions above. Let $v = \hat{\mathrm{VaR}}_{\hat{\eta}}(\alpha)$, and pick $(y_0, z_0) \in \mathrm{Supp}(\hat{\eta})$.

To start, assume that $z_0 \in [0, y_0 - v)$. Fix $t \in (-z_0, y_0 - z_0)$, and consider the measure

$$\begin{aligned} \eta_\varepsilon(A) =& \hat{\eta}(A) - \hat{\eta}(A \cap B_\varepsilon(y_0, z_0)) + \\ & + \hat{\eta}\left\{(y, z) \in B_\varepsilon(y_0, z_0) : (y, z + t) \in A\right\}, \qquad A \in \mathcal{B}_{\mathbb{R}^2}, \end{aligned}$$

where $B_\varepsilon(y_0, z_0)$ is the open ball of radius ε and center at (y_0, z_0). For sufficiently small $\varepsilon > 0$, $\eta_\varepsilon \in \mathcal{M}$ and $\hat{\mathrm{VaR}}_{\eta_\varepsilon}(\alpha) \le \hat{\mathrm{VaR}}_{\hat{\eta}}(\alpha)$. Therefore, optimality of $\hat{\eta}$ implies

$$\hat{P}(\eta_\varepsilon) \ge \hat{P}(\hat{\eta}). \tag{19}$$

For $i \in \{1, 2, \ldots, m\}$, we have

$$\begin{aligned} \int_{\mathbb{R}^2} z^i d\eta_\varepsilon =& \int_{\mathbb{R}^2} z^i d\hat{\eta} - \int_{B_\varepsilon(y_0, z_0)} z^i d\hat{\eta} + \int_{B_\varepsilon(y_0, z_0)} (z + t)^i d\hat{\eta} = \\ &= \int_{\mathbb{R}^2} z^i d\hat{\eta} + \left((z_0 + t)^i - z_0^i\right) \hat{\eta}\left(B_\varepsilon(y_0, z_0)\right) + o\left(\hat{\eta}\left(B_\varepsilon(y_0, z_0)\right)\right). \end{aligned}$$

Thus, differentiability of γ implies

$$\begin{aligned} \hat{P}(\eta_\varepsilon) =& \hat{P}(\hat{\eta}) + \sum_{i=1}^m \left.\frac{\partial \gamma}{\partial x_i}\right|_{x = \int_{\mathbb{R}^2}(z, z^2, \ldots, z^m) d\hat{\eta}} \times \left((z_0 + t)^i - z_0^i\right) \times \hat{\eta}\left(B_\varepsilon(y_0, z_0)\right) + \\ & + o\left(\hat{\eta}\left(B_\varepsilon(y_0, z_0)\right)\right) = \\ =& \hat{P}(\hat{\eta}) + \left(Q_{\hat{\eta}}(z_0 + t) - Q_{\hat{\eta}}(z_0)\right) \times \hat{\eta}\left(B_\varepsilon(y_0, z_0)\right) + \\ & + o\left(\hat{\eta}\left(B_\varepsilon(y_0, z_0)\right)\right), \end{aligned} \tag{20}$$

and hence, inequality (19) implies

$$Q_{\hat{\eta}}(z_0 + t) \ge Q_{\hat{\eta}}(z_0), \qquad \forall z_0 + t \in (0, y_0).$$

By continuity of $Q_{\hat{\eta}}$, this is

$$z_0 = \arg\min_{z \in [0, y_0]} Q_{\hat{\eta}}(z).$$

If, instead, we assume that $z_0 \in [y_0 - v, y_0]$, we still have $\eta_\varepsilon \in \mathcal{M}$ and $\hat{\mathrm{VaR}}_{\eta_\varepsilon}(\alpha) \leq \hat{\mathrm{VaR}}_{\hat{\eta}}(\alpha)$, provided $t \in (y_0 - v - z_0, y_0 - z_0)$ and $\varepsilon > 0$ is sufficiently small. The same argument as above shows that in this case we have

$$z_0 = \arg \min_{z \in [y_0 - v, y_0]} Q_{\hat{\eta}}(z).$$

Thus, we proved that

$$\mathrm{Supp}(\hat{\eta}) \subset \left\{ (y,z) : y \geq 0, z = \arg \min_{t \in [0,y]} Q_{\hat{\eta}}(t) \text{ or } z = \arg \min_{t \in [y-v,y]} Q_{\hat{\eta}}(t) \right\}. \tag{21}$$

Since $\zeta \mapsto Q_{\hat{\eta}}(\zeta)$ is a (nonconstant) polynomial, it has only finitely many local minima in $[0, +\infty)$. Let $0 \leq c_1 < c_2 < \ldots < c_p$ be the local minimizers of $\zeta \mapsto Q_{\hat{\eta}}(\zeta)$ in $[0, +\infty)$. Then, (21) implies that the support of $\hat{\eta}$ is contained in the set

$$\left(\bigcup_{i=1}^{p} \{(y, c_i) : y \geq c_i\} \right) \cup \{(y,y) : y \geq 0\} \cup \{(y, y-v) : y \geq v\}.$$

Due to Definition 4, this means that the measure $\hat{\eta}$ admits a representation

$$\begin{aligned} \hat{\eta}(A) = & \sum_{i=1}^{p} \int_{c_i}^{+\infty} m_i(y) \chi_A(y, c_i) dF(y) + \\ & + \int_{0}^{+\infty} m_{p+1}(y) \chi_A(y,y) dF(y) + \\ & + \int_{v}^{+\infty} m_{p+2}(y) \chi_A(y, y-v) dF(y), \qquad \forall A \in \mathcal{B}_{\mathbb{R}^2}, \end{aligned} \tag{22}$$

where $m_i,\ i = 1, 2, \ldots, (p+2)$ are measurable nonnegative functions such that

$$\Pr \left\{ \sum_{i=1}^{p+2} m_i(Y) = 1 \right\} = 1,$$

and F is the distribution function of Y.

Fix $(y_1, z_1), (y_2, z_2) \in \mathrm{Supp}(\hat{\eta})$ such that

$$y_1 > v, \quad y_2 > v, \quad z_1 = \arg \min_{z \in [0, y_1]} Q_{\hat{\eta}}(z), \quad z_2 = \arg \min_{z \in [y_2 - v, y_2]} Q_{\hat{\eta}}(z). \tag{23}$$

Fix

$$t_1 \in (y_1 - z_1 - v, y_1 - z_1), \qquad t_2 \in (-z_2, y_2 - z_2), \tag{24}$$

and consider the measure

$$\begin{aligned}\eta_\varepsilon(A) =&\hat\eta(A) - \hat\eta\left(A \cap B_\varepsilon(y_1,z_1)\right) + \\ &+ \hat\eta\left\{(y,z)\in B_\varepsilon(y_1,z_1) : (y, z+t_1)\in A\right\} - \\ &- \frac{\hat\eta\left(B_\varepsilon(y_1,z_1)\right)}{\hat\eta\left(B_\varepsilon(y_2,z_2)\right)}\hat\eta\left(A\cap B_\varepsilon(y_2,z_2)\right) + \\ &+ \frac{\hat\eta\left(B_\varepsilon(y_1,z_1)\right)}{\hat\eta\left(B_\varepsilon(y_2,z_2)\right)}\hat\eta\left\{(y,z)\in B_\varepsilon(y_1,z_1) : (y,z+t_1)\in A\right\}, \quad A\in\mathcal{B}_{\mathbb{R}^2}.\end{aligned}$$

For every sufficiently small $\varepsilon > 0$ we have $\eta_\varepsilon \in \mathcal{M}$ and the argument used to obtain (20) shows that for this new measure we have

$$\begin{aligned}\hat P(\eta_\varepsilon) =&\hat P(\hat\eta) + \left(Q_{\hat\eta}(z_1+t_1) - Q_{\hat\eta}(z_1) + Q_{\hat\eta}(z_2+t_2) - Q_{\hat\eta}(z_2)\right)\hat\eta\left(B_\varepsilon(y_1,z_1)\right) + \\ &+ o\left(\hat\eta\left(B_\varepsilon(y_1,z_1)\right)\right).\end{aligned}$$

Thus, optimality of $\hat\eta$ implies

$$Q_{\hat\eta}(z_1+t_1) - Q_{\hat\eta}(z_1) + Q_{\hat\eta}(z_2+t_2) - Q_{\hat\eta}(z_2) \geq 0.$$

By picking z_1+t_1 close to $\arg\min_{z\in[y_1-v,y_1]} Q_{\hat\eta}(z)$, z_2+t_2 close to $\arg\min_{z\in[0,y_2]} Q_{\hat\eta}(z)$, and using (23), we see that this implies

$$\min_{z\in[y_1-v,y_1]} Q_{\hat\eta}(z) - \min_{z\in[0,y_1]} Q_{\hat\eta}(z) \geq \min_{z\in[y_2-v,y_2]} Q_{\hat\eta}(z) - \min_{z\in[0,y_2]} Q_{\hat\eta}(z)$$

Since this holds for any pair $(y_1,z_1),(y_2,z_2)\in \mathrm{Supp}(\hat\eta)$ satisfying (23), we conclude that there is a constant $C\in[0,+\infty)$ such that

$$Q_{\hat\eta}(z_1) = \min_{z\in[0,y_1]} Q_{\hat\eta}(z) \leq \min_{z\in[y_1-v,y_1]} Q_{\hat\eta}(z) - C$$

for every $(y_1,z_1)\in\mathrm{Supp}(\hat\eta)$ such that $z_1 < y_1 - v$, and

$$Q_{\hat\eta}(z_2) = \min_{z\in[(y_2-v)^+,y_2]} Q_{\hat\eta}(z) \leq \min_{z\in[0,y_2]} Q_{\hat\eta}(z) + C$$

for every $(y_2,z_2)\in\mathrm{Supp}(\hat\eta)$ such that $z_2 \geq y_2 - v$.

Fix C satisfying these conditions. Let $y>0$ be a Lebesgue point of the functions $m_i f$, $i=1,2,\ldots,(p+2)$, where $f=F'$ is the probability density of Y. Using the representation (22) and the results above, we see that

If $\min_{z\in[y-v,y]} Q_{\hat\eta}(z) - \min_{z\in[0,y]} Q_{\hat\eta}(z) > C$, then $m_{p+1}(y) = m_{p+2}(y) = 0$ and $m_i(y)=0$ for every $i\in\{1,2,\ldots,p\}$ such that $c_i \geq y-v$;

If $\min_{z\in[y-v,y]} Q_{\hat{\eta}}(z) - \min_{z\in[0,y]} Q_{\hat{\eta}}(z) < C$, then $m_i(y) = 0$ for every $i \in \{1, 2, \ldots, p\}$ such that $c_i < y - v$.

