Chapter 1
Generic Hamiltonian dynamical systems: an overview

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Abstract We present for a general audience the state of the art on the generic properties of C^2 Hamiltonian dynamical systems.

1.1 Introduction and main definitions

Hamiltonian systems form a fundamental subclass of dynamical systems. Their importance follows from the vast range of applications throughout different branches of science. Generic properties of such systems are thus of great interest since they give us the "typical" behaviour (in some appropriate sense) that one could expect from the class of models at hand (cf. [38]). There are, of course, considerable limitations to the amount of information one can extract from a specific system by looking at generic cases. Nevertheless, it is of great utility to learn that a selected model can be slightly perturbed in order to obtain dynamics we understand in a reasonable way.

1.1.1 Residual sets and generic properties

A residual set is a countable intersection of dense open sets. The elements of a residual set are called generic. A property that holds within a residual set is also refered as generic.

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A Baire space is a topological space with the property that residual sets are dense. The space of C^s, $s \in \mathbb{N} \cup \{0\}$, functions on a manifold is Baire.

1.1.2 Hamiltonian dynamics

Let M be a $2d$-dimensional smooth manifold endowed with a symplectic structure, i.e. a closed and nondegenerate 2-form ω. The pair (M, ω) is called a symplectic manifold which is also a volume manifold by Liouville's theorem. Let μ be the so-called Lebesgue measure associated to the volume form $\omega^d = \omega \wedge \cdots \wedge \omega$.

A diffeomorphism $g\colon (M, \omega) \to (N, \omega')$ between two symplectic manifolds is called a symplectomorphism if $g^*\omega' = \omega$. The action of a diffeomorphism on a 2-form is given by the pull-back $(g^*\omega')(X,Y) = \omega'(g_*X, g_*Y)$. Here X and Y are vector fields on M and the push-forward $g_*X = DgX$ is a vector field on N. Notice that a symplectomorphism $g\colon M \to M$ preserves the Lebesgue measure μ since $g^*\omega^d = \omega^d$.

For any smooth Hamiltonian function $H\colon M \to \mathbb{R}$ there is a corresponding Hamiltonian vector field $X_H\colon M \to TM$ determined by $\iota_{X_H}\omega = dH$ being exact, where $\iota_v \omega = \omega(v, \cdot)$ is a 1-form. Notice that H is C^s iff X_H is C^{s-1}. The Hamiltonian vector field generates the Hamiltonian flow, a smooth 1-parameter group of symplectomorphisms φ_H^t on M satisfying $\frac{d}{dt}\varphi_H^t = X_H \circ \varphi_H^t$ and $\varphi_H^0 = \mathrm{id}$. Since $dH(X_H) = \omega(X_H, X_H) = 0$, X_H is tangent to the energy level sets $H^{-1}(\{e\})$, for some energy value $e \in H(M)$.

If $v \in T_xH^{-1}(\{e\})$, i.e. $dH(v)(x) = \omega(X_H, v)(x) = 0$, then its push-forward by φ_H^t is again tangent to $H^{-1}(\{e\})$ on $\varphi_H^t(x)$ since

$$dH(D\varphi_H^t v)(\varphi_H^t(x)) = \omega(X_H, D\varphi_H^t v)(\varphi_H^t(x)) = {\varphi_H^t}^*\omega(X_H, v)(x) = 0.$$

We consider also the tangent flow $D\varphi_H^t\colon TM \to TM$ that satisfies the linear variational equation (the linearized differential equation)

$$\frac{d}{dt} D\varphi_H^t = DX_H(\varphi_H^t) D\varphi_H^t$$

with $DX_H\colon M \to TTM$.

We say that x is a *regular* point if $dH(x) \neq 0$ (x is not critical). We denote the set of regular points by $\mathscr{R}(H)$ and the set of critical points by $\mathrm{Crit}(H)$. We call $H^{-1}(\{e\})$ a regular energy level of H if $H^{-1}(\{e\}) \cap \mathrm{Crit}(H) = \emptyset$. A regular energy surface is a connected component of a regular energy level.

Given any regular energy level or surface $\mathscr{E}$, we induce a volume form $\omega_{\mathscr{E}}$ on the $(2d-1)$-dimensional manifold $\mathscr{E}$ in the following way. For each $x \in \mathscr{E}$,

$$\omega_{\mathscr{E}}(x) = \iota_Y \omega^d(x) \quad \text{on } T_x\mathscr{E}$$

defines a $(2d-1)$ non-degenerate form if $Y \in T_x M$ satisfies $dH(Y)(x) = 1$. Notice that this definition does not depend on Y (up to normalization) as long as it is transversal to $\mathscr{E}$ at x. Moreover,

$$dH(D\varphi_H^t Y)(\varphi_H^t(x)) = d(H \circ \varphi_H^t)(Y)(x) = 1.$$

Thus, $\omega_{\mathscr{E}}$ is φ_H^t-invariant, and the measure $\mu_{\mathscr{E}}$ induced by $\omega_{\mathscr{E}}$ is again invariant. In order to obtain finite measures, we need to consider compact energy levels.

On the manifold M we also fix any Riemannian structure which induces a norm $\|\cdot\|$ on the fibers $T_x M$. We will use the standard norm of a bounded linear map A given by $\|A\| = \sup_{\|v\|=1} \|A\,v\|$ and also the co-norm defined by $\mathbf{m}(A) = \|A^{-1}\|^{-1}$.

