

ON THE SPACE DISCRETIZATION OF PDES WITH UNBOUNDED COEFFICIENTS ARISING IN FINANCIAL MATHEMATICS – THE CASE OF ONE SPATIAL DIMENSION

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ABSTRACT. We study the space discretization of the Cauchy problem for a second order linear parabolic PDE, with one spatial dimension and unbounded time and space-dependent coefficients. The PDE free term and the initial data are also allowed to grow. Under the assumption that the PDE does not degenerate, the problem's weak solution is approximated in space, with finite-difference methods. The rate of convergence is estimated. A numerical example is given in order to illustrate the theoretical results.

1. INTRODUCTION AND CLASSICAL RESULTS

In this article, we make use of finite-difference methods to approximate in space the weak solution of the Cauchy problem

$$(1.1) \quad Lu - u_t + f = 0 \text{ in } Q, \quad u(0, x) = g(x) \text{ in } \mathbb{R},$$

where $Q = [0, T] \times \mathbb{R}$, with T a positive constant, L is the second-order partial differential operator with real coefficients

$$(1.2) \quad L(t, x) = a(t, x) \frac{\partial^2}{\partial x^2} + b(t, x) \frac{\partial}{\partial x} + c(t, x),$$

for each $t \in [0, T]$ uniformly elliptic in the space variable, and f and g are given real-valued functions. We allow the growth in space of the first and second-order coefficients in L (linear and quadratic growth, respectively), and of the data f and g (polynomial growth).

This article follows previous works by the same authors ([6, 7]), where the same approach is utilized for the numerical approximation of the more general case of multidimensional PDEs. In the present article, by considering the special case of one dimension in space, a stronger convergence result holds. In particular, the same order of accuracy is obtained under regularity assumptions weaker than those required in [6, 7] for the corresponding convergence result.

Linear parabolic PDE problems arise in Financial Mathematics (see, e.g., [10]), and there lies our main motivation.

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The finite-difference method for approximating PDE is a mature area of research.* We refer to [17], for the numerical approximation study of the Cauchy problem for general multidimensional linear parabolic PDEs of order $m \geq 2$, with bounded time and space-dependent coefficients. The approach is pursued in the framework of the classical approach.

Also, there has been a long and extensive research on the application of finite-difference methods to financial option pricing.† Nevertheless, most studies concerning the numerical approximation of PDE problems in Finance consider the particular case where the PDE coefficients are constant (see, e.g., [1, 2, 5, 16]). This occurs, namely, in option pricing under Black-Scholes stochastic model (in one or several dimensions), when the asset appreciation rate and volatility are taken constant.

In this article, we deal with the challenge posed by the unboundedness of the PDE coefficients, under the strong assumption that the PDE does not degenerate.

We consider problem (1.1) in the framework of the variational approach, and impose weak regularity over the operator's coefficients and the data f and g . We make use of the L^2 theory of solvability of linear PDEs in weighted Sobolev spaces and, in particular, consider the deterministic one spatial dimension special case of a class of weighted Sobolev spaces introduced by O. G. Purtukhia [12, 13, 14, 15], and further generalized by I. Gyöngy and N. V. Krylov [9], for the treatment of linear SPDEs. By considering discrete versions of these spaces, we set an appropriate discretized framework and investigate the spatial approximation of the PDE problem's weak solution. In order to facilitate the study, we make use of basic one-step finite-difference schemes. We conclude the article by presenting a numerical example that illustrates our analytic and numerical results.

We briefly review some facts on the solvability of problem (1.1). We first introduce the above mentioned class of weighted Sobolev spaces.‡

Let U be a domain in \mathbb{R} , i.e., an open subset of \mathbb{R} . Let $r > 0$, $\rho > 0$ be smooth functions in U and $m \geq 0$ an integer. The weighted Sobolev space $W^{m,2}(r, \rho)(U)$ is the Banach Space of all locally integrable functions $v : U \rightarrow \mathbb{R}$ such that for each integer $\alpha \geq 0$, with $\alpha \leq m$, $D^\alpha v$ exists in the weak sense, and

$$\|v\|_{W^{m,2}(r,\rho)(U)} := \left(\sum_{\alpha \leq m} \int_U r^2 |\rho^\alpha D^\alpha v|^2 dx \right)^{1/2}$$

is finite. Endowed with the inner product

$$(v, w)_{W^{m,2}(r,\rho)(U)} := \sum_{\alpha \leq m} \int_U r^2 \rho^{2\alpha} D^\alpha v D^\alpha w dx,$$

for all $v, w \in W^{m,2}(r, \rho)(U)$, which generates the norm, $W^{m,2}(r, \rho)(U)$ is a Hilbert space.

