# Ruin Probabilities in the context of the Winner's Curse

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## 1 Introduction

The winner's curse is a tendency for the winning bid in an auction to exceed the intrinsic value or true worth of an item. The gap in auctioned versus intrinsic value can typically be attributed to incomplete information, emotions, or a variety of other subjective factors that may influence bidders.

As we mentioned before, the current research about the winner's curse in the insurance business is scarce, but there is a growing importance of this phenomenon in the insurance market nowadays.

It is probable that within the next 10 or 20 years most insurance policies will be sold entirely on online insurance comparison websites. These websites appeared on the internet a few years ago and can show quotes for many different insurers at the same time.

The ubiquity of these aggregator sites advertising and the ease of access to a huge range of premium quotes means that client behaviour is changing and most policyholders are tempted to check the competitiveness of their renewal quotes every year.

In this sense, selling insurance on those websites becomes a reverse auction process, where clients will look for the lowest prices to pay for their policies, and therefore the company which offer such prices *wins* the auction. However, it is likely that the winning company is the one that made the worst estimation of their premium and will therefore collect a small premium for the protection provided. In the long run, this can lead to insolvency, because the insurance company can quickly pick up large volumes of unprofitable and undesirable business.

The General Insurance Research Organization, some years ago, treated this problem so seriously that it set up a special group of actuaries to examine this topic and prepare a comprehensive report, which can be found in GIRO (2009).

In the context of the winner's curse, it is important to make realistic assumptions of asymmetry of information and experience in the market:

- Insurers have different possibilities to observe the market movements of each other (they can model the competitors' tariffs based on the information they can gather).

- Auctions have several turns.

- Consumers take into account other factors besides the price of the policy (like the reputation of the insurer).
- Different levels of competition and market share among the insurers.

To price properly the offered products on aggregators, the insurance company has to effectively estimate the possible loss. Here, estimating the loss is the most important problem to solve. One natural tool for this purpose comes from calculating VaR as in Palmowski (2017).

However, we are going to develop a new approach that relies on ruin probabilities in order to assess the losses and the evolution of surplus of the insurer in a winner's curse scenario.

## 1.1 Ruin probability

In ruin theory, there are several important models that have been addressed in the literature in the past years. One of them is the Sparre–Andersen risk model, which considers the insurer's surplus at a fixed time $t > 0$ as a function of three quantities: the amount of surplus at time 0, the amount of premium income received up to time $t$ and the amount paid out in claims up to time $t$:

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \tag{1.1}$$

where $U(t)$ is the insurer's surplus, $u$ is the surplus at time 0, $c$ is the insurer's rate of premium income per unit time, which is assumed to be received continuously, $\{N(t)\}_{t\geq 0}$ be a counting process for the number of claims and $\{X_i\}_{i=1}^{\infty}$ are the individual claim amounts, modeled as a sequence of independent and identically distributed random variables.

## 1.2 The Winner's Curse

In the context of the Winner's Curse, the premium income of a given insurance company becomes a random process: a counting process $\{M(t)\}_{t\geq 0}$ for the number of premiums received, with individual premium amounts $\{Y_j\}_{j=1}^{\infty}$, where each individual premium value is the result of a winning sale at the insurance aggregator website.

In other words, the $M(t)$ process can model insurance attempts (e.g. in some browsers / insurance comparison websites). In this case if $p$ is the probability of winning the auction (a person chooses our offer), then for each $Y_j$ we may have $P(Y_j > 0) = p$ and $P(Y_j = 0) = 1 - p$ (i.e. lost in the bidding and therefore there is no premium).

Therefore, the premium income on a surplus process up to time $t$ can be written as $\sum_{j=1}^{M(t)} Y_j$, and the resulting modified surplus process becomes

$$U(t) = u + \sum_{j=1}^{M(t)} Y_j - \sum_{i=1}^{N(t)} X_i, \tag{1.2}$$

Denote by $\{T_i\}_{i=1}^{\infty}$ the times of claim arrivals and by $\{Z_j\}_{j=1}^{\infty}$ the times of premium arrivals. Moreover, denote the interclaim times by $W_i = T_i - T_{i-1}, i \geq 2$, with $W_1 = T_1$. Likewise, denote the times between premium arrivals by $V_j = Z_j - Z_{j-1}, i \geq 2$, with $V_1 = Z_1$.