The symplectic structure guarantees by Darboux theorem the existence of an atlas $\{h_j \colon U_j \to \mathbb{R}^{2d}\}$ satisfying $h_j^* \omega_0 = \omega$ with

$$\omega_0 = \sum_{i=1}^{d} dy_i \wedge dy_{d+i}. \tag{1.1}$$

On the other hand, when dealing with volume manifolds (N, Ω) of dimension p, Moser's theorem [30] gives an atlas $\{h_j \colon U_j \to \mathbb{R}^p\}$ such that $h_j^*(dy_1 \wedge \cdots \wedge dy_p) = \Omega$.

For more on the general symplectic and Hamiltonian theories, see e.g. [1].

1.1.3 Our setting

In the following we will always assume that M is a $2d$-dimensional compact smooth symplectic manifold with a smooth boundary ∂M (including the case $\partial M = \emptyset$) and $d \geq 2$. Furthermore, C^s Hamiltonians are real-valued functions on M that are constant on each connected component of ∂M. We denote by $C^s(M)$ the set of C^s Hamiltonians. This set is endowed with the C^2-topology.

Under these conditions, the Hamiltonian flow is globally defined with respect to time because H is constant on the components of ∂M or, equivalently, X_H is tangent to ∂M.

1.1.4 Transversal linear Poincaré flow

Given any regular point x we take the orthogonal splitting $T_x M = \mathbb{R} X_H(x) \oplus N_x$, where $N_x = (\mathbb{R} X_H(x))^{\perp}$ is the normal fiber at x. Consider the automorphism of vector bundles

$$\begin{aligned} D\varphi_H^t \colon T_{\mathscr{R}} M &\to T_{\mathscr{R}} M \\ (x, v) &\mapsto (\varphi_H^t(x), D\varphi_H^t(x)\, v). \end{aligned} \tag{1.2}$$

Of course, in general, the subbundle $N_{\mathscr{R}}$ is not $D\varphi_H^t$-invariant. So we relate to the $D\varphi_H^t$-invariant quotient space $\widetilde{N}_{\mathscr{R}} = T_{\mathscr{R}}M/\mathbb{R}X_H(\mathscr{R})$ with an isomorphism $\phi_1 : N_{\mathscr{R}} \to \widetilde{N}_{\mathscr{R}}$. The unique map

$$P_H^t : N_{\mathscr{R}} \to N_{\mathscr{R}}$$

such that $\phi_1 \circ P_H^t = D\varphi_H^t \circ \phi_1$ is called the *linear Poincaré flow* for H. Denoting by $\Pi_x : T_xM \to N_x$ the canonical orthogonal projection, the linear map $P_H^t(x) : N_x \to N_{\varphi_H^t(x)}$ is

$$P_H^t(x)\,v = \Pi_{\varphi_H^t(x)} \circ D\varphi_H^t(x)\,v.$$

We now consider

$$\mathscr{N}_x = N_x \cap T_xH^{-1}(e),$$

where $T_xH^{-1}(e) = \ker dH(x)$ is the tangent space to the energy level set with $e = H(x)$. Thus, $\mathscr{N}_{\mathscr{R}}$ is invariant under P_H^t. So we define the map

$$\Phi_H^t : \mathscr{N}_{\mathscr{R}} \to \mathscr{N}_{\mathscr{R}}, \qquad \Phi_H^t = P_H^t|_{\mathscr{N}_{\mathscr{R}}},$$

called the *transversal linear Poincaré flow* for H such that

$$\Phi_H^t(x) : \mathscr{N}_x \to \mathscr{N}_{\varphi_H^t(x)}, \quad \Phi_H^t(x)\,v = \Pi_{\varphi_H^t(x)} \circ D\varphi_H^t(x)\,v$$

is a linear symplectomorphism for the symplectic form induced on $\mathscr{N}_{\mathscr{R}}$ by ω.

1.1.5 Oseledets theorem

Take $H \in C^2(M)$. Since the time-1 map of any tangent flow derived from a Hamiltonian vector field is measure preserving, we obtain a version of Oseledets theorem for Hamiltonian systems. Given a point $x \in M$ we say that x is *Oseledets regular* if there exists a splitting $T_xM = E_x^1 \oplus \dots E_x^{k(x)}$ and numbers $\lambda^1(x) \geq \cdots \geq \lambda^{k(x)}(x)$ such that for any (non-zero) vector $v \in E_x^i$ we have

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|D\varphi_H^t(x)\,v\| = \lambda^i(x).$$

The Oseledets theorem [32] asserts that Oseledets regular points form a η-full measure set for any φ_H^t-invariant probability measure η.

Moreover,

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \sin \alpha_t = 0, \tag{1.3}$$

where α_t is the angle at time t between any subspaces of the splitting.

The splitting of the tangent bundle is called *Oseledets splitting* and the real numbers $\lambda^i(H,x)$ are called the *Lyapunov exponents*. The full measure set of the *Oseledets points* is denoted by $\mathscr{O}(H) = \mathscr{O}$.

The vector field direction $\mathbb{R}X_H(x)$ is trivially an Oseledets's direction with zero Lyapunov exponent.

If $x \in \mathscr{R} \cap \mathscr{O}$ and $\lambda^i(x) \neq 0$, the Oseledets splitting on T_xM induces a $\Phi_H^t(x)$-invariant splitting on $\mathscr{N}_x$ where $\mathscr{N}_x^i = \Pi_x(E_x^i)$.

The next lemma makes explicit that the dynamics of $D\varphi_H^t$ and Φ_H^t are coherent so that the Lyapunov exponents for both cases are related. The proof uses (1.3).

