Numerical evaluation of continuous time ruin probabilities for a portfolio with credibility updated premiums

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Abstract

The probability of ruin in continuous and finite time is numerically evaluated in a classical risk process where the premium can be updated according to credibility models and therefore change from year to year. A major consideration in the development of this approach is that it should be easily applicable to large portofolios. Our method uses as a first tool the model developed by Afonso *et al.* (2009), which is quite flexible and allows premuims to change annually. We extend that model by introducing a credibility approach to experience rating.

We consider a portfolio of risks which satisfy the assumptions of the Bühlmann (1967, 1969) or Bühlmann and Straub (1970) credibility models. We compute finite time ruin probabilities for different scenarios and compare with those when a fixed premium is considered.

Keywords: Probability of ruin; finite time ruin probability; credibility premiums; Bühlmann's model; Bühlmann-Straub's model; large portfolios; numerical evaluation.

1 Introduction

We compute finite time ruin probabilities for a continuous time compound Poisson risk model for a portfolio of risks where the premiums can be updated according to the Bühlmann (1967, 1969) or Bühlmann and Straub (1970) credibility models. We compare these results with those when the total net premium is fixed.

Our approach is based on the framework developed by Afonso *et al.* (2009), which uses a combination of simulation and approximation to calculate the finite and continuous time ruin probability for a risk model where the premium is updated from year to year but is kept constant during the year.

The problem of calculating the probability of ruin for a risk process where the premium is updated according to a credibility model has been considered by Dubey (1977) and Tsai and Parker (2004). The former contains some interesting theoretical results, the latter focuses on numerical results for a discrete time model where the premium is updated according to Bühlmann's credibility model.

In the next section we set out our basic methodology and introduce assumptions and definitions.

In Section 3 we describe a portfolio which satisfies the assumptions of Bühlmann's credibility model. We show how to calculate the probability of ruin in continuous and finite time using two approaches to the calculation of the annual premium: a 'classical' approach, where important characteristics of the portfolio are assumed to be known with certainty and the premium does not change from year to year, and a credibility approach where the net premium is updated using Bühlmann's credibility model. Numerical results are presented for the portfolio.

Section 4 follows a similar pattern to Section 3, but the portfolio is now set up to satisfy the assumptions of the Bühlmann–Straub credibility model.

The major question of interest is, 'How is the probability of ruin affected if we update premiums according to a credibility model?' We can pose this question in terms of the portfolio as a whole, or in terms of each individual risk within the portfolio. It is not possible to give a definitive answer to either question, but our examples provide some insight and the methodology set out in this paper provides a way of investigating this question further.

2 Basic framework

In this section we summarize briefly our basic framework for the calculation of the probability of ruin in finite and continuous time. We do this in the context of a single risk, but in our major applications in Sections 3 and 4 we will extend it to a portfolio of risks. Full details of the basic methodology for a single risk can be found in Afonso *et al.* (2009).

Consider the surplus process for a single risk over an n-year period. We denote by $S(t)$ the aggregate claims up to time t, so that $S(0) = 0$, and by Y_i the aggregate claims in year i, $i = 1, \ldots, n$, so that $Y_i = S(i) - S(i-1)$. We assume that $\{Y_i\}_{i=1}^{n}$ is a sequence of *i.i.d.* random variables, each with a compound Poisson distribution whose first three moments exist.

Let P_i denote the premium charged in year i and let $U(t)$ denote the insurer's surplus at time t, $0 \leq t \leq n$. We assume premiums are received continuously at a constant rate throughout each year. The initial surplus, u $(= U(0))$, and the initial premium, P_1, are known. For $i = 2, \ldots, n$, we assume that P_i is a function of $\{U(j)\}_{j=1}^{i-1}$, the surplus at the end of each of the preceding years. For any time t, $0 \leq t \leq n$, $U(t)$ is calculated as follows:

$$U(t) = u + \sum_{j=1}^{i-1} P_j + (t - i + 1)P_i - S(t) \tag{2.1}$$

where i is the integer such that $t \in [i-1, i)$, and where $\sum_{j=1}^{0} P_j = 0$.

For $i \geq 2$, the surplus level $U(i)$, and hence the premium P_i, are random variables. Where we wish to refer to a particular realization of these variables, we will use the lower case letters $u(i)$ and p_i, respectively.

We denote by $\psi(u, n)$ the probability of ruin in continuous time within n years. We also need the probability of ruin within year i, given the surplus $u(i-1)$ at the start of the year, the surplus $u(i)$ at the end of the year and the rate of premium income p_i during the year, where $u(i-1), u(i) \geq 0$; this is denoted $\psi(u(i-1), 1, u(i))$. We can calculate, more precisely, estimate, $\psi(u, n)$, as follows:

(a) We simulate the annual aggregate claims, $Y_1, Y_2, \ldots, Y_n$, N times, approximating the distribution of each Y_i by a translated gamma distribution with matching moments.