Suppose that for some $i \in \{1, 2, \ldots, p\}$, we have

$$\Pr\{m_i(Y) > 0 \text{ and } m_{p+1}(Y) > 0\} > 0.$$

The results above show that this implies

$$\Pr\{Q_{\hat{\eta}}(Y) = Q_{\hat{\eta}}(c_i)\} > 0 \quad \text{or} \quad \Pr\{Q_{\hat{\eta}}(Y) = Q_{\hat{\eta}}(c_i) + C\} > 0,$$

which is a contradiction because, since $Q_{\hat{\eta}}$ is a nonconstant polynomial, the sets

$$\{y : Q_{\hat{\eta}}(y) = Q_{\hat{\eta}}(c_i)\}, \qquad \{y : Q_{\hat{\eta}}(y) = Q_{\hat{\eta}}(c_i) + C\}$$

must be finite. Hence we must have

$$\Pr\{m_i(Y) > 0 \text{ and } m_{p+1}(Y) > 0\} = 0, \qquad \text{for } i = 1, 2, \ldots, p.$$

A similar argument shows that

$$\Pr\{m_i(Y) > 0 \text{ and } m_{p+2}(Y) > 0\} = 0, \qquad \text{for } i = 1, 2, \ldots, p;$$
$$\Pr\{m_{p+1}(Y) > 0 \text{ and } m_{p+2}(Y) > 0\} = 0.$$

Now, suppose there are distinct $i, j \in \{1, 2, \ldots, p\}$ such that

$$\Pr\{m_i(Y) > 0 \text{ and } m_j(Y) > 0\} > 0.$$

without loss of generality, we may assume that there is a interval $[a, b]$ such that

$$\Pr\{Y \in [a, b],\ m_i(Y) > 0 \text{ and } m_j(Y) > 0\} > 0;$$
$$c_i + v \notin (a, b), \qquad c_j + v \notin (a, b).$$

There exists $t \in (a, b)$ such that

$$\int_a^t (m_i(y) + m_j(y))\, dF(y) = \int_a^b m_i(y) dF(y).$$

Let $\tilde{m}_i$, $\tilde{m}_j$ denote the functions

$$\tilde{m}_i(y) = \begin{cases} m_i(y) + m_j(y), & \text{for } y \in [a, t], \\ 0, & \text{for } y \in (t, b], \\ m_i(y), & \text{for } y \notin [a, b], \end{cases}$$

$$\tilde{m}_j(y) = \begin{cases} 0, & \text{for } y \in [a, t], \\ m_i(y) + m_j(y), & \text{for } y \in (t, b], \\ m_j(y), & \text{for } y \notin [a, b]. \end{cases}$$

Thus, $\Pr\{Y \in [a,b],\ \tilde{m}_i(Y) > 0 \text{ and } \tilde{m}_j(Y) > 0\} = 0$ and it can be checked that the measure $\tilde{\eta}$ obtained by substituting $\tilde{m}_i$, $\tilde{m}_j$ for m_i, m_j in the representation (22) satisfies

$$\int_{\mathbb{R}^2} (z, z^2, \ldots, z^m) d\tilde{\eta} = \int_{\mathbb{R}^2} (z, z^2, \ldots, z^m) d\hat{\eta}, \qquad \mathrm{VaR}_{\tilde{\eta}}(\alpha) = \mathrm{VaR}_{\hat{\eta}}(\alpha).$$

Thus, we proved that m_i, $i \in \{1, 2, \ldots, p+2\}$ can be chosen such that

$$\Pr\{m_i(Y) > 0 \text{ for more than one } i\} = 0.$$

In that case the function

$$Z(y) = \sum_{i=1}^{p} c_i \chi_{\{m_i=1\}}(y) + y\chi_{\{m_{p+1}=1\}}(y) + (y-v)\chi_{\{m_{p+2}=1\}}(y)$$

satisfies

$$Z \in \mathcal{Z}, \qquad \mathrm{VaR}_{Y-Z}(\alpha) = \hat{\mathrm{VaR}}_{\hat{\eta}}(\alpha), \qquad P(Z) = \hat{P}(\hat{\eta}),$$

and all the conditions stated in the Proposition.[1] ■

Proposition 2 shows that the solution of Problem 1 is in general a nonmonotonic discontinuous function. Figure 1 shows one possible optimal treaty for the case where the premium calculation principle depends in the first four moments of the ceded risk.

For a premium of this type, Q_Z is a forth-degree polynomial with positive principal coefficient. Therefore, Q_Z may have up to two local minima in $[0, +\infty)$, denoted by c_1, c_2. If $c_1 > 0$, $c_2 > c_1 + v$ and $Q_Z(c_2) < Q_Z(c_1)$, we have

$$\arg \min_{\zeta \in [(y-v)^+, y]} Q_Z(\zeta) = \begin{cases} y, & \text{for } y \le c_1, \\ c_1 & \text{for } y \in [c_1, c_1+v], \\ y-v, & \text{for } y \in [c_1+v, y_1), \\ y, & \text{for } y \in (y_1, c_2], \\ c_2 & \text{for } y \in [c_2, c_2+v], \\ y-v & \text{for } y \ge c_2+v. \end{cases}$$

Since Q_Z is strictly increasing in $[c_2, +\infty)$ and $\lim_{\zeta \to +\infty} Q_Z(\zeta) = +\infty$, we see that there is a unique $y_2 > c_2 + v$ such that $Q_Z(y_2 - v) = Q_Z(c_2) + C$. Therefore, the optimal treaty Z should have discontinuities at the points y_1, y_2, as shown in the picture.

[1] Close examination of this proof shows what happens when Assumption 2 is lifted: the solution is similar with the single difference that atoms may have to be "split" at discontinuity points of the optimal treaty. I.e., if y_0 is a point of discontinuity of the optimal treaty Z and $\Pr\{Y = y_0\} > 0$, then $\Pr\{Z \in \{Z(y_0^-), Z(y_0^+)\} | Y = y_0\} = 1$ but we may have $0 < \Pr\{Z = Z(y_0^-) | Y = y_0\} < 1$.

The same observation holds for all solutions in this paper.

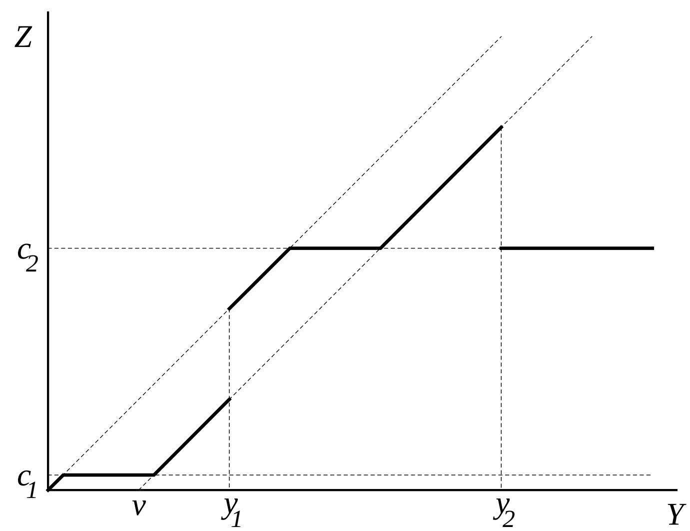

Figure 1: Optimal policy (full line) for a reinsurance premium depending on the first four moments.

Notice that the discontinuity at the point y_2 depends both on the risk measure (VaR) and the premium principle, while the discontinuity at the point y_1 is due to the particular premium calculation principle being considered. Thus, in order to have well behaved solutions, one should look not only for "good" risk measures but also for "good" premium principles.

Below we provide the solutions for some important particular principles.

The expected value principle

Here we present the solution of Problem 1 when the risk measure is VaR and the reinsurance premium is a functional of type

$$P(Z) = (1+\beta)E[Z], \tag{25}$$

where $\beta > 0$ is a constant.

Proposition 3 *If the reinsurance premium is of type* (25)*, then a solution for Problem 1 is:*

$$Z(y) = \begin{cases} y - \mathrm{VaR}_Y(\alpha + \frac{1}{1+\beta}), & \text{if } y \in \left[\mathrm{VaR}_Y(\alpha + \frac{1}{1+\beta}), \mathrm{VaR}_Y(\alpha)\right], \\ 0, & \text{if } y \notin \left[\mathrm{VaR}_Y(\alpha + \frac{1}{1+\beta}), \mathrm{VaR}_Y(\alpha)\right], \end{cases} \tag{26}$$

taking $\mathrm{VaR}_Y(\alpha + \frac{1}{1+\beta}) = 0$ *whenever* $\alpha + \frac{1}{1+\beta} \geq 1$. $\square$

Proof. For a functional of type (25), the expression (18) reduces to $Q_\eta(\zeta) = (1+\beta)\zeta$. Hence,

$$\arg\min_{\zeta\in[0,y]} Q_{Z_v}(\zeta) = 0, \qquad \arg\min_{\zeta\in[(y-v)^+,y]} Q_{Z_v}(\zeta) = (y-v)^+$$

and the map $y \mapsto \left(\min_{\zeta\in[(y-v)^+,y]} Q_{Z_v}(\zeta) - \min_{\zeta\in[0,y]} Q_{Z_v}(\zeta)\right)$ is obviously strictly monotonic increasing in $[v, +\infty)$. Therefore, Proposition 2 states that the optimal treaty is of type

$$Z(y) = \begin{cases} y - v, & y \in [v, \mathrm{VaR}_Y(\alpha)]; \\ 0, & y \notin [v, \mathrm{VaR}_Y(\alpha)], \end{cases}$$

and Problem 1 reduces to finding the optimal parameter $v \in [0, \mathrm{VaR}_Y(\alpha)]$. To do this, check that

$$\mathrm{VaR}_{R_Z}(\alpha) = P(Z) + v = (1+\beta)\int_v^{\mathrm{VaR}_Y(\alpha)} (y-v)f(y)dy + v.$$

Elementary calculus shows that

$$\frac{\partial}{\partial v}\mathrm{VaR}_{R_Z}(\alpha) = 1 - (1+\beta)(1-\alpha-F(v)),$$

and hence $v = \mathrm{VaR}_Y(\alpha + \frac{1}{1+\beta})$. $\blacksquare$

Variance-related principles

Now, we present the solution of Problem 1 when the reinsurance premium is a functional of type

$$P(Z) = E[Z] + g(\mathrm{var}[Z]), \tag{27}$$

where $g : [0, +\infty) \mapsto [0, +\infty)$ is a continuous function, differentiable in $(0, +\infty)$, and $g'(x) > 0 \ \forall x > 0$.

Proposition 4 *If the reinsurance premium is a functional of type* (27) *and* g' *is bounded in a neighborhood of zero then Problem 1 admits a solution:*

$$Z(y) = \begin{cases} y - v, & y \in [v, \mathrm{VaR}_Y(\alpha)], \\ 0, & \text{if } y \notin [v, \mathrm{VaR}_Y(\alpha)], \end{cases} \tag{28}$$

where $v \in [0, \mathrm{VaR_Y}(\alpha)]$ *is either zero or a solution of the equation*

$$g'(\mathrm{var}[Z]) = \frac{1}{2E[Z]}. \tag{29}$$

If g' *is unbounded in any neighborhood of zero, then either Problem 1 admits a solution of type* (28) *or the solution is* $Z \equiv 0$ *(no reinsurance at all).* □

Proof. For a functional of type (27), the expression (18) reduces to a second degree polynomial $Q_\eta(\zeta) = a_1\zeta + a_2\zeta^2$, with $a_1 \in \mathbb{R}$, $a_2 > 0$. Hence, Q_η admits one unique minimizer $z_1 \in [0, +\infty)$. Also, the map $y \mapsto \left(\min_{\zeta \in [(y-v)^+, y]} Q_\eta(\zeta) - \min_{\zeta \in [0,y]} Q_\eta(\zeta) \right)$ is monotonic increasing. Therefore, Proposition 2 states that there exists an optimal treaty of type

$$Z(y) = \begin{cases} y, & \text{if } y \le z_1, \\ z_1, & \text{if } y \in [z_1, z_1 + v], \\ y - v, & \text{if } y \in [z_1 + v, \mathrm{VaR}_Y(\alpha)], \\ z_1, & \text{if } y > \mathrm{VaR}_Y(\alpha), \end{cases} \tag{30}$$

and Problem 1 reduces to finding the optimal parameters $z_1 \ge 0$, $v \ge 0$ with $z_1 + v \le \mathrm{VaR}_Y(\alpha)$. This is an optimization problem in $\mathbb{R}^2$ with linear constraints that can be solved by standard Karush-Khun-Tucker conditions.

Indeed, for a treaty of type (30), we have

$$\begin{aligned} E[Z^k] = &\int_0^{z_1} y^k f(y)dy + z_1^k \int_{z_1}^{z_1+v} f(y)dy + \\ &+ \int_{z_1+v}^{\mathrm{VaR}_Y(\alpha)} (y-v)^k f(y)dy + z_1^k \int_{\mathrm{VaR}_Y(\alpha)}^{+\infty} f(y)dy, \qquad k \in \mathbb{N}, \end{aligned}$$

and hence

$$\begin{aligned} \frac{\partial}{\partial z_1} E[Z^k] &= k z_1^{k-1}(F(z_1+v) - F(z_1) + \alpha), \\ \frac{\partial}{\partial v} E[Z^k] &= -k \int_{z_1+v}^{\mathrm{VaR}_Y(\alpha)} (y-v)^{k-1} f(y)dy. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{\partial}{\partial z_1} \mathrm{VaR}_{R_Z}(\alpha) = \frac{\partial}{\partial z_1} \left(E[Z] + g(\mathrm{var}[Z]) + v \right) = \\ &= (F(z_1+v) - F(v) + \alpha)(1 - 2g'(\mathrm{var}[Z])(E[Z] - z_1)), \\ &\frac{\partial}{\partial v} \mathrm{VaR}_{R_Z}(\alpha) = \alpha + F(z_1+v) - \\ &- 2g'(\mathrm{var}[Z]) \left(\int_{z_1+v}^{\mathrm{VaR}_Y(\alpha)} (y-v) f(y)dy - E[Z](1 - \alpha - F(z_1+v)) \right). \end{aligned}$$

Hence, the cost function and the constraints are continuously differentiable and, solving the Karush-Khun-Tucker conditions, we see that if g' is bounded in some neighborhood of zero, then the optimal parameters must solve either $v = 0$ or $z_1 = 0$ and (29). If g' is unbounded in any neighborhood of zero, then the parameters $z_1 = 0$, $v = \mathrm{VaR}_Y(\alpha)$ also define a candidate minimizer of $\mathrm{VaR}_{R_Z}(\alpha)$. ■

4.2 Risk adjusted premium principles

When the insurer's risk measure is VaR and the premium calculation principle is of type (11), the solution of Problem 1 is as follows.