Lemma 1 ([8]). *Given $x \in \mathscr{R} \cap \mathscr{O}$, the Lyapunov exponents of the Φ_H^t-invariant decomposition are equal to the ones of the $D\varphi_H^t$-invariant decomposition.*

We now restate the Oseledets theorem for the dynamic cocycle Φ_H^t: For μ-a.e. $x \in M$ there exists a splitting of the normal bundle $\mathscr{N}_x = \mathscr{N}_x^1 \oplus \cdots \oplus \mathscr{N}_x^{k(x)}$ and numbers $\lambda^1(x) \geq \cdots \geq \lambda^{k(x)}(x)$ such that for any (non-zero) vector $v \in \mathscr{N}_x^i$ we have

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \|\Phi_H^t(x)\, v\| = \lambda^i(x).$$

Observe that there exist at most $2d-2$ different exponents for Φ_H^t. Moreover, the Lyapunov exponents of Φ_H^t are symmetric (i.e. if λ is one the the exponents, then $-\lambda$ is also one of the exponents and their multiplicity is the same). Finally, $\dim(\mathscr{N}_x^+) = \dim(\mathscr{N}_x^-)$ and since $\dim(\mathscr{N}_x^+)$ is even we obtain that $\dim(\mathscr{N}_x^0)$ is also even.

1.1.6 Hyperbolicity and dominated splitting

Let $H \in C^2(M)$. Given any compact and φ_H^t-invariant set $\Lambda \subset H^{-1}(e)$, we say that Λ is a *hyperbolic set* for φ_H^t if there exist $m \in \mathbb{N}$ and a $D\varphi_H^t$-invariant splitting $T_\Lambda H^{-1}(e) = E_\Lambda^+ \oplus E_\Lambda^- \oplus E_\Lambda$ such that for all $x \in \Lambda$ we have:

- $\|D\varphi_H^m(x)|_{E_x^-}\| \leq \frac{1}{2}$ (uniform contraction),
- $\|D\varphi_H^{-m}(x)|_{E_x^+}\| \leq \frac{1}{2}$ (uniform expansion),
- E includes the directions of the vector field and of the gradient of H.

Similarly, we can define a hyperbolic structure for the transversal linear Poincaré flow Φ_H^t. We say that Λ is hyperbolic for Φ_H^t on Λ if $\Phi_H^t|_\Lambda$ is a hyperbolic vector bundle automorphism. The next lemma relates the hyperbolicity for Φ_H^t with the hyperbolicity for φ_H^t. It is an immediate consequence of a result by Doering [22] for the linear Poincaré flow extended to our Hamiltonian setting and the transversal linear Poincaré flow.

Lemma 2. *Let Λ be an φ_H^t-invariant and compact set. Then Λ is hyperbolic for φ_H^t iff Λ is hyperbolic for Φ_H^t.*

We now consider a weaker form of hyperbolicity. Let $\Lambda \subset M$ be an φ_H^t-invariant set and $m \in \mathbb{N}$. A splitting of the bundle $\mathscr{N}_\Lambda = \mathscr{N}_\Lambda^1 \oplus \mathscr{N}_\Lambda^2$ is an *m-dominated splitting* for the transversal linear Poincaré flow if it is Φ_H^t-invariant and continuous such that

$$\frac{\|\Phi_H^m(x)|\mathscr{N}_x^2\|}{\mathbf{m}(\Phi_H^m(x)|\mathscr{N}_x^1)} \leq \frac{1}{2}, \qquad \text{for all } x \in \Lambda. \tag{1.4}$$

We call $\mathscr{N}_\Lambda = \mathscr{N}_\Lambda^1 \oplus \mathscr{N}_\Lambda^2$ a *dominated splitting* if it is m-dominated for some $m \in \mathbb{N}$.

If Λ has a dominated splitting, then we may extend the splitting to its closure, except to critical points. Moreover, the angle between $\mathscr{N}^1$ and $\mathscr{N}^2$ is bounded away from zero on Λ. Under the four-dimensional assumption the decomposition is unique. For more details about dominated splitting see [20].

We say that a dominated splitting $\mathscr{N}_\Lambda = \mathscr{N}_\Lambda^- \oplus \mathscr{N}_\Lambda^0 \oplus \mathscr{N}_\Lambda^+$ over the set Λ is *partially hyperbolic* if the bundle $\mathscr{N}_\Lambda^-$ is uniformly contractive and the bundle $\mathscr{N}_\Lambda^+$ is uniformly expanding.

The proof of the next lemma (see [8, Lemma 2.6]) hints to the fact that the four-dimensional setting is crucial in obtaining hyperbolicity from the dominated splitting structure.

Lemma 3. *Let $H \in C^2(M)$ and a regular energy surface $\mathscr{E}$. If $\Lambda \subset \mathscr{E}$ has a dominated splitting for Φ_H^t, then $\overline{\Lambda}$ is hyperbolic.*

Actually, the previous lemma is a version of the following general fact proved in [18, Theorem 11] which we trivially adapt for Hamiltonians.

Theorem 1. *Let $H \in C^2(M)$ and let $\mathscr{N}_\Lambda = \mathscr{N}_\Lambda^1 \oplus \mathscr{N}_\Lambda^2$ be a dominated splitting over a φ_H^t-invariant set Λ. Assume that $\dim \mathscr{N}_\Lambda^1 \leq \dim \mathscr{N}_\Lambda^2$ and let $\mathscr{N}_\Lambda^+ = \mathscr{N}_\Lambda^1$. Then $\mathscr{N}_\Lambda^2$ splits invariantly as $\mathscr{N}_\Lambda^0 \oplus \mathscr{N}_\Lambda^-$ with $\dim \mathscr{N}_\Lambda^+ = \mathscr{N}_\Lambda^-$, and the splitting $\mathscr{N}_\Lambda = \mathscr{N}_\Lambda^+ \oplus \mathscr{N}_\Lambda^0 \oplus \mathscr{N}_\Lambda^-$ is partially hyperbolic.*

1.1.7 Elliptic, parabolic and hyperbolic closed orbits

Let $\Gamma \subset M$ be a closed orbit of least period τ. The characteristic multipliers of Γ are the eigenvalues of $\Phi_H^\tau(p)$, which are independent of the point $p \in \Gamma$. We say that Γ is

- *k- elliptic* iff $2k$ characteristic multipliers are simple, non-real and of modulus 1;
- *parabolic* iff the characteristic multipliers are real and of modulus 1;
- *hyperbolic* iff the characteristic multipliers have modulus different from 1.

