Chapter 1
KAM theory as a limit of renormalization

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Abstract This is a brief survey of recent results on the KAM stability of quasiperiodic dynamics using renormalization of vector fields.

1.1 Introduction

For thirty years renormalization ideas have been used in the theory of dynamical systems. After the pioneering work of Feigenbaum [6] in the late 1970's, there has been a number of different applications of renormalization techniques. Its core concept is rescaling. That is, rescaling of space by zooming in a region in phase space; rescaling of time by considering a different time frame, as it takes longer to return to the region. Complicated dynamical behaviour can then turn out to be simpler in the new renormalized system. If by iterating the rescaling one gets convergence, it is a clear hint that the system looks the same in smaller scales. Moreover, if this self similarity is in some sense trivial, one can then hope to prove conjugacy between the systems.

The connection between KAM and renormalization theories has been realized for quite some time. Renormalization approach to KAM has several important advantages. First of all, it provides a unified setting which allows to deal with both the cases of smooth KAM-type invariant tori and non-smooth critical tori. Secondly, the proofs based on renormalizations are conceptually very simple and give a different perspective on the problem of small divisors. For the continuous-time situation, several KAM results for small-divisor problems in quasiperiodic motion have been obtained by studying the stability of trivial fixed sets of renormalization operators (cf. e.g. [16, 22, 23, 19, 7]). There was however a relevant restriction when dealing with multiple frequencies. Because renormalization methods rely fundamentally on

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the continued fractions expansion of the frequency vector, the lack of a multidimensional version of continued fractions was the reason for failing to replicate KAM in its full generality. This limitation was recently overcome in [12] by adapting Lagarias' algorithm [21] and deriving estimates for multidimensional continued fractions (MCF) expansions of diophantine vectors.

In the case of Hamiltonian systems with two degrees of freedom MacKay proposed in the early 1980's a renormalization scheme for the construction of KAM invariant tori [27] (see also [29, 30, 31]). The scheme was realized for the construction of invariant curves for two-dimensional conservative maps of the cylinder. An important feature of MacKay's approach is the analysis of both smooth KAM invariant curves and so-called critical curves corresponding to critical values of a parameter above which invariant curves no longer exist. From the point of view of renormalization theory the KAM curves correspond to a trivial linear fixed point for the renormalization transformations, while critical curves give rise to very complicated fixed points with nontrivial critical behavior. MacKay's renormalization scheme was carried out only for a small class of Diophantine rotation numbers with periodic continued fraction expansion (such as the golden mean). Khanin and Sinai studied a different renormalization scheme for general Diophantine rotation numbers [14]. Both of the above early approaches were based on renormalization for maps or their generating functions. Essentially, the renormalization transformations are defined in the space of pairs of mappings which, being iterates of the same map, commute with each other. These commutativity conditions cause difficult technical problems, and led MacKay [28] to propose the development of alternative renormalization schemes acting directly on vector fields. The same idea was realized by Koch [16] who proves a KAM type result for analytic perturbations of linear Hamiltonians $H^0(x,y) = \omega \cdot y$, for frequencies ω which are eigenvectors of hyperbolic matrices in $\mathrm{SL}(2,\mathbb{Z})$ with only one unstable direction. Notice that the set of such frequencies has zero Lebesgue measure and in the case $d = 2$ corresponds to vectors with a quadratic irrational slope. Further improvements and applications of Koch's techniques appeared in [1, 17, 22, 23, 7], emphasizing the connection between KAM and renormalization theories.

Other renormalization ideas have appeared in the context of the stability of invariant tori for nearly integrable Hamiltonian systems inspired by quantum field theory and an analogy with KAM theory (see e.g. [2], and [8, 9] where it is used a graph representation of the invariant tori in terms of Feynman diagrams).

In section 1.2 we describe a multidimensional continued fractions scheme, which gives estimates to be used in the renormalization. In the remaining sections we include examples of systems and several KAM-type results obtained by renormalization. In particular, in section 1.3 we give a sketch of the proof of almost reducibility for analytic linear skew-product flows (cf. [5]). In section 1.4 we study local conjugacy classes for toroidal flows. Finally, in section 1.5 we present the main ideas for the renormalization proof of the "classical" KAM theorem in the context of Hamiltonian dynamics.

Throughout this text we denote by $\mathrm{Homeo}(M)$ and $\mathrm{Diff}^r(M)$, $r \in \mathbb{N} \cup \{\infty, \omega\}$, the set of homeomorphisms and C^r-diffeomorphisms on M. Moreover, we add a subscript 0 to distinguish the case of homotopic to the identity maps. Finally, $\mathrm{Vect}^r(M)$ stands for the set of C^r-vector fields on M. Recall that the transformation of an arbitrary vector field X on a manifold M by $\psi \in \mathrm{Diff}(M)$ is given by

$$\psi^* X = D\psi \circ \psi^{-1} \cdot X \circ \psi^{-1}. \tag{1.1}$$

1.2 Multidimensional continued fractions

An essential ingredient of the renormalization scheme is a continued fractions decomposition of vectors, relating the number-theoretical properties of the frequencies and the conjugacy smoothness.

In this section we present the multidimensional continued fractions algorithm introduced in [12] following ideas of Dani [4], Lagarias [21] and Kleinbock-Margulis [15]. In addition, we define the class of diophantine vectors from the properties of the continued fractions expansion.