Proposition 5 *If the premium calculation principle is a functional of type* (11) *then, Problem 1 admits a solution*

$$Z(y) = \begin{cases} y, & \text{if } y \leq \mathrm{VaR}_Y(\alpha), \\ 0 & \text{if } y > \mathrm{VaR}_Y(\alpha). \ \square \end{cases} \tag{31}$$

Proof. Let $\hat{\eta} \in \mathcal{M}$ be a solution of Problem 2.

Let $v = \hat{\mathrm{VaR}}_{\hat{\eta}}(\alpha)$ and suppose there is a point $(y_0, z_0) \in \mathrm{Supp}(\hat{\eta})$ with $0 < z_0 < y_0 - v$. Consider the measure

$$\begin{aligned} \eta_\varepsilon(A) =& \hat{\eta}(A) - \hat{\eta}(A \cap B_\varepsilon(y_0, z_0)) + \\ &+ \hat{\eta}\left\{(y, z) \in B_\varepsilon(y_0, z_0) : (y, z - z_0 + \varepsilon) \in A\right\}, \qquad A \in \mathcal{B}_{\mathbb{R}^2}. \end{aligned}$$

Then, for sufficiently small $\varepsilon > 0$

$$\begin{aligned} &\eta_\varepsilon\{(y, z) : z > t\} \leq \hat{\eta}\{(y, z) : z > t\} && \text{for } t \leq 2\varepsilon \text{ or } t \geq z_0 - \varepsilon; \\ &\eta_\varepsilon\{(y, z) : z > t\} < \hat{\eta}\{(y, z) : z > t\} && \text{for } t \in (2\varepsilon, z_0 - \varepsilon), \end{aligned}$$

i.e., $\hat{P}(\eta_\varepsilon) < \hat{P}(\hat{\eta})$. Since $\hat{\mathrm{VaR}}_{\eta_\varepsilon}(\alpha) = \hat{\mathrm{VaR}}_{\hat{\eta}}(\alpha)$, this is a contradiction to the optimality of $\hat{\eta}$. Hence $\hat{\eta}$ must satisfy $\hat{\eta}\{(y, z) : 0 < z < y - v\} = 0$. A similar argument shows that $\hat{\eta}$ must satisfy $\hat{\eta}\{(y, z) : z > 0, z > y - v\} = 0$, and therefore, $\hat{\eta}$ must be concentrated in the set

$$\{(y, 0) : y \geq 0\} \cup \{(y, y - v) : y \geq v\}.$$

Now, suppose there are points $(y_1,0),(y_2,y_2-v)\in \mathrm{Supp}(\hat\eta)$ with $v<y_1<y_2$, and consider the measure

$$\begin{aligned}\eta_\varepsilon(A) =&\hat\eta(A)-\hat\eta(A\cap\{(y,0):|y-y_1|<\varepsilon\})+\\ &+\hat\eta\{(y,0):|y-y_1|<\varepsilon,(y,y-v)\in A\}-\\ &-\frac{\hat\eta\{(y,0):|y-y_1|<\varepsilon\}}{\hat\eta\{(y,y-v):|y-y_2|<\varepsilon\}}\hat\eta(A\cap\{(y,y-v):|y-y_2|<\varepsilon\})+\\ &+\frac{\hat\eta\{(y,0):|y-y_1|<\varepsilon\}}{\hat\eta\{(y,y-v):|y-y_2|<\varepsilon\}}\hat\eta\{(y,y-v):|y-y_2|<\varepsilon,(y,0)\in A\},\qquad A\in\mathcal{B}_{\mathbb{R}^2}.\end{aligned}$$

Then, $\hat{\mathrm{VaR}}_{\eta_\varepsilon}(\alpha)=\hat{\mathrm{VaR}}_{\hat\eta}(\alpha)$ and, for sufficiently small $\varepsilon>0$ we have

$$\begin{aligned}&\eta_\varepsilon\{(y,z):z>t\}\le\hat\eta\{(y,z):z>t\}\qquad \text{for } t\in[0,y_1-v+\varepsilon)\cup(y_2-v-\varepsilon,+\infty),\\ &\eta_\varepsilon\{(y,z):z>t\}<\hat\eta\{(y,z):z>t\}\qquad \text{for } t\in[y_1-v+\varepsilon,y_2-v-\varepsilon],\end{aligned}$$

which implies $\hat P(\eta_\varepsilon)<\hat P(\hat\eta)$, a contradiction to the optimality of $\hat\eta$.

This shows that $\hat\eta$ is actually a (deterministic) treaty of type

$$Z(y)=\begin{cases} y-v, & \text{for } y\in[v,\mathrm{VaR}_Y(\alpha)],\\ 0 & \text{for } y\notin[v,\mathrm{VaR}_Y(\alpha)],\end{cases}\tag{32}$$

with $v\le \mathrm{VaR}_Y(\alpha)$.

We conclude the proof by showing that $v=0$. To see this, notice that, for a treaty of type (32), we have

$$P(Z)=\int_v^{\mathrm{VaR}_Y(\alpha)} w(1-\alpha-F(t))dt.$$

Hence

$$\frac{\partial}{\partial v}\mathrm{VaR}_{R_Z}(\alpha)=1-w(1-\alpha-F(v))\ge 0\qquad \text{a.e. } v\in[0,\mathrm{VaR}_Y(\alpha)],$$

i.e., $\mathrm{VaR}_{R_Z}(\alpha)$ is a monotonically increasing function of v and hence the minimum is attained at $v=0$. ■

5 The CTE risk measure

In this section, we solve Problem 1 when the insurer's risk measure is the CTE.

5.1 Premiums depending on moments

Suppose the premium calculation principle is of type (10).

We use the same notation as in Section 4.1. In particular, for each $\eta \in \mathcal{M}$ we consider the polynomial function $Q_\eta(\zeta) = \sum_{i=1}^{m} \frac{\partial\gamma}{\partial x_i}\zeta^i$ (equality (18)).

Proposition 6 *(under Assumptions 1, 2 and 3):*

Let $\hat\eta$ solve Problem 2. If γ is differentiable at $\hat x = \int_{\mathbb{R}^2} \left(z, z^2, \dots z^m\right) d\hat\eta$, $\nabla\gamma(\hat x) \neq 0$ and $\nabla\gamma(\hat x) \neq \left(\frac{1}{\alpha}, 0, \dots, 0\right)$, then Problem 1 admits an optimal treaty $\hat Z \in \mathcal{Z}$ such that, for each claim size $y > 0$ we have:

$$\hat Z(y) = \arg\min_{\zeta\in[0,(y-v)^+]} \left(Q_{\hat Z}(\zeta) - \frac{\zeta}{\alpha}\right) \quad \textit{or} \quad \hat Z(y) = \arg\min_{\zeta\in[(y-v)^+,y]} Q_{\hat Z}(\zeta),$$

with $v = \mathrm{VaR}_{Y-\hat Z}(\alpha)$. Further, there is a constant $C \in \mathbb{R}$ such that

$$\begin{aligned}
\hat Z(y) &= \arg\min_{\zeta\in[(y-v)^+,y]} Q_{\hat Z}(\zeta), \\
&\qquad \textit{if} \min_{\zeta\in[(y-v)^+,y]} Q_{\hat Z}(\zeta) - \min_{\zeta\in[0,(y-v)^+]} \left(Q_{\hat Z}(\zeta) + \frac{y-\zeta}{\alpha}\right) < C; \\
\hat Z(y) &= \arg\min_{\zeta\in[0,(y-v)^+]} \left(Q_{\hat Z}(\zeta) + \frac{y-\zeta}{\alpha}\right), \\
&\qquad \textit{if} \min_{\zeta\in[(y-v)^+,y]} Q_{\hat Z}(\zeta) - \min_{\zeta\in[0,(y-v)^+]} \left(Q_{\hat Z}(\zeta) + \frac{y-\zeta}{\alpha}\right) > C. \ \square
\end{aligned}$$

Proof. This is similar to the proof of Proposition 2, with some adaptations due to the tail component of the risk measure.

Fix $\hat\eta \in \mathcal{M}$, a solution of Problem 2 satisfying the assumptions above. Let $v = \hat{\mathrm{VaR}}_{\hat\eta}(\alpha)$, and pick $(y_0, z_0) \in \mathrm{Supp}(\hat\eta)$.

To start, assume that $z_0 \in [0, y_0 - v)$. Fix $t \in (-z_0, y_0 - v - z_0)$, and consider the measure

$$\begin{aligned}
\eta_\varepsilon(A) =& \hat\eta(A) - \hat\eta(A \cap B_\varepsilon(y_0, z_0)) + \\
&+ \hat\eta\left\{(y,z) \in B_\varepsilon(y_0,z_0) : (y, z+t) \in A\right\}, \qquad A \in \mathcal{B}_{\mathbb{R}^2}.
\end{aligned}$$

For sufficiently small $\varepsilon > 0$, we have $\eta_\varepsilon \in \mathcal{M}$, $\hat{\mathrm{VaR}}_{\eta_\varepsilon}(\alpha) = \hat{\mathrm{VaR}}_{\hat\eta}(\alpha)$, $\eta_\varepsilon\{y - z > v\} =$

$\hat{\eta}\{y - z > v\}$. It is possible to check that

$$\begin{aligned}
&\hat{P}(\eta_\varepsilon) + \hat{\text{CTE}}_{\eta_\varepsilon}(\alpha) = \\
=&\hat{P}(\hat{\eta}) + \hat{\text{CTE}}_{\hat{\eta}}(\alpha) + \\
&+ \left(Q_{\hat{\eta}}(z_0 + t) + \frac{y_0 - z_0 - t}{\alpha} - \left(Q_{\hat{\eta}}(z_0) + \frac{y_0 - z_0}{\alpha} \right) \right) \hat{\eta}\left(B_\varepsilon(y_0, z_0)\right) + \\
&+ o\left(\hat{\eta}\left(B_\varepsilon(y_0, z_0)\right)\right).
\end{aligned}$$

Hence, the argument used in the proof of Proposition 2 shows that

$$Q_{\hat{\eta}}(z_0) - \frac{z_0}{\alpha} = \min_{z \in [0, y_0 - v]} \left(Q_{\hat{\eta}}(z) - \frac{z}{\alpha} \right) \qquad \text{whenever } z_0 \in [0, y_0 - v). \tag{33}$$

Similarly, assuming $z_0 \in (y_0 - v, y_0]$ and taking $t \in (y_0 - v - z_0, y_0 - z_0)$, we can show that

$$Q_{\hat{\eta}}(z_0) = \min_{z \in [y_0 - v, y_0]} Q_{\hat{\eta}}(z) \qquad \text{whenever } z_0 \in (y_0 - v, y_0]. \tag{34}$$

Since $\zeta \mapsto Q_{\hat{\eta}}(\zeta)$, $\zeta \mapsto Q_{\hat{\eta}}(\zeta) - \frac{\zeta}{\alpha}$ are (nonconstant) polynomials, they have only finitely many local minima in $[0, +\infty)$. Let $c_1, c_2, \ldots, c_p$ be the local minimizers of $\zeta \mapsto Q_{\hat{\eta}}(\zeta) - \frac{\zeta}{\alpha}$ in $[0, +\infty)$, $c_{p+1}, c_{p+2}, \ldots, c_{p+q}$ be the local minimizers of $\zeta \mapsto Q_{\hat{\eta}}(\zeta)$ in $[0, +\infty)$. The results (33) and (34) show that the support of $\hat{\eta}$ must be contained in the set

$$\begin{aligned}
\left(\bigcup_{i=1}^{p} \{(y, c_i) : y \geq c_i + v\} \right) \cup \left(\bigcup_{i=p+1}^{p+q} \{(y, c_i) : c_i \leq y \leq c_i + v\} \right) \cup \\
\cup \{(y, y) : y \geq 0\} \cup \{(y, y - v) : y \geq v\}.
\end{aligned}$$