Notation. In the sequel, when $U = \mathbb{R}$ the argument in the function space notation is dropped. For instance, we denote $W^{m,2}(r, \rho)(\mathbb{R}) =: W^{m,2}(r, \rho)$.

We make some assumptions on the behaviour of the weight functions r and ρ (see [9]).

* See [17] for a brief history of the finite-difference method, and also for the references of the original publications and further major contributions.

† The references for the seminal applications of finite differences to financial problems, and further major research can be tracked down in the review paper [3].

‡ We refer to [9] for a complete description of this class of spaces.

Assumption 1. Let $m \geq 0$ be an integer, and $r > 0$ and $\rho > 0$ smooth functions on \mathbb{R} . There exists a constant K such that

- (1) $|D^\alpha \rho| \leq K \rho^{1-\alpha}$ for all α such that $\alpha \leq m - 1$ if $m \geq 2$;
- (2) $|D^\alpha r| \leq K \frac{r}{\rho^\alpha}$ for all α such that $\alpha \leq m$;
- (3) $\sup_{|x-y|<\varepsilon} \left(\frac{r(x)}{r(y)} + \frac{\rho(x)}{\rho(y)} \right) = K$ for some $\varepsilon > 0$, $x, y \in \mathbb{R}$.

Now, we switch point of view and consider the functions $w : Q \rightarrow \mathbb{R}$ as functions on $[0, T]$ with values in \mathbb{R}^∞ such that, for all $t \in [0, T]$, $w(t) := \{w(t, x) : x \in \mathbb{R}\}$.

We impose a coercivity condition over the operator (1.2), and make assumptions on the growth and regularity of the operator's coefficients and also on the regularity of the free data f and g (see [9]):

Assumption 2. Let $r > 0$ and $\rho > 0$ be smooth functions on \mathbb{R} , and $m \geq 0$ an integer.

- (1) There exists a constant $\lambda > 0$ such that $a(t, x) \geq \lambda \rho^2(x)$, for all $t \geq 0$, $x \in \mathbb{R}$;
- (2) The coefficients in L and their derivatives in x up to the order m are measurable functions in $[0, T] \times \mathbb{R}$ such that

$$|D_x^\alpha a| \leq K \rho^{2-\alpha} \forall \alpha \leq m \vee 1, \quad |D_x^\alpha b| \leq K \rho^{1-\alpha}, \quad |D_x^\alpha c| \leq K \forall \alpha \leq m,$$
 for any $t \in [0, T]$, $x \in \mathbb{R}$, with K a constant and D_x^α denoting the α^{th} partial derivative operator with respect to x ;
- (3) $f \in L^2([0, T]; W^{m-1,2}(r, \rho))$ and $g \in W^{m,2}(r, \rho)$.

Notation. We use the notation $W^{-1,2}(r, \rho) := (W^{1,2}(r, \rho))^*$, where $(W^{1,2}(r, \rho))^*$ is the dual of $W^{1,2}(r, \rho)$.

We define the generalized solution of problem (1.1).

Definition 1. We say that $u \in C([0, T]; W^{0,2}(r, \rho))$ is a generalized solution of (1.1) on $[0, T]$ if

- (1) $u \in L^2([0, T]; W^{1,2}(r, \rho))$;
- (2) For every $t \in [0, T]$

$$\begin{aligned} (u(t), \varphi) &= (g, \varphi) + \int_0^t \{ - (a(s)D_x u(s), D_x \varphi) \\ &\quad + (b(s)D_x u(s) - D_x a(s)D_x u(s), \varphi) \\ &\quad + (c(s)u(s), \varphi) + \langle f(s), \varphi \rangle \} ds \end{aligned}$$

holds for all $\varphi \in C_0^\infty$.

Notation. The notation (\cdot, \cdot) in the above definition stands for the inner product in $W^{0,2}(r, \rho)$.

