Two-dimensional ruin problems for a renewal risk process with investments and proportional reinsurance: exact and asymptotic results

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Abstract

We study the joint ruin problem for two insurance companies that divide between them claims, premia and investments on a risky asset (stock). Modelling the risk of the insurance companies by renewal jump diffusion processes and the investment by a geometric Brownian motion, we investigate the asymptotic behavior of ruin probabilities. We consider semi-exponential and regularly varying claim sizes. We also assume that inter-arrival times are phase-type.
Keywords: Renewal risk model, ruin probability, two-dimensional risk process, Laplace transform, Tauberian theorem, jump-diffusion process, risky investment, geometric Brownian motion.

1 Introduction

In this paper we consider two-dimensional risk model which starts from the initial capital (u_1, u_2) and in which two companies split the amount they pay out of each claim in fixed proportions δ_1 and δ_2 ($\delta_1 + \delta_2 = 1$), and receive premiums at rates c_1 and c_2, respectively. Moreover, both of them may continuously invest their reserves into a risky asset with a price that follows a geometric Brownian motion with drift a and volatilities σ. This gives that surplus process of the portfolios is given by:

$$U_i(t) = u_i + c_i t + \delta_i \left(a \int_0^t U_i(s)ds + \sigma \int_0^t U_i(s)dB_s \right) - \delta_i S(t), \quad i = 1, 2, \tag{1.1}$$

where $S(t)$ is the aggregate claims process $\sum_{k=1}^{N(t)} = X_k$. The renewal process $N(t)$ represents the number of claims occurred up to time t. The claims X_k are and independent of the claim arrival times T_k. The claim inter-arrival times are denoted

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$W_k = T_k - T_{k-1}, k \geq 1$. The claim amounts follow a distribution with density f_X and c.d.f. F_X.

Remark 1.1. From (1.1) we can see that

$$dU_i(t) = c + \delta_i(aU_i(t)dt + \sigma U_i(t)dB_t) - \delta_i dS(t).$$

By Itô's formula:

$$U_i(t) = e^{\triangle(t)}\left[u + c\int_0^t e^{-\triangle(s)}ds - \delta_i\int_0^t e^{-\triangle(s)}dS(s)\right],$$

where $\triangle(t) = \delta_i((a - \frac{\sigma^2}{2})t + \sigma B_t)$. Thus

$$\int_0^t e^{-\triangle(s)}dS(s) = \sum_{k=1}^{N(t)} X_k e^{-\triangle(T_k)},$$

and therefore

$$U_i(t) = e^{\triangle(t)}\left[u + c\int_0^t e^{-\triangle(s)}ds - \delta_i\sum_{k=1}^{N(t)} X_k e^{-\triangle(T_k)}\right].$$

In terms of ruin problems, as it will be evident later, the ruin probabilities are the same $U_i(t)$ and $\frac{U_i(t)}{\delta_i}$. For the latter process $\delta_1 = \delta_2 = 1$ which will be assumed from now on.

We will also assume that the claim inter-arrival times W_k have a phase-type probability density function f_W that satisfies the following ordinary differential equation with constant coefficients (as in Albrecher *et al* (2012)):

$$\mathcal{L}\left(\frac{d}{dt}\right)f_W(t) = \sum_{j=0}^{n}\alpha_j\frac{d^j}{dt^j}f_W(t) = \prod_{i=1}^{n}\left(\frac{d}{dt} + \beta_i\right)f_W(t) = 0,$$

with homogeneous or nonhomogeneous conditions

$$\begin{aligned}
f_W^k(0) = 0 \ \ \text{(homogeneous)} \qquad & k = 0, \ldots, n-2,\\
f_W^{n-1}(0) = \alpha_0; \qquad & \\
\text{or} \qquad & \\
f_W^k(0) = M_k \ \ \text{(nonhomogeneous)} \qquad & k = 0, \ldots, n-2,\\
f_W^{n-1}(0) = \alpha_0. \qquad &
\end{aligned}$$

Let $\mathcal{L}^*$ denote the adjoint of $\mathcal{L}$ that describes f_W

$$\mathcal{L}^*\left(\frac{d}{dt}\right)f_W(t) = \sum_{j=0}^{n}(-1)^j\alpha_j\frac{d^j}{dt^j}f_W(t) = \prod_{i=1}^{n}\left(-\frac{d}{dt} + \beta_i\right)f_W(t).$$