This means that the measure $\hat{\eta}$ admits a representation

$$\begin{aligned}
\hat{\eta}(A) =& \sum_{i=1}^{p} \int_{c_i + v}^{+\infty} m_i(y) \chi_A(y, c_i) dF(y) + \\
&+ \sum_{i=p+1}^{p+q} \int_{c_i}^{c_i + v} m_i(y) \chi_A(y, c_i) dF(y) + \\
&+ \int_0^{+\infty} m_{p+q+1}(y) \chi_A(y, y) dF(y) + \\
&+ \int_v^{+\infty} m_{p+q+2}(y) \chi_A(y, y - v) dF(y), \qquad \forall A \in \mathcal{B}_{\mathbb{R}^2},
\end{aligned} \tag{35}$$

where $m_i,\ i = 1, 2, \ldots, (p + q + 2)$ are measurable nonnegative functions such that

$$\Pr\left\{ \sum_{i=1}^{p+q+2} m_i(Y) = 1 \right\} = 1.$$

For each $A \in \mathcal{B}_{\mathbb{R}^2}$ and $i = 1, 2, \ldots, (p+q+2)$, let

$$\chi_{A,i}(y) = \begin{cases} \chi_{A\cap\{(y,z):0\le z<y-v\}}(y, c_i), & \text{for } i = 1, 2, \ldots, p; \\ \chi_{A\cap\{(y,z):(y-v)^+\le z\le y\}}(y, c_i), & \text{for } i = p+1, p+2, \ldots, p+q; \\ \chi_{A\cap\{(y,z):0\le z\le y\}}(y, y), & \text{for } i = p+q+1; \\ \chi_{A\cap\{(y,z):0\le z\le y\}}(y, y-v), & \text{for } i = p+q+2. \end{cases}$$

This allows us to write the representation (35) in the more compact form

$$\hat{\eta}(A) = \sum_{i=1}^{p+q+2} \int_0^{+\infty} m_i(y)\chi_{A,i}(y)\, dF(y), \qquad \forall A \in \mathcal{B}_{\mathbb{R}^2}.$$

Fix $y_1, y_2 > v$ and suppose there are $i_1 \in \{1, 2, \ldots, p\}$, $i_2 \in \{p+1, p+2, \ldots, p+q+2\}$ such that

$$\int_{y_1-\varepsilon}^{y_1+\varepsilon} m_{i_1}\chi_{\mathbb{R}^2,i_1} dF > 0, \quad \int_{y_2-\varepsilon}^{y_2+\varepsilon} m_{i_2}\chi_{\mathbb{R}^2,i_2} dF > 0, \qquad \forall\varepsilon > 0.$$

Fix $z_1 \in (y_1 - v, y_1)$, $z_2 \in [0, y_2 - v)$ and consider the measure

$$\begin{aligned} \eta_\varepsilon(A) =& \hat{\eta}(A) - \int_{y_1-\varepsilon}^{y_1+\varepsilon} m_{i_1}(y)\chi_{A,i_1}(y) dF(y) + \\ &+ \int_{y_1-\varepsilon}^{y_1+\varepsilon} m_{i_1}(y)\chi_{\mathbb{R}^2,i_1}(y)\chi_A(y, z_1) dF(y) - \\ &- \frac{\int_{y_1-\varepsilon}^{y_1+\varepsilon} m_{i_1}\chi_{\mathbb{R}^2,i_1} dF}{\int_{y_2-\varepsilon}^{y_2+\varepsilon} m_{i_2}\chi_{\mathbb{R}^2,i_2} dF} \int_{y_2-\varepsilon}^{y_2+\varepsilon} m_{i_2}(y)\chi_{\mathbb{R}^2,i_2}(y)\chi_A(y, c_{i_2}) dF(y) + \\ &+ \frac{\int_{y_1-\varepsilon}^{y_1+\varepsilon} m_{i_1}\chi_{\mathbb{R}^2,i_1} dF}{\int_{y_2-\varepsilon}^{y_2+\varepsilon} m_{i_2}\chi_{\mathbb{R}^2,i_2} dF} \int_{y_2-\varepsilon}^{y_2+\varepsilon} m_{i_2}(y)\chi_{\mathbb{R}^2,i_2}(y)\chi_A(y, z_2) dF(y), \quad A \in \mathcal{B}_{\mathbb{R}^2}. \end{aligned}$$

For every sufficiently small $\varepsilon > 0$ we have $\eta_\varepsilon \in \mathcal{M}$, $\hat{\text{VaR}}_{\eta_\varepsilon}(\alpha) = \hat{\text{VaR}}_{\hat{\eta}}(\alpha)$, $\eta_\varepsilon\{y - z > v\} = \hat{\eta}\{y - z > v\}$ and

$$\begin{aligned} &\hat{P}(\eta_\varepsilon) + \hat{\text{CTE}}_{\eta_\varepsilon}(\alpha) = \\ =&\hat{P}(\hat{\eta}) + \hat{\text{CTE}}_{\hat{\eta}}(\alpha) + \\ &+ \left(Q_{\hat{\eta}}(z_1) - Q_{\hat{\eta}}(c_{i_1}) - \frac{y_1 - c_{i_1}}{\alpha} + Q_{\hat{\eta}}(z_2) + \frac{y_2 - z_2}{\alpha} - Q_{\hat{\eta}}(c_{i_2})\right) \times \\ &\quad \times \int_{y_1-\varepsilon}^{y_1+\varepsilon} m_{i_1}\chi_{\mathbb{R}^2,i_1} dF + o\left(\int_{y_1-\varepsilon}^{y_1+\varepsilon} m_{i_1}\chi_{\mathbb{R}^2,i_1} dF\right), \end{aligned} \tag{36}$$

with $c_{i_2} = y_2$ in the case $i_2 = p + q + 1$ or $c_{i_2} = y_2 - v$ in the case $i_2 = p + q + 2$.

Choosing z_1 close to $\arg\min_{z\in[y_1-v,y_1]} Q_{\hat{\eta}}(z)$, and z_2 close to $\arg\min_{z\in[0,y_2-v]} \left(Q_{\hat{\eta}}(z) - \frac{z}{\alpha}\right)$, we can use (36) to prove that

$$\min_{z\in[y_1-v,y_1]} Q_{\hat{\eta}}(z) - \min_{z\in[0,y_1-v]} \left(Q_{\hat{\eta}}(z) + \frac{y_1 - z}{\alpha}\right) \geq$$

$$\geq \min_{z\in[y_2-v,y_2]} Q_{\hat{\eta}}(z) - \min_{z\in[0,y_2-v]} \left(Q_{\hat{\eta}}(z) + \frac{y_2 - z}{\alpha}\right).$$

From this point the proof is identical to the proof of Proposition 2. ■

By Proposition 6, optimal treaties may theoretically have discontinuities, depending on the premium calculation principle. Below we show that at least for the expected value principle and variance-related principles, this is not the case.

The expected value principle

If the premium is computed by the expected value principle is (25), Proposition 6 takes the corollary:

Proposition 7 *Assuming that the premium calculation principle is* (25)*:*

(*a*) *If* $1 + \beta < \frac{1}{\alpha}$, *then any stop-loss treaty* $Z(y) = (y - v)^+$ *with* $v \in F^{-1}\left\{\frac{\beta}{1+\beta}\right\}$ *is a solution for Problem 1.*

(*b*) *If* $1 + \beta > \frac{1}{\alpha}$, *then* $Z \equiv 0$ *(no reinsurance) is a solution for Problem 1.*

(*c*) *If* $1 + \beta = \frac{1}{\alpha}$, *then any treaty satisfying* $0 \leq Z(y) \leq (y - \mathrm{VaR}_Y(\alpha))^+ \ \forall y \geq 0$ *is a solution for Problem 1.* □

Proof. For the expected value principle (25), we have $Q_Z(t) = (1 + \beta)t$.

Consider the case when $1 + \beta < \frac{1}{\alpha}$. Then,

$$\arg\min_{t\in[0,(y-v)^+]} \left(Q(t) + \frac{y - t}{\alpha}\right) = (y - v)^+ = \arg\min_{t\in[(y-v)^+,y]} Q(t).$$

Hence Proposition 6 guarantees that Problem 1 admits a solution of type $Z(y) = (y - v)^+$ and it only remains to find the optimal value of the parameter v.

For a stop-loss treaty, we have

$$\mathrm{CTE}_{R_Z}(\alpha) = \begin{cases} (1+\beta)\int_v^{+\infty}(y-v)dF(y) + v, & \text{for } v \leq \mathrm{VaR}_Y(\alpha), \\ (1+\beta)\int_v^{+\infty}(y-v)dF(y) + \\ \quad +\frac{1}{\alpha}\left(\int_{\mathrm{VaR}_Y(\alpha)}^{v} ydF(y) + \int_v^{+\infty} vdF(y)\right), & \text{for } v \geq \mathrm{VaR}_Y(\alpha). \end{cases}$$

By elementary calculus, we see that any v satisfying $F(v) = \frac{\beta}{1+\beta}$ is optimal.

In the case when $1+\beta > \frac{1}{\alpha}$, we have

$$\arg \min_{t\in[0,(y-v)^+]} \left(Q(t) + \frac{y-t}{\alpha} \right) = 0, \qquad \arg \min_{t\in[(y-v)^+,y]} Q(t) = (y-v)^+,$$
$$\min_{t\in[0,(y-v)^+]} \left(Q(t) + \frac{y-t}{\alpha} \right) = \frac{y}{\alpha}, \qquad \min_{t\in[(y-v)^+,y]} Q(t) = (1+\beta)(y-v)^+.$$

Hence, Proposition 6 states that there is an optimal treaty of type

$$Z(y) = \begin{cases} y - v, & \text{for } y \in [v, V]; \\ 0 & \text{for } y \notin [v, V], \end{cases}$$

with $0 \le v \le V < +\infty$. For a treaty of this type:

$$\mathrm{CTE}_{R_Z}(\alpha) = \begin{cases} (1+\beta)\int_v^V (y-v)dF(y) + \frac{\alpha - \int_V^{+\infty} dF(y)}{\alpha} v + \\ \quad + \frac{1}{\alpha}\int_V^{+\infty} y dF(y), & \text{for } v \le \mathrm{VaR}_Y(\alpha), \\ (1+\beta)\int_v^V (y-v)dF(y) + \\ \quad + \frac{1}{\alpha}\int_{\mathrm{VaR}_Y(\alpha)}^v y dF(y) + \frac{1}{\alpha}\int_v^V v dF(y) + \\ \quad + \frac{1}{\alpha}\int_V^{+\infty} y dF(y), & \text{for } v \ge \mathrm{VaR}_Y(\alpha). \end{cases}$$

Therefore, the minimum is attained with $v = V$, i.e., $Z \equiv 0$.

Finally, consider the case $1+\beta = \frac{1}{\alpha}$. The proof of Proposition 6 shows that any solution of Problem 2 must satisfy

$$\eta\{(y,z) : z \le (y - \hat{\mathrm{VaR}}_\eta(\alpha))^+\} = 1. \tag{37}$$

Further, $1+\beta = \frac{1}{\alpha}$ implies that, for any pair of random treaties $\eta_1, \eta_2 \in \mathcal{M}$ satisfying $\hat{\mathrm{VaR}}_{\eta_1}(\alpha) = \hat{\mathrm{VaR}}_{\eta_2}(\alpha)$ and (37), we have

$$\hat{P}(\hat{\eta}_1) + \hat{\mathrm{CTE}}_{\eta_1}(\alpha) = \hat{P}(\hat{\eta}_2) + \hat{\mathrm{CTE}}_{\eta_2}(\alpha).$$

The result follows from the fact that the optimal treaty in the class of stop-loss treaties is $Z(y) = (y - \mathrm{VaR}_Y(\alpha))^+$ when $1+\beta = \frac{1}{\alpha}$. ■

Variance-related principles

Now, we present the solution of Problem 1 when the reinsurance premium is a functional of type 27.

Proposition 8 *Suppose the reinsurance premium is a functional of type* (27).