1.2.1 Flow on homogeneous space

Denote by $G = \mathrm{SL}(d,\mathbb{R})$, $\Gamma = \mathrm{SL}(d,\mathbb{Z})$ and take a fundamental domain $\mathscr{F} \subset G$ of the homogeneous space $\Gamma \backslash G$ (the space of d-dimensional non-degenerate unimodular lattices). On $\mathscr{F}$ consider the flow:

$$\Phi^t \colon \mathscr{F} \to \mathscr{F}, \quad M \mapsto P(t) M E^t, \tag{1.2}$$

where

$$E^t = \mathrm{diag}(\mathrm{e}^{-t}, \ldots, \mathrm{e}^{-t}, \mathrm{e}^{(d-1)t}) \in G$$

and $P(t)$ is the unique family in Γ that keeps $\Phi^t M$ in $\mathscr{F}$ for every $t \geq 0$.

Let $\omega = (\alpha, 1) \in \mathbb{R}^d$. We are interested in the orbit under Φ^t of the matrix

$$M_\omega = \begin{pmatrix} I & \alpha \\ 0 & 1 \end{pmatrix}. \tag{1.3}$$

1.2.2 Growth of the flow

Let the function $\delta \colon \Gamma \backslash G \to \mathbb{R}^+$ measuring the shortest vector in the lattice M be

$$\delta(M) = \inf_{k \in \mathbb{Z}^d \setminus \{0\}} \| {}^\top k M \|, \tag{1.4}$$

where $\|\cdot\|$ stands for the ℓ_1-norm (in the following we will make use of the corresponding matrix norm taken as the usual operator norm). Notice that $\delta(\Phi^t M_\omega) = \delta(M_\omega E^t)$.

Proposition 1 ([12]). *There exist* $C_1, C_2 > 0$ *such that for all* $t \geq 0$

$$\|\Phi^t M_\omega\| \leq \frac{C_1}{\delta(\Phi^t M_\omega)^{d-1}} \quad \textit{and} \quad \|(\Phi^t M_\omega)^{-1}\| \leq \frac{C_2}{\delta(\Phi^t M_\omega)}. \tag{1.5}$$

1.2.3 Stopping times

Consider a sequence of times, called *stopping times*,

$$t_0 = 0 < t_1 < t_2 < \cdots \to +\infty \tag{1.6}$$

such that the matrices $P(t)$ in (1.2) satisfy

$$P_n := P(t_n) \neq P(t_{n-1}), \tag{1.7}$$

with $n \in \mathbb{N}$. We also set $P_0 = P(t_0) = I$. The sequence of matrices $P_n \in \mathrm{SL}(d, \mathbb{Z})$ are the rational approximates of ω, called the *multidimensional continued fractions expansion*. In addition we define the transfer matrices

$$T_n = P_n P_{n-1}^{-1}, \quad n \in \mathbb{N}, \quad \text{and} \quad T_0 = I. \tag{1.8}$$

The flow of M_ω taken at the time sequence is thus the sequence of matrices

$$M_n := \Phi^{t_n} M_\omega = P_n M_\omega E^{t_n}. \tag{1.9}$$

Using some properties of the flow, the above can be decomposed (see [12]) into

$$M_n = \begin{pmatrix} I & \alpha_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta_n & 0 \\ {}^\top\beta_n & \gamma_n \end{pmatrix} \tag{1.10}$$

with γ_n being the d-th component of the vector $\mathrm{e}^{(d-1)t_n} P_n \omega$.

Define $\omega_n = (\alpha_n, 1)$, $\omega_0 = \omega$ and, for $n \in \mathbb{N}$,

$$\omega_n = \eta_n T_n \omega_{n-1}, \tag{1.11}$$

where η_n is a normalization factor.

If $d = 2$ there exists a sequence of stopping times (called Hermitte critical times) that gives an accelerated version of the standard continued fractions of a number α [21].

1.2.4 Resonance cone

Given resonance widths σ, i.e. a sequence $\sigma\colon \mathbb{N}_0 \to \mathbb{R}^+$, define the resonant cones to be

$$I_n^+ = \{k \in \mathbb{Z}^d \colon |k \cdot \omega_n| \le \sigma_n \|k\|\}. \tag{1.12}$$

In addition, let

$$A_n = \sup_{k \in I_n^+ \setminus \{0\}} \frac{\| {}^\top T_{n+1}^{-1} k\|}{\|k\|}. \tag{1.13}$$

Proposition 2 ([26]). *There is $c > 0$ such that for any $n \in \mathbb{N}_0$*

$$A_n \le c\, \mathrm{e}^{-\delta t_{n+1}} \frac{\sigma_n \mathrm{e}^{d\,\delta t_{n+1}} + 1}{\delta(M_n)^{d-1} \delta(M_{n+1})}, \tag{1.14}$$

where $\delta t_{n+1} = t_{n+1} - t_n$.

1.2.5 Diophantine vectors

A vector $\omega \in \mathbb{R}^d$ is Diophantine with exponent $\beta \ge 0$ if there is a constant $C > 0$ such that

$$|\omega \cdot k| > \frac{C}{\|k\|^{d-1+\beta}}.$$

It is a well known fact that the sets $DC(\beta)$ of Diophantine vectors with exponent $\beta > 0$ are of full Lebesgue measure [3]. On the other hand, the set $DC(0)$ has zero Lebesgue measure. A vector is said to be diophantine if it belongs to $DC = \cup_{\beta \ge 0} DC(\beta)$. The next proposition gives us a complete characterization of diophantine vectors in terms of the behaviour of the flow Φ^t of M_ω.