We call $d-1$-elliptic orbits total elliptic. In case $d = 2$ we have that 1-elliptic are total.

It is clear that under small perturbations, d-elliptic and hyperbolic orbits are stable whilst parabolic ones are unstable.

We refer to a point in a closed orbit as periodic. Periodic points are classified in the same way as the respective closed orbit.

1.1.8 Perturbation lemmas

We include here several perturbation results in our setting. The first is the celebrated Pugh's closing lemma [37, §9]:

Theorem 2 (Pugh's closing lemma). *If $\varepsilon > 0$ and $x \in M$ is a recurrent point for the flow φ_H^t associated to $H \in C^2(M)$, then there exists $\widetilde{H} \in C^2(M)$ ε-C^2-close to H such that x is a periodic point for $\varphi_{\widetilde{H}}^t$.*

An important upgrade is the Arnaud's closing lemma [4]. It states that the orbit of a non-wandering point can be approximated for a very long time by a closed orbit of a nearby Hamiltonian.

Theorem 3 (Arnaud's closing lemma). *Let $H \in C^s(M)$, $2 \leq s \leq \infty$, a non-wandering point $x \in M$ and $\varepsilon, r, \tau > 0$. Then, we can find $\widetilde{H} \in C^s(M)$ ε-C^2-close to H, a closed orbit Γ of $\widetilde{H}$ with least period ℓ, $p \in \Gamma$ and a map $g\colon [0,\tau] \to [0,\ell]$ close to the identity such that:*

- $\operatorname{dist}\left(\varphi_H^t(x), \varphi_{\widetilde{H}}^{g(t)}(p)\right) < r$, $0 \leq t \leq \tau$, *and*
- $H = \widetilde{H}$ *on* $M \setminus A$, *where* $A = \bigcup_{0 \leq t \leq \ell}\left(B(p,r) \cap B(\varphi_{\widetilde{H}}^t(p), r)\right)$.

The next theorem is a version of Franks' lemma for Hamiltonians proved by Vivier [41]. Roughly, it says that we can realize a Hamiltonian corresponding to a given perturbation of the transversal linear Poincaré flow. It is proved for $2d$-dimensional manifolds with $d \geq 2$.

Theorem 4 (Vivier's lemma). *Let $H \in C^s(M)$, $2 \leq s \leq \infty$, $\varepsilon, \tau > 0$ and $x \in M$. There exists $\delta > 0$ such that for any flowbox V of an injective arc of orbit $\Sigma = \varphi_H^{[0,t]}(x)$, $t \geq \tau$, and a transversal symplectic δ-perturbation F of $\Phi_H^t(x)$, there is $\widetilde{H} \in C^{\max\{2,s-1\}}(M)$ ε-C^2-close to H satisfying:*

- $\Phi_{\widetilde{H}}^t(x) = F$,
- $H = \widetilde{H}$ *on* $\Sigma \cup (M \setminus V)$.

In order to perform local perturbations to our original Hamiltonians, we need an improved version of a lemma by Robinson [39] that provides us with symplectic flowbox coordinates. Consider the canonical symplectic form on $\mathbb{R}^{2d}$ given by ω_0 as in (1.1). The Hamiltonian vector field of any smooth $H\colon \mathbb{R}^{2d} \to \mathbb{R}$ is then

$$X_H = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \nabla H,$$

where I is the $d \times d$ identity matrix. Let the Hamiltonian function $H_0\colon \mathbb{R}^{2d} \to \mathbb{R}$ be given by $y \mapsto y_{d+1}$, so that

$$X_{H_0} = \frac{\partial}{\partial y_1}.$$

Theorem 5 (Symplectic flowbox coordinates [8]). *Let $H \in C^s(M)$, $2 \leq s \leq \infty$, and $x \in M$. If $x \notin \operatorname{Crit}(M)$, there exists a neighborhood U of x and a local C^{s-1}-symplectomorphism $g\colon (U,\omega) \to (\mathbb{R}^{2d}, \omega_0)$ such that $H = H_0 \circ g$ on U.*

1.2 Abundance of zero Lyapunov exponents away from hyperbolicity

The computation of Lyapunov exponents is one of the main problems in the modern theory of dynamical systems. They give us fundamental information on the asymptotic exponential behaviour of the linearized system. It is therefore important to understand these objects in order to study the time evolution of orbits. In particular, Pesin's theory deals with non-vanishing Lyapunov exponents systems (non-uniformly hyperbolic). This setting jointly with a C^α regularity, $\alpha > 0$, of the tangent map allows us to derive a very complete geometric picture of the dynamics (stable/unstable invariant manifolds). On the other hand, if we aim at understanding both local and global dynamics, the presence of zero Lyapunov exponents creates lots of obstacles. An example is the case of conservative systems: using enough differentiability, the celebrated KAM theory guarantees persistence of invariant quasiperiodic motion on tori yielding zero Lyapunov exponents.

