Applied Probability Trust (9 January 2008)

EULER, LAMBERT, AND THE LAMBERT W FUNCTION TODAY

P. B. BRITO,* *UECE, ISEG, Technical University of Lisbon*

F. FABIÃO,** *CEMAPRE, ISEG, Technical University of Lisbon*

A. STAUBYN,*** *Matemática Aplicada/IISA, Instituto Superior de Agronomia, Technical University of Lisbon*

Abstract

The Lambert W function has found applications in an extraordinary number of scientific fields. In this paper we present a short historical review, a brief description of the function, and a survey of some of its applications.

Keywords: Lambert W function; delay differential equation

2000 Mathematics Subject Classification: Primary 33E20

Secondary 34K06

1. Introduction

On the celebration of the 300th anniversary of Leonhard Euler's birth, we offer an historical look at a well-known mathematical function.

The Lambert W function dates back to Johann Lambert (1728-1777) and Leonhard Euler (1707-1783). These two mathematicians developed a series solution for the trinomial equation, but left it unnamed. The series was christened Lambert W function two centuries later, when it was included in the algebraic system Maple.

The Lambert W function, represented by $W(z)$, is defined as the inverse of the

* Postal address: Rua Miguel Lupi 1249-078 Lisboa, Portugal. Email address: pbrito@iseg.utl.pt
** Postal address: Rua do Quelhas 1200-781 Lisboa, Portugal. Email address: mabiao@iseg.utl.pt
*** Postal address: 1349-017 Lisboa, Portugal. Email address: staubyn@isa.utl.pt

function $f(z) = ze^z$, satisfying

$$W(z)e^{W(z)} = z.$$

According to [17], the mathematical history of $W(z)$ begins in 1758 when Lambert solved the trinomial equation

$$x = q + x^m,$$

subsequently transformed by Euler into the form

$$x^\alpha - x^\beta = (\alpha - \beta)\, v x^{\alpha+\beta}.$$

After expansion in series, a new function, $T(z)$, known as *tree function,* was obtained as

$$T(z) = \sum_{n\geq 1} n^{n-1} \frac{z^n}{n!}. \tag{1}$$

According to [32], in 1844 Eisenstein noticed that the tree function, was a generating function that generates *rooted labelled trees,* and satisfies the functional equation

$$T(z) = ze^{T(z)}. \tag{2}$$

Thus, the tree function $T(z)$ is related to the W-Lambert function by $T(z) = -W(-z)$.

In [27, 41], the designation of *Cayley trees* is associated with the function that satisfies the relation (2). The *Cayley function*, $C(z)$, is introduced as the function satisfying the functional equation

$$C(z) = ze^{C(z)}, \tag{3}$$

which suggests that the tree function $T(z)$ is an example of a Cayley function, named after the mathematician Arthur Cayley who enumerated rooted labelled trees in the nineteenth century. Consequently, the two functions, W and T, have found relevant applications in combinatorics, particulary in enumerating trees.

It is also interesting to mention the reason for choosing the letter W for the Lambert function. The reason for the choice of the letter W to designate the Lambert function, when it was included in Maple, see [17], is not entirely clear. However, "... fortuitously, the letter W has additional significance because of the pioneering work on many aspects of W by Wright". Edward Maitland Wright [42, 43] made quite significant contributions to Delay Differential Equations (DDEs).

From 1949 [42] to 1959 [43], Wright obtained relevant results in the solution of $ze^z = a$, which is the characteristic equation associated with the DDE $x' = Bx\,(t-r)$. Assuming that $x(t) = Ce^{\lambda t}$ is a solution for some value of λ we obtain after substitution a transcendental equation $ze^z = a$ where $z = \lambda r$ and $a = rB$. As Wright states in [43], the contribution of Hayes in 1950 [30] was also undeniably important in the study of that transcendental equation. Today, a more appealing solution involving the operator W can be expressed by $z = W(a)$.

Furthermore, in 1973, Crowley, Fritsch and Shafer [22] presented an algorithm, the 443 algorithm, for solving the transcendental equation $we^w = x$. Recently, this algorithm has been superseded by algorithm 743 [5].

As mentioned in [15], the W function is sometimes known as the *omega function* because of several contributions where the Greek letter ω is used. Values of the Lambert W function can be found at `functions.wolfram.com`, and an informative poster is available at `www.orcca.on.ca/LambertW`.