Finally, we state the existence and uniqueness of the solution of problem (1.1) (see, e.g., [9, 11]).

Theorem 1. *Under (1)–(2) in Assumption 1, with $m + 1$ in place of m , and (1)–(3) in Assumption 2, problem (1.1) admits a unique generalized solution u on $[0, T]$. Moreover $u \in C([0, T]; W^{m,2}(r, \rho)) \cap L^2([0, T]; W^{m+1,2}(r, \rho))$ and*

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|u(t)\|_{W^{m,2}(r, \rho)}^2 + \int_0^T \|u(t)\|_{W^{m+1,2}(r, \rho)}^2 dt \\ & \leq N \left(\|g\|_{W^{m,2}(r, \rho)}^2 + \int_0^T \|f(t)\|_{W^{m-1,2}(r, \rho)}^2 dt \right), \end{aligned}$$

with N a constant.

2. MAIN RESULTS

We now proceed to the discretization of problem (1.1) in the space-variable. We define the h -grid on R , with $h \in (0, 1]$,

$$(2.1) \quad Z_h = \{x \in R : x = nh, \quad n = 0, \pm 1, \pm 2, \dots\}.$$

Denote

$$(2.2) \quad \partial^+ u = \partial^+ u(t, x) = h^{-1}(u(t, x+h) - u(t, x))$$

and

$$(2.3) \quad \partial^- u = \partial^- u(t, x) = h^{-1}(u(t, x) - u(t, x-h)),$$

for every $x \in Z_h$, the forward and backward difference quotients in space, respectively. Define the discrete operator

$$L_h(t, x) = a(t, x)\partial^- \partial^+ + b(t, x)\partial^+ + c(t, x).$$

We consider the discrete problem

$$(2.4) \quad L_h u - u_t + f_h = 0 \quad \text{in } Q(h), \quad u(0, x) = g_h(x) \quad \text{in } Z_h,$$

where $Q(h) = [0, T] \times Z_h$, with $T \in (0, \infty)$, and f_h and g_h are functions such that $f_h : Q(h) \rightarrow R$ and $g_h : Z_h \rightarrow R$.

For functions $v : Z_h \rightarrow R$, we introduce the discrete version of the weighted Sobolev space $W^{0,2}(r, \rho)$:

$$l^{0,2}(r) = \{v : |v|_{l^{0,2}(r)} < \infty\},$$

where the norm $|v|_{l^{0,2}(r)}$ is defined by

$$|v|_{l^{0,2}(r)} = \left(\sum_{x \in Z_h} r^2(x) |v(x)|^2 h \right)^{1/2}.$$

Define the inner product

$$(v, w)_{l^{0,2}(r)} = \sum_{x \in Z_h} r^2(x) v(x) w(x) h,$$

for any $v, w \in l^{0,2}(r)$, which induces the above norm. Endowed with the inner product, the space $l^{0,2}(r)$ is clearly a Hilbert space.

For functions $v : Z_h \rightarrow R$, we introduce also the discrete version of the weighted Sobolev space $W^{1,2}(r, \rho)$:

$$l^{1,2}(r, \rho) = \{v : |v|_{l^{1,2}(r, \rho)} < \infty\},$$

with the norm $|v|_{l^{1,2}(r, \rho)}$ defined by

$$|v|_{l^{1,2}(r, \rho)}^2 = |v|_{l^{0,2}(r)}^2 + |\rho \partial^+ v|_{l^{0,2}(r)}^2.$$

We endow $l^{1,2}(r, \rho)$ with the inner product, inducing the above norm,

$$(v, w)_{l^{1,2}(r, \rho)} = (v, w)_{l^{0,2}(r)} + (\rho \partial^+ v, \rho \partial^+ w)_{l^{0,2}(r)},$$

for any functions v, w in $l^{1,2}(r, \rho)$.