Proposition 3 ([26]). *Let $\beta \ge 0$. Then, $\omega \in DC(\beta)$ iff there is $C' > 0$ such that*

$$\delta(\Phi^t M_\omega) > C' \mathrm{e}^{-\theta t}, \quad t \ge 0,$$

with $\theta = \beta/(d+\beta)$.

Proposition 4 ([12]). *If $\omega \in DC(\beta)$, $\beta \ge 0$, there are constants $c_i > 0$ such that, for any stopping-time sequence $t\colon \mathbb{N}_0 \to \mathbb{R}$,*

$$\|M_n\| \le c_1 \exp[(d-1)\theta t_n], \tag{1.15}$$
$$\|M_n^{-1}\| \le c_2 \exp(\theta t_n), \tag{1.16}$$
$$\|T_n\| \le c_5 \exp[(1-\theta)\delta t_n + d\,\theta\, t_n], \tag{1.17}$$
$$\|T_n^{-1}\| \le c_6 \exp[(d-1)(1-\theta)\delta t_n + d\,\theta\, t_n], \tag{1.18}$$

where $\delta t_n = t_n - t_{n-1}$ and $\theta = \beta/(d+\beta)$.

Proposition 5 ([26]). *If $\omega \in DC(\beta)$, $\beta \geq 0$, then there is $c > 0$ such that for any $n \in \mathbb{N}_0$,*

$$A_n \leq c\,\mathrm{e}^{-(1-\theta)\delta t_{n+1}+d\theta t_n}\left(\sigma_n \mathrm{e}^{d\delta t_{n+1}}+1\right). \tag{1.19}$$

Possible choices are $t_n = c_1(1+\beta)^n$ and $\sigma_n = e^{-c_2(1+\beta)^n}$ with some $c_i > 0$.

Here we have only discussed the case of Diophantine frequency vectors. However, renormalization can be used for a larger class of vectors, cf. e.g. [24, 25, 26, 18, 20, 10, 11].

1.3 Almost reducibitity of linear skew-product flows

In this section we deal with skew-product vector fields, which are linear differential equations of dimension two, with quasiperiodic coefficients. This is a generalization of the classical Floquet theory. Our goal is to present the main ideas behind renormalization for this kind of dynamics. We present a sketch of a proof on almost reducibility of these systems.

1.3.1 Skew-product vector fields

Consider the manifold $M = \mathbb{T}^d \times \mathrm{SL}(2,\mathbb{R})$. Let $\mathrm{Vect}^r_{sw}(M)$ be the set of C^r-vector fields on M of the form:

$$X(x,y) = (\omega, f(x)y), \quad (x,y) \in M, \tag{1.20}$$

where $\omega \in \mathbb{R}^d \setminus \{0\}$ and $f \in C^r(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$. We will use the following notation

$$X = (\omega, f).$$

Each element of $\mathrm{Vect}^r_{sw}(M)$ generates a skew-product flow on M, i.e. a flow of the type

$$\phi^t(x,y) = (x+\omega t, \Phi^t(x)y),$$

where $\Phi^t \colon \mathbb{T}^d \to \mathrm{SL}(2,\mathbb{R})$.

As we want to preserve the space $\mathrm{Vect}^r_{sw}(M)$ under coordinate changes, we consider the set $\mathrm{Diff}^{r+1}_{sw}(M)$ of

$$\psi(x,y) = (Tx, F(x)y), \quad (x,y) \in M, \tag{1.21}$$

where $F \in C^{r+1}(\mathbb{T}^d, \mathrm{SL}(2,\mathbb{R}))$ and $T \in \mathrm{SL}(d,\mathbb{Z})$ is a linear automorphism of the torus. For simplicity, we write

$$\psi = (T, F).$$

A vector field in the new coordinates is then given by the formula

$$\psi^* X(x,y) = (T\omega, L_\omega F(T^{-1}x)\cdot F(T^{-1}x)^{-1}y + \mathrm{Ad}_{F(T^{-1}x)} f(T^{-1}x)\cdot y), \tag{1.22}$$

where

$$L_\omega = \omega\cdot D = \sum_i \omega_i \partial/\partial x_i \tag{1.23}$$

and $\mathrm{Ad}_A b = AbA^{-1}$.

1.3.2 Fibered rotation number

Consider the natural projection $p\colon \mathbb{R}^2\setminus\{0\}\to\mathbb{T}^1$ given by the argument of a vector. The fibered rotation number of the flow generated by $X=(\omega,f)\in\mathrm{Vect}^0_{sw}(M)$ is defined to be

$$\rho(X) = \lim_{t\to+\infty} p\left(\frac{\int_0^t f\circ\phi^s(x,y)\, v\, ds}{t}\right)$$

for $(x,y)\in M$ and $v\in\mathbb{R}^2\setminus\{0\}$. This measures the asymptotic frequency of rotation of the fiber flow in $\mathbb{R}^2$. We will be interested in vector fields for which ρ exists at any point and direction v.

1.3.3 Almost reducibility

In some cases it is possible to find a diffeomorphism that simplifies X, in particular reducing it to a "constant" vector field. More precisely, we have the following definition.

1. $X\in\mathrm{Vect}^r_{sw}(M)$ is C^s*-conjugated to* $Y\in\mathrm{Vect}^r_{sw}(M)$ if there is $\psi\in\mathrm{Diff}^s_{sw}(M)$ such that $\psi^*X=Y$.
2. X is C^s*-reducible* if its C^s-conjugacy class contains a vector field $Z=(\omega,u)$, with $u\in\mathrm{sl}(2,\mathbb{R})$.
3. X is C^s*-almost reducible* if the closure of its C^s-conjugacy class contains a vector field $Z=(\omega,u)$, with $u\in\mathrm{sl}(2,\mathbb{R})$.