If $Z \equiv 0$ *(no reinsurance at all) is not optimal, then Problem 1 admits a solution:*

$$Z(y) = \begin{cases} 0, & \text{for } y \in [0, v], \\ y - v, & \text{for } y \in [v, v + c], \\ c, & \text{for } y \geq v + c, \end{cases} \tag{38}$$

where c, v *are constants satisfying*

$$c = E[Z] + \frac{1}{2g'(\mathrm{var}[Z])} \frac{1-\alpha}{\alpha}, \tag{39}$$

$$\frac{\alpha - \int_{v+c}^{+\infty} dF}{2g'(\mathrm{var}[Z])\alpha} = \int_v^{v+c} \left(y - v - \left(E[Z] - \frac{1}{2g'(\mathrm{var}[Z])} \right) \right) dF, \tag{40}$$

$$0 \leq v \leq \mathrm{VaR}_Y(\alpha). \ \square$$

Proof. For the functional (27), we have

$$Q_Z(\zeta) = \left(1 - 2E[Z]g'(\mathrm{var}[Z])\right)\zeta + g'(\mathrm{var}[Z])\zeta^2.$$

Hence, the unique minimizer of $Q_Z : [0, +\infty) \mapsto \mathbb{R}$ is

$$c_1 = \left(E[Z] - \frac{1}{2g'(\mathrm{var}[Z])} \right)^+, \tag{41}$$

and the unique minimizer of $\zeta \mapsto Q_Z(\zeta) + \frac{y-\zeta}{\alpha}$ is

$$c_2 = E[Z] + \frac{1}{2g'(\mathrm{var}[Z])} \frac{1-\alpha}{\alpha}. \tag{42}$$

The map $y \mapsto \left(\min_{\zeta \in [(y-v)^+, y]} Q_\eta(\zeta) - \min_{\zeta \in [0, (y-v)^+]} Q_\eta(\zeta) + \frac{y-\zeta}{\alpha} \right)$ is monotonic increasing in $[v, +\infty)$. Therefore, Proposition 6 states that there exists an optimal treaty of type

$$Z(y) = \begin{cases} y, & \text{for } y \leq c_1, \\ c_1, & \text{for } y \in [c_1, c_1 + v], \\ y - v, & \text{for } y \in [c_1 + v, V], \\ c_2, & \text{for } y > V, \end{cases} \tag{43}$$

with $0 \leq v \leq \mathrm{VaR}_Y(\alpha) \leq V$, $v + c_2 \leq V$. Thus, Problem 1 reduces to finding the optimal parameters c_1, c_2, v, V.

For a treaty of type (43), we have

$$
\begin{aligned}
E[Z^k] = & \int_0^{c_1} y^k dF(y) + c_1^k \int_{c_1}^{v+c_1} dF(y) + \\
& + \int_{v+c_1}^{V} (y-v)^k dF(y) + c_2^k \int_V^{+\infty} dF(y), \qquad k \in \mathbb{N}, \\
\mathrm{CTE}_{R_Z}(\alpha) = & E[Z] + g(\mathrm{var}[Z]) + \frac{\alpha - \int_V^{+\infty} dF}{\alpha} v + \frac{1}{\alpha} \int_V^{+\infty} (y - c_2) dF(y).
\end{aligned}
$$

Let $\hat{c}_1$, $\hat{c}_2$, $\hat{v}$, $\hat{V}$ denote optimal parameters and, for each $V \geq \max\{\hat{v} + \hat{c}_2, \mathrm{VaR}_Y(\alpha)\}$, let Z_V and CTE_V denote the treaty and the corresponding CTE with parameters $\hat{c}_1$, $\hat{c}_2$, $\hat{v}$, V. Then

$$
\begin{aligned}
\mathrm{CTE}_V - \mathrm{CTE}_{\hat{V}} = & \int_{\hat{V}}^{V} g'(\mathrm{var}[Z_t])(t - \hat{v} - \hat{c}_2) \times \\
& \times \left(t - \hat{v} + \hat{c}_2 - 2 \left(E[Z_t] + \frac{1}{2g'(\mathrm{var}[Z_t])} \frac{1-\alpha}{\alpha} \right) \right) dF(t) dt.
\end{aligned}
$$

Due to (42), this is

$$
\mathrm{CTE}_V - \mathrm{CTE}_{\hat{V}} = \int_{\hat{V}}^{V} g'(\mathrm{var}[Z_t])(t - \hat{v} - \hat{c}_2)^2 dF(t) + o(|F(V) - F(\hat{V})|),
$$

and therefore, optimality implies $\hat{V} = \max\{\hat{v} + \hat{c}_2, \mathrm{VaR}_Y(\alpha)\}$.

Also, for the optimal value of the parameters, we have

$$
\begin{aligned}
\frac{\partial \mathrm{CTE}_{R_Z}(\alpha)}{\partial v} = & \left(2g'(\mathrm{var}[Z]) E[Z] - 1\right) \int_{\hat{v}+\hat{c}_1}^{\hat{V}} dF - \\
& - 2g'(\mathrm{var}[Z]) \int_{\hat{v}+\hat{c}_1}^{\hat{V}} (y - \hat{v}) dF(y) + \frac{\alpha - \int_{\hat{V}}^{+\infty} dF}{\alpha}
\end{aligned}
\qquad (44)
$$

If $\hat{v} + \hat{c}_2 \leq \mathrm{VaR}_Y(\alpha)$, then $\hat{V} = \mathrm{VaR}_Y(\alpha)$ and (44) reduces to

$$
\begin{aligned}
\frac{\partial \mathrm{CTE}_{R_Z}(\alpha)}{\partial v} = & \left(2g'(\mathrm{var}[Z]) E[Z] - 1\right) \int_{\hat{v}+\hat{c}_1}^{\mathrm{VaR}_Y(\alpha)} dF - \\
& - 2g'(\mathrm{var}[Z]) \int_{\hat{v}+\hat{c}_1}^{\mathrm{VaR}_Y(\alpha)} (y - \hat{v}) dF(y) \leq 0,
\end{aligned}
$$

with strict inequality holding whenever $\hat{v} + \hat{c}_1 < \mathrm{VaR}_Y(\alpha)$. Thus, $\hat{v} + \hat{c}_2 \leq \mathrm{VaR}_Y(\alpha)$ implies $Z \equiv 0$.

Thus, if $Z \equiv 0$ is not optimal, we must have $\hat{V} = \hat{v} + \hat{c}_2$. Suppose that this is the case and $\hat{c}_1 > 0$. Then, using (41) and rearranging, (44) reduces to

$$
\frac{\partial \mathrm{CTE}_{R_Z}(\alpha)}{\partial v} = 2g'(\mathrm{var}[Z]) \int_0^{\hat{c}_1} (y - \hat{c}_1) dF(y).
$$

If $\int_0^{\hat{c}_1} dF = 0$, then the result follows immediately from the translation invariance of $X \mapsto \mathrm{CTE}_X(\alpha)$ and $Z \mapsto P(Z)$. If $\int_0^{\hat{c}_1} dF > 0$, then $\frac{\partial \mathrm{CTE}_{R_Z}(\alpha)(Z)}{\partial v} < 0$ and hence $\hat{v} = \mathrm{VaR}_Y(\alpha)$. Notice that (41) and (42) imply

$$\begin{aligned}\hat{c}_2 =& \hat{c}_1 + \frac{1}{2g'(\mathrm{var}[Z])}\frac{1}{\alpha} = \hat{c}_1 + \frac{1}{\alpha}(E[Z] - \hat{c}_1) = \\ =& \hat{c}_1 + \frac{1}{\alpha}\left(\int_0^{\hat{c}_1}(y-\hat{c}_1)dF(y) + \int_{\hat{v}+\hat{c}_1}^{\hat{v}+\hat{c}_2}(y-\hat{v}-\hat{c}_2)dF(y) + \int_{\hat{v}+\hat{c}_1}^{+\infty}(\hat{c}_2-\hat{c}_1)dF(y)\right).\end{aligned}$$

This is

$$\left(\alpha - \int_{\hat{v}+\hat{c}_1}^{+\infty} dF(y)\right)(\hat{c}_2 - \hat{c}_1) = \int_0^{\hat{c}_1}(y-\hat{c}_1)dF(y) + \int_{\hat{v}+\hat{c}_1}^{\hat{v}+\hat{c}_2}(y-\hat{v}-\hat{c}_2)dF(y).$$

If $\hat{v} = \mathrm{VaR}_Y(\alpha)$, the left-hand side is non-negative while $\int_0^{\hat{c}_1} dF > 0$ makes the right-hand side strictly negative. This contradiction proves that $\hat{c}_1 = 0$.

Finally, optimality of Z implies $\frac{\partial \mathrm{CTE}_{R_Z}(\alpha)}{\partial v} = 0$. Using (44), this reduces to (40). ■

5.2 Risk adjusted premium principles

In this section we solve Problem 1 when the insurer's risk measure is CTE and premium calculation principle is of type (11).

Proposition 9 *If the insurer's risk measure is* CTE *and the reinsurance premium is computed by a functional* (11)*, and the equation* $w(t) = \frac{t}{\alpha}$ *admits a solution in the interval* $(0, 1 - F(0))$*, then Problem 1 admits a solution*

$$Z(y) = \begin{cases} y, & \text{for } y \le V, \\ V, & \text{for } y \ge V, \end{cases}$$

where V *is any solution of* $w(1 - F(V)) = \frac{1-F(V)}{\alpha}$.

If $w(t) \le \frac{t}{\alpha}$ *for every* $t \in (0, 1 - F(0))$*, then the treaty* $Z \equiv Y$ *(cedence of all risk) is optimal.*

If $w(t) \ge \frac{t}{\alpha}$ *for every* $t \in (0, 1 - F(0))$*, then the treaty* $Z \equiv 0$ *(no reinsurance) is optimal.* □

Proof. Continuity and monotonicity of w imply absolute continuity. Therefore the derivative w' exists at almost every point of $[0, 1]$, and $w(x) = \int_0^x w'(t)dt$, $\forall x \in [0, 1]$. Due to concavity of w, w' coincides almost everywhere with a monotonic nonincreasing function, left- and right-derivatives exist at every point and $w'_-(x) \ge w'_+(x)$ for every $x \in (0, 1)$.

Fix $\hat{\eta} \in \mathcal{M}$, a solution for Problem 2. Let $v = \hat{\mathrm{VaR}}_{\hat{\eta}}(\alpha)$, fix (y_0, z_0) such that $0 \leq z_0 < y_0 - v$, $\theta \in (0, y_0 - z_0 - v)$ and consider the measure

$$\begin{aligned}\tilde{\eta}(A) =& \hat{\eta}(A) - \hat{\eta}\left(B_\varepsilon(y_0, z_0) \cap A\right) + \\ & + \hat{\eta}\left\{(y, z) \in B_\varepsilon(y_0, z_0) : (y, z + \theta) \in A\right\}, \qquad \forall A \in \mathcal{B}_{\mathbb{R}^2}.\end{aligned}$$

For every sufficiently small $\varepsilon > 0$ we have $\tilde{\eta} \in \mathcal{M}$, $\hat{\mathrm{VaR}}_{\tilde{\eta}}(\alpha) = \hat{\mathrm{VaR}}_{\hat{\eta}}(\alpha)$. Optimality of $\hat{\eta}$ and monotonicity of w imply that

$$\begin{aligned}0 \leq & \hat{P}(\tilde{\eta}) + \hat{\mathrm{CTE}}_{\tilde{\eta}}(\alpha) - (\hat{P}(\hat{\eta}) + \hat{\mathrm{CTE}}_{\hat{\eta}}(\alpha)) = \\ = & \int_{z_0 - \varepsilon}^{z_0 + \theta + \varepsilon} \left(w(\tilde{\eta}\{z > t\}) - w(\hat{\eta}\{z > t\})\right) dt - \frac{\theta \hat{\eta}(B_\varepsilon(y_0, z_0))}{\alpha} \leq \\ \leq & \int_{z_0 - \varepsilon}^{z_0 + \theta + \varepsilon} \left(w(\hat{\eta}\{z > t\} + \hat{\eta}(B_\varepsilon(y_0, z_0))) - w(\hat{\eta}\{z > t\})\right) dt - \frac{\theta \hat{\eta}(B_\varepsilon(y_0, z_0))}{\alpha}.\end{aligned}$$

Due to concavity of w, this implies

$$\begin{aligned}0 \leq & \left(w(\hat{\eta}\{z > z_0 + \theta + \varepsilon\} + \hat{\eta}(B_\varepsilon(y_0, z_0))) - w(\hat{\eta}\{z > z_0 + \theta + \varepsilon\})\right)(\theta + 2\varepsilon) - \\ & - \frac{\theta \hat{\eta}(B_\varepsilon(y_0, z_0))}{\alpha}.\end{aligned}$$

dividing by $\theta \hat{\eta}(B_\varepsilon(y_0, z_0))$, this implies

$$0 \leq \frac{w(\hat{\eta}\{z > z_0 + \theta + \varepsilon\} + \varepsilon \hat{\eta}(B_\varepsilon(y_0, z_0))) - w(\hat{\eta}\{z > z_0 + \theta + \varepsilon\})}{\varepsilon \hat{\eta}(B_\varepsilon(y_0, z_0))} \frac{\theta + 2\varepsilon}{\theta} - \frac{1}{\alpha}.$$

Therefore

$$\lim_{\theta \to 0^+} \lim_{\varepsilon \to 0^+} \frac{w(\hat{\eta}\{z > z_0 + \theta + \varepsilon\} + \hat{\eta}(B_\varepsilon(y_0, z_0))) - w(\hat{\eta}\{z > z_0 + \theta + \varepsilon\})}{\hat{\eta}(B_\varepsilon(y_0, z_0))} \frac{\theta + 2\varepsilon}{\theta} \geq \frac{1}{\alpha},$$

which implies $w'_-(\hat{\eta}\{z > z_0\}) \geq \frac{1}{\alpha}$.