Assume that the interclaim times $W_i$ follow a distribution $K_1$ with density $k_1$, and the claim sizes $X_i$ follow a distribution $P_1$ with density $p_1$. In the same way, assume that the times between premium arrivals $V_j$ follow a distribution $K_2$ with density $k_2$, and the premium sizes $Y_j$ follow a distribution $P_2$ with density $p_2$.

A representation of the modified surplus process (1.2) is given in Figure 1.

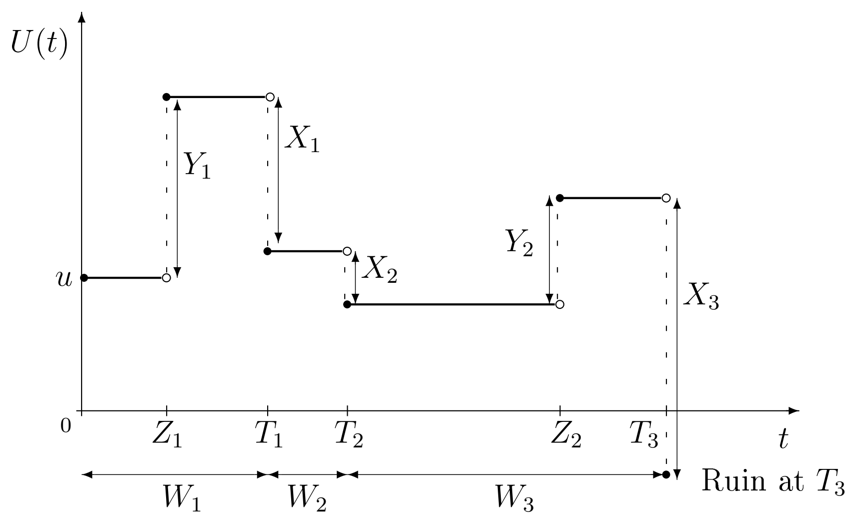


Figure 1: The modified surplus process.

In order to avoid certain ruin, we must impose a premium income condition of the form

$$\sum_{\text{premiums } Y_j \text{ in } W_i} E(Y_j) > E(X_i) \tag{1.3}$$

We define the ruin time $\tau = \inf\{t \geq 0 : U(t) < 0\}$, and the ruin probability as

$$\psi(u) = P(\tau < \infty \mid U(0) = u), u \geq 0. \tag{1.4}$$

**Theorem 1.1.** *The ruin probability* (1.4) *satisfies the following renewal equation*

$$\begin{aligned}\psi(u) &= P_{-1,2}\int_0^\infty \psi(u+y)p_2(y)dy + \\ &P_{1,-2}\left[\int_0^u \psi(u-x)p_1(x)dx + 1 - P_1(u)\right] + \\ &P_{1,2}\int_0^\infty \left[\int_0^{u+y} \psi(u+y-x)p_1(x)dx + 1 - P_1(u+y)\right] p_2(y)dy \end{aligned} \quad (1.5)$$

*where*

$$\begin{aligned} P_{-1,2} &= \int_0^\infty k_2(t)(1-K_1(t))dt \\ P_{1,-2} &= \int_0^\infty k_1(t)(1-K_2(t))dt \\ P_{1,2} &= \int_0^\infty k_1(t)k_2(t)dt \end{aligned}$$

*Proof.* By conditioning on the time and the amount of the first event, which can be a claim, a premium arrival or both claim and premium, we have the following cases to consider:

1. First premium income arrives before the first claim.

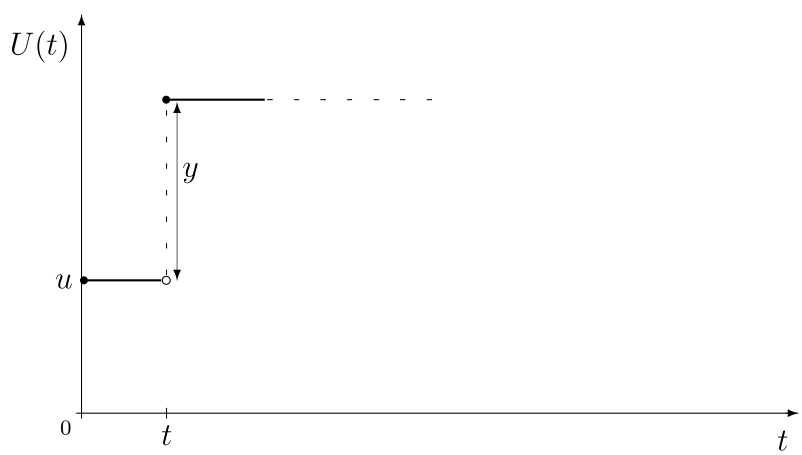


Figure 2: First premium before first claim.