Theorem 1. *Let* $\omega\in\mathbb{R}^d$ *be Diophantine and* $C>0$*. There is* $\varepsilon>0$ *such that if* $f\in C^\omega(\mathbb{T}^d,\mathrm{sl}(2,\mathbb{R}))$ *is* ε*-*C^ω*-close to constant and* $|\rho(\omega,f)|<C$*, then* (ω,f) *is* C^ω*-almost reducible.*

Notice that ε does not depend on the arithmetical properties of the rotation number. In the remaining part of this section we present the main steps towards the proof of the above theorem.

1.3.4 Non-homotopic to the identity diffeomorphism

Given $m \in \mathbb{Z}^d$, we will also be interested in the following transformation of coordinates:

$$\psi_m = (I, R_m)$$

where $R_m\colon \mathbb{T}^d \to \mathrm{SO}(2,\mathbb{R})$ is

$$R_m(x) = \begin{bmatrix} \cos(2\pi m\cdot x) & -\sin(2\pi m\cdot x) \\ \sin(2\pi m\cdot x) & \cos(2\pi m\cdot x) \end{bmatrix}.$$

The action on a vector field $X = (\omega, f)$ is given by

$$\psi_m^* X = \left(\omega, 2\pi m\cdot\omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \mathrm{Ad}_{R_m} f\right).$$

In particular, the rotation number is changed as

$$\rho(\psi_m^* X) = \rho(X) - \frac{1}{2} m\cdot\omega.$$

1.3.5 Lifts and complexification

Let $r > 0$ and consider the domain

$$\mathscr{D}_r = \{x \in \mathbb{C}^d \colon \|\operatorname{Im} x\| < r/2\pi\} \tag{1.24}$$

for the norm $\|z\| = \sum_i |z_i|$ on $\mathbb{C}^d$. Take a real-analytic map

$$F\colon \mathscr{D}_r \to \mathrm{SL}(2,\mathbb{C}),$$

$\mathbb{Z}^d$-periodic, on the form of the Fourier series

$$F(x) = \sum_{k\in\mathbb{Z}^d} F_k \mathrm{e}^{2\pi \mathrm{i} k\cdot x} \tag{1.25}$$

with $F_k \in \mathrm{SL}(2,\mathbb{C})$. The Banach spaces $\mathscr{A}_r$ and $\mathscr{A}_r'$ are the subspaces such that the respective norms

$$\|F\|_r = \sum_{k\in\mathbb{Z}^d} \|F_k\|\, \mathrm{e}^{r\|k\|}, \tag{1.26}$$

$$\|F\|_r' = \sum_{k\in\mathbb{Z}^d} (1 + 2\pi\|k\|)\, \|F_k\|\, \mathrm{e}^{r\|k\|} \tag{1.27}$$

are finite. Here and in the following we use the matrix norm $\|A\| = \max_j \sum_i |A_{i,j}|$ for any square matrix A with entries $A_{i,j}$.

Similarly, define the space $\mathfrak{a}_r$ of real-analytic functions $\mathscr{D}_r \to \mathrm{sl}(2,\mathbb{C})$, $\mathbb{Z}^d$-periodic and on the form of Fourier series, having the same type of bounded norm as (1.26). We are interested in vector fields that can be written as

$$X(x,y) = (\omega, f(x)y), \quad (x,y) \in \mathscr{D}_r \times \mathrm{SL}(2,\mathbb{C}). \tag{1.28}$$

The space of such vector fields is denoted by V_r whenever f is in $\mathfrak{a}_r$. The norm on this space is defined to be

$$\|X\|_r = \|\omega\| + \|f\|_r. \tag{1.29}$$

1.3.6 Uniformization

The theorem below states the existence of a nonlinear change of coordinates isotopic to the identity that cancels the

$$I^- = \{k \in \mathbb{Z}^d : |k \cdot \omega| > \sigma \|k\|\}$$

Fourier modes of a sufficiently close to constant $X \in V_r$, with $\sigma > 0$. We are only eliminating the far from resonance modes, this way avoiding the complications usually related to small divisors.

Let $u \in \mathrm{sl}(2,\mathbb{C})$ and

$$B_r(u,\varepsilon) = \{f \in \mathfrak{a}_r : \|f - u\|_r < \varepsilon\}$$

where

$$\varepsilon = \frac{C\sigma^2}{\|\omega\| + \|u\|}. \tag{1.30}$$

In order to simplify notations, here and in the following C stands for some positive universal constant, not necessarily the same.

Theorem 2. *Let $|\rho| \le \sigma/4$ and $u \in \mathrm{sl}(2,\mathbb{R})$ with eigenvalues $\pm \mathrm{i}\rho$. There is an analytic map $\mathfrak{U} \colon B_r(u,\varepsilon) \to \mathscr{A}'_r$ such that*

$$\mathbb{I}^- \psi^*(X) = 0 \quad \textit{where} \quad \psi = (I, \mathfrak{U}(f))$$

and

$$\begin{aligned} \|\mathfrak{U}(f) - I\|'_r &\le \frac{C}{\sigma} \|\mathbb{I}^- X\|_r \\ \|\psi^* X - \mathbb{E}X\|_r &\le C \|(\mathbb{I} - \mathbb{E})X\|_r. \end{aligned} \tag{1.31}$$

Moreover, $\mathfrak{U}(f) \colon \mathbb{R}^d \to \mathrm{SL}(2,\mathbb{R})$.