By taking $\theta \in (-z_0, 0)$, a similar argument shows that $w'_+(\hat{\eta}\{z \geq z_0\}) \leq \frac{1}{\alpha}$. Thus, we have

$$w'(x) \geq \frac{1}{\alpha}, \qquad \text{for a.e. } x < \hat{\eta}\{z > z_0\} \tag{45}$$

$$w'(x) \leq \frac{1}{\alpha}, \qquad \text{for a.e. } x > \hat{\eta}\{z \geq z_0\} \tag{46}$$

By taking (y_0, z_0) with $z_0 > y_0 - v$ and using again the same argument, we see that optimality of $\hat{\eta}$ implies $\hat{\eta}\{z > (y - v)^+\} = 0$.

Now, suppose there exists $0 \leq z_1 < z_2$ satisfying (45), (46). Then,

$$w'(x) = \frac{1}{\alpha}, \qquad \text{a.e. } x \in [\hat{\eta}\{z \geq z_2\}, \hat{\eta}\{z > z_1\}]. \tag{47}$$

Pick $y_2 > y_1 > z_2 + v$, let $B = [y_1, y_2] \times [z_1, z_2)$, and consider the measure

$$\tilde{\eta}(A) = \hat{\eta}(A) - \hat{\eta}(A \cap B) + \hat{\eta}\left\{(y,z) \in B : (y, z_2) \in A\right\}, \quad \forall A \in \mathcal{B}_{\mathbb{R}^2}.$$

Due to (47), we have

$$\begin{aligned}
&\hat{\mathrm{CTE}}_{\tilde{\eta}}(\alpha) - \hat{\mathrm{CTE}}_{\hat{\eta}}(\alpha) = \\
=&\int_{z_1}^{z_2} w\left(\hat{\eta}\{z > t\} + \hat{\eta}\{z_1 \le z \le t, y_1 \le y \le y_2\}\right) - w\left(\hat{\eta}\{z > t\}\right)\, dt - \\
&- \frac{1}{\alpha}\int_B (z_2 - z) d\hat{\eta} = \\
=&\frac{1}{\alpha}\left(\int_{z_1}^{z_2} \hat{\eta}\{z_1 \le z \le t, y_1 \le y \le y_2\} dt - \int_B (z_2 - z) d\hat{\eta}\right) = 0,
\end{aligned}$$

Thus, there is an optimal treaty

$$Z(y) = \begin{cases} 0, & \text{for } y \le v, \\ y - v, & \text{for } y \in [v, V], \\ c, & \text{for } y > V, \end{cases} \tag{48}$$

with $c > 0$, $v = \mathrm{VaR}_{Y-Z}(\alpha) \le \mathrm{VaR}_Y(\alpha)$, $V \ge \max\{\mathrm{VaR}_Y(\alpha), v + c\}$. Thus the problem reduces to finding the optimal value for the parameters c, v, V.

For the treaty (48), we have

$$\begin{aligned}
\mathrm{CTE}_{R_Z}(\alpha) = &\int_0^c w(1 - F(v + t))dt + \int_c^{V-v} w(F(V) - F(v + t))dt + \\
&+ v + \frac{1}{\alpha}\int_V^{+\infty} (y - v - c) dF(y).
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial}{\partial c}\mathrm{CTE}_{R_Z}(\alpha) =& w(1 - F(v + c)) - w(F(V) - F(v + c)) - \frac{1 - F(V)}{\alpha} = \\
=& w(1 - F(v + c)) - 1 + \frac{F(V) - F(v + c)}{\alpha} - \int_0^{F(V) - F(v+c)} w'(t) dt.
\end{aligned}$$

Due to (45), this is strictly negative except if $F(v+c) = 1$. Thus, we may chose $c = V - v$, which implies

$$\frac{\partial}{\partial v}\mathrm{CTE}_{R_Z}(\alpha) = 1 - w(1 - F(v)) \ge 0,$$

i.e., $\mathrm{CTE}_{R_Z}(\alpha)$ is monotonically increasing with v and hence we can set $v = 0$. Thus, $\mathrm{CTE}_{R_Z}(\alpha) = \int_0^V w(1 - F(t))dt + \frac{1}{\alpha}\int_V^{+\infty} (y - V) df(y)$ and

$$\frac{d}{dV}\mathrm{CTE}_{R_Z}(\alpha) = w(1 - F(V)) - \frac{1 - F(V)}{\alpha},$$

which concludes the proof. ■

6 The CTE^- risk measure

The CTE^- risk measure behaves in an intermediate way between VaR and CTE. This is not surprising if we notice that, contrary to VaR, the CTE^- takes into account the tail of the distribution but with a smaller weight than the CTE.

In this section, we present the solutions for Problem 1 when the risk measure is CTE^-. Since the proofs are similar to the CTE case, we omit them.

6.1 Premiums depending on moments

Proposition 10 *(under Assumptions 1, 2 and 3):*

Let $\hat\eta$ solve Problem 2 and let $v = \hat{\mathrm{VaR}}_{\hat\eta}(\alpha)$. If γ is differentiable at $\hat x = \int_{\mathbb{R}^2} \left(z, z^2, \ldots z^m\right) d\hat\eta$, $\nabla\gamma(\hat x) \neq 0$ and $\nabla\gamma(\hat x) \neq \left(\frac{1}{\hat\eta\{y-z\geq v\}}, 0, \ldots, 0\right)$, then Problem 1 admits an optimal treaty $\hat Z \in \mathcal{Z}$ such that, for each claim size $y > 0$ we have:

$$\hat Z(y) = \arg \min_{\zeta\in[0,(y-v)^+]} \left(Q_{\hat Z}(\zeta) - \frac{\zeta}{\hat\eta\{y-z\geq v\}} \right) \qquad \textit{or}$$

$$\hat Z(y) = \arg \min_{\zeta\in[(y-v)^+,y]} Q_{\hat Z}(\zeta).$$

Further, there is a constant $C \in \mathbb{R}$ such that

$$\begin{aligned} \hat Z(y) &= \arg \min_{\zeta\in[(y-v)^+,y]} Q_{\hat Z}(\zeta), \\ &\quad \textit{if} \min_{\zeta\in[(y-v)^+,y]} Q_{\hat Z}(\zeta) - \min_{\zeta\in[0,(y-v)^+]} \left(Q_{\hat Z}(\zeta) + \frac{y-\zeta}{\hat\eta\{y-z\geq v\}} \right) < C; \\ \hat Z(y) &= \arg \min_{\zeta\in[0,(y-v)^+]} \left(Q_{\hat Z}(\zeta) + \frac{y-\zeta}{\hat\eta\{y-z\geq v\}} \right), \\ &\quad \textit{if} \min_{\zeta\in[(y-v)^+,y]} Q_{\hat Z}(\zeta) - \min_{\zeta\in[0,(y-v)^+]} \left(Q_{\hat Z}(\zeta) + \frac{y-\zeta}{\hat\eta\{y-z\geq v\}} \right) > C. \ \square \end{aligned}$$

As in the case of VaR and CTE, we present the specialization of this result to the expected value principle and to loadings depending on the variance:

Proposition 11 *Suppose that the premium calculation principle is* (25).

If $1 + \beta \leq \frac{1}{1-F(0)}$, then the optimal treaty is $Z \equiv Y$ (cedence of all risk).

If $1 + \beta > \frac{1}{1-F(0)}$, then the optimal treaty is

$$Z(y) = \begin{cases} 0, & \textit{for } y \notin [v, \mathrm{VaR}_Y(\alpha)], \\ y - v, & \textit{for } y \in [v, \mathrm{VaR}_Y(\alpha)], \end{cases}$$

with $v = \mathrm{VaR}_{Y-Z}(\alpha) \leq \mathrm{VaR}_Y(\alpha)$ depending on the particular distribution of Y. $\square$

Proposition 12 *Suppose that the reinsurance premium is a functional of type* (27)*. If* $Z \equiv 0$ *(no reinsurance at all) is not optimal, then Problem 1 admits a solution:*

$$Z(y) = \begin{cases} 0, & \text{for } y \in [0, v], \\ y - v, & \text{for } y \in [v, \mathrm{VaR}_Y(\alpha)], \\ c, & \text{for } y > \mathrm{VaR}_Y(\alpha), \end{cases} \tag{49}$$

with $c \in (0, \mathrm{VaR}_Y(\alpha) - v)$ *satisfying*

$$c = E[Z] + \frac{1}{2g'(\mathrm{var}[Z])} \frac{F(v)}{1 - F(v)}, \tag{50}$$

with $v = \mathrm{VaR}_{Y-Z}(\alpha) \in [0, \mathrm{VaR}_Y(\alpha) - c_2)$ *depending on the density* f*.* □

Risk adjusted premium principles

Proposition 13 *If the insurer's risk measure is* CTE^-*, the reinsurance premium is computed by a functional* (11)*, and* $w(\alpha) > \frac{\alpha}{1-F(0)}$*, then Problem 1 admits a solution*

$$Z(y) = \begin{cases} y, & \text{for } y \leq \mathrm{VaR}_Y(\alpha), \\ c, & \text{for } y > \mathrm{VaR}_Y(\alpha), \end{cases}$$

where $c < \mathrm{VaR}_Y(\alpha)$ *solves*

$$w(1 - F(c)) - w(1 - \alpha - F(c)) - \frac{\alpha}{1 - F(0)} = 0, \tag{51}$$

with $c = 0$ *if this equation does not admit a solution in* $[0, \mathrm{VaR}_Y(\alpha))$*.*

If $w(\alpha) \leq \frac{\alpha}{1-F(0)}$*, then Problem 1 admits a solution*

$$Z(y) = \begin{cases} y, & \text{for } y \leq V, \\ V, & \text{for } y > V, \end{cases}$$

where $V \geq \mathrm{VaR}_Y(\alpha)$ *solves*

$$w(1 - F(V)) = \frac{1 - F(V)}{1 - F(0)}, \tag{52}$$

with $V = +\infty$ *(i.e.* $Z \equiv Y$*) if this equation has no solution in* $[\mathrm{VaR}_Y(\alpha), +\infty)$*.* □

7 Discussion

Since the results for the CTE^{-} risk measure are intermediate between the results for VaR and the results for CTE, we center our discussion on these two extreme cases.

The most evident feature of the optimal treaties obtained in Section 4 is the presence of discontinuities. This means that these contracts cannot be applied in the real world because any claim amount close to a point of discontinuity would lead to a conflict between insurer and reinsurer.

Not all discontinuities are caused by the choice of risk measures: ultimately, the optimal treaty depends both on the risk measure and the premium calculation principle. In the case of the premium depending on moments, any discontinuities involving values of to be refunded above $Y - \mathrm{VaR}_{Y-Z}(\alpha)$ are due to the localization of the local minima of the polynomial $Q_Z(\zeta)$ and hence depend on the premium principle being used. It should be noted that for the most important principles of this type (expected value and variance-related principles), no such discontinuities occur.

On the other hand, optimal treaties for the VaR criterion always exhibit a discontinuity where the amount to be refunded drops from $Y - \mathrm{VaR}_{Y-Z}(\alpha)$, for claim values smaller than the discontinuity point, to a smaller constant for any claim value greater than the discontinuity point. This type of discontinuity is a general property of VaR, irrespective of the premium principle.

As pointed before, it is arguable that since discontinuous treaties have no practical application, they should be ruled out a priori by constraints on the class of admissible treaties. To this we reply that the discontinuity of the latter type should be taken as very strong evidence in support of the often quoted critique that VaR leads to excessive focus on small claims disregarding large losses.