In this section we study the dependence of the Lyapunov exponents on the dynamics of Hamiltonian flows. For a survey of the theory see [18] and references therein. In Theorem 6 we state that zero Lyapunov exponents for four-dimensional Hamiltonian systems are very common, at least for a C^2-residual subset. This picture changes radically for the C^∞ topology, the setting of most common Hamiltonian systems coming from applications. In this case Markus and Meyer showed that there exists a residual of C^∞ Hamiltonians neither integrable nor ergodic [28].

Theorem 6 ([8]). *Let $d = 2$. For a C^2-generic Hamiltonian $H \in C^2(M)$, the union of the regular energy surfaces $\mathscr{E}$ that are either Anosov or have zero Lyapunov exponents $\mu_{\mathscr{E}}$-a.e. for the Hamiltonian flow, forms an open μ-mod 0 and dense subset of M.*

Geodesic flows on negative curvature surfaces are well-known systems yielding Anosov energy levels. An example of a mechanical system which is Anosov on each positive energy level was obtained by Hunt and MacKay [25].

Another dichotomy result for the transversal linear Poincaré flow on the tangent bundle is the following:

Theorem 7 ([8]). *Let $d = 2$. There exists a C^2-dense subset $\mathfrak{D}$ of $C^2(M)$ such that, if $H \in \mathfrak{D}$, there exists an invariant decomposition $M = D \cup Z \pmod 0$ satisfying:*

- *$D = \bigcup_{n\in\mathbb{N}} D_{m_n}$, where D_{m_n} is a set with m_n-dominated splitting for the transversal linear Poincaré flow of H,*
- *the Hamiltonian flow of H has zero Lyapunov exponents for $x \in Z$.*

The proof of the above theorems is based on a result that allows us to decay the Lyapunov exponents of points without dominated splitting. This is possible by first constructing a local perturbation in the coordinates given by Lemma 5, that mixes the transversal directions of non-zero Lyapunov exponents along an orbit segment. Thus the effects of contraction and expansion average out.

The following problem is the generalization of the recent result by Bochi [15] to our context.

Open problem 1. *Show that Theorem 7 holds for $d > 2$.*

1.3 Denseness of elliptic points away from hyperbolicity

In this section we recall a related C^2-generic dichotomy by Newhouse [31]: for a C^2-generic Hamiltonian, an energy surface through any $p \in M$ is Anosov or is in the closure of 1-elliptical periodic orbits.

The Newhouse dichotomy was first proved for C^1-generic symplectomorphisms in [31], and extensions have appeared afterwards [3, 40, 24]. Those were all done for discrete-time dynamics.

Theorem 8 ([9]). *Let $d = 2$. Given $\varepsilon > 0$ and an open subset $U \subset M$, if $H \in C^2(M)$ has a far from Anosov regular energy surface intersecting U, then there is $\widetilde{H} \in C^\infty(M)$ ε-C^2-close to H having a closed elliptic orbit through U.*

The above theorem is proved in [9] (see [10] for divergence-free 3-flows) by looking first at the case of hyperbolic closed orbits with a small angle between the stable and unstable directions. Those are then showed to become elliptic by a small perturbation. On the other hand, for hyperbolic closed orbits with large angles and without dominated splitting, an adaptation of Mañé's perturbation techniques [10] leads again to elliptic orbits by a perturbation. The remaining case of hyperbolic closed orbits with dominated splitting and large angle is not true generically (as the case of parabolic ones).

As an almost direct consequence we arrive at the Newhouse dichotomy for four-dimensional Hamiltonians. Recall that for a C^2-generic Hamiltonian all but finitely many points are regular.

Theorem 9 ([9]). *Let $d = 2$. For a C^2-generic $H \in C^2(M)$, the union of the Anosov regular energy surfaces and the closed elliptic orbits, forms a dense subset of M.*

Open problem 2. *Prove the related result for $d > 2$: For a C^2-generic Hamiltonian, the union of the partially hyperbolic regular energy surfaces and the closed elliptic orbits, forms a dense subset of M.*

1.4 Star energy surfaces

Consider the set $\mathscr{M} = M \times C^2(M)$ endowed with the standard product topology. Given $(p,H) \in \mathscr{M}$, we denote by $\mathscr{E}_{p,H}$ the energy surface in $H^{-1}(H(p))$ containing p. We say that $\mathscr{E}_{p,H}$ is a *star energy surface* if it is regular and there exists a neighbourhood $\mathscr{U}$ of (p,H) such that all energy surfaces $\mathscr{E}_{\widetilde{p},\widetilde{H}}$, with $(\widetilde{p},\widetilde{H}) \in \mathscr{U}$, are regular and have all closed orbits hyperbolic.

Denote by $\mathscr{G}$ the set of $(p,H) \in \mathscr{M}$ such that $\mathscr{E}_{p,H}$ is star, and by $\mathscr{A}$ if $\mathscr{E}_{p,H}$ is Anosov. If there exists a homeomorphism between $\mathscr{E}_{p,H}$ and any nearby $\mathscr{E}_{\widetilde{p},\widetilde{H}}$ preserving orbits and their orientations, we say that (p,H) is *structurally stable*, i.e. $(p,H) \in \mathscr{S}$.

The next theorem is classical in the theory of dynamical systems, namely Anosov systems are open and structurally stable (see e.g. [13]).

Theorem 10. *Let $d \geq 2$. $\mathscr{A}$ is open and $\mathscr{A} \subset \mathscr{S}$.*

In the $d = 2$ case, there is already a good characterization of Anosov energy surfaces.