1.3.7 Proof of Theorem 2

We now prove Theorem 2.

1.3.7.1 Homotopy method

The coordinate transformation ψ will be determined by some U in

$$\mathscr{B}_\delta = \left\{ U \in \mathbb{I}^- \mathscr{A}_r' \colon \|U - I\|_r' < \delta \right\},$$

for

$$\delta = C\varepsilon/\sigma < 1. \tag{1.32}$$

Define the operator

$$\begin{aligned} \mathscr{F} \colon \mathscr{B}_\delta &\to \mathbb{I}^- \mathscr{A}_r \\ U &\mapsto \mathbb{I}^-(L_\omega U \cdot U^{-1} + \mathrm{Ad}_U f). \end{aligned} \tag{1.33}$$

If U is real-analytic, then $\mathscr{F}(U)$ is also real-analytic. The derivative of $\mathscr{F}$ at U is the linear map from $\mathbb{I}^- \mathscr{A}_r'$ to $\mathbb{I}^- \mathscr{A}_r$ given by

$$D\mathscr{F}(U)H = \mathbb{I}^-(L_\omega H - L_\omega U \cdot U^{-1} H - \mathrm{Ad}_U f \cdot H + H f) U^{-1}. \tag{1.34}$$

We want to find a solution of

$$\mathscr{F}(U_t) = (1-t)\mathscr{F}(U_0), \tag{1.35}$$

with $0 \le t \le 1$ and initial condition $U_0 = I$. Differentiating the above equation with respect to t, we get

$$D\mathscr{F}(U_t) \frac{dU_t}{dt} = -\mathscr{F}(I). \tag{1.36}$$

Proposition 6. *There is $\delta > 0$ such that if $U \in \mathscr{B}_\delta$, then $D\mathscr{F}(U)^{-1} \colon \mathbb{I}^- \mathscr{A}_r \to \mathbb{I}^- \mathscr{A}_r'$ is bounded and*

$$\|D\mathscr{F}(U)^{-1}\| < \delta/\varepsilon.$$

From the above proposition (to be proved in Section 1.3.7.2) we integrate (1.36) with respect to t, obtaining the integral equation:

$$U_t = I - \int_0^t D\mathscr{F}(U_s)^{-1} \mathscr{F}(I)\, ds. \tag{1.37}$$

In order to check that $U_t \in \mathscr{B}_\delta$ for any $0 \le t \le 1$, we estimate its norm:

$$\begin{aligned} \|U_t - I\|_r' &\le t \sup_{v \in \mathscr{B}_\delta} \|D\mathscr{F}(v)^{-1} \mathscr{F}(I)\|_r' \\ &\le t \sup_{v \in \mathscr{B}_\delta} \|D\mathscr{F}(v)^{-1}\| \, \|\mathbb{I}^- f\|_r < t\delta \|\mathbb{I}^- f\|_r/\varepsilon, \end{aligned} \tag{1.38}$$

so, $\|U_t - I\|'_r < \delta$. Therefore, the solution of (1.35) exists in $\mathscr{B}_\delta$ and is given by (1.37). Moreover, if X is real-analytic, then U_t takes real values for real arguments.

In view of

$$\mathbb{I}^+(\mathrm{Ad}_U\, f - u) = \mathbb{I}^+\left[(U-I)f(U^{-1}-I) + (U-I)\widetilde{f} + \widetilde{f}(U^{-1}-I) + \widetilde{f}\right], \tag{1.39}$$

where $\widetilde{f} = f - u$, we get

$$\begin{aligned}\|U_t^* X - \mathbb{E}X\|_r \leq & \|\mathbb{I}^+ L_\omega(U-I)\cdot(U^{-1}-I)\|_r + \|\mathbb{I}^+(\mathrm{Ad}_U\, f - u)\|_r + (1-t)\|\mathbb{I}^- f\|_r \\ \leq & 2\|\omega\|\,\|U\|_r\|U-I\|_r\|U-I\|'_r + 2\|U\|\,(\|u\| + \|\widetilde{f}\|)\|U-I\|_r^2 \\ & + \|\widetilde{f}\|_r(1+2\|U\|_r)\|U-I\|_r + \|\widetilde{f}\|_r + (1-t)\|\mathbb{I}^- f\|_r \\ \leq & (3-t)\|\widetilde{f}\|_r.\end{aligned} \tag{1.40}$$

Theorem 2 corresponds to the case $t = 1$.

1.3.7.2 Proof of Proposition 6

Lemma 1. *$D\mathscr{F}(I)^{-1} \colon \mathbb{I}^-\mathscr{A}_r \to \mathbb{I}^-\mathscr{A}'_r$ is bounded and*

$$\|D\mathscr{F}(I)^{-1}\| < \frac{5}{\sigma - 10\|(\mathbb{I}-\mathbb{E})f\|_r}. \tag{1.41}$$

Proof. Let $g = (\mathbb{I}-\mathbb{E})f$. From (1.34) one has

$$\begin{aligned} D\mathscr{F}(I)H &= \mathbb{I}^-(L_\omega + \mathrm{ad}_f)H \\ &= \left[\mathbb{I} + \mathbb{I}^-\,\mathrm{ad}_g\,(L_\omega + \mathrm{ad}_u)^{-1}\right](L_\omega + \mathrm{ad}_u)H,\end{aligned} \tag{1.42}$$

where $\mathrm{ad}_b A = Ab - bA$. Thus, the inverse of this operator, if it exists, is given by

$$D\mathscr{F}(I)^{-1} = (L_\omega + \mathrm{ad}_u)^{-1}\left[\mathbb{I} + \mathbb{I}^-\,\mathrm{ad}_g\,(L_\omega + \mathrm{ad}_u)^{-1}\right]^{-1}. \tag{1.43}$$