(b) Given the values of the annual aggregate claims for simulation j, we can calculate successively $u(1), p_2, u(2), p_3, \dots, u(n-1), p_n, u(n)$. For that simulation we then compute the probability of ruin, denoted as $\psi_j(u, n)$.

(c) If $u(i) < 0$ for any i, we set $\psi_j(u, n) = 1$ for that simulation and move to the next simulation.

(i) If $u(i) \geq 0$ for each i, we calculate the probability of ruin $\psi(u(i-1), 1, u(i))$ within each year $[i-1, i]$ conditional on the process starting at $u(i-1)$ and ending at $u(i)$. Afonso *et al.* (2009) show how to approximate it using a translated gamma process approximation to the continuous time surplus process within the year. For this simulation, we set

$$\psi_j(u, n) = 1 - \Pi_{i=1}^{n}(1 - \psi(u(i-1), 1, u(i)))$$

(ii) Our estimate of $\psi(u, n)$ is then the mean of the estimates from each simulation, $\{\psi_j(u, n)\}_{j=1}^{N}$, and we can also calculate the standard error of this estimate.

3 Ruin probability with Bühlmann's credibility model

3.1 Preliminaries

In this section we discuss the effect on the probability of ruin for a portfolio of risks of updating premiums according to the Bühlmann credibility model. We do this through a numerical example based on a portfolio of five risks. The portfolio is specified in Section 3.2. In Section 3.3 we show how to calculate the probability of ruin in the case where the premiums, collective and individual, are not updated annually. This is a 'classical' risk theory approach. In Section 3.4 we show how to calculate the probability of ruin in the case where the individual premiums, and hence the collective premium, are updated annually using Bühlmann's credibility model. The numerical results are discussed in Section 3.5.

3.2 The portfolio

Our portfolio is specified as follows:

- The annual aggregate claims from risk k, $k = 1, 2, \dots, 5$, in year i, denoted Y_{ki}, have a compound Poisson distribution. The Poisson parameter is λ_{ki} and individual claim amounts have a lognormal distribution with parameters θ_k and 0.97411 (so that, for example, the mean of a single claim is $\exp(\theta_k + 0.97411/2)$.
- For each of the five risks, the parameter θ_k has the value indicated in Table 3.1.

Risk k	1	2	3	4	5
θ_k	0.1	0.1	0.2	0.2	0.4

Table 3.1: Values of the risk parameter, θ_k.

- The annual aggregate claims from different risks are independent.
- The annual aggregate claims from the same risk in different years are independent.
- We suppose that we already have five years' data for this portfolio and that we measure time in years from when the portfolio was first insured, so that 'now' is time 5, which is the start of the sixth year.
- We are interested in the probability of ruin in continuous time over the next 10 years, that is, between times 5 and 15.
- There is an initial surplus, u, currently available for the whole portfolio. When considering single risks, we assume an initial surplus $u/5$ is assigned to each risk.
- We consider two models for the Poisson parameters, λ_{ki}:

  N1 λ_{ki} is constant and equal to 1 000 each year for each risk. In this case we denote the common value λ;

  N2 λ_{ki} is a random variable and $\{\{\lambda_{ki}\}_{k=1}^{5}\}_{i=1}^{n}$ is a set of *i.i.d.* random variables, each with a $U(800, 1\,200)$ distribution.

Note that this portfolio has been constructed to satisfy the assumptions of the Bühlmann credibility model. For scenario N2:

$$\begin{aligned} \mathrm{E}[Y_{ki}|\theta_k] &= \mathrm{E}[\ \mathrm{E}[Y_{ki}|\theta_k]|\lambda_{ki}] \\ &= \mathrm{E}[\lambda_{ki}]\exp(\theta_k + 0.97441/2) \\ &= 1000\exp(\theta_k + 0.97441/2) \end{aligned}$$

$$\begin{aligned} \mathrm{V}[X_{ki}|\theta_k] &= \mathrm{V}[\mathrm{E}[(Y_{ki}|\theta_k)|\lambda_{ki}]] + \mathrm{E}[\mathrm{V}[(Y_{ki}|\theta_k)|\lambda_{ki}]] \\ &= \mathrm{V}[\ \lambda_{ki}\exp(\theta_k + 0.97441/2)] + \mathrm{E}[\ \lambda_{ki}\exp(2\theta_k + 0.97441)] \\ &= \exp(2\theta_k + 0.97441)(\mathrm{V}[\ \lambda_{ki}] + \mathrm{E}[\ \lambda_{ki}]) \\ &= \frac{43\,000}{3}\exp(2\theta_k + 0.97441) \end{aligned}$$

so that both $\mathrm{E}[Y_{ki}|\theta_k]$ and $\mathrm{V}[Y_{ki}|\theta_k]$ are some functions of the risk parameter θ_k, as required. Note that scenario N1 is a special case of scenario N2 with $\mathrm{V}[\ \lambda_{ki}] = 0$.