Theorem 11 ([13]). *$\mathscr{G} = \mathscr{A} = \mathscr{S}$ for $d = 2$.*

In rough terms the proof of the previous theorem goes as follows. By Lemma 3, in the four-dimensional context, dominated splitting is tantamount to hyperbolicity. So, we are left to show that in the absence of domination it is possible to create a non-hyperbolic closed orbit by an arbitrary small C^2 perturbation of the Hamiltonian.

Assume that we do not have dominated splitting (cannot be Anosov) and we still have the star property. We claim that we must be far from systems exhibiting elliptic closed orbits, and moreover we must have good uniform constants of hyperbolicity over closed orbits. Since we do not have domination, we use the ideas from the proof of Theorem 6 to obtain an Oseledets regular point with (almost) zero exponents. Then, the closing lemma (Theorem 3) produce a closed orbit without good constants of hyperbolicity, contradicting our assumption.

We say that (p,H) is *isolated in the boundary of $\mathscr{A}$* if $\mathscr{E}_{p,H}$ is not Anosov but any nearby $\mathscr{E}_{\widetilde{p},\widetilde{H}}$ such that $H \neq \widetilde{H}$ or $\widetilde{p} \notin \mathscr{E}_{p,H}$ is Anosov. As a consequence of Theorem 11, we obtain the following.

Corollary 1. *Let $d = 2$. The boundary of $\mathscr{A}$ has no isolated points.*

Open problem 3. *Show that Theorem 11 holds for $d > 2$.*

1.5 Robust transitivity

We say that a dynamical system is *transitive* if it has a dense orbit. Moreover, it is C^r*-robustly transitive* if in addition any arbitrarily C^r-close system is transitive.

Theorem 12 (Horita-Tahzibi [24]). *Any robustly transitive symplectomorphism defined in a compact symplectic manifold is partially hyperbolic.*

Working in the Hamiltonian context, we have that a regular energy surface is *transitive* if it has a dense orbit, and it is *robustly transitive* if the restriction of any sufficiently C^2-close Hamiltonian to a nearby regular energy surface is still transitive.

Theorem 13 (Vivier [41]). *Let $d = 2$. Any Hamiltonian admitting a robustly transitive regular energy surface is Anosov on that surface.*

We observe that the proof of this theorem uses the Hamiltonian version of Franks' lemma (Lemma 4).

It is easy to see that Theorem 8 also implies Theorem 13. In fact, if a regular energy surface $\mathscr{E}$ of $H \in C^2(M)$ is far from Anosov, then by Theorem 8 there exists a C^2-close C^∞-Hamiltonian with an elliptic closed orbit on a nearby regular energy surface. This invalidates the chance of robust transitivity for H according to a KAM-type criterium (see [41, Corollary 9]).

Taking into account Theorem 1 we get the following question.

Open problem 4. *Let $d > 2$. Show that if a Hamiltonian admits a robustly transitive regular energy surface, then it is partially hyperbolic there.*

1.6 Genericity of dense orbits

It follows from Poincaré's recurrence theorem that, in the volume-preserving context, almost any point is recurrent. However, the points can be restricted to some region of the manifold both for the past and for the future. The problem of knowing if a given dynamical system exhibits only one "piece" or, in other words, if there is any dense orbit, is a central problem in the modern theory of dynamical systems. A partial answer to this problem was given by Bonatti and Crovisier in [19] for the volume-preserving discrete-time case and by the same authors and Arnaud in the symplectomorphism framework [5]. They proved that for some C^1-residual subset any map has a dense orbit.

In the continuous-time case the first author proved in [7] the corresponding version for divergence-free flows, and recently Ferreira announced the following result.

Theorem 14 ([23]). *For a C^2-generic Hamiltonian H and $e \in H(M)$, we have that $H^{-1}(\{e\})$ has a transitive energy surface.*

Theorem 14 is a central tool in order to obtain important results in the generic theory of Hamiltonians (e.g. Open Problems 2, 3 and 4).

The main tool to conclude the proof of the previous result is the next theorem, a version for Hamiltonians of the connecting lemma for pseudo-orbits.

We say that the numbers $\sigma_1, ..., \sigma_{2d}$ satisfy a *trivial resonance relation* if

$$\sigma_i = \prod_{j=1}^{2d} \sigma_j^{k_j}, \quad i = 1, \dots, 2d,$$

where $k_i \in \mathbb{N}$ such that either $k_i \neq 1$ or there exists $j \neq i$ verifying $k_j \neq 0$.

Theorem 15. *Let $(p,H) \in \mathscr{M}$ such that $\mathscr{E}_{p,H} \subset H^{-1}(\{p\})$ is a regular surface. Suppose that every closed orbit there has a trivial resonance relation between the Flo-*

quet exponents. Then, for any $x, y \in \mathscr{E}_{p,H}$ connectected by a pseudo-orbit, there is a C^2-nearby $\widetilde{H}$ and $t > 0$ such that $\varphi^t_{\widetilde{H}}(x) = y$.

1.7 On Palis' conjecture

It is known from Peixoto's work [35, 36] that structurally stable flows on surfaces form a dense open set. A few years later Palis formulated the following conjecture for general dynamical systems defined on a closed manifold (flows, diffeomorphisms, or even more general transformations). Any system can always be C^1 approximated by another one which is uniformly hyperbolic or else it exhibits either a homoclinic tangency or a heterodimensional cycle [34].

In the conservative setting a more accurate result holds. In fact, Bessa and Rocha recently proved that any volume-preserving diffeomorphism of dimension $d \geq 3$ (or symplectomorphism of dimension $d \geq 4$) can be C^1 approximated by a volume-preserving (symplectic) diffeomorphism which is Anosov or else it exhibits a heterodimensional cycle [12].