Several ruin problems could be considered:

1. The first time τ_{or} when (at least) one insurance company is ruined, that is, the exit time of $(U_1(t), U_2(t))$ from the positive quadrant

$$\tau_{or}(u_1, u_2) := \inf\{t \geq 0 : U_1(t) < 0 \ \text{ or } \ U_2(t) < 0\}. \tag{1.2}$$

2. The first time τ_{sim} when the insurance companies experience simultaneous ruin, that is, the entrance time of $(U_1(t), U_2(t))$ into the negative quadrant

$$\tau_{sim}(u_1, u_2) := \inf\{t \geq 0 : U_1(t) < 0 \ \text{ and } \ U_2(t) < 0\}. \tag{1.3}$$

The associated ultimate/perpetual ruin probabilities will be respectively denoted by $\psi_{or}(u_1, u_2)$ and $\psi_{sim}(u_1, u_2)$

$$\begin{aligned} \psi(u_1, u_2) &= P(\tau_{or}(u_1, u_2) < \infty), \\ \psi_{sim}(u_1, u_2) &= P(\tau_{sim}(u_1, u_2) < \infty). \end{aligned}$$

Letting $\tau_i(u_i) = \inf\{t \geq 0 : U_i(t) < 0\}, \ i = 1, 2$, we also consider

$$\psi_{and}(u_1, u_2) = P(\tau_1(u_1) < \infty \ \text{ and } \ \tau_2(u_2)) < \infty). \tag{1.4}$$

Denoting $\psi_i(u_i) = P(\tau_i(u_i) < \infty)$, the ruin probability of U_i when $U_i(0) = u_i$, it clearly holds that

$$\psi_{sim}(u_1, u_2) \leq \psi_{and}(u_1, u_2) = \psi_1(u_1) + \psi_2(u_2) - \psi(u_1, u_2).$$

Therefore it is clear that the crucial is $\psi(u_1, u_2)$ on which we will focus from now on. In this paper we start from construction partial differential equation for ψ. Uisng Laplace transform method and properly formulated heavy-side principle we will derive the asymptotics pf $\psi(u, uv)$ as $u \to \infty$ for fixed proportion of initial capitals $v > 1$. We will consider semexponential regime of claim distributions and regularly-varying one.

The paper is organized as follows. In Section 2 we will construct differential equation for ψ. Later, in Section **??**, we derive equation for the laplace transform for $\phi(u)$ which allows to get above mentioned asymptotics (see Section **??**). We conclude our paper with the Section **??** concerning examples and numerical analysis.

2 Integro-differential equations

In this section we will obtain integro-differential equations satisfied by the ruin probabilities ψ.

2.1 Unperturbed Case

We start from the case when $\sigma = 0$, that is, we assume in this subsection that the two insurance companies do not invest on the risky asset. The surplus processes become then:

$$U_i(t) = u_i + c_i t - S(t), \quad i = 1, 2. \tag{2.1}$$

Considering the time and the amount of the first claim, we can obtain a renewal equation that is satisfied by ψ:

$$\psi(u_1, u_2) = \int_0^\infty f_W(t) \left[\int_0^\infty f_X(x) \psi(u_1 + c_1 t - x, u_2 + c_2 t - x) dx \right] dt, \tag{2.2}$$

which becomes

$$\begin{aligned} \psi(u_1, u_2) \quad = \quad & \int_0^\infty f_W(t) \left[\int_0^{\min\{u_1 + c_1 t, u_2 + c_2 t\}} f_X(x) \psi(u_1 + c_1 t - x, u_2 + c_2 t - x) dx \right. \\ & \left. + \int_{\min\{u_1 + c_1 t, u_2 + c_2 t\}}^\infty f_X(x) dx \right] dt. \end{aligned}$$

Assume from now on, without loss of generality, that the initial capital of the first company is smaller than the initial surplus of the second, but in contrast the first company charges higher premiums than the second. This is,

$$u_1 < u_2 \quad \text{and} \quad c_1 > c_2.$$

If no claim arrives the two surpluses meet at the time

$$T = \frac{u_2 - u_1}{c_1 - c_2};$$

see Figure 2.1. Note that

$$\min\{u_1 + c_1 t, u_2 + c_2 t\} = \begin{cases} u_1 + c_1 t & \text{if } t \leq T, \\ u_2 + c_2 t & \text{if } t > T. \end{cases}$$