By looking at the spectral properties of the operator $(2\pi\mathrm{i}k\cdot\omega I + \mathrm{ad}_u)$, with the spectrum of ad_u being $\{0, \pm 4\pi\mathrm{i}\rho\}$, it is possible to write

$$(L_\omega + \mathrm{ad}_u)H(x) = \sum_{k\in I^-} S\Lambda_k S^{-1} H_k \mathrm{e}^{2\pi\mathrm{i}k\cdot x} \tag{1.44}$$

where

$$\Lambda_k = (2\pi\mathrm{i})\,\mathrm{diag}(k\cdot\omega, k\cdot\omega, k\cdot\omega + 2\rho, k\cdot\omega - 2\rho) \tag{1.45}$$

and

$$S = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & \mathrm{i} & -\mathrm{i} \\ -1 & 0 & \mathrm{i} & -\mathrm{i} \\ 0 & 1 & 1 & 1 \end{bmatrix}. \tag{1.46}$$

So, we have the linear map from $\mathbb{I}^-\mathscr{A}_r$ to $\mathbb{I}^-\mathscr{A}_r'$,

$$(L_\omega + \mathrm{ad}_u)^{-1} F(x) = \sum_{k\in I^-} S\Lambda_k^{-1} S^{-1} F_k \mathrm{e}^{2\pi\mathrm{i} k\cdot x}. \tag{1.47}$$

Now, for $k \in I^-$,

$$\begin{aligned} \|(L_\omega + \mathrm{ad}_u)^{-1} F\|_r' &\leq \frac{4}{2\pi} \sum_{k\in I^-} \frac{1+2\pi\|k\|}{|k\cdot\omega|} \|F_k\| \mathrm{e}^{r\|k\|} \\ &< \frac{5}{\sigma}\|F\|_r. \end{aligned} \tag{1.48}$$

It is possible to bound from above the norm of ad_g by $2\|g\|_r$. Therefore,

$$\|\mathbb{I}^- \mathrm{ad}_g (L_\omega + \mathrm{ad}_u)^{-1}\| < \frac{10}{\sigma}\|g\|_r < 1,$$

and

$$\left\| \left[\mathbb{I} + \mathbb{I}^- \mathrm{ad}_g (L_\omega + \mathrm{ad}_u)^{-1}\right]^{-1} \right\| < \frac{1}{1 - \frac{10}{\sigma}\|g\|_r}.$$

The statement of the lemma is now immediate.

As r is constant, in the following we drop it from our notations.

Lemma 2. *Given $U \in \mathscr{B}_\delta$, the linear operator $D\mathscr{F}(U) - D\mathscr{F}(I)$ mapping $\mathbb{I}^-\mathscr{A}_r'$ into $\mathbb{I}^-\mathscr{A}_r$, is bounded and*

$$\|D\mathscr{F}(U) - D\mathscr{F}(I)\| < 2\|U\| \left[\|\omega\|(1+2\|U\|) + 2\|f\|(1+\|U\|+\|U\|^2)\right] \|U - I\|. \tag{1.49}$$

Proof. In view of (1.34), we have

$$\begin{aligned} [D\mathscr{F}(U) - D\mathscr{F}(I)]\, H =& \mathbb{I}^- L_\omega H \cdot (U^{-1} - I) - L_\omega U \cdot U^{-1} H U^{-1} \\ &+ H f (U^{-1} - I) + fH - \mathrm{Ad}_U f \cdot HU^{-1}. \end{aligned} \tag{1.50}$$

It is possible to estimate the norms of the above terms by

$$\begin{aligned} \|L_\omega H \cdot (U^{-1} - I)\| &\leq \|\omega\|\,\|U^{-1} - I\|\|H\|', \\ \|L_\omega U \cdot U^{-1} H U^{-1}\| &\leq \|\omega\|\,\|U^{-1}\|^2\|U - I\|'\|H\|, \\ \|H f (U^{-1} - I)\| &\leq \|f\|\|U^{-1} - I\|\|H\|, \\ \|fH - \mathrm{Ad}_U f \cdot HU^{-1}\| &= \|fH(U^{-1} - I) + f(U^{-1} - I)HU^{-1} + (U^{-1} - I) f U^{-1} H U^{-1}\| \\ &\leq \|f\|\,(1 + \|U^{-1}\| + \|U^{-1}\|^2)\|U^{-1} - I\|\|H\|. \end{aligned} \tag{1.51}$$

Finally, notice that $\|U^{-1}-I\| \leq \|U^{-1}\|\,\|U-I\| \leq 2\|U\|\,\|U-I\|$.