We can simulate the aggregate claims for each of the five risks and for each of the 15 years, 10 in the future and five in the past. It is convenient to do this by assuming each Y_{ki} has a translated gamma distribution, it's simple and a good fit [please see Afonso (2008)]. The steps in this simulation process are first to simulate a value for λ_{ki} (if necessary), to calculate the parameters of the translated gamma distribution which matches the first three moments of Y_{ki} (given the value of λ_{ki}) and finally simulating the value of (the translated gamma approximation to) Y_{ki}.

3.3 The classical approach

We now put ourselves in the place of the actuary setting premiums for each of the five risks for each future year for this portfolio. We assume in this subsection that the actuary knows (precisely) the expected value of the aggregate claims for the portfolio each year, $\mathrm{E}[\sum_{k=1}^{5} Y_{ki}]$. The premium charged each year for this portfolio is P, where:

$$P = (1 + \zeta(u))\mathrm{E}\left[\sum_{k=1}^{5} Y_{ki}\right]$$

where the premium loading factor, $\zeta(u)$, is a function of the initial surplus, u, and is taken from Table 3.2. Note that since under scenario N2 the expected value of the Poisson parameter is the same as the constant value for N1, the value of $\mathrm{E}\left[\sum_{k=1}^{5} Y_{ki}\right]$ is the same under N1 and N2. Hence the premium is the same in both cases. The loading factors, $\zeta(u)$, have been chosen so that the probability of ultimate ruin for the portfolio under N1 is approximately 0.01. This eases the comparison of results for different initial surpluses. See Afonso *et al.* (2009). The premium for the portfolio does not change from year to year and is received continuously at constant rate. Where we require a premium for each individual risk in the year, we assume this is $P/5$.

u	250	300	350	400	450
$\zeta(u)$	0.0539	0.0432	0.0359	0.0305	0.0265

Table 3.2: Premium loading factors for Section 3.3.

To calculate/estimate the probability of ruin within 10 years for this portfolio, we simulate the future aggregate claims a large number of times, 50 000 in our example, and use the methodology outlined in Section 2 applied to the total aggregate claims for the portfolio each year. We do this for the two scenarios N1 (fixed λ_{ki}) and N2 (variable λ_{ki}). By applying this methodology to the simulated aggregate claims for each risk, and assuming an initial surplus $u/5$ and annual premium $P/5$, we can estimate the probability of ruin within 10 years for each risk. The numerical results based on this approach are set out in Table 3.3 in the columns headed P1 N1 and P1 N2. Note that for the portfolio the values of $\psi(u, 10)$ for P1 are close to 0.01 as intended.

We refer to the approach in this subsection as the 'classical approach' since it has many elements of classical risk/ruin theory: some properties of the aggregate claims distribution are known with certainty (the mean in our case), the premiums are constant and any data is ignored in terms of decision making.

3.4 The credibility approach

In this subsection we assume the actuary takes a different, and more realistic, approach. The actuary has no specific information about any parameter values. The aggregate annual claims for each of the five risks for each of the past five years are known now and, with each successive year, the aggregate claims for that year for each risk are known at the end of that year.

The actuary assumes that these five risks satisfy the assumptions of the Bühlmann credibility model, as set out, for example, in Norberg (1979), Section 3C, and updates the annual net premium for each of these risks in accordance with this model.

Let P_{ki} denote the premium charged for risk k, $k = 1, 2, \ldots, 5$, at the start of year i, $i = 6, 7, \ldots, 15$. We assume that this premium has a constant loading factor, $\zeta(u)$, which depends on the initial surplus and is the same for each of the risks in each year. We denote by Π_{ki} the corresponding net premium, so that:

$$\Pi_{ki} = P_{ki}/(1 + \zeta(u))$$

Then the annual net premium is calculated as follows:

$$\Pi_{ki} = \hat{Z}_{i-1}\overline{Y}_{k,i-1} + (1 - \hat{Z}_{i-1})\hat{E}_{i-1},$$

where:

$$\begin{aligned}
\overline{Y}_{k,i-1} &= \sum_{j=1}^{i-1} Y_{kj}/(i-1), \\
\hat{E}_{i-1} &= \sum_{k=1}^{5} \overline{Y}_{k,i-1}/5, \\
\hat{Z}_{i-1} &= (i-1)/(i-1+\hat{\sigma}^2/\hat{\tau}^2), \\
\hat{\sigma}^2 &= \frac{1}{5}\sum_{k=1}^{5} \frac{1}{(i-2)} \sum_{j=1}^{i-1} (Y_{kj} - \overline{Y}_{k,i-1})^2, \\
\hat{\tau}^2 &= \max\left(\frac{1}{4}\sum_{k=1}^{5}(\overline{Y}_k - \overline{Y})^2 - \frac{\hat{\sigma}^2}{5}, 0\right).
\end{aligned}$$

This is the Bühlmann credibility premium with the usual estimators for the structural parameters. See, for example, Norberg (1979), Section 3D.