In respect to the two-dimensional area-preserving discrete-time case, we have the following.

Theorem 16. *Any area-preserving diffeomorphism in a compact surface can always be C^1 approximated by another area-preserving diffeomorphism which is either Anosov or it exhibits a homoclinic tangency.*

Proof. By Newhouse's dichotomy [31] for a C^1-dense subset $\mathscr{D}$ of the Baire space of area-preserving diffeomorphisms endowed with the C^1-topology, we have that: if $f \in \mathscr{D}$, then f is Anosov or the elliptic points of f are dense in the manifold. It is sufficient to show that if f is in the C^1-interior of the complementary set of Anosov maps, we can C^1-approximate f by an area-preserving diffeomorphism g displaying a homoclinic tangency.

Now, we choose one elliptic point p for f. Since the C^2 area-preserving diffeomorphisms are C^1-dense in the C^1 area-preserving diffeomorphisms [42] and the elliptic points are stable, we can C^1-approximate f by $f_0 \in C^2$ such that the analytic continuation p_0 of p is elliptic. Now, since f_0 is of class C^2, we use the weak pasting lemma for diffeomorphisms [2] to create an invariant curve for some area-preserving diffeomorphism f_1 arbitrarily close to f_0. Finally, [29] is used to obtain persistence of homoclinic tangencies for g arbitrarily close to f_1.

Taking into account the previous result, we believe that the following result should hold.

Open problem 5. *Let $d = 2$. Given $H \in C^2(M)$, $e \in H(M)$ and $\varepsilon > 0$, then there exists $\widetilde{H}$ ε-C^2-close to H such that some regular energy surface in $\widetilde{H}^{-1}(\{e\})$ is Anosov or else it contains a homoclinic tangency associated to some hyperbolic closed orbit.*

Open problem 6. *Let $d > 2$. Given $H \in C^2(M)$, $e \in H(M)$ and $\varepsilon > 0$, then there exists $\widetilde{H}$ ε-C^2-close to H such that some regular energy surface in $\widetilde{H}^{-1}(\{e\})$ is Anosov or else it contains a heterodimensional cycle.*

1.8 Subclasses of Hamiltonian systems

There are many subclasses of $C^2(M)$ for which it would be very interesting to find generic properties. We will only brifly mention below two of them, because of their high importance in many branches of science: mechanical systems and geodesic flows.

Let Q be a d-dimensional smooth compact manifold and take there the local coordinates $q = (q_1, \ldots, q_d)$. We can write any $\sigma \in T_q^*Q$ as $\sigma = p \cdot dq$ where $p \in \mathbb{R}^d$ and $dq = (dq_1, \ldots, dq_d)$. Therefore, local coordinates on the cotangent bundle $M = T^*Q$ are given by (q, p). Notice that $\omega = dq \wedge dp$ is a symplectic form defined locally on M. For these local coordinates a mechanical system is a Hamiltonian $H \in C^\infty(T^*M)$ given by $H = T + V$, where T is the kinetic energy and $V\colon Q \to \mathbb{R}$ the potential. The function T is chosen to be homogeneous of degree 2, i.e. $T = \frac{1}{2}\langle p, p\rangle_q$. This is the general setting of most classical mechanics.

The results in the previous sections do not hold if we restrict to mechanical systems, because we would need to perturb in the same class, i.e. on the Riemannian metric $\langle \cdot, \cdot \rangle$ or on the potential V. It is thus an open question whether any sort of generic property would remain true in this context. In particular, we have the following question.

Open problem 7. *Can we C^2 approximate any given mechanical system by another mechanical system which has the dichotomy in Theorem 6?*

A somewhat first step would be to deal with a simpler situation:

Open problem 8. *Let Q be a closed surface. Given a C^2 Hamiltonian on T^*Q of the form $H = T$, is there V arbitrarly C^2 small such that $\widetilde{H} = T + V$ has the above mentioned dichotomy?*

Geodesic flows on the unit tangent bundle $M = SQ$ are a particular example of Hamiltonian mechanical systems, given by $H = T$. It would be of great interest to answer related questions specifically for those systems.