It can be easily checked that $l^{1,2}(r, \rho)$ is a reflexive and separable Banach space, continuously and densely embedded into the Hilbert space $l^{0,2}(r)$.[§]

We switch our viewpoint and consider the functions $w : Q(h) \rightarrow R$ as functions on $[0, T]$ with values in R^∞ , defined by

$$w(t) = \{w(t, x) : x \in Z_h\}.$$

for all $t \in [0, T]$. For these functions, we consider the space

$$C([0, T]; l^{0,2}(r))$$

of continuous $l^{0,2}(r)$ -valued functions on $[0, T]$, and the spaces

$$L^2([0, T]; l^{m,2}(r, \rho)) = \left\{ w : [0, T] \rightarrow l^{m,2}(r, \rho) : \int_0^T |w(t)|_{l^{m,2}(r, \rho)}^2 dt < \infty \right\},$$

with $m = 0, 1$.

Notation. We identify $l^{0,2}(r, \rho)$ with $l^{0,2}(r)$.

We make some assumptions over the regularity of the data f_h and g_h in (2.4).

Assumption 3. Let $r > 0$ be a smooth function on R .

- (1) $f_h \in L^2([0, T]; l^{0,2}(r))$;
- (2) $g_h \in l^{0,2}(r)$.

We define the generalized solution of problem (2.4).

Definition 2. We say that $u \in C([0, T]; l^{0,2}(r)) \cap L^2([0, T]; l^{1,2}(r, \rho))$ is a generalized solution of (2.4) if, for all $t \in [0, T]$,

$$(u(t), \varphi) = (g_h, \varphi) + \int_0^t \left\{ - (a(s) \partial^+ u(s), \partial^+ \varphi) + (b(s) \partial^+ u(s) - \partial^+ a(s) \partial^+ u(s), \varphi) \right. \\ \left. + (c(s) u(s), \varphi) + \langle f_h(s), \varphi \rangle \right\} ds$$

holds for all $\varphi \in l^{1,2}(r, \rho)$.

Notation. In the above definition, (\cdot, \cdot) denotes the inner product in $l^{0,2}(r)$.

Next, we establish the existence and uniqueness of the generalized solution of discrete problem (2.4), and determine an estimate for the solution. With this result, we show that the numerical scheme is stable, i.e., informally, that the solution of the discrete problem remains bounded independently of the space-step h .

Theorem 2. *Under (1)–(2) in Assumption 2 and (1)–(2) in Assumption 3, problem (2.4) has a unique generalized solution u in $[0, T]$. Moreover*

$$\sup_{0 \leq t \leq T} |u(t)|_{l^{0,2}(r)}^2 + \int_0^T |u(t)|_{l^{1,2}(r, \rho)}^2 dt \leq N \left(|g_h|_{l^{0,2}(r)}^2 + \int_0^T |f_h(t)|_{l^{0,2}(r)}^2 dt \right),$$

with N a constant independent of h .

[§]We refer to [7], where this is proved for the more general case where $l^{0,2}(r)$ and $l^{1,2}(r, \rho)$ are spaces of real-valued functions on a d -dimensional grid.

Finally, we state a result on the convergence of the discrete problem's generalized solution to the PDE problem's generalized solution, and compute a convergence rate (the accuracy of the approximation is of order 1). The result is obtained by imposing that the weights ρ are bounded from below by a positive constant. Note that this amounts to assume that the weights ρ are increasing functions of $|x|$, which is precisely the case we are studying.

Theorem 3. *Assume that the hypotheses of Theorems 1 and 2 are satisfied. Assume additionally that (3) in Assumption 1 holds and that $\rho(x) \geq C$ on R , with $C > 0$ a constant. Denote by u the solution of (1.1) in Theorem 1 and by u_h the solution of (2.4) in Theorem 2. Assume also that $u \in L^2([0, T]; W^{3,2}(r, \rho))$. Then*

$$\begin{aligned} & \sup_{0 \leq t \leq T} |u(t) - u_h(t)|_{l^{0,2}(r)}^2 + \int_0^T |u(t) - u_h(t)|_{l^{1,2}(r,\rho)}^2 dt \\ & \leq h^2 N \int_0^T |u(t)|_{W^{3,2}(r,\rho)}^2 dt + N \left(|g - g_h|_{l^{0,2}(r)}^2 + \int_0^T |f(t) - f_h(t)|_{l^{0,2}(r)}^2 dt \right), \end{aligned}$$

for some constant N independent of h .

3. SOME DETAILS

Theorem 2 is obtained by proving that the discretized problem (2.4) can be cast in a general initial-value problem for linear evolution equations in abstract spaces, and then making use of an existence and uniqueness result for this general problem (see, e.g., [11] for the latter result).