This leads us to write the renewal equation for ψ in the following way:

$$\begin{aligned} \psi(u_1, u_2) \quad = \quad & \int_0^T f_W(t) \left[\int_0^{u_1 + c_1 t} f_X(x) \psi(u_1 + c_1 t - x, u_2 + c_2 t - x) dx \right. \\ & \left. + \int_{u_1 + c_1 t}^\infty f_X(x) dx \right] dt + \\ & \int_T^\infty f_W(t) \left[\int_0^{u_2 + c_2 t} f_X(x) \psi(u_1 + c_1 t - x, u_2 + c_2 t - x) dx \right. \\ & \left. + \int_{u_2 + c_2 t}^\infty f_X(x) dx \right] dt = I_1 + I_2. \end{aligned}$$

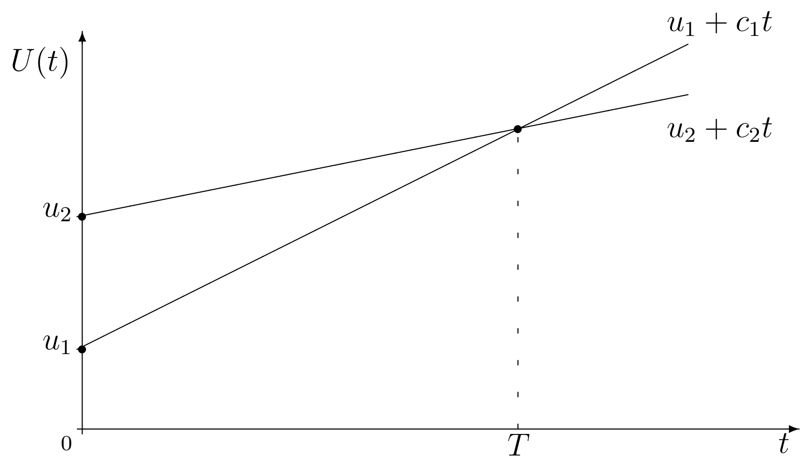

Figure 1: Crossing of boundary

We can perform the change of variables $s = u_1 + c_1 t$ in the first integral I_1 and the change $s = u_2 + c_2 t$ in the second I_2. This gives

$$\begin{aligned}
\psi(u_1, u_2) \quad = \quad & \frac{1}{c_1} \int_{u_1}^{u_1 + c_1 T} f_W\left(\frac{s - u_1}{c_1}\right) \left[\int_0^s f_X(x) \psi\left(s - x, \frac{c_2}{c_1} s - x + u_2 - \frac{c_2}{c_1} u_1\right) dx \right. \\
& \left. + \int_s^\infty f_X(x) dx \right] ds + \\
& \int_{u_2 + c_2 T}^\infty f_W\left(\frac{s - u_2}{c_2}\right) \left[\int_0^s f_X(x) \psi\left(\frac{c_1}{c_2} s - x + u_1 - \frac{c_1}{c_2} u_2, s - x\right) dx \right. \\
& \left. + \int_s^\infty f_X(x) dx \right] ds = I_1 + I_2,
\end{aligned}$$

where $u_1 + c_1 T = u_2 + c_2 T = \dfrac{u_2 c_1 - u_1 c_2}{c_1 - c_2}$.

Define the operator $\mathcal{A} = c_1 \dfrac{\partial}{\partial u_1} + c_2 \dfrac{\partial}{\partial u_2}$. We apply the operator $\mathcal{A}$ to the integrals I_1 and I_2:

$$
\begin{aligned}
\mathcal{A}\{I_1\} &= \mathcal{A}\left\{\frac{1}{c_1}\int_{u_1}^{u_1+c_1T} f_W\left(\frac{s-u_1}{c_1}\right)\left[\int_0^s f_X(x)\psi\left(s-x,\frac{c_2}{c_1}s-x+u_2-\frac{c_2}{c_1}u_1\right)dx\right.\right.\\
&\quad \left.\left.+\int_s^\infty f_X(x)dx\right]dt\right\}\\
&= -f_W(0)\left[\int_0^{u_1} f_X(x)\psi(u_1-x,u_2-x)dx+\int_{u_1}^\infty f_X(x)dx\right]\\
&\quad +\frac{1}{c_1}f_W(T)\left[\int_0^{u_1+c_1T} f_X(x)\psi(u_1+c_1T-x,u_2+c_2T-x)dx\right.\\
&\quad \left.+\int_{u_1+c_1T}^\infty f_X(x)dx\right](\underbrace{\mathcal{A}(u_1+c_1T)}_{=0})\\
&\quad +\frac{1}{c_1}\int_{u_1}^{u_1+c_1T}\mathcal{A}\left\{f_W\left(\frac{s-u_1}{c_1}\right)\right\}\left[\int_0^s f_X(x)\psi\left(s-x,\frac{c_2}{c_1}s-x+u_2-\frac{c_2}{c_1}u_1\right)dx\right.\\
&\quad \left.+\int_s^\infty f_X(x)dx\right]dt\\
&\quad +\frac{1}{c_1}\int_{u_1}^{u_1+c_1T} f_W\left(\frac{s-u_1}{c_1}\right)\left[\int_0^s f_X(x)\underbrace{\mathcal{A}\left\{\psi\left(s-x,\frac{c_2}{c_1}s-x+u_2-\frac{c_2}{c_1}u_1\right)\right\}}_{=0}dx\right]dt\\
&= -f_W(0)\left[\int_0^{u_1} f_X(x)\psi(u_1-x,u_2-x)dx+\int_{u_1}^\infty f_X(x)dx\right]\\
&\quad +\frac{1}{c_1}\int_{u_1}^{u_1+c_1T}\mathcal{A}\left\{f_W\left(\frac{s-u_1}{c_1}\right)\right\}\left[\int_0^s f_X(x)\psi\left(s-x,\frac{c_2}{c_1}s-x+u_2+\frac{c_2}{c_1}u_1\right)dx\right.\\
&\quad \left.+\int_s^\infty f_X(x)dx\right]dt.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathcal{A}\{I_2\} &= \frac{1}{c_2}\int_{u_2+c_2T}^\infty \mathcal{A}\left\{f_W\left(\frac{s-u_2}{c_2}\right)\right\}\left[\int_0^s f_X(x)\psi\left(\frac{c_1}{c_2}s-x+u_1+\frac{c_1}{c_2}u_2,s-x\right)dx\right.\\
&\quad \left.+\int_s^\infty f_X(x)dx\right]dt.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathcal{A}\{\psi(u_1,u_2)\} &= -f_W(0)\left[\int_0^{u_1} f_X(x)\psi(u_1-x,u_2-x)dx + \int_{u_1}^{\infty} f_X(x)dx\right] \\
&+\frac{1}{c_1}\int_{u_1}^{u_1+c_1T} \mathcal{A}\left\{f_W\left(\frac{s-u_1}{c_1}\right)\right\}\left[\int_0^s f_X(x)\psi\left(s-x,\frac{c_2}{c_1}s-x+u_2+\frac{c_2}{c_1}u_1\right)dx\right. \\
&\left.+\int_s^{\infty} f_X(x)dx\right]dt \\
&+\frac{1}{c_2}\int_{u_2+c_2T}^{\infty} \mathcal{A}\left\{f_W\left(\frac{s-u_2}{c_2}\right)\right\}\left[\int_0^s f_X(x)\psi\left(\frac{c_1}{c_2}s-x+u_1+\frac{c_1}{c_2}u_2,s-x\right)dx\right. \\
&\left.+\int_s^{\infty} f_X(x)dx\right]dt.
\end{aligned}
$$

Denoting $\mathcal{A}^j = \mathcal{A}\circ\ldots\circ\mathcal{A}$ we have,

$$
\begin{aligned}
\mathcal{A}^j\left\{f_W\left(\frac{s-u_1}{c_1}\right)\right\} &= (-1)^j f_W^{(j)}\left(\frac{s-u_1}{c_1}\right), \quad \mathcal{A}^j\left\{f_W\left(\frac{s-u_1}{c_1}\right)\right\}\Big|_{s=u_1} = (-1)^j f_W^{(j)}(0) \\
\mathcal{A}^j\left\{f_W\left(\frac{s-u_2}{c_2}\right)\right\} &= (-1)^j f_W^{(j)}\left(\frac{s-u_2}{c_2}\right), \quad \mathcal{A}^j\left\{f_W\left(\frac{s-u_2}{c_2}\right)\right\}\Big|_{s=u_2} = (-1)^j f_W^{(j)}(0).
\end{aligned}
$$