$$I_1 = \int_0^\infty k_2(t)(1-K_1(t)) \int_0^\infty \psi(u+y)p_2(y)dydt \quad (1.6)$$

2. First claim arrives before the first premium income.

- Ruin does not occur

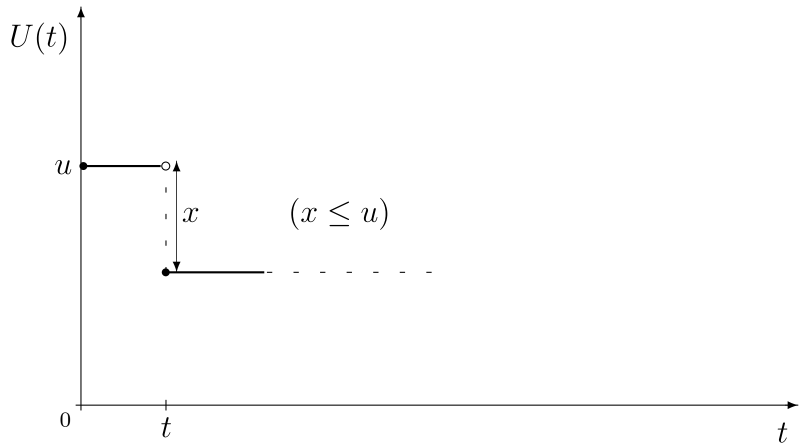


Figure 3: First claim before first premium and no ruin.

$$I_{2.1} = \int_0^\infty k_1(t)(1 - K_2(t)) \int_0^u \psi(u - x) p_1(x) dx dt \tag{1.7}$$

- Ruin occurs

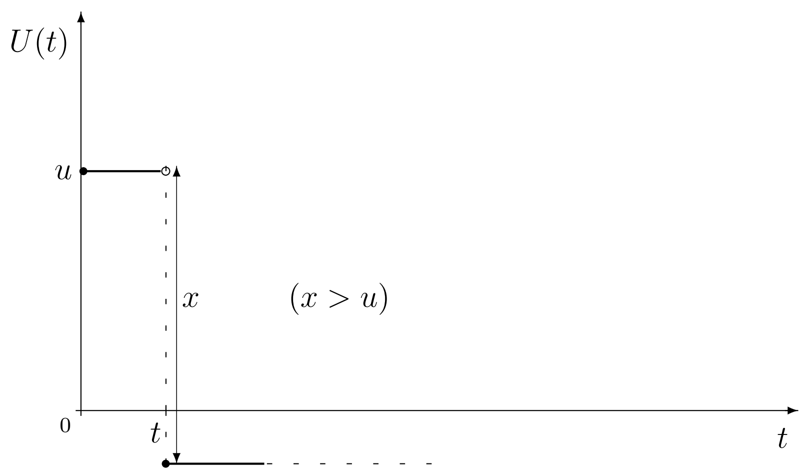


Figure 4: First claim before first premium and ruin.

$$I_{2.2} = \int_0^\infty k_1(t)(1 - K_2(t))(1 - P_1(u)) dx dt \tag{1.8}$$

3. First premium income and claim occurs simultaneously.

- Ruin does not occur

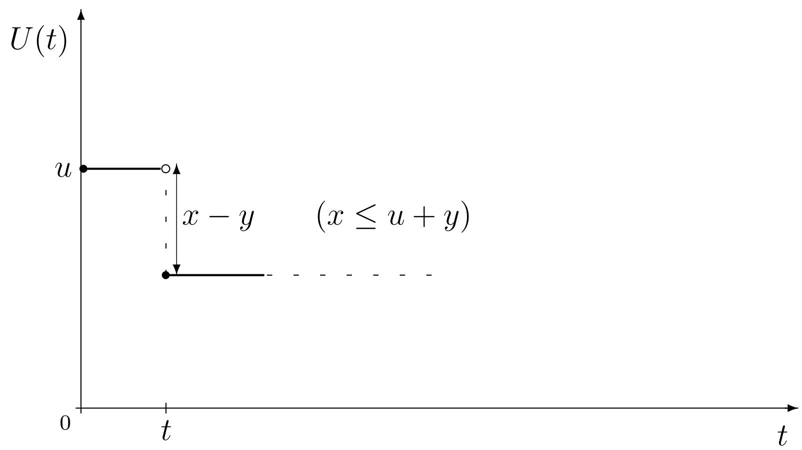


Figure 5: First claim and premium simultaneously and no ruin.