2. The two real branches of the Lambert W function

The Lambert W function is a multivalued function displaying two real branches. The figure below represents the two real branches of $W(x)$. If x is real in the interval $-\frac{1}{e} < x < 0$, there are two real values for $W(x)$. The branch satisfying $W(x) \geq -1$, is defined as the *principal branch* of the W function and is denoted by $W(x)$ or $W_0(x)$. The branch satisfying $W(x) \leq -1$ is denoted by $W_{-1}(x)$. An explanation of this notation can be found in [17].

If x is real, we have the following result on the existence of real values for $W(x)e^{W(x)} = x$: if $x \geq 0$ then there is an unique real root which is positive except for $W(0) = 0$; if $-\frac{1}{e} < x < 0$ then there are two negative real roots, $W_0(x)$ and $W_{-1}(x)$; if $x = -\frac{1}{e}$ then there is only one negative real solution, $W_0(-\frac{1}{e}) = W_{-1}(-\frac{1}{e}) = -1$ and if $x < -\frac{1}{e}$ then there are no real solutions.

The main properties and calculus of W are given in [20, 21], in particular expressions

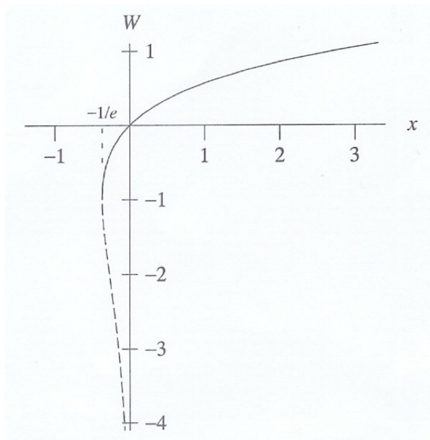

FIGURE 1: The two real branches of $W(x)$: $W_0(x)$ (—) and $W_{-1}(x)$ (- - -).

for its derivative, for integrals containing W, and asymptotic expansions of the complex branches of W.

We conclude this section noting that

$$W(x) = \sum_{n\geq 1} \frac{(-n)^{n-1}}{n!} x^n \tag{4}$$

is a convergent series for $|x| < \frac{1}{e}$.

3. Applications of W: a short survey

Knowledge on the Lambert W function, as a mathematical tool, has allowed the derivation of closed form solutions for models in numerous scientific disciplines, for which explicit or exact solutions were not known, and alternative iterative methods or approximate solutions had been used.

Mathematics, physics, biology, geology, engineering, and even risk theory, are some areas in which the usefulness of the Lambert W function has been proved. In physics, we point out the applications of W to optics [21], particle physics [6], general relativity [13], and geophysics [9]. In engineering, we may mention electronics [12, 37], and acoustics [11]. Other applications of W can be found in biochemistry [36], geology [35], risk theory [2], technological systems [25], and in information theory [40].

In the field of mathematics, the applications of the Lambert W function are also very rich. Corless, Gonnet, Hare and Jeffrey [15] highlight the study of real values of $W(x)$ and its manipulation in Maple.

Shagi-Di Shih [39] uses the inverse of the function xe^x to represent periodic orbits of two nonlinear relaxation oscillations systems. The main contribution of that paper consists in studying the singular behaviour of each function $W(-k, x)$ at the branch point $x = -e^{-1}$, where $W(-k, x)$ are denoted as `LambertW(k,x)` in Maple.

Corless et al. [16] have discussed the relationship that exists between the inverse of $y^\alpha e^y$ and the *Stirling numbers of the first and second kind*, respectively, $\left[{n \atop m}\right]$ and $\left\{{n \atop m}\right\}$. On this issue see [33], where a definition of the Stirling numbers based on the analysis of the transformation from a polynomial expressed in powers of k to a polynomial expressed in terms of binomial coefficients is presented. Stirling numbers are the coefficients involved in this transformation. In particular, Stirling numbers of the first kind are used to convert from binomial coefficients to powers,

$$n!\binom{x}{n} = \sum_k (-1)^{n-k} \left[{n \atop k}\right] x^k,$$

while Stirling numbers of the second kind are used to convert from powers to binomial coefficients,

$$x^n = \sum_k \left\{{n \atop k}\right\} \binom{x}{k} k!.$$

Following Knuth's notation [34], the Stirling numbers of the first kind, $\left[{n \atop k}\right]$, should be called Stirling cycle numbers, while Stirling numbers of the second kind, $\left\{{n \atop k}\right\}$, should be called Stirling subset numbers.