Theorem 3 is proved owing both to the stability properties of the numerical scheme (Theorem 2), and to its consistency properties. The following result concerns the consistency, and asserts that the difference quotients approximate the partial derivatives (with accuracy of order 1).

Proposition 1. *Assume that (1)–(3) in Assumption 1 are satisfied and that $\rho(x) \geq C$ on R , with $C > 0$ a constant. Let $u(t) \in W^{2,2}(r, \rho)$, $v(t) \in W^{3,2}(r, \rho)$, for all $t \in [0, T]$. Then there exists a constant N independent of h such that*

$$\begin{aligned} (1) \quad & \sum_{x \in Z_h} r^2(x) \left| \frac{\partial}{\partial x} u(t, x) - \partial^+ u(t, x) \right|^2 \rho^2(x) h \leq h^2 N |u(t)|_{W^{2,2}(r,\rho)}^2, \\ (2) \quad & \sum_{x \in Z_h} r^2(x) \left| \frac{\partial^2}{\partial x^2} v(t, x) - \partial^- \partial^+ v(t, x) \right|^2 \rho^4(x) h \leq h^2 N |v(t)|_{W^{3,2}(r,\rho)}^2, \end{aligned}$$

for all $t \in [0, T]$.

We stress that the estimates in Proposition 1 and in Theorem 3 are obtained under regularity assumptions weaker than those in the correspondent consistency and convergence results in [6, 7] for the more general case of multidimensional PDEs.

For a complete description of the discretization study presented in this article, we refer to [8].

4. A NUMERICAL EXAMPLE

In this section we illustrate the theoretical results and numerical scheme referred above. We consider problem (1.1) with the following coefficient functions

$$a(t, x) = (1 + t)(1 + x^2), \quad b(t, x) = 2tx, \quad \text{and} \quad c(t, x) = \sin(tx),$$

and with

$$f(t, x) = t(1 + x^3) \quad \text{and} \quad g(x) = (1 + x^3) \sin(2\pi x).$$

Then, problem (1.1) reads

$$(4.1) \quad Lu - u_t + t(1 + x^3) = 0 \quad \text{in } Q, \quad u(0, x) = (1 + x^3) \sin(2\pi x) \quad \text{in } \mathbb{R},$$

where

$$L(t, x) = (1 + t)(1 + x^2) \frac{\partial^2}{\partial x^2} + 2tx \frac{\partial}{\partial x} + \sin(tx).$$

We proceed in two steps:

- we observe that the assumptions of Theorems 1, 2 and 3 are satisfied and so their conclusions hold for problem (4.1) and its space-discretized version presented below
- we make a computational simulation using the method of lines.

4.1. Analytic and numerical solution of (4.1). Take $m = 2$, and consider the functional settings of Section 1 and of Section 2 as underlying adequately the study of (4.1) and of the corresponding space-discretized version, respectively.

Consider the weight functions (particular cases taken from [9], citing [12])

$$(4.2) \quad \begin{aligned} r(x) &= (1 + x^2)^\beta, \quad \text{with } \beta < -11/4 \\ \rho(x) &= (1 + x^2)^{1/2}. \end{aligned}$$

It is easy to see that:

- r and ρ satisfy Assumption 1 (with $m = 3$ as required in Theorem 1);
- the coefficient functions $a(t, x)$, $b(t, x)$ and $c(t, x)$ considered above satisfy (1) and (2) in Assumption 2 and the functions f and g satisfy (3).

Then the assumptions of Theorem 1 are satisfied and its conclusion holds for problem (4.1).

We now discretize problem (4.1) on the h -grid Z_h on \mathbb{R} defined by (2.1). For $x = nh \in Z_h$ and $t \in [0, T]$, define

$f_h(t, x) := f(t, nh) = t(1 + (nh)^3)$ and $g_h(x) := g(nh) = (1 + (nh)^3) \sin(2\pi nh)$, discrete versions of f and g , respectively. We then obtain the family of ordinary differential equations in the time variable t

$$(4.3) \quad L_h u(t, nh) - u_t(t, nh) + t(1 + (nh)^3) = 0 \quad \text{in } Q(h) = [0, T] \times Z_h$$

satisfying

$$(4.4) \quad u(0, nh) = (1 + (nh)^3) \sin(2\pi nh) \quad \text{in } Z_h,$$

where the operator L_h is defined by

$$L_h(t, x) = a(t, x) \partial^+ \partial^- + b(t, x) \partial^+ + c(t, x),$$

with ∂^+ and ∂^- the forward and backward difference quotients in space defined, respectively, by (2.2) and (2.3).