$$I_{3.1} = \int_0^\infty k_1(t)k_2(t) \int_0^\infty \int_0^{u+y} \psi(u+y-x)p_1(x)p_2(y)dxdydt \tag{1.9}$$

- Ruin occurs

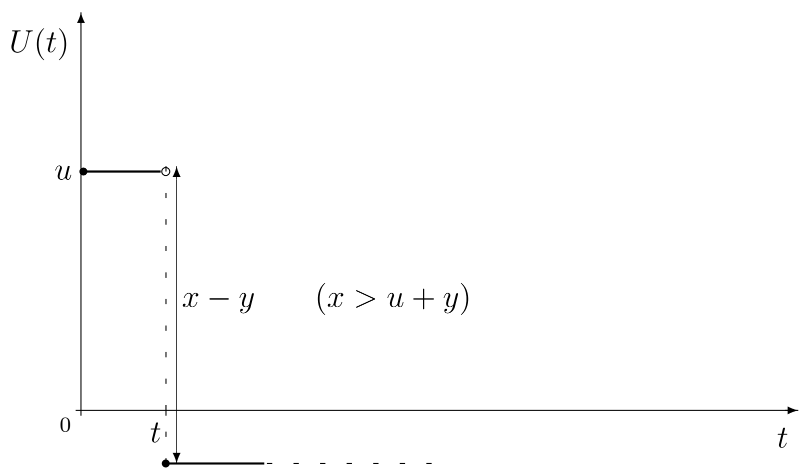


Figure 6: First claim and premium simultaneously and ruin.

$$I_{3.2} = \int_0^\infty k_1(t)k_2(t) \int_0^\infty (1-P_1(u+y))p_2(y)dydt \tag{1.10}$$

Finally, from (1.6), (1.7), (1.8), (1.9) and (1.10), we have

$$\psi(u) = I_1 + I_{2.1} + I_{2.2} + I_{3.1} + I_{3.2} \tag{1.11}$$

Since, on each case, the integrals with respect to time and amounts (claim or premium) are independent, we can do some rearrangement and renaming of terms, to obtain (1.5).

$\square$

**Example 1.1.** *Assume the especial case when both the distributions of the claims amounts and the premium amounts are exponential:*

$$\text{Claims}: \quad P_1(x) = 1 - e^{-\beta_1 x}, \quad p_1(x) = \beta_1 e^{-\beta_1 x} \tag{1.12}$$
$$\text{Premiums}: \quad P_2(x) = 1 - e^{-\beta_2 x}, \quad p_2(x) = \beta_2 e^{-\beta_2 x} \tag{1.13}$$

**Theorem 1.2.** *Under the conditions of the example 1.1, the ruin probability satisfies the following integro-differential equation*

$$\frac{d}{du}\psi(u)+(\beta_1-\beta_1 P_{1,-2}+\beta_2 P_{-1,2})\psi(u) = (\beta_2^2 P_{-1,2}+\beta_1\beta_2 P_{1,2}+\beta_1\beta_2 P_{-1,2})\int_u^\infty \psi(y)e^{-\beta_2(y-u)}dy \tag{1.14}$$

*Proof.* The proof follows by taking a derivative of $\psi(u)$ with respect to $u$ using the expression (1.5) and rearranging terms. $\square$

We can denote

$$\begin{aligned} A &= \beta_1 - \beta_1 P_{1,-2} + \beta_2 P_{-1,2} \\ B &= \beta_2^2 P_{-1,2} + \beta_1\beta_2 P_{1,2} + \beta_1\beta_2 P_{-1,2} = (\beta_2 P_{-1,2} + \beta_1 P_{1,2} + \beta_1 P_{-1,2})\beta_2 = \widetilde{B}\beta_2 \\ W(u) &= \int_u^\infty \psi(y)\beta_2 e^{-\beta_2(y-u)}dy \end{aligned}$$

Then equation (1.14) becomes

$$\frac{d}{du}\psi(u) + A\psi(u) = \widetilde{B}W(u) \tag{1.15}$$

Our objective now is to find and expression for $\psi$. Since we do not know the value of $\psi$ at any particular value of $u$, then there are no known boundary conditions for the integro-differential equation (1.15).