The *unwinding number* definition proposed by Corless, Hare and Jeffrey in [18], enables the establishment of a precise distinction between the different branches of the Lambert function. This contribution is fundamental in obtaining an asymptotic expansion for all branches of $W(z)$ and for all z [20], and has also led to a better solution for transcendental equations [19].

It is a curious coincidence that, two centuries ago, Euler and Lambert were studying the solution of particular algebraic equations, while, in 1998, the Lambert W function was again used in solving some transcendental equations, see [19].

More recently, the Lambert W function has been applied to Infinite Exponentials [28], and Graph Theory [26].

However, as noted earlier, it is in the field of time-delay dynamics that the modern Lambert W function has found a special area of application. Two of the most recent contributions to this field are [1, 38].

4. Delay differential equations and the Lambert W−function

Delay differential equations were introduced by Condorcet and Laplace in the eighteenth century. *Delay systems* are sometimes called *hereditary systems*, *retard equations*, or *differential-difference equations*. Literature surveys can be found in [23] and [4], and the classics [7, 24, 29].

During the eighteenth century, Condorcet and Laplace were studying delay differential equations while Euler and Lambert were considering certain other equations. Why did their results cross only two centuries later? One good reason is that the vehicle that joins the two fields is the characteristic equation associated with $x' = Bx(t-r)$. If we assume a solution of type $e^{\lambda t}$, then we have

$$\lambda - Be^{-\lambda r} = 0 \Leftrightarrow \lambda r e^{\lambda r} = rB \Leftrightarrow \lambda r = W\left(rB\right).$$

The solution of the DDE can then be expressed as

$$x(t) = Ce^{\frac{W(rB)}{r}t},$$

satisfying $x' = Bx(t-r)$.

If we introduce an initial function to obtain a unique solution, then the problem is less simple, as can be seen in [3]. The authors point out the advantage of using the Lambert function to derive a closed form solution to the linear DDE system.

Recently, the authors of [14] examined the solutions of the matrix equation $S\exp(S) = A$, which is the analogous to the previous equation $\lambda r e^{\lambda r} = rB$. The first simple equation that the authors studied, $y'(t) = Ay(t-1)$, is very useful in mathematical biology models because the delay logistic equation has this particular DDE form, after linearization at the equilibrium point [10]. Though efficient numerical computational routines are available, e.g. in `DynPac` or `dde23`, the authors were concerned with finding analytical solution methods by exploring the relationships between the matrix function $W_k(A)$ and the solutions of $S\exp(S) = A$.

Heffernan and Corless [31] have instead examined some computer algebra approaches for solving DDEs. The main methods discussed are the Method of Steps, the Laplace Transform method, and, more recently, the Least Squares method. It is interesting to note that this last method has been used in [3], where the authors use the Lambert W function concept to generalize the state transition matrix concept from Ordinary Differential Equations (ODEs) to DDEs.

A more specific work which relates the Lambert W function with the discussion of the roots of the characteristic equation is presented by Boivin and Zhong [8]. In Bellman and Cooke [7] we can find the use of expansions of the form

$$u(t) = \sum_r p_r(t) e^{s_r t}$$

for the solution $u(t)$ of

$$\begin{aligned} a_0 u'(t) + b_0 u(t) + b_1 u(t-w) &= 0 \quad t > w,\ a_0 \neq 0 \\ u(t) &= g(t), \quad 0 \leq t \leq w, \end{aligned}$$

where the sum is taken over all the characteristic roots s_r. The subject of the asymptotic location of the zeros of the characteristic function

$$h(s) = a_0 s + b_0 + b_1 e^{-ws},$$

has since been extensively studied. In [8] the authors study the characteristic roots of equation $y'(t) = ay(t-1)$ by using the Lambert W function, to obtain an asymptotic distribution of the roots that provide a stability result on the completeness of exponential systems.

Acknowledgements

We are grateful for the very helpful comments and suggestions by an anonymous referee. The first author acknowledges the *Fundação para a Ciência e Tecnologia* (FCT) for financial support, under the Multi-annual Funding Project (POCTI/0436/2003). The second author also acknowledges partial financial support from FCT, under program FEDER/POCI 2010.