Proposition 6 now follows from $\|U\| < 1+\delta$ and

$$\begin{aligned}\|D\mathscr{F}(U)^{-1}\| &\leq \left(\|D\mathscr{F}(I)^{-1}\|^{-1} - \|D\mathscr{F}(U) - D\mathscr{F}(I)\|\right)^{-1} \\ &< \left\{\sigma/5 - \varepsilon - 2\delta\|U\|\left[\|\omega\|(1+2\|U\|) + 2\|f\|(1+\|U\|+\|U\|^2)\right]\right\}^{-1} \\ &< \{\sigma/5 - \varepsilon - C\,\delta(\|\omega\|+\|f\|)\}^{-1}.\end{aligned} \tag{1.52}$$

Therefore, for δ and ε as in (1.32) and (1.30), respectively,

$$\|D\mathscr{F}(U)^{-1}\| < \frac{\delta}{\varepsilon}. \tag{1.53}$$

1.3.8 Rescaling

The rescaling that we are interested comes from the continued fractions expansion of ω. That is, we want to use skew diffeomorphisms of the type (T_n, I) where T_n are as in section 1.2. Futhermore, we rescale time by η_n.

Applying the rescaling to a vector field X with no I^- Fourier modes has the effect of improving its analyticity radius and thus C^ω-approximating X to a constant by a factor of order e^{-C/A_n}.

1.3.9 One-step renormalization operator

The renormalization step is briefly summarized below.

1. Let $m = \arg\min\{|k\cdot\omega + 2\rho| : k \in I^-\}$. So, $\|m\| \leq \frac{C|\rho|}{1-\sigma}$ and $|\rho'| = |\rho - \frac{1}{2}m\cdot\omega| \leq \sigma/4$.
2. Use ψ_m^* to obtain a vector field with rotation number ρ'. The C^ω-distance between the vector field and a constant will be increased by a factor $e^{C\|m\|}$.
3. Eliminate the modes in I^-.
4. Use the rescaling introduced in 1.3.8.

After one step, the vector field will get C^ω-closer to constant if the norm improvement by the rescaling overcomes the opposite effect by ψ_m^*. This indeed holds for ω Diophantine, using the bounds obtained at the end of section 1.2.

Notice that $\|m\|$ only depends on $|\rho|$ and σ. On the other hand, σ is chosen at each step according to the arithmetic properties of ω.

By iterating the renormalization step we are able to show convergence to a trivial limit set, namely a set of constant vector fields. That is, the renormalization contracts a small neighbourhood around that set. We remark that the diameter of that neighbourhood does not depend on the arithmetical properties of ρ, but only on $|\rho|$.

1.4 Conjugacy classes of torus translations

Consider the d-torus $\mathbb{T}^d$. We want to study flows on this manifold. Define the rotation vector of a flow ϕ^t at each $x \in \mathbb{T}^d$ to be the asymptotic direction of the corresponding orbit of the lift $\Phi^t(x)$ to the universal cover:

$$\operatorname{rot}(\phi)(x) = \lim_{t\to+\infty} \frac{\Phi^t(x) - x}{t}, \tag{1.54}$$

if the limit exists. If the rotation vector exists at x for a flow ϕ^t generated by a vector field X on $\mathbb{T}^d$ (i.e. $\frac{d}{dt}\phi^t = X \circ \phi^t$), it is the time average of the vector field along the orbit:

$$\operatorname{rot}(\phi)(x) = \lim_{t\to+\infty} \frac{1}{t} \int_0^t X \circ \phi^s(x) ds. \tag{1.55}$$

When the rotation vector exists for all $x \in \mathbb{T}^d$, the rotation set of ϕ is

$$\operatorname{rot}(\phi) = \{\operatorname{rot}(\phi)(x) \colon x \in \mathbb{T}^d\}. \tag{1.56}$$

Lemma 3 ([26]). *Let $h \in \mathrm{Homeo}_0(\mathbb{T}^d)$, $\lambda \neq 0$ and $T \in \mathrm{GL}(d,\mathbb{Z})$. If $\operatorname{rot}(\phi) \neq \emptyset$, then*

$$\operatorname{rot}(h^{-1} \circ \phi \circ h) = \operatorname{rot}(\phi) \quad \textit{and} \quad \operatorname{rot}(T^{-1} \circ \phi^{\lambda\cdot} \circ T) = \lambda T^{-1} \operatorname{rot}(\phi). \tag{1.57}$$

Proposition 7 ([26]). *Let ϕ^t be the flow generated by $X \in \mathrm{Vect}^0(\mathbb{T}^d)$ and $\omega \in \mathbb{R}^d$. If $\operatorname{rot}\phi = \{\omega\}$, then*

$$\|\mathbb{E}X - \omega\| \leq d\|X - \mathbb{E}X\|_{C^0}, \tag{1.58}$$

where $\mathbb{E}X = \int_{\mathbb{T}^d} X\, dm$ and m denotes the Lebesgue measure on $\mathbb{T}^d$.

We will be interested in vector fields generating flows that possess the same rotation vector for all orbits. Hence, for a vector field X we will write $\operatorname{rot} X$ to mean the unique rotation vector associated to the flow generated by X.

The C^ω-conjugacy classes of constant Diophantine vector fields can be described, at least locally, by the rotation vector.

Theorem 3 ([26]). *Let $\omega \in \mathbb{R}^d$ be Diophantine. If X is a real-analytic vector field on $\mathbb{T}^d$ sufficiently C^ω-close to constant with unique rotation vector ω, then there exists $h \in \mathrm{Diff}_0^\omega(\mathbb{T}^d)$ such that $h^*(X) = \omega$. The conjugacy h depends analytically on X.*

A proof of the above theorem is obtained by comparing the renormalization orbits of X and ω. They get close to each other exponentially fast, and from that we are able to construct an analytic conjugacy.