However, we could proceed by taking Laplace transforms on (1.15) to obtain $\psi(u)$.

**Theorem 1.3.** *The Laplace transform of $\psi(u)$ is given by*

$$\widehat{\psi}(s) = \frac{\psi(0)(s-\beta_2)}{s^2 - (\beta_2 - A)s - (A - \widetilde{B})\beta_2} \tag{1.16}$$

*Proof.* Taking Laplace transforms at both sides of (1.15) gives

$$s\widehat{\psi}(s) - \psi(0) + A\widehat{\psi}(s) = \widetilde{B}\widehat{W}(s)$$

In general we have that

$$\widehat{\left(\int_u^\infty \psi(y)p(y-u)dy\right)}(s) = \widehat{\psi}(s)\widehat{p}(-s)$$

Therefore,

$$\widehat{W}(s) = \widehat{\psi}(s)\frac{\beta_2}{\beta_2 - s}$$

Rearranging terms, the result follows. □

## 2 Zbyszek's notes/thoughts/references

Firstly, we agreed that wining the bid corresponds to insuring and loosing the bid corresponds to erasing the trial to insure. Hence the best counting premium process $M(t)$ is a Poisson process and bidding in this case corresponds to thinning property of the Poisson process. That is, with certain probability we accept a Poisson point which is equivalent to winning the bid and with complementing probability we erase a Poisson point which is equivalent to loosing the bid. The other model assumptions are made for convenience and in future we plan to analyse more complex models. Therefore we assume that $M$ is a Poisson process, $N$ is a renewal process and everything (claims, premiums, arrivals processes) is independent of each other and constructed on a common probability space. We assume that premiums and claims re i.i.d. random variables.

I start from my main doubt concerning above Theorem 1.1. $\psi(u)$ is a ruin probability when the first arrival epoch of premium and claim is the same like the ones appearing after that. Now, in the proof of Theorem 1.1, in (1.6), we put $\psi(u+y)$ on its rhs but this could be true only when remaining time to the nearest claim arrival will be the same like the others. Hence only when claim arrival process $N$ is Poisson process.

Let me first then focus on this case only. If $M$ and $N$ are both independent Poisson process than adding them will produce again Poisson process $\tilde{N}$ with intensity $\lambda_1 + \lambda_2$. In this case,

$$U(t) = u - \sum_{i=1}^{\tilde{N}(t)} \tilde{X}_i,$$

where

$$\tilde{X}_i = \frac{\lambda_1}{\lambda_1+\lambda_2}X_i - \frac{\lambda_2}{\lambda_1+\lambda_2}Y_i$$

and we probably end up with so-called risk process with two-sided jumps; see e.g. Li (2000) and references therein. In above mentioned paper, it is assumed that premiums are of the phase-type. The difference is that, in their model there is additional drift $ct$ in $U(t)$ for $c > 0$ and they have additional perturbation. Still, I believe that many arguments will remain unchanged. Similar arguments one can find in Labbe (2011).

For this special case, $U(t)$ is a difference of independent spectrally positive Lévy process ([premium compound process) and compound process (being a claim process) though. If we assume additionally that claims are of the phase-type then $U(t)$ is a special case of the process analyzed in Asmussen *et al.* (2004) - compare with eq. (12) there. In this case the ultimate ruin probability is known and its given in Lemma 1 and eq. (19) of Asmussen *et al.* (2004) (by taking $a \to 0$). See also Example on the top of page 88, where the case of exponential claims and premiums is handled and Mordecki (2001).

So to some extend, the case of two Poisson process $M$ and $N$ is slightly risk to focus on. It could be still great to handle it as an example. apart from it, I believe that Thm. 1.1 and equation appearing there is new hence even this single result seems to be publishable.