Using the adjoint to $\mathcal{A}$ operator $\mathcal{L}^*$ (in L^2 Hilbert space) we can observe that:

$$
\begin{aligned}
\mathcal{L}^*(\mathcal{A})\left\{f_W\left(\frac{s-u_1}{c_1}\right)\right\} &= \mathcal{L}^*\left(c_1\frac{\partial}{\partial u_1}\right)\left\{f_W\left(\frac{s-u_1}{c_1}\right)\right\} = 0, \\
\mathcal{L}^*(\mathcal{A})\left\{f_W\left(\frac{s-u_2}{c_2}\right)\right\} &= \mathcal{L}^*\left(c_2\frac{\partial}{\partial u_2}\right)\left\{f_W\left(\frac{s-u_2}{c_2}\right)\right\} = 0.
\end{aligned}
$$

Let

$$
\begin{aligned}
W(u_1,u_2) &= \int_0^{u_1} f_X(x)\psi(u_1-x,u_2-x)dx + \int_{u_1}^{\infty} f_X(x)dx \\
&= \int_0^{u_1} f_X(x)\psi(u_1-x,u_2-x)dx + \overline{F}_X(u_1) = \int_0^{\infty} f_X(x)\psi(u_1-x,u_2-x)dx.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathcal{A}^j\{W(u_1,u_2)\} &= \int_0^{\infty} f_X(x)\mathcal{A}^j\{\psi(u_1-x,u_2-x)\}dx \\
&= \int_0^{u_1} f_X(x)\mathcal{A}^j\{\psi(u_1-x,u_2-x)\}dx, \quad j\geq 1.
\end{aligned}
$$

The above allow us to write

$$
\begin{aligned}
\mathcal{A}^j\{\psi(u_1,u_2)\} &= \sum_{k=0}^{j-1}(-1)^{j-k} f_W^{j-1-k}(0)\mathcal{A}^k\{W(u_1,u_2)\} \\
&+\frac{1}{c_1}\int_{u_1}^{u_1+c_1T}\mathcal{A}^j\left\{f_W\left(\frac{s-u_1}{c_1}\right)\right\}\left[\int_0^s f_X(x)\psi\left(s-x,\frac{c_2}{c_1}s-x+u_2+\frac{c_2}{c_1}u_1\right)dx\right. \\
&+\left.\int_s^\infty f_X(x)dx\right]dt \\
&+\frac{1}{c_2}\int_{u_2+c_2T}^{\infty}\mathcal{A}^j\left\{f_W\left(\frac{s-u_2}{c_2}\right)\right\}\left[\int_0^s f_X(x)\psi\left(\frac{c_1}{c_2}s-x+u_1+\frac{c_1}{c_2}u_2,s-x\right)dx\right. \\
&+\left.\int_s^\infty f_X(x)dx\right]dt.
\end{aligned}
$$

Applying the operator $\mathcal{L}^*(\mathcal{A})$ to $\psi(u_1,u_2)$ we obtain

$$
\begin{aligned}
\mathcal{L}^*(\mathcal{A})\{\psi(u_1,u_2)\} &= \sum_{j=0}^{n}\alpha_j(-1)^j\mathcal{A}\{\psi(u_1,u_2)\} \\
&= \sum_{j=0}^{n}\alpha_j(-1)^j\left[\sum_{k=0}^{j-1}(-1)^{j-k}f_W^{j-1-k}(0)\mathcal{A}^k\{W(u_1,u_2)\}\right] \\
&+\frac{1}{c_1}\int_{u_1}^{u_1+c_1T}\underbrace{\mathcal{L}^*(\mathcal{A})\left\{f_W\left(\frac{s-u_1}{c_1}\right)\right\}}_{0}\left[\int_0^s f_X(x)\psi\left(s-x,\frac{c_2}{c_1}s-x+u_2+\frac{c_2}{c_1}u_1\right)dx\right. \\
&+\left.\int_s^\infty f_X(x)dx\right]dt \\
&+\frac{1}{c_2}\int_{u_2+c_2T}^{\infty}\underbrace{\mathcal{L}^*(\mathcal{A})\left\{f_W\left(\frac{s-u_2}{c_2}\right)\right\}}_{=0}\left[\int_0^s f_X(x)\psi\left(\frac{c_1}{c_2}s-x+u_1+\frac{c_1}{c_2}u_2,s-x\right)dx\right. \\
&+\left.\int_s^\infty f_X(x)dx\right]dt \\
&= \sum_{k=0}^{n-1}\left(\sum_{j=k+1}^{n}\widetilde{\alpha}_j(-1)^{j-k}f_W^{j-1-k}(0)\right)\mathcal{A}^k\{W(u_1,u_2)\} \\
&= \sum_{k=0}^{n-1}\left(\sum_{j=k+1}^{n}\widetilde{\alpha}_j(-1)^{j-k}f_W^{j-1-k}(0)\right)\int_0^\infty f_X(x)\mathcal{A}^k\{\psi(u_1-x,u_2-x)\}dx \\
&= \sum_{k=0}^{n-1}\left(\sum_{j=k+1}^{n}\widetilde{\alpha}_j(-1)^{j-k}f_W^{j-1-k}(0)\right)\int_0^{u_1} f_X(x)\mathcal{A}^k\{\psi(u_1-x,u_2-x)\}dx \\
&+\left(\sum_{j=1}^{n}\widetilde{\alpha}_j(-1)^j f_W^{j-1}(0)\right)\overline{F}_X(u_1)
\end{aligned}
$$