Note that the total net premium for year i, $\sum_{k=1}^{5} \Pi_{ki}$, is equal to $\sum_{k=1}^{5} \sum_{j=1}^{i-1} Y_{kj}/(i-1)$, which is the natural estimate of the mean annual aggregate claims for the portfolio based on the data observed so far.

For each scenario N1 and N2 we calculate/estimate the probability of ruin within 10 years for this portfolio and also for each risk separately by simulating the past and future annual aggregate claims 50 000 times. Here we use the methodology outlined in Section 2 with the annual premiums updated as described above. The numerical results based on this approach are set out in Table 3.3 in the columns headed P2 N1 and P2 N2.

3.5 Results

Table 3.3 shows numerical results for both the 'classical' and credility approaches under scenarios N1 and N2. These results are estimates of the probability of ruin, $\psi(u, 10)$, and the standard deviation of each estimate, SD$[\psi(u, 10)]$, for each individual risk and for the portfolio, for different values of the inital surplus, u. The same set of 50 000 simulations of $\{Y_{ki}\}$ for N1 and N2 were used to calculate the probability of ruin for cases P1 and P2.

We make the following comments on the results in Table 3.3:

(i) The pattern of results in the table, for the portfolio and for the individual risks, is the same for all values of u.

(ii) Comparing portfolio values for scenarios P1 (classical) and P2 (credibility), we see that the probability of ruin is higher for P2 – relatively much higher for N1 (fixed Poisson parameter) than N2 (variable Poisson parameter).

(iii) The portfolio values for scenario P1 N2 are very much higher than for P1 N1. We would expect this. The extra variability resulting from the variable Poisson parameter has not been offset by any increase in the premium. The model for a variable Poisson parameter, based on the uniform distribution, may not be reasonable in practice. However, the increase in the values from P1 to P2 is a reminder that the 'classical' assumption of a fixed (and known) Poisson parameter may be very misleading. See Daykin *et al.* (1996), page 329.

(iv) For scenarios P1, the results for individual risks vary widely. This is because the risks are different – different expected claim amounts – but have been assigned, somewhat arbitrarily, the same premium, $P/5$, and initial surplus, $u/5$. It is noticeable that under P2, credibility adjusted premiums, the values of $\psi(u, 10)$ are all much closer to each other. In other words, the credibility adjustment is working quickly to assign an appropriate premium to each risk.

(v) The standard deviations of $\psi(u, 10)$ are all very small.

A more detailed analysis of those simulations leading to 'end of year', rather than 'within year', ruin shows that:

(a) The credibility premium in the year before ruin is always less than the fixed premium, by about 1%, and that the aggregate claims in the year of ruin are on average 10% higher than expected. This sheds light on point (ii) above.

(b) The proportion of $\psi(u, 10)$ due to 'end of year', rather than 'within year', ruin is similar for P1 and P2 but decreases with the initial surplus, ranging from 0.926 ($u = 250$) to 0.672 ($u = 450$) for N1 and from 0.291 ($u = 250$) to 0.113 ($u = 450$) for N2.

(c) The average time to ruin increases a little with u but is similar for different combinations of P1, P2, N1 and N2, ranging from 1 to 1.5 years.

See Afonso (2008) for full details.