This is particularly obvious when the reinsurance premium is computed by a risk adjusted principle and all claims below $\mathrm{VaR}_Y(\alpha)$ are fully refunded while claims above $\mathrm{VaR}_Y(\alpha)$ are not refunded at all. This is a self-defeating strategy because small claims are less than the reinsurance premium. Hence the VaR-minimizing strategy only affords protection for the events when the claim amount is greater than the reinsurance premium but less than $\mathrm{VaR}_Y(\alpha)$, leaving the insurer in a worse situation in all other events, as shown in the following proposition.

Proposition 14 *(under Assumptions 1, 2):*
If Y is not identically zero then, for any $\varepsilon > 0$ and any $\alpha \in (0,1)$ there exists a coher-

ent premium calculation principle such that the optimal treaty for the VaR(α) *criterion satisfies:*

$$\Pr\{R_Z < Y - \varepsilon\} = 0, \qquad \Pr\{R_Z \leq Y\} < \varepsilon. \ \square$$

Proof. Consider a premium calculation principle of type (11) and let Z denote the optimal treaty for this premium and the VaR(α) risk measure. Due to Proposition 5, the difference between the net risk after optimal reinsurance and the risk without reinsurance is

$$R_Z - Y = P(Z) - Z = \int_0^{\mathrm{VaR}_Y(\alpha)} w(1-\alpha-F(t))\, dt - Y\chi_{\{Y \leq \mathrm{VaR}_Y(\alpha)\}}.$$

Consider the sequence of functions $\{w_n(t) = t^{\frac{1}{n}}\}_{n\in\mathbb{N}}$. The proof follows from the fact that $\lim \int_0^{\mathrm{VaR}_Y(\alpha)} w_n(1-\alpha-F(t))\, dt = \mathrm{VaR}_Y(\alpha)$. ■

Notice that the insurer is expected to make a profit only in the events when the claim amount is not large. Thus, by buying the optimal reinsurance for the VaR criterion, the insurer is essentially renouncing to the opportunity of making profits without gaining protection against large claims. To see this, notice that the net revenue of the insurer after reinsurance is the random variable $c - R_Z$, where c is the gross revenue of insurance premiums. Thus, if $c < \mathrm{VaR}_Y(\alpha) - \varepsilon$ (which holds for small α, provided the distribution of Y has a tail), then Proposition 14 shows that a VaR(α)-minimizing insurer prefers a strategy that yields negative net revenue with probability equal to one, to alternative profitable strategies with the same tail distribution!

Now, if we consider CTE instead of VaR, the optimal treaties are monotonic continuous, at least for the expected value, variance-related and risk adjusted premium principles. For these cases, the optimal treaties are either stop-loss or stop-loss with a ceiling, which at first sight seem quite reasonable treaties. However, notice that for risk adjusted premium principles, the retention threshold is always zero. This suggests that although CTE takes into account the tail of the claim-size distribution, it does not confer it enough weight. Indeed, CTE satisfies a weaker version of Proposition 14:

Proposition 15 *Suppose that Y is an integrable continuous random variable.*

For any $\varepsilon > 0$ and any $\alpha \in (0,1)$ there exists a coherent premium calculation principle such that the optimal treaty for the CTE(α) *criterion satisfies:*

$$\Pr\{R_Z < Y - \varepsilon\} = 0, \qquad \Pr\{R_Z > Y\} > 1 - \alpha - \varepsilon. \ \square$$

Proof. Consider a sequence of premium calculation principles

$$P_n(Z) = \int_0^{+\infty} (\Pr\{Z > t\})^{\frac{1}{n}} dt, \qquad n \in \mathbb{N},$$

and let $Z_n = \min\{Y, V_n\}$ denote the corresponding optimal treaties, given by Proposition 9. The proof follows from the fact that the sequence $R_{Z_n} = Y + P_n(Z_n) - Z_n$, converges uniformly from below to $R_Z = \max\{Y, \mathrm{VaR}_Y(\alpha)\}$. ■

Like in the VaR case, this implies that a CTE(α)-minimizing insurer may prefer a strategy that yields negative net revenue with probability equal to one, to alternative profitable strategies.

The proofs of Propositions 14 and 15 are obtained by picking sequences $\{w_n\}_{n\in\mathbb{N}}$ converging pointwise to 1 in $(0, 1]$, i.e., by picking sequences of premium principles with increasing loading. Thus, the results above show that, by failing to give sufficient weight to the tail of the claim-size distribution, CTE (and, a fortiori, VaR) contain a built-in lack of sensitivity to the premium loading. This pushes the insurer to buy roughly the same reinsurance (exactly the same, in the case of VaR), even when the loading is so high as to make reinsurance virtually unaffordable.

So, there is strong evidence that all the risk measures examined in this paper fail to weight correctly the tail events of the claim-size distribution and are not a sound basis for choice of reinsurance strategies.

The fact that these measures lead in some cases to nonprofitable strategies also suggests that a "good" optimization criterion must necessarily take into account other parameters reflecting the overall situation of the insurer. In some way, some measure of the mid- and long-term fitness of the firm should have some bearing in the choice of the reinsurance strategy.

In particular, it is an open question to known in which way different premium revenues and/or different values of reserves held by the insurer should influence the choice of reinsurance, for a given claim-size distribution.

It can be argued that such an approach requires elaborate models of the firm, creating difficulties and opaqueness that override any advantage over the inherently simple quantile-based models. It is our opinion, simplicity does not always mean a better understanding of the issues under consideration and our results shows that, at least when applied to reinsurance, quantile risk measures simply do not fulfill their purported function.

Let us mention that the adjustment coefficient seems to have some qualities that quantile measures lack. It is related to the probability of eventual ruin in infinite time by the

Lundberg inequality and so is related to the long-term viability of the insurer. It is criticized precisely on the grounds of its relation to infinite-time survivability: it assumes that the same scenario (i.e., distribution of claim-amounts) occurs in every period in the future, which is unrealistic, and is concerned with putative events far off in the future, which are of little concern to the present decision-maker.

In our opinion, the adjustment coefficient is not a tool for planning into the far future, let alone infinite horizon. It should be viewed as a tool to take present decisions leaving some margin for future decisions (i.e., striking some balance between present priorities and highly uncertain future). Notice that positive probability of survival in infinite time requires positive expected growth of reserves. Thus, the adjustment coefficient contains a built-in trade-off between safety and profitability. For this reason, strategies that maximize the adjustment coefficient must provide a positive expected net revenue, avoiding some of the shortcomings of quantile measures.

Also, optimal treaties maximizing the adjustment coefficient seem to have good properties (see Guerra and Centeno (2008) and Guerra and Centeno (2010)). In particular, if the premium is computed by a variance related principle, then the optimal solution is a convex increasing function which splits the tail of the claim-size distribution between insurer and reinsurer.

Appendix: Proof of Proposition 1

As stated in the main text, Proposition 1 follows immediately from compactness of the space $\mathcal{M}$ and lower semicontinuity of the functionals (13)–(17). In this appendix we provide a rigorous proof of these properties, one by one.

Compactness of $\mathcal{M}$

We introduce the short notation

$$\langle \eta, g \rangle = \int_{\mathbb{R}^2} g d\eta, \qquad \eta \in \mathcal{M},\ g \in C_c.$$

The set C_c provided with the topology of uniform convergence admits a countable dense subset $\{g_n\}_{n\in\mathbb{N}}$. Since $|\langle \eta, g \rangle| \leq \max_{x\in\mathbb{R}^2} |g(x)| < +\infty$ holds for every $\eta \in \mathcal{M}$, $g \in C_c$, it follows that every sequence $\{\langle \eta_n, g \rangle\}_{n\in\mathbb{N}}$ is a real bounded sequence and therefore contains a convergent subsequence. Thus, we can pick $\left\{\eta_{n_k^1}\right\}_{k\in\mathbb{N}}$, a subsequence of $\{\eta_n \in \mathcal{M}\}_{n\in\mathbb{N}}$

such that $\left\{\left\langle \eta_{n_k^1}, g_1 \right\rangle\right\}_{k\in\mathbb{N}}$ is convergent. Repeating the same argument, for each $i \in \mathbb{N}$ we can pick $\left\{\eta_{n_k^{i+1}}\right\}_{k\in\mathbb{N}}$, a subsequence of $\left\{\eta_{n_k^i}\right\}_{k\in\mathbb{N}}$ such that $\left\{\left\langle \eta_{n_k^{i+1}}, g_{i+1} \right\rangle\right\}_{k\in\mathbb{N}}$ converges. It follows that $\left\{\eta_{n_k^k}\right\}_{k\in\mathbb{N}}$ is a subsequence of $\{\eta_n\}_{n\in\mathbb{N}}$ such that all the sequences

$$\left\{\left\langle \eta_{n_k^k}, g_i \right\rangle\right\}_{k\in\mathbb{N}}, \qquad i \in \mathbb{N}$$

converge. To see that all the sequences

$$\left\{\left\langle \eta_{n_k^k}, g \right\rangle\right\}_{k\in\mathbb{N}}, \qquad g \in C_c$$

converge, notice that

$$\begin{aligned}
&\left|\left\langle \eta_{n_k^k}, g \right\rangle - \left\langle \eta_{n_m^m}, g \right\rangle\right| \leq \\
&\leq \left|\left\langle \eta_{n_k^k}, g_i \right\rangle - \left\langle \eta_{n_m^m}, g_i \right\rangle\right| + \left|\left\langle \eta_{n_k^k}, g - g_i \right\rangle\right| + \left|\left\langle \eta_{n_m^m}, g - g_i \right\rangle\right| \leq \\
&\leq \left|\left\langle \eta_{n_k^k}, g_i \right\rangle - \left\langle \eta_{n_m^m}, g_i \right\rangle\right| + 2 \max_{x\in\mathbb{R}^2} \left|g(x) - g_i(x)\right|.
\end{aligned}$$

Since $\{g_i\}_{i\in\mathbb{N}}$ is dense in C_c, we see that $\left\{\left\langle \eta_{n_k^k}, g \right\rangle\right\}_{k\in\mathbb{N}}$ is a Cauchy sequence and therefore it is convergent.

This shows that the map $g \mapsto \lim \left\langle \eta_{n_k^k}, g \right\rangle$ is a well defined positive bounded linear functional in C_c. Thus, the Riesz representation theorem (see, e.g. Rudin (1987)) states that there is one unique regular positive measure η satisfying

$$\langle \eta, g \rangle = \lim \left\langle \eta_{n_k^k}, g \right\rangle, \qquad \forall g \in C_c.$$

Let $B = \{(y,z) : 0 \leq z \leq y\}$. Since $\left\langle \eta_{n_k^k}, g \right\rangle = 0$ holds whenever $\mathrm{Supp}(g) \cap B = \emptyset$, it is clear that $\eta\left(\mathbb{R}^2 \setminus B\right) = 0$. Thus, in order to show that $\eta \in \mathcal{M}$, we only need to show that $\eta(A \times \mathbb{R}) = \Pr\{Y \in A\}$ holds for every open set $A \subset \mathbb{R}$.