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References

1. R. Abraham and J. E. Marsden, Foundations of Mechanics, Benjamin-Cummings, 2nd edition, 1978.
2. A. Arbieto and C. Matheus, *A pasting lemma and some applications for conservative systems.* Ergod. Th. & Dynam. Sys. **27**, 5 (2007), 1399–1417.
3. M.-C. Arnaud, *The generic symplectic C^1-diffeomorphisms of four-dimensional symplectic manifolds are hyperbolic, partially hyperbolic or have a completely elliptic periodic point*, Ergod. Th. & Dynam. Sys. **22**, 6 (2002), 1621–1639.
4. M.-C. Arnaud, *Le "closing lemma" en topologie C^1*, Mém. S.M.F. **74**, (1998), 1–120.
5. Arnaud, M.-C., C. Bonatti and S. Crovisier *Dynamiques symplectiques génériques*, Ergod. Th. & Dynam. Sys., 25 (2005), 1401–1436.
6. M. Bessa. *The Lyapunov exponents of generic zero divergence* 3*-dimensional vector fields*, Erg. Theor. Dyn. Syst., 27:1445–1472, 2007.
7. M. Bessa. *Generic incompressible flows are topological mixing*, C. R. Acad. Sci. Paris, Ser. I 346 1169–1174, 2008..
8. M. Bessa and J. Lopes Dias, *Generic dynamics of* 4*-dimensional C^2 Hamiltonian systems*, Commun. Math. Phys **281**, 1, 597–619, 2008.
9. M. Bessa and J. Lopes Dias, *Hamiltonian elliptic dynamics on symplectic 4-manifolds*, Proc. Amer. Math. Soc. **137**, 585–592, 2009.
10. M. Bessa and P. Duarte, *Abundance of elliptical dynamics on conservative* 3*-flows*, Dynamical Systems **23**, (4), 409–424, 2008.
11. M. Bessa and J. Rocha, *On C^1-robust transitivity of volume-preserving flows*, Jour. of Diff. Equat. **245**, (11), 3127–3143, 2008.
12. M. Bessa and J. Rocha, *Anosov versus Heterodimensional cycles: A C^1 dichotomy for conservative maps*, `http://cmup.fc.up.pt/cmup/bessa/` Preprint, 2009.
13. M. Bessa, C. Ferreira and J. Rocha, *On the stability of the set of hyperbolic closed orbits of a Hamiltonian*, Math. Proc. Cambridge Phil. Soc. (to appear).
14. J. Bochi. *Genericity of zero Lyapunov exponents*, Erg. Theor. Dyn. Syst., 22:1667–1696, 2002.
15. J. Bochi. *C^1-generic symplectic diffeomorphisms: partial hyperbolicity and zero center Lyapunov exponents*, Journal of the Inst. of Math. Jussieu (2010) 9(1), 49–93.
16. J. Bochi and B. Fayad. *Dichotomies between uniform hyperbolicity and zero Lyapunov exponents for $SL(2,R)$ cocycles*, Bull. Braz. Math. Soc., 37:307–349, 2006.
17. J. Bochi and M. Viana. *The Lyapunov exponents of generic volume preserving and symplectic maps*, Ann. Math., 161:1423-1485, 2005.
18. J. Bochi and M. Viana. *Lyapunov exponents: How frequently are dynamical systems hyperbolic?* in Advances in Dynamical Systems. Cambridge Univ. Press, 2004.
19. C. Bonatti and S. Crovisier. *Récurrence et généricité*, Inventiones Math. 158 (2004), 33–104.
20. C. Bonatti, L. Díaz and M. Viana. *Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective*, Encycl. of Math. Sc. 102. Math. Phys. 3. Springer-Verlag, 2005.
21. R. Bowen. *Equilibrium states and ergodic theory of Anosov diffeomorphisms*, Lect. Notes in Math. 470, Springer-Verlag, 1975.
22. C. Doering. *Persistently transitive vector fields on three-dimensional manifolds*, Proceedings on Dynamical Systems and Bifurcation Theory, 160:59–89, 1987.
23. C. Ferreira. *Genericity of transitive energy levels for Hamiltonians*, In Preparation, 2010.
24. V. Horita and A. Tahzibi. *Partial hyperbolicity for symplectic diffeomorphisms*, Ann. Inst. H. Poincaré Anal. Non Linéaire **23**, 5 (2006), 641–661.
25. T. Hunt and R. S. MacKay. *Anosov parameter values for the triple linkage and a physical system with a uniformly chaotic attractor*, Nonlinearity, 16:1499-1510, 2003.
26. R. Mañé. *Oseledec's theorem from generic viewpoint*, Proceedings of the international Congress of Mathematicians, Warszawa, 2, 1259–1276, 1983.

27. R. Mañé. *The Lyapunov exponents of generic area preserving diffeomorphisms*, International Conference on Dynamical Systems (Montevideo, 1995), *Res. Notes Math. Ser.*, 362:110–119, 1996.
28. L. Markus and K. R. Meyer. *Generic Hamiltonian Dynamical Systems are neither Integrable nor Ergodic*, Memoirs AMS, 144, 1974.
29. L. Mora and N. Romero, *Persistence of homoclinic tangencies for area-preserving maps*, Annales fac. sc. Toulouse, 6 (4) (1997) 711–725.
30. J. Moser. *On the volume elements on a manifold*, Trans. Amer. Math. Soc., 120:286–294, 1965.
31. S. Newhouse. *Quasi-elliptic periodic points in conservative dynamical systems*, Am. J. Math., 99:1061–1087, 1977.
32. V. I. Oseledets. *A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc., 19:197–231, 1968.
33. J. Oxtoby and S. Ulam, *Measure-preserving homeomorphisms and metrical transitivity*, Ann. of Math. 42 (1941) 874–920.
34. J. Palis, *Open questions leading to a global perspective in dynamics*, Nonlinearity **21** (2008), 37–43.
35. M. Peixoto, *On structural stability*, Annals of Math. **69** (1959), 199–222.
36. M. Peixoto, *Structural stability on two-dimensional manifolds*, Topology **1** (1962), 101–120.
37. C. Pugh and C. Robinson, *The C^1 closing lemma, including Hamiltonians*, Ergod. Th. & Dynam. Sys. **3** (1983), 261–313.
38. C. Robinson, *Generic properties of conservative systems* I and II, Am. J. Math. **92** (1970), 562–603 and 897–906.
39. C. Robinson. *Lectures on Hamiltonian Systems*, Monograf. Mat. IMPA, 1971.
40. R. Saghin and Z. Xia, *Partial Hyperbolicity or dense elliptical periodic points for C^1-generic symplectic diffeomorphisms*, Trans. A.M.S. **358**, 11 (2006), 5119–5138.
41. T. Vivier, *Robustly transitive* 3*-dimensional regular energy surfaces are Anosov*, Institut de Mathématiques de Bourgogne, Dijon, Preprint 412 (2005). http://math.u-bourgogne.fr/topo/prepub/pre05.html
42. E. Zehnder, *Note on smoothing symplectic and volume-preserving diffeomorphisms*, In Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976, volume 597 *Lecture Notes in Math.*, 828–854. Springer, Berlin, 1977.