with $\widetilde{\alpha}_j = \alpha_j(-1)^j$. In summary, we can see that $\psi(u_1, u_2)$ satisfies the following integro-differential equation

$$\mathcal{L}^*(\mathcal{A})\{\psi(u_1,u_2)\} = \mathcal{Q}(\mathcal{A})\left\{\int_0^\infty \psi(u_1-x,u_2-x)f_X(x)dx\right\}, \tag{2.3}$$

where $\mathcal{Q}(x) = \sum_{k=0}^{n-1} Q_k x^k$ for

$$Q_k = \sum_{j=k+1}^{n} \widetilde{\alpha}_j(-1)^{j-k} f_W^{j-k-1}(0).$$

2.2 Perturbed Case

We assume in this subsection that the two insurance companies invest on the risky asset, that is, $\sigma > 0$. The surplus are given in (1.1).

Define the infinitesimal generator of three-dimensional Markov risk process $)(U_1(t), U_2(t), Z(t))$:

$$\overline{\mathcal{A}} := \frac{\partial}{\partial t} + \sum_{i=1}^{2}\left((c_i + au_i)\frac{\partial}{\partial u_i} + \frac{\sigma^2}{2}u_i^2\frac{\partial^2}{\partial u_i^2}\right),$$

where $Z(t)$ is a age inter-arrival time measuring how how much time has passsed from the last claim arrival.

We will assume from now that the claim sizes have continuous density. Then following straightforward arguments we can show that the probability of ruin $\psi(u_1, u_2)$ is in the domain $\mathcal{D}(\overline{\mathcal{A}})$ and as the exit probability it is harmonic. This means that it satisfies the following integro-differential equation:

$$\overline{\mathcal{A}}\psi(u_1,u_2) = 0.$$

After applying adjoin operator we will end up at the generalization of (2.3):

$$\mathcal{L}^*(\mathcal{A})\{\psi(u_1,u_2)\} = \mathcal{Q}(\mathcal{A})\left\{\int_0^\infty \psi(u_1-x,u_2-x)f_X(x)dx\right\} \tag{2.4}$$

for

$$\mathcal{A} := \sum_{i=1}^{2}\left((c_i + au_i)\frac{\partial}{\partial u_i} + \frac{\sigma^2}{2}u_i^2\frac{\partial^2}{\partial u_i^2}\right).$$

Above equation could be rewritten in the following way:

$$\begin{aligned}
\prod_{i=1}^{n}(-\mathcal{A}+\beta_i)\psi(u_1,u_2) &= \mathcal{Q}(\mathcal{A})\int_0^\infty \psi(u_1-x,u_2-x)f_X(x)dx, \\
&= \sum_{k=0}^{n-1} Q_k \int_0^{u_1} \frac{\partial}{\partial u_1}\psi(u_1-x,u_2-x)f_X(x)dx + \\
&\quad Q_0\int_{u_1}^\infty f_X(x)dx. \qquad (2.5)
\end{aligned}$$

We have also the following boundary condition:

$$\lim_{u_1\to\infty} \psi(u_1, u_2) = 0.$$

For homogeneous conditions we obtain

$$Q_k = \sum_{j=k+1}^{n} \alpha_j (-1)^{j-k} f_W^{j-k-1}(0) = 0,\ k = 1, \dots, n-1$$

and $Q_0 = (-1)^n f_W^{n-1}(0) = (-1)^n \alpha_0 = \prod_{i=1}^n (-\beta_i)$.