k	u	P1 N1		P2 N1		P1 N2		P2 N2	
		$\psi(u,10)$	SD$[\psi(u,10)]$	$\psi(u,10)$	SD$[\psi(u,10)]$	$\psi(u,10)$	SD$[\psi(u,10)]$	$\psi(u,10)$	SD$[\psi(u,10)]$
1	50	0.048	7.94E-08	0.322	2.17E-06	0.181	1.77E-06	0.493	4.12E-06
2	50	0.048	7.77E-08	0.325	2.18E-06	0.183	1.78E-06	0.494	4.13E-06
3	50	0.321	1.83E-06	0.378	2.30E-06	0.540	4.10E-06	0.585	4.03E-06
4	50	0.321	1.82E-06	0.377	2.28E-06	0.541	4.10E-06	0.589	4.02E-06
5	50	1.000	1.60E-08	0.477	2.34E-06	1.000	1.64E-08	0.731	3.24E-06
Port.	**250**	**0.009**	**3.81E-08**	**0.013**	**7.66E-08**	**0.171**	**2.38E-06**	**0.204**	**2.81E-06**
1	60	0.034	6.66E-08	0.334	2.61E-06	0.176	1.90E-06	0.520	4.28E-06
2	60	0.034	6.52E-08	0.337	2.63E-06	0.178	1.92E-06	0.521	4.29E-06
3	60	0.326	2.25E-06	0.394	2.76E-06	0.570	4.23E-06	0.616	4.08E-06
4	60	0.325	2.22E-06	0.393	2.75E-06	0.570	4.22E-06	0.621	4.07E-06
5	60	1.000	1.60E-08	0.501	2.77E-06	1.000	1.60E-08	0.760	3.12E-06
Port.	**300**	**0.010**	**7.18E-08**	**0.015**	**1.40E-07**	**0.217**	**3.00E-06**	**0.251**	**3.38E-06**
1	70	0.024	5.37E-08	0.343	2.95E-06	0.169	1.97E-06	0.536	4.38E-06
2	70	0.023	5.26E-08	0.345	2.97E-06	0.171	1.98E-06	0.540	4.39E-06
3	70	0.327	2.57E-06	0.407	3.12E-06	0.590	4.28E-06	0.637	4.08E-06
4	70	0.326	2.55E-06	0.406	3.11E-06	0.591	4.27E-06	0.642	4.07E-06
5	70	1.000	1.60E-08	0.519	3.10E-06	1.000	1.60E-08	0.778	3.03E-06
Port.	**350**	**0.011**	**1.06E-07**	**0.018**	**2.02E-07**	**0.254**	**3.44E-06**	**0.288**	**3.78E-06**
1	80	0.016	4.21E-08	0.349	3.21E-06	0.162	1.99E-06	0.547	4.44E-06
2	80	0.016	4.14E-08	0.351	3.23E-06	0.164	2.00E-06	0.551	4.46E-06
3	80	0.327	2.82E-06	0.416	3.39E-06	0.604	4.31E-06	0.651	4.07E-06
4	80	0.326	2.80E-06	0.415	3.38E-06	0.606	4.29E-06	0.657	4.05E-06
5	80	1.000	1.60E-08	0.532	3.35E-06	1.000	1.60E-08	0.791	2.95E-06
Port.	**400**	**0.012**	**1.38E-07**	**0.020**	**2.57E-07**	**0.282**	**3.73E-06**	**0.316**	**4.01E-06**
1	90	0.011	3.25E-08	0.352	3.40E-06	0.155	1.98E-06	0.554	4.48E-06
2	90	0.011	3.20E-08	0.354	3.41E-06	0.156	1.99E-06	0.558	4.48E-06
3	90	0.324	3.00E-06	0.422	3.60E-06	0.613	4.33E-06	0.660	4.13E-06
4	90	0.325	2.99E-06	0.421	3.58E-06	0.615	4.31E-06	0.667	4.04E-06
5	90	1.000	1.60E-08	0.542	3.53E-06	1.000	1.60E-08	0.799	2.91E-06
Port.	**450**	**0.013**	**1.68E-07**	**0.022**	**3.04E-07**	**0.304**	**3.94E-06**	**0.337**	**4.19E-06**

Table 3.3: Section 3: estimates and standard deviations of $\psi(u,10)$.

4 Ruin Probability with the Bühlmann–Straub credibility model

4.1 The portfolio

In this section we consider the calculation of the probability of ruin, in continuous and finite time, for a portfolio of risks which satisfy the assumptions of the Bühlmann–Straub credibility model. Our approach is similar to that used in Section 3 – a numerical study based on a specified portfolio – except that we will present numerical results only for the portfolio, and not for the individual risks. We consider the portfolio to be more interesting than the individual risks. We start by specifying our portfolio.

We have a portfolio of five risks, for each of which we have five years' past claims data. Time is (again) measured in years from when the data were collected so 'now' is time 5. For risk k, $k = 1, 2, \ldots, 5$, claims data for year i, $i = 1, 2, \ldots$, consists of the total aggregate claims, Y_{ki}, and an associated 'risk volume', or weight, w_{ki}. The scaled aggregate claims, Y_{ki}/w_{ki}, is denoted X_{ki}. We assume that the risk volumes for future years, $i = 6, 7, \ldots$, are non–random and known at the start of the relevant year.