I suggest though to focus on the case when $N$ is a genera renewal process. I will follow very nice idea of Dong *et al.* (2013) and observe the risk process $U(t)$ only at epochs $T_i$ of claim arrivals. Since $M$ is still Poisson process, by lack of memory of the exponential interarrival times of $M$, the ruin probability for $U$ is the same as the one for

$$\overline{U}(t) = u + t - \sum_{i=1}^{\overline{N}(t)} X_i,$$

where $\overline{N}$ is a renewal process with $k$th interarrival time $\overline{W}_k = \sum_{l=M(T_{k-1})}^{M(T_k)} Y_l$. Indeed, we have

$$\begin{aligned} \psi(u) &= P\left(\exists t \geq 0:\ U(t) < 0\right) = P\left(\exists j \in \mathbb{N}:\ u + \sum_{l=1}^{M(T_j)} Y_l - \sum i = 1^j X_i < 0\right) \\ &= P\left(\exists t \geq 0:\ \overline{U}(t) < 0\right). \end{aligned}$$

In this way, we end up with the classical Sparre-Andersen risk model.

Our next step will be to put our problem into the st-up of Bergel *et al.* (2015) and Bergel *et al.* (2016). To do so, we will choose parameters of our starting model in such a way, that the generic $\overline{W}$ will be phase-type. Observe that the Laplace transform of $W$ equals

$$E[e^{-s\overline{W}}] = E[e^{-s\sum_{l=1}^{M(T_1)} Y_l}] = \int_0^\infty \sum_{m=0}^{\infty} \frac{(\lambda_2 t)^m}{m!} e^{-\lambda_2 t} (E[e^{-sY_1}])^m k_1(t)\, dt = \hat{k}_1(\lambda_2(1-\hat{p}_2(s))),$$

where $\hat{k}_1$ is the Laplace transform of the claim interarrival time $k_1$ and $\hat{p}_2$ is the Laplace transform of the generic premium.

Note that if $k_1$ and $p_2$ have rational Laplace transform so this is the same for $\overline{W}_1$. We will prove that if $k_1$ is a phase type and $p_2$ is a phase-type then $\overline{W}_1$ is a phase-type. We also prove that if $k_1$ and $p_2$ are exponential then $\overline{W}_1$ is a mixture of exponentials.

Indeed, ...

# References


H. Albrecher, C. Constantinescu and E. Thomann (2012). *Asymptotic results for renewal risk models with risky investments.* Stochastic Processes and their Applications, **122**(11), 3767-3789.

R. G. Alcoforado, A. I. Bergel, R. M. Cardoso, A. D. Egídio dos Reis and E. V. Rodríguez–Martínez (2021) *"Ruin and Dividend Measures in the Renewal Dual Risk Model", Methodology and Computing in Applied Probability*, doi: 10.1007/s11009-021-09876-4.

Soren Asmussen, Florin Avram, Martijn R. Pistorius (2004). Russian and American put options under exponential phase-type Lévy models. Stochastic Processes and their Applications 109, 79–111.

A. I. Bergel and Z. Palmowski (2021) *"Two-dimensional ruin problems for a renewal risk process with investments and proportional reinsurance: exact and asymptotic results"* (preprint).

A. I. Bergel, A. D. Egídio dos Reis and E. V. Rodríguez–Martínez (2017) *"On dividends in the Phase–Type dual risk model", Scandinavian Actuarial Journal*, **2017**(9), 761-784.

A. I. Bergel and A. D. Egídio dos Reis (2016) *"Ruin problems in the generalized Erlang(n) risk model", European Actuarial Journal*, **6**(1), 257-275.

A. I. Bergel and A. D. Egídio dos Reis (2015) *"Further developments in the Erlang(n) risk process", Scandinavian Actuarial Journal*, **2015**(1), 32-48.

Hua Dong, Zaiming Liu (2013). The ruin problem in a renewal risk model with two-sided jumps. Mathematical and Computer Modelling 57, 800–811.

GIRO Working Party (2009). *Winner's curse: the unmodelled impact of competition.* Report of the Winner's Curse GIRO Working Party.

Chantal Labbé, Hristo S. Sendov, Kristina P. Sendova (2011). The Gerber-Shiu function and the generalized Cramér-Lundberg model. Applied Mathematics and Computation 218, 3035–3056.

Mordecki, E.r (2001). Ruin probabilities for Lévy processes with mixed-exponential negative jumps. Theory Probab. Appl.

Zhimin Zhang , Hu Yang, and Shuanming Li. The perturbed compound Poisson risk model with two-sided jumps.

Z. Palmowski (2017). *A note on VaR for the winner's curse.* Economics of the 21st Century, **3**(15).

Z. Palmowski and I. Czarna (2011). *Ruin probability with Parisian delay for a spectrally negative Levy risk process.* Journal of Applied Probability, **48**(4), 984-1002.