Therefore equation (2.5) becomes

$$\prod_{i=1}^{n}(-\mathcal{A}+\beta_i)\psi(u_1,u_2) = \prod_{i=1}^{n}(-\beta_i)\left(\int_0^{u_1} \psi(u_1-x, u_2-x) f_X(x)dx + \int_{u_1}^{\infty} f_X(x)dx\right)$$

which can be rewritten in the following way:

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k \alpha_k \mathcal{A}^k \psi(u_1,u_2) \quad &- \quad \prod_{i=1}^{n}(-\beta_i)\int_0^{u_1} \psi(u_1-x,u_2-x) f_X(x)dx \\ &= \quad \prod_{i=1}^{n}(-\beta_i)\overline{F}_X(u_1)\, dx \end{aligned} \tag{2.6}$$

with $\overline{F}_X(x) = 1 - F_X(x)$.

3 Laplace transform

To get asymptotics of the ruin probability $\psi(u_1, u_2)$ it is very natural assume that both insurance companies have comparable initial capital since they are connected with each other via proportional reinsurance (they divide premia and claims in fixed proportion). Therefore it is reasonable to assume that

$$u_2 = u_1 v \tag{3.1}$$

for fixed constant $v > 1$. Taking $u_1 = u$ the ruin probability $\psi(u_1, u_2)$ is a function

$$\phi(u) := \psi(u, vu). \tag{3.2}$$

We will choose at the beginning the specific proportion:

$$v = \frac{c_1}{c_2} > 1/$$

and will assume that $\sigma = 0$. Then

$$c_1\phi'(u) = \mathcal{A}\psi(u, vu).$$

We denote by $\hat{g}(s) = \int_0^\infty e^{-sx} g(x)dx$ the Laplace transform of general function g. Applying this Laplace transform to the integro-differential equation (2.6) and then we obtain:

$$\sum_{k=0}^{n}(-1)^k \alpha_k \widehat{\mathcal{A}^k \phi}(s) - \prod_{i=1}^{n}(-\beta_i) H(s) = \prod_{i=1}^{n}(-\beta_i)\widehat{\overline{F}}_X(s).$$

where $H(s)$ would be the partial Laplace transform of

$$\int_0^u \psi(u-x, vu-x) f_X(x)dx$$

and $\widehat{\mathcal{A}^k \phi}(s) = \widehat{\mathcal{A}}\widehat{\mathcal{A}^{k-1}\phi}(s)$ with

$$\widehat{\mathcal{A}}\widehat{g}(s) = c_1 s \widehat{g}(s).$$

References

Albrecher, H., Constantinescu, C. and Thomann, E. (2012). Asymptotic results for renewal risk models with risky investments. *Stochastic Processes and their Applications*, **122**(11), 3767-3789.

Avram, F., Palmowski, Z. and Pistorius, M. (2008a). Exit problem of a two dimensional risk process from the quadrant: exact and asymptotic results. *Annals of Applied Probability*, **18**(6), 2421-2449.

Avram, F., Palmowski, Z. and Pistorius, M. (2008b). A two-dimensional ruin problem on the positive quadrant. *Insurance: Mathematics and Economics*, **42**, 227-234.

Constantinescu, C. and Thomann, E. (2013). Martingales in renewal jump-difussion processes. *Annals of Applied Probability* (submitted).

Hu, Z. and Jiang, B. (2013). On joint ruin probabilities of a two-dimensional risk model with constant interest rate. *Journal of Applied Probability*, **50**, 309-322.

Jiang, T. and Yan, H. (2006). The finite-time ruin probability for the jump-diffusion model with constant interest force. *Acta Mathematicae Applicatae Sinica*, **22**, 171-176.

Tang, Q. and Yuan, Z. (2014). Randomly weighted sums of subexponential random variables with application to capital allocation. *Extremes*, **17**, 467-493.

Wang, W. and Zhang, Y. (2011). Ruin probabilities of a bidimensional risk model with investment. *Statistic and Probability Letters*, **82**, 130-138.