It is clear that $f_h \in L^2([0, T]; l^{0;2}(r))$ and $g_h \in l^{0;2}(r)$ and so Assumption 3 is satisfied. Then, we can apply Theorem 2 and so its conclusion holds for problem (4.3)–(4.4).

Notice that $\rho(x) = (1 + x^2)^{1/2} \geq 1$ on \mathbb{R} . Under the above setting we can apply Theorem 3. But before that observe that

- by the conclusion of Theorem 1 applied to our example, the generalized solution u of problem (4.1) satisfies

$$\int_0^T |u(t)|_{W^{3,2}(r,\rho)}^2 dt \leq N \left(|g|_{W^{2,2}(r,\rho)}^2 + \int_0^T |f(t)|_{W^{1,2}(r,\rho)}^2 dt \right);$$

- due to the chosen approximation

$$|g - g_h|_{l^{0,2}(r)}^2 + \int_0^T |f(t) - f_h(t)|_{l^{0,2}(r)}^2 dt = 0.$$

Owing to these facts, we have the following upper bound for the estimate in Theorem 3

$$(4.5) \quad \begin{aligned} h^2 N \int_0^T |u(t)|_{W^{3,2}(r,\rho)}^2 dt + N \left(|g - g_h|_{l^{0,2}(r)}^2 + \int_0^T |f(t) - f_h(t)|_{l^{0,2}(r)}^2 dt \right) \\ \leq h^2 N \left(|g|_{W^{2,2}(r,\rho)}^2 + \int_0^T |f(t)|_{W^{1,2}(r,\rho)}^2 dt \right). \end{aligned}$$

Thus, applying Theorem 3 and using the upper bound (4.5), a result concerning the convergence of the discrete problem's generalized solution to the PDE problem's generalized solution holds. This result, which falls in spirit of Theorem 3, is stated next for the sake of completeness.

Proposition 2. *Denote by u the solution of problem (4.1) and by u_h the solution of problem (4.3)–(4.4). Then*

$$\begin{aligned} \sup_{0 \leq t \leq T} |u(t) - u_h(t)|_{l^{0,2}(r)}^2 + \int_0^T |u(t) - u_h(t)|_{l^{1,2}(r,\rho)}^2 dt \\ \leq h^2 N \left(|g|_{W^{2,2}(r,\rho)}^2 + \int_0^T |f(t)|_{W^{1,2}(r,\rho)}^2 dt \right), \end{aligned}$$

for some constant N independent of h .

4.2. Computational simulation. Now we implement our example computationally. In order to do so we introduce some boundary conditions.

Let $0 \leq x \leq 1$ and take $h > 0$ such that $h \times M = 1$ for some integer $M > 1$. Define the h -grid in $[0, 1]$:

$$Z_h^* = \{0, h, 2h, \dots, 1\}.$$

Impose the boundary conditions:

$$u(t, 0) = 0 \quad \text{and} \quad u(t, 1) = 0.$$

The discrete problem (4.3)–(4.4) can be written

– for $1 \leq n \leq M - 1$,

$$\begin{aligned} u_t(t, nh) &= \left(\frac{a(t, nh)}{h^2} + \frac{b(t, nh)}{h} \right) u(t, (n+1)h) \\ &\quad + \left(c(t, nh) - \frac{2a(t, nh)}{h^2} - \frac{b(t, nh)}{h} \right) u(t, nh) \\ &\quad + \frac{a(t, nh)}{h^2} u(t, (n-1)h) + f(t, nh) \end{aligned}$$

$$u(0, nh) = (1 + (nh)^3) \sin(2\pi nh),$$

– for $n = 0$ and $n = M$,

$$u(t, 0) = u(t, 1) = 0.$$

Figure 1 is a representation of the solution of the discrete problem, for $h = 0.01$. The numerical scheme was implemented making use of the software *Mathematica*, version 7.0.0.

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