References

[1] AMANN, A., JUST, W. AND SCHOLL, E. (2007). Some basic remarks on eigenmode expansions of time-delay dynamics. *Physica A* **373,** 191–202.

[2] AMBAGASPITIYA, R. S. AND BALAKRISHNAN, N. (1994). On the compound generalized poisson distributions. *Astin Bulletin* **24,** 255–263.

[3] ASL, F. M. AND ULSOY, A. G. (2003). Analysis of a system of linear delay differential equations. *Journal of Dynamic Systems, Measurement, and Control* **125,** 215–223.

[4] BAKER, C. T. H., PAUL, C. A. H. AND WILLÉ, D. R. (1995). A bibliography on the numerical solution of delay differential equations. *MCCM Numerical Analysis Report* **269,** 1–51.

[5] BARRY, D. A., BARRY, S. J. AND CULLIGAN-HENSLEY, P. J. (1995). Algorithm 743: Wapr: A Fortran routine for calculating real values of the W-function. *ACM Trans. on Math. Software* **21,** 172–181.

[6] BARTOLINI, R., BOTMAN, J. I. M., MIGLIORATI, M. AND THOMAS, C. (2002). Longitudinal beam distribution, in a storage ring with pure inductance described by the Lambert W function. In *Proceedings of EPAC.* pp. 1565–1567.

[7] BELLMAN, R. AND COOKE, K. L. (1963). *Differential Difference Equations.* Academic Press, New York.

[8] BOIVIN, A. AND ZHONG, H. (2007). Completeness of systems of complex exponentials and the Lambert W functions. *Trans. Amer. Math. Soc.* **359,** 1829–1849.

[9] BÖNI, P., BRAUN, A. AND BRIGGS, K. M. (2002). Analytical solution to Matthew's and Blakeslee's critical dislocation formation thickness of epitaxially grown thin films. *Journal of Crystal Growth* **241,** 231–234.

[10] BRAUER, F. AND CASTILLO-CHÁVEZ, C. (2001). *Mathematical Models in Population Biology and Epidemiology.* Springer, New York.

[11] CASEY, M. A. (2001). Reduced-rank spectra and minimum entropy priors for generalized sound recognition. In *Proceedings of the Workshop on Consistent and Reliable Cues for Sound Analysis.* EUROSPEECH.

[12] CHAPEAU-BLONDEAU, F. AND MONIR, A. (2002). Numerical evaluation of the Lambert W function and application to generation of generalized Gaussian noise with exponent 1/2. *IEEE Transactions on Signal Processing* **50,** 2160–2165.

[13] COLL, B. (1998). A universal law of gravitational deformation for general relativity. In *Proc. of the Spanish Relavistic Meeting.* EREs. Salamanca, Spain.

[14] CORLESS, R. M., DING, H., HIGHAM, N. J. AND JEFFREY, D. J. (2007). The solution of $S\exp(S) = A$ is not always the Lambert W function of A. In *Proceedings of ISSAC'07.*

[15] CORLESS, R. M., GONNET, G. H., HARE, D. E. G. AND JEFFREY, D. J. (1993). Lambert's W function in Maple. *Maple Tech. Newsletter* **9,** 12–22.

[16] CORLESS, R. M., GONNET, G. H., HARE, D. E. G. AND JEFFREY, D. J. (1995). Sur l'inversion de $y^\alpha e^y$ au moyen des nombres de Stirling associés. *Comptes Rendus de L'Académie des Sciences, Paris* **320,** 1449–1452.

[17] CORLESS, R. M., GONNET, G. H., HARE, D. E. G., JEFFREY, D. J. AND KNUTH, D. E. (1996). On the Lambert W function. *Advances in Computational Mathematics* **5,** 329–359.

[18] CORLESS, R. M., HARE, D. E. G. AND JEFFREY, D. J. (1996). Unwinding the branches of the Lambert W function. *The Mathematical Scientist* **21,** 1–7.

[19] CORLESS, R. M., HARE, D. E. G. AND JEFFREY, D. J. (1998). Exact rational solutions of a transcendental equation. *C. R. Math. Rep. Acad. Sci. Canada* **20,** 71–76.