We assume:

- The annual aggregate claims from risk k, $k = 1, 2, \ldots, 5$, in year i, Y_{ki}, have a compound Poisson distribution. The Poisson parameter is $w_{ki}\lambda_{ki}$ and individual claim amounts have a lognormal distribution with parameters θ_k and 0.97411.
- The parameters λ_{ki} are constant and equal to 10 for all risks and all years.
- For each of the five risks, the parameter θ_k has the value indicated in Table 3.1.
- The annual aggregate claims from different risks are independent.
- The annual aggregate claims from the same risk in different years are independent.
- There is an initial surplus, u, currently available for the whole portfolio.
- We are interested in the probability of ruin in continuous time over the next 10 years, $\psi(u, 10)$.
- We consider three cases for the risk volume, see Table 4.1 for values.

W1 $w_{1,i} = 30, w_{2,i} = 250, w_{3,i} = 60, w_{4,i} = 120, w_{5,i} = 40$. The risk volumes vary among the risks but are constant for each year. The 'dominant' risk, risk 2, has a small risk parameter, θ_k.

W2 $w_{1,i} = 30, w_{2,i} = 40, w_{3,i} = 60, w_{4,i} = 120, w_{5,i} = 250$. The risk volumes vary among the risks but are constant for each year. The 'dominant' risk, risk 5, has a large risk parameter, θ_k.

W3 For this case, the risk volumes have been generated by assuming

$$w_{k,i} = U(0.5w^1_{k,i}, 1.5w^1_{k,i})$$

where $w^1_{k,i}$ is the risk volume of case W1.

		Risk					
Risk Volume	Year	1	2	3	4	5	Total
W1	$1, \cdots, 15$	30	250	60	120	40	500
W2	$1, \cdots, 15$	30	40	60	120	250	500
W3	1	44	277	34	66	33	454
	2	20	298	67	170	24	579
	3	17	217	90	171	45	540
	4	16	287	62	157	22	544
	5	15	153	77	168	50	463
	6	30	250	60	120	40	500
	7	20	337	71	76	28	532
	8	31	354	71	117	38	611
	9	25	288	52	105	38	508
	10	30	203	62	144	35	474
	11	29	313	88	104	42	576
	12	43	148	37	101	41	370
	13	42	289	56	91	31	509
	14	20	318	87	157	43	625
	15	15	360	69	105	59	608

Table 4.1: Section 4: Risk volumes, w_{ki}, by case, risk and year.

Note that this portfolio has been constructed to satisfy the assumptions of the Bühlmann–Straub credibility model, since:

$$\begin{aligned} \mathrm{E}[X_{ki}|\theta_k] &= \frac{\mathrm{E}[Y_{ki}|\theta_k]}{w_{ki}} \\ &= \lambda_{ki}\exp(\theta_k + 0.97441/2) \\ &= 10\exp(\theta_k + 0.97441/2) \end{aligned}$$

$$\begin{aligned} \mathrm{V}[X_{ki}|\theta_k] &= \frac{w_{ki}\lambda_{ki}\exp(2\theta_k + 0.97441)}{w_{ki}^2} \\ &= 10\exp(2\theta_k + 0.97441)/w_{ki} \end{aligned}$$

so that both $\mathrm{E}[X_{ki}|\theta_k]$ and $w_{ki}\mathrm{V}[X_{ki}|\theta_k]$ are some functions of the risk parameter θ_k, as required.

Throughout Section 4 we will assume that the premium loading factor applied to annual net premiums calculated for the portfolio is always 10%.

As in Section 3, we will calculate $\psi(u, 10)$ using different approaches to the calculation of the annual premium for the portfolio. In this section we will use a 'classical' approach, where the actuary knows the expected value of the aggregate annual claims for each of the five risks, an 'intermediate' approach, where the actuary knows the structure of the portfolio, and a credibility approach, where the net premium is updated at the start of each year according to the Bühlmann–Straub credibility model. These approaches are described in Sections 4.2 4.3 and 4.4, respectively. An important distinction between these approaches is that for the 'classical' and 'intermediate' approaches the actuary has some prior information about the portfolio and takes no account of the data. For the credibility approach, the actuary´s only information about the portfolio comes from the data itself. The numerical results are presented and discussed in Section 4.5.

The calculation of $\psi(u, 10)$ proceeds as in Section 3. For each year i, $i = 6, 7, \ldots, 15$, and risk k, $k = 1, \ldots, 5$, we simulate Y_{ki} by simulating from a translated gamma distribution with the same first three moments. The surplus at the start of the year, $u(i-1)$, the values of the corresponding risk volumes, w_{ki}, and the total gross premium to be charged in the year, p_i, are all known at the start of the year. The surplus at the end of the year is $u(i)$, where:

$$u(i) = u(i-1) + p_i - \sum_{k=1}^{5} Y_{ki}.$$

If $u(i-1)$ is negative, ruin has occurred. If $u(i)$ is non–negative, we can calculate the probability of ruin within the year, $\psi(u(i-1), 1, u(i))$, by approximating the distribution of the total aggregate claims in year i, $\sum_{k=1}^{5} Y_{ki}$, by a translated gamma distribution with the same first three moments. We can then calculate $\psi(u, 10)$ as in Section 2.