Fix an open set $A \subset \mathbb{R}$, a small $\varepsilon > 0$, and pick a compact set $B \subset A$ such that $\Pr\{Y \in A \setminus B\} < \varepsilon$. There is a function $g \in C_c$ such that

$$\chi_{\{(y,z):y\in B, 0\leq z\leq y\}} \leq g \leq \chi_{A\times\mathbb{R}}.$$

Then,

$$\lim \eta_{n_k^k}(A \times \mathbb{R}) \geq \lim \left\langle \eta_{n_k^k}, g \right\rangle = \langle \eta, g \rangle.$$

By taking a sequence g_k converging monotonically from below to $\chi_{A\times\mathbb{R}}$, we see that $\lim \eta_{n_k^k}(A \times \mathbb{R}) \geq \eta(A \times \mathbb{R})$. Also, $\lim \eta_{n_k^k}(A \times \mathbb{R}) \leq \lim \left\langle \eta_{n_k^k}, g \right\rangle + \varepsilon = \langle \eta, g \rangle + \varepsilon \leq \eta(A \times \mathbb{R}) + \varepsilon$. Making ε go to zero, we see that $\lim \eta_{n_k^k}(A \times \mathbb{R}) \leq \eta(A \times \mathbb{R})$.■

To prove lower semicontinuity of the risk measures, we use the following lemma:

Lemma 1 *Consider a sequence* $\{\eta_n \in \mathcal{M}\}_{n\in\mathbb{N}}$ *converging to* $\eta \in \mathcal{M}$*, continuous functions* $g : \mathbb{R}^2 \mapsto [0, +\infty[$*,* $h : \mathbb{R}^2 \mapsto \mathbb{R}$ *and a sequence* $\{a_n \in \mathbb{R}\}_{n\in\mathbb{N}}$ *converging to* $a \in \mathbb{R}$*. Then,*

$$\int_{\{(y,z):h(y,z)>a\}} g d\eta \leq \liminf \int_{\{(y,z):h(y,z)>a_n\}} g d\eta_n. \ \square$$

Proof. Fix $\varepsilon > 0$ and pick $\phi \in C_c$ such that $0 \leq \phi \leq \chi_{\{h>a+\varepsilon\}}$. Then,

$$\liminf \int_{\{h>a_n\}} g d\eta_n \geq \liminf \int_{\{h>a_n\}} \phi g d\eta_n = \liminf \int \phi g d\eta_n = \int \phi g d\eta.$$

Making ε go to zero it is possible to pick a monotonically increasing sequence $\{\phi_n\}_{n\in\mathbb{N}}$ converging to $\chi_{\{h>a\}}$. Using Lebesgue's monotone convergence theorem, we obtain the desired inequality. ■

Lower semicontinuity of $\eta \mapsto \hat{\mathrm{VaR}}_\eta(\alpha)$

Fix a sequence $\{\eta_n \in \mathcal{M}\}_{n\in\mathbb{N}}$ converging to $\eta \in \mathcal{M}$. Without loss of generality, we may assume that the sequence $\{\mathrm{VaR}_{\eta_n}(\alpha)\}_{n\in\mathbb{N}}$ converges (take a subsequence if necessary). By Lemma 1, we have

$$\eta\{y - z > \lim \mathrm{VaR}_{\eta_n}(\alpha)\} \leq \liminf \eta_n\{y - z > \mathrm{VaR}_{\eta_n}(\alpha)\} \leq \alpha.$$

This shows that $\mathrm{VaR}_\eta(\alpha) \leq \lim \mathrm{VaR}_{\eta_n}(\alpha)$ and hence $\eta \mapsto \mathrm{VaR}_\eta(\alpha)$ is lower semicontinuous.■

Lower semicontinuity of $\eta \mapsto \hat{\mathrm{CTE}}_\eta(\alpha)$

Suppose that $\lim \mathrm{CTE}_{\eta_n}(\alpha) < \mathrm{CTE}_\eta(\alpha)$. Then, we can pick $\varepsilon > 0$ such that $\mathrm{CTE}_{\eta_n}(\alpha) + \varepsilon < \mathrm{CTE}_\eta(\alpha)$ for every sufficiently large $n \in \mathbb{N}$. This is

$$\begin{aligned} &\frac{\alpha - \eta\{y - z > V\}}{\alpha} V + \frac{\int_{\{y-z>V\}} (y - z) d\eta}{\alpha} - \\ &- \frac{\alpha - \eta_n\{y - z > V_n\}}{\alpha} V_n - \frac{\int_{\{y-z>V_n\}} (y - z) d\eta_n}{\alpha} > \varepsilon, \end{aligned}$$

where $V = \mathrm{VaR}_\eta(\alpha)$, $V_n = \mathrm{VaR}_{\eta_n}(\alpha)$. Rearranging, the inequality above becomes

$$\begin{aligned} &(\alpha - \eta_n\{y - z > V_n\})(V - V_n) + \\ &+ \int_{\{y-z>V\}} y - z - V d\eta - \int_{\{y-z>V_n\}} y - z - V d\eta_n > \alpha\varepsilon. \end{aligned}$$

This implies

$$(\alpha - \eta_n\{y - z > V_n\})(V - V_n) +$$
$$+ \int_{\{y-z>V\}} y - z - (V - \frac{\alpha\varepsilon}{2}) d\eta - \int_{\{y-z>V_n\}} y - z - (V - \frac{\alpha\varepsilon}{2}) d\eta_n > \frac{\alpha\varepsilon}{2}.$$

Due to Lemma 1 and lower semicontinuity of $\eta \mapsto \mathrm{VaR}_\eta(\alpha)$, the limit of the left-hand term cannot be greater than zero while the right-hand side is strictly positive. This shows that $\lim \mathrm{CTE}_{\eta_n}(\alpha) \geq \mathrm{CTE}_\eta(\alpha)$ must hold and hence $\eta \mapsto \mathrm{CTE}_\eta(\alpha)$ is lower semicontinuous.■

Lower semicontinuity of $\eta \mapsto \hat{\mathrm{CTE}}_\eta^-(\alpha)$

Let V, V_n be as above. Fix $\delta > 0$ and notice that

$$\eta\{y - z \leq V + \delta\} \geq 1 - \alpha, \qquad \eta_n\{y - z < \lim V_k - \delta\} \leq 1 - \alpha,$$

for every sufficiently large $n \in \mathbb{N}$. It follows that

$$\begin{aligned}
&\eta\{V + \delta < y - z < \lim V_k - \delta\} = \\
=&\eta\{y - z < \lim V_k - \delta\} - \eta\{y - z \leq V + \delta\} \leq \\
\leq&\limsup \eta_n\{y - z < \lim V_k - \delta\} - \eta\{y - z \leq V + \delta <\} \leq \\
\leq&1 - \alpha - (1 - \alpha) = 0.
\end{aligned}$$

By making δ go to zero, we obtain

$$\eta\{\mathrm{VaR}_\eta(\alpha) < y - z < \limsup \mathrm{VaR}_{\eta_k}(\alpha)\} = 0. \tag{53}$$

Due to (53), we have

$$\begin{aligned}
CTE_\eta^-(\alpha) =& \frac{\int_{\{y-z\geq V\}}(y-z)d\eta}{\eta\{y - z \geq V\}} = \frac{\eta\{y - z = V\}V + \int_{\{y-z\geq \lim V_n\}}(y-z)d\eta}{\eta\{y - z = V\} + \eta\{y - z \geq \lim V_n\}} = \\
=& \frac{\eta\{y - z = V\}V + \eta\{y - z \geq \lim V_n\}(\lim V_n - \varepsilon)}{\eta\{y - z = V\} + \eta\{y - z \geq \lim V_n\}} + \\
&+ \frac{\int_{\{y-z>\lim V_n\}}(y - z - (\lim V_n - \varepsilon))d\eta}{\eta\{y - z = V\} + \eta\{y - z \geq \lim V_n\}}.
\end{aligned}$$

Takin $\varepsilon > 0$, using Lemma 1 and then making $\varepsilon \to 0^+$, we obtain

$$\begin{aligned}
&CTE_\eta^-(\alpha) \leq \\
\leq& \frac{\eta\{y - z = V\}V}{\eta\{y - z = V\} + \eta\{y - z \geq \lim V_n\}} + \\
&+ \frac{\eta\{y - z \geq \lim V_n\}\lim V_n + \lim_{k\to\infty} \int_{\{y-z>V_k\}}(y - z - \lim V_n)d\eta_k}{\eta\{y - z = V\} + \eta\{y - z \geq \lim V_n\}}.
\end{aligned} \tag{54}$$

Now,

$$\eta\{y-z\geq \lim V_n\}\lim V_n+\lim_{k\to\infty}\int_{\{y-z>V_k\}}(y-z-\lim V_n)d\eta_k=$$

$$=\lim V_n\left(\eta\{y-z\geq \lim V_n\}-\lim_{k\to\infty}\eta_k\{y-z\geq V_k\}\right)+$$

$$+\lim \eta_k\{y-z\geq V_k\}V_k+\lim_{k\to\infty}\int_{\{y-z>V_k\}}(y-z-V_k)d\eta_k=$$

$$=\lim V_n\left(\eta\{y-z\geq \lim V_n\}-\lim_{k\to\infty}\eta_k\{y-z\geq V_k\}\right)+\lim_{k\to\infty}\int_{\{y-z\geq V_k\}}(y-z)d\eta_k.$$

Hence, the right-hand-side of (54) is a convex combination of numbers no greater than $\lim \frac{\int_{\{y-z\geq V_n\}}(y-z)d\eta_n}{\eta_n\{y-z\geq V_n\}}$. This proves that $\mathrm{CTE}_\eta^-(\alpha)\leq \lim \mathrm{CTE}_{\eta_n}^-(\alpha)$, therefore $\eta\mapsto \mathrm{CTE}_\eta^-(\alpha)$ is lower semicontinuous.■

Lower semicontinuity of $\hat{P}(\eta)=\gamma\left(\int_{\mathbb{R}^2}(z,z^2,\ldots,z^m)d\eta\right)$

For each $M\in(0,+\infty)$ let $g_M\in C_c$ denote a function taking values in the interval $[0,1]$ such that $g_M(y,z)=1$ whenever $0\leq z\leq y\leq M$. Fix $\eta\in\mathcal{M}$ and a sequence $\{\eta_n\in\mathcal{M}\}_{n\in\mathbb{N}}$, converging to η. Then, for every $M\in(0,+\infty)$, we have

$$\liminf_{n\to\infty}\langle\eta_n,z^m\rangle\geq\liminf_{n\to\infty}\langle\eta_n,z^m g_M\rangle=\langle\eta,z^m g_M\rangle.$$

Making M go to ∞ and using Lebesgue's monotone convergence theorem, we see that $\liminf_{n\to\infty}\langle\eta_n,z^m\rangle\geq\langle\eta,z^m\rangle$, i.e., the map $\eta\mapsto\langle\eta,z^m\rangle$ is lower semicontinuous.
If $\lim_{n\to\infty}\langle\eta_n,z^m\rangle=+\infty$ then, uniform convergence of γ implies $\lim\hat{P}(\eta_n)=+\infty\geq\hat{P}(\eta)$. Suppose instead that $\{\langle\eta_n,z^m\rangle\}_{n\in\mathbb{N}}$ is bounded. For each $M\in(0,+\infty)$, we have

$$\langle\eta_n,z^m\rangle\geq\int_{\{(y,z):y\geq M\}}z^m d\eta_n\geq M\int_{\{(y,z):y\geq M\}}z^{m-1}d\eta_n.$$

Hence,

$$\left\langle\eta_n,z^{m-1}\right\rangle=\left\langle\eta_n,z^{m-1}g_M\right\rangle+\left\langle\eta_n,z^{m-1}(1-g_M)\right\rangle\leq$$

$$\leq\left\langle\eta_n,z^{m-1}g_M\right\rangle+\int_{\{(y,z):y\geq M\}}z^{m-1}d\eta_n\leq\left\langle\eta_n,z^{m-1}g_M\right\rangle+\frac{Const.}{M}.$$

Since $\lim_{n\to\infty}\left\langle\eta_n,z^{m-1}g_M\right\rangle=\left\langle\eta,z^{m-1}g_M\right\rangle\leq\left\langle\eta,z^{m-1}\right\rangle$ and M can be made arbitrarily large, it follows that

$$\lim_{n\to\infty}\left\langle\eta_n,z^{m-1}\right\rangle=\left\langle\eta,z^{m-1}\right\rangle.$$

Thus, using the continuity and monotonicity of γ, we obtain

$$\begin{aligned}\liminf_{n\to\infty}\hat{P}(\eta_n)=&\gamma\left(\langle\eta,z\rangle,\ldots,\left\langle\eta,z^{m-1}\right\rangle,\liminf_{n\to\infty}\langle\eta_k,z^m\rangle\right)\geq\\ \geq&\gamma\left(\langle\eta,z\rangle,\ldots,\left\langle\eta,z^{m-1}\right\rangle,\langle\eta,z^m\rangle\right)=\hat{P}(\eta),\end{aligned}$$

which proves lower semicontinuity of (16).■

Lower semicontinuity of $\hat{P}(\eta)=\int_0^{+\infty}w(\eta\{z>t\})dt$

Fix $\eta\in\mathcal{M}$ and a sequence $\{\eta_n\in\mathcal{M}\}_{n\in\mathbb{N}}$, converging to η. For each $b\in(0,+\infty)$, we have

$$\lim\eta_n(\mathbb{R}\times[0,b])=\eta(\mathbb{R}\times[0,b]).$$

Also, for every $M\in(0,+\infty)$, we have

$$\hat{P}(\eta_n)\geq\int_0^M w\left(1-\eta_n(\mathbb{R}\times[0,t]\right)dt,$$

hence, the dominated convergence theorem guarantees

$$\liminf_{n\to\infty}\hat{P}(\eta_n)\geq\int_0^M w\left(\eta(\mathbb{R}\times(t,+\infty)\right)dt.$$

Thus, making $M\to+\infty$, one obtains $\liminf_{n\to\infty}\hat{P}(\eta_n)\geq\hat{P}(\eta)$, which proves lower semicontinuity of (17).■

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