[20] CORLESS, R. M., JEFFREY, D. J. AND KNUTH, D. E. (1997). A sequence of series for the Lambert W function. In *Proceedings of ISSAC'97.* pp. 197–204.

[21] CORLESS, R. M., JEFFREY, D. J. AND VALLURI, S. R. (2000). Some applications of the Lambert W function to physics. *Can. J. Phys.* **78,** 823–831.

[22] CROWLEY, W. P., FRITSCH, F. N. AND SHAFER, R. E. (1973). Algorithm 443: solution of the transcendental equation $we^w = x$. *Comm. ACM* **16,** 123–124.

[23] DAMBRINE, M., GOUAISBANT, F., PERRUQUETI, W. AND RICHARD, J. P. (2000). Systems with delays: an overview of some recent advances. *SACTA* **3,** 3–23.

[24] DIEKMANN, O., GILS, S. A. V., LUNEL, S. M. V. AND WALTHER, H. O. (1995). *Delay Equations: Functional, Complex, and Nonlinear Analysis.* Springer, New York.

[25] DITZEL, M. AND SERDIJN, W. A. (2001). Optimal energy assignment for frequency selective fading channels. *Ubicom-Publication* **24,** 1–5.

[26] FLAJOLET, P., SALVY, B. AND SCHAEFFER, G. (2004). Airy phenomena and analytic combinatorics of connected graphs. *The Electronic Journal of Combinatorics* **11,** 1–30.

[27] FLAJOLET, P. AND SEDGEWICK, R. (1996). *An Introduction to the Analysis of Algorithms.* Addison-Wesley.

[28] GALIDAKIS, I. N. (2004). On an application of Lambert's W function to infinite exponentials. *Complex Variables* **11,** 759–780.

[29] HALE, J. K. (1977). *Theory of Functional Differential Equations.* Springer.

[30] HAYES, N. D. (1950). Roots of the transcendental equation associated with a certain difference-differential equation. *J. London Math. Soc.* **25,** 226–232.

[31] HEFFERNAN, J. M. AND CORLESS, R. M. (2006). Solving some delay differential equations with computer algebra. *Mathematical Scientist* **31,** 21–34.

[32] JANSON, S., KNUTH, D., LUCZAK, T. AND PITTEL, B. (1993). The birth of the giant component. *Random Structures and Algorithms* **4,** 233–358.

[33] KNUTH, D. (1973). *The Art of Computer Programming. vol I. Fundamental Algorithms.* Addison-Wesley, Massachusetts.

[34] KNUTH, D. (1992). Two notes on notation. *Amer. Math. Monthly* **99,** 403–422.

[35] LIU, G. AND TAO, G. (2002). A new algorithm for borehole electromagnetic measurements employing Lambert W function. *Mathematical Geology* **34,** 529–542.

[36] MENDOZA, C. AND SCHNEL, S. (1997). Closed form solution for time-dependent enzyme kinetics. *J. Theor. Biol.* **187,** 207–212.

[37] MUCI, J., ORTIZ-CONDE, A. AND SANCHEZ, F. J. G. (2000). Exact analytical solutions of the forward non-ideal diode equation with series and shunt parasitic resistances. *Solid-State Electronics* **44,** 1861–1864.

[38] NELSON, P. W., YI, S. AND ULSOY, A. G. (2007). Analysis of systems of linear delay differential equations using the matrix Lambert function and the Laplace transformation. *Automatica* **forthcoming,**.

[39] SHIH, S. D. (2002). On periodic orbits of relaxation oscillations. *Taiwanese Journal of Mathematics* **6,** 205–234.

[40] SZPANKOWSKI, W. (1998). On asymptotics of certain recurrences arising in universal coding. *Problems of Information Transmission* **34,** 55–61.

[41] SZPANKOWSKI, W. (2001). *Average Case Analysis of Algorithms on Sequences.* Series in Discrete Mathematics and Optimization. Wiley-Interscience.

[42] WRIGHT, E. M. (1949). The linear difference-differential equation with constant coefficients. In *Proc. Royal Soc. Edinburgh.* vol. A62. pp. 387–393.

[43] WRIGHT, E. M. (1959). Solution of the equation $ze^z = a$. In *Proc. Royal Soc. Edinburgh.* vol. A65. pp. 193–203.