4.2 The classical approach

For the classical approach we assume our actuary knows precisely the values of the expected unscaled annual aggregate claims for each of the five risks, $\mathrm{E}[X_{k\bullet}]$. This is a slightly stronger assumption than in Section 3.3, where we assumed that only the expected annual aggregate claims for the whole portfolio was known.

The gross annual premium for the whole portfolio in year i, $i = 6, 7, \ldots, 15$, is given by:

$$(1 + 0.1) \sum_{k=1}^{5} w_{ki} \mathrm{E}[X_{ki}].$$

4.3 The intermediate approach

For the intermediate approach we assume the actuary knows the structure of the portfolio in the sense that (s)he knows that

- The underlying Poisson parameter for all risks in all years, before scaling by the risk volume, is 10.
- 40% of the risks have expected claim amount $\exp(0.1 + 0.97441/2)$.
- 40% of the risks have expected claim amount $\exp(0.2 + 0.97441/2)$.
- 20% of the risks have expected claim amount $\exp(0.4 + 0.97441/2)$.

However, the actuary does not know which risk has which expected claim amount (and does not learn by looking at the data).

Hence, in year i, the total premium charged is p_i, where:

$$p_i = (1{+}0.1) \left(\sum_{k=1}^{5} 10 w_{ki} \right) (0.4[\exp(0.1 + 0.97441/2) + \exp(0.2 + 0.97441/2)] + 0.2 \exp(0.4 + 0.97441/2)) .$$

4.4 The credibility approach

For the credibility approach, we assume the actuary knows only the past aggregate claims and corresponding risk volumes for each of the five risks (as well as the risk volumes at the start of each future year). In particular, the actuary does not know the risk parameter, θ_k, for risk k. The actuary assumes the risks satisfy all the conditions for the Bühlmann–Straub credibility model, and updates each year the net premium for each risk accordingly.

The gross annual premium for the whole portfolio in year i, $i = 6, 7, \ldots, 15$, is given by:

$$(1 + 0.1) \sum_{k=1}^{5} w_{ki} \, P_{ki}^{C}$$

where P_{ki}^{C} is the net credibility premium for risk k in year i. This premium is calculated as follows:

$$
\begin{aligned}
P_{ki}^{C} &= \hat{z}_k X_k + (1-\hat{z}_k)\hat{\mu} \\
X_k &= \sum_{j=1}^{i-1} \frac{w_{kj}}{w_{k\bullet}} X_{kj} \\
w_{k\bullet} &= \sum_{l=1}^{i-1} w_{kl} \\
\hat{z}_k &= \sum_{l=1}^{i-1} w_{kl} \Big/ \left(\sum_{l=1}^{i-1} w_{kl} + \frac{\hat{\sigma}^2}{\hat{\tau}^2} \right) \\
\hat{\mu} &= \sum_{k=1}^{5} \frac{z_k}{z_{\bullet}} X_k
\end{aligned}
$$

$$
\begin{aligned}
z_{\bullet} &= \sum_{k=1}^{5} z_k \\
\hat{\sigma}^2 &= \frac{1}{5} \sum_{k=1}^{5} \frac{1}{(i-2)} \sum_{j=1}^{i-1} w_{kj} (X_{kj} - \overline{X}_k)^2 \\
\hat{\tau}^2 &= \max\left(c \left\{ \frac{5}{4} \sum_{k=1}^{5} \frac{w_{k\bullet}}{w_{\bullet\bullet}} (\overline{X}_k - \overline{X})^2 - \frac{r\hat{\sigma}^2}{w_{\bullet\bullet}} \right\}, 0 \right) \\
c &= \frac{4}{5} \left\{ \sum_{k=1}^{5} \frac{w_{k\bullet}}{w_{\bullet\bullet}} \left(1 - \frac{w_{k\bullet}}{w_{\bullet\bullet}} \right) \right\}^{-1} \\
\overline{X} &= \sum_{k=1}^{5} \frac{w_{k\bullet}}{w_{\bullet\bullet}} X_k \\
w_{\bullet\bullet} &= \sum_{k=1}^{5} w_{k\bullet}
\end{aligned}
$$

It can be seen that we are using the usual estimators within the Bühlmann–Straub model. See, for example, Bühlmann and Gisler (2005), Theorem 4.2 and Section 4.8.