1.5 Invariant tori in phase space

Let $B \subset \mathbb{R}^d$, $d \geq 2$, be an open set containing the origin, and let H^0 be a real-analytic Hamiltonian function

$$H^0(x,y) = \omega \cdot y + \frac{1}{2} {}^\top yQy, \qquad (x,y) \in \mathbb{T}^d \times B, \tag{1.59}$$

with $\omega \in \mathbb{R}^d$ and a real symmetric $d \times d$ matrix Q. H^0 is said to be non-degenerate if $\det Q \neq 0$.

Theorem 4 ([13]). *Suppose H^0 is non-degenerate and ω is Diophantine. If H is a real-analytic Hamiltonian on $\mathbb{T}^d \times B$ sufficiently close to H^0, then the Hamiltonian flow of H leaves invariant a Lagrangian d-dim torus where it is analytically conjugated to the linear flow $\phi_t(x) = x + t\omega$ on $\mathbb{T}^d$, $t \geq 0$. The conjugacy depends analytically on H.*

Hamiltonian vector fields involve more complicated analysis than torus flows since there is extra dynamics on the action direction and we need to preserve the symplectic structure. Our goal is to find an analytic embedding $\mathbb{T}^d \to \mathbb{T}^d \times B$ that conjugates the Hamiltonian flow to the linear flow on the torus given by ω.

We do not work directly with vector fields, instead we renormalize Hamiltonian functions

$$H(x,y) = H^0(x,y) + F(x,y), \qquad (x,y) \in \mathbb{T}^d \times B$$

where F is a sufficiently small analytic perturbation. Using a rescaling of time we may assume that $\omega = (\alpha, 1)$. The perturbation F is decomposed in a Taylor-Fourier series

$$F(x,y) = \sum_{k,\nu} F_{k,\nu} y_1^{\nu_1} \dots y_d^{\nu_d} \mathrm{e}^{2\pi \mathrm{i} k \cdot x}$$

where the sum is taken over $k \in \mathbb{Z}^d$ and $\nu_i \in \mathbb{N} \cup \{0\}$. By the analyticity of F, its modes decay exponentially as $\|k\| \to +\infty$ for fixed ν.

Renormalization is an iterative scheme that at each step produces a new Hamiltonian. Suppose that after the $(n-1)$-th step the Hamiltonian is of the form

$$H_{n-1}(x,y) = \omega_{n-1} \cdot y + \frac{1}{2} {}^\top yQ_{n-1}y + F_{n-1}(x,y) \tag{1.60}$$

where Q_{n-1} is a symmetric matrix with non-zero determinant. Moreover, we assume that F_{n-1} only contains Taylor-Fourier modes in I^+_{n-1}, i.e. satisfying

$$|\omega_{n-1} \cdot k| \leq \sigma_{n-1} \|k\| \quad \text{or} \quad \|\nu\| \geq \tau_{n-1} \|k\|$$

for some $\sigma_{n-1}, \tau_{n-1} > 0$. So, the n-th step is defined by the following operations:

1. Apply a linear operator corresponding to an affine symplectic transformation given by
$$(x,y) \mapsto (T_n{}^{-1} x, {}^\top T_n y + b_n)$$

for some fixed vector b_n.
2. Rescale the action in order to "zoom in" around the invariant torus.
3. Rescale time (energy) to ensure that the frequency vector is of the form $\omega_n = (\alpha_n, 1)$.
4. Eliminate the (irrelevant) constant mode of the Hamiltonian.
5. Eliminate all the modes outside the resonant cone I_n^+ (thus avoiding dealing with small divisors) by a close to the identity symplectomorphism.

The first transformation above has a conjugate action

$$k \mapsto {}^{\top}T_n{}^{-1}k.$$

It follows from the hyperbolicity of T_n that this transformation contracts I_{n-1}^+ if σ_{n-1} and τ_{n-1}^{-1} are small enough. This significantly improves the analyticity domain in the x direction which implies the decrease of the estimates for the corresponding modes. As a result, all modes with $k \neq 0$ become smaller.

Besides the (trivial) case $(k, \nu) = (0, 0)$ which is dealt by operation (4) above, we control the size of the remaining $k = 0$ modes in different ways. The case

$$S := \sum_i \nu_i = 1$$

(corresponding to the linear term in the action y) is eliminated by a proper choice of the affine parameter b_n depending on Q_{n-1} and the perturbation. That is, b_n is used to eliminate an unstable direction related to frequency vectors. The quadratic term in the action ($S = 2$) is included in the new symmetric matrix Q_n which has again non-zero determinant and becomes smaller due to the action rescaling. Finally, we show that the action rescaling is also responsible for the decrease of the higher terms $S \geq 3$.

The overall consequence of the iterative scheme just described is that it converges to a limit set of Hamiltonians of the type

$$y \mapsto v \cdot y.$$

That is, the "limit" is a degenerate linear function of the action, and from that we show the existence of an ω-invariant torus for the initial Hamiltonian. To prove convergence we need to find proper choices of σ_n and τ_n as well as of stopping times t_n, which turns out to be possible for Diophantine ω. Roughly, too small values of σ_{n-1} and τ_{n-1}^{-1} make harder to eliminate modes as they are "too" resonant. On the other hand, large values imply that T_n does not contract I_{n-1}^+. Similarly, large $t_n - t_{n-1}$ improve the hyperbolicity of the matrices T_n but worsen the estimates on their norms and consequently enlarge the perturbation.

Acknowledgements Most of the work presented in this text was done in collaboration with K. Khanin and J. Marklof. The author was partially supported by Fundação para a Ciência e a Tecnologia through the Program POCI 2010.

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