4.5 Numerical results

Table 4.2 shows (estimated) values of $\psi(u, 10)$, together with the corresponding standard errors of the estimates, for selected values of the initial surplus, u, and for:

- Three cases for the risk volumes, W1, W2 and W3.
- Three approaches to the calculation of the net premiums: the classical approach, labelled P3, the intermediate approach, labelled P4, and the credibility approach, labelled P5.

RiskVolume	u	P3 $\psi(u,10)$	P3 SD[$\psi(u,10)$]	P4 $\psi(u,10)$	P4 SD[$\psi(u,10)$]	P5 $\psi(u,10)$	P5 SD[$\psi(u,10)$]
W1	80	0.0597	4.53E-08	0.0206	5.00E-09	0.0624	6.03E-08
	90	0.0429	3.11E-08	0.0131	2.67E-09	0.0452	4.29E-08
	100	0.0308	2.11E-08	0.0083	1.41E-09	0.0329	3.02E-08
	110	0.0221	1.42E-08	0.0053	7.33E-10	0.0239	2.12E-08
	120	0.0159	9.47E-09	0.0034	3.80E-10	0.0174	1.49E-08
	130	0.0114	6.30E-09	0.0021	1.96E-10	0.0127	1.04E-08
W2	80	0.0838	6.38E-08	0.7710	1.70E-06	0.0865	8.17E-08
	90	0.0628	4.66E-08	0.7477	1.94E-06	0.0653	6.12E-08
	100	0.0470	3.34E-08	0.7250	2.16E-06	0.0493	4.51E-08
	110	0.0352	2.37E-08	0.7031	2.37E-06	0.0373	3.30E-08
	120	0.0264	1.66E-08	0.6817	2.57E-06	0.0282	2.39E-08
	130	0.0197	1.16E-08	0.6609	2.75E-06	0.0213	1.73E-08
W3	80	0.0569	4.56E-08	0.0197	5.07E-09	0.0595	6.02E-08
	90	0.0409	3.06E-08	0.0125	2.64E-09	0.0431	4.21E-08
	100	0.0294	2.03E-08	0.0079	1.36E-09	0.0313	2.93E-08
	110	0.0211	1.34E-08	0.0050	6.99E-10	0.0228	2.04E-08
	120	0.0151	8.78E-09	0.0032	3.56E-10	0.0166	1.42E-08
	130	0.0109	5.74E-09	0.0020	1.80E-10	0.0121	9.93E-09

Table 4.2: Section 4: Estimates and standard deviations of $\psi(u, 10)$.

All the values are based on the same set of 50 000 simulations of the scaled aggregate annual claims.

It can be seen from Table 4.1 that the risk volumes for scenario W3 are broadly similar to, but more variable than, those for W1. From Table 4.2 we can see that this variability of the risk volumes has a negligible effect on the probability of ruin. The results in Table 4.2 have very small standard errors, as was the case in Section 3.

One difference between the results in Table 3.3 and those in Table 4.2 is that in Section 3 we chose the premium loading factors so that, for a given initial surplus, the probability of ruin in the classical case with fixed Poisson parameter was approximately 0.01. This is not the case in Section 4, where the premium loading factor is always 10%. Consequently, in Table 4.2 $\psi(u, 10)$ is always a decreasing function of u.

The important features of the results in Table 4.2 are:

(i) The results for the intermediate case, P4, are good for W1 and W3, but poor for W2. This because in this case the actuary is lucky that both W1 and W2 give higher weights to the risks with lower expected claim amounts – recall that our actuary does not know the expected claim amount for the individual risks and does not learn from the data.

(ii) The results for the classical and credibility cases are very similar, with the credibility case always giving a slightly higher ruin probability. For the classical case, our actuary has precise knowledge of the expected claim amounts for each of the risks. For the credibility case, the five years of data are sufficient to allow our actuary to calculate appropriate premiums.

More detailed analysis of the simulations which lead to end of year ruin show that for all scenarios:

(a) End of year ruin occurs almost always within the first two years.

(b) The average aggregate claims in the year of ruin is higher than the overall expected aggregate claims.

(c) For the credibility case, P5, the average net premium in the year of ruin is lower than the overall expected aggregate claims. This implies that in these cases ruin occurs when a 'bad' year (higher than expected claims) follows one or more 'good' years.

See Afonso (2008) for more details relating to cases P4 and P5.

5 Concluding remark

One of our objectives has been to devise a methodology which can be used to calculate the probability of ruin for large portfolios. See Afonso *et al.* (2009). To ease the presentation we have illustrated our methodology using portfolios with just five risks. Increasing the number of risks would increase the time needed to produce results, but only linearly. On the other hand, increasing the size of the portfolios by increasing the Poisson parameter for the expected number of claims would have no effect on